Essential University Physics

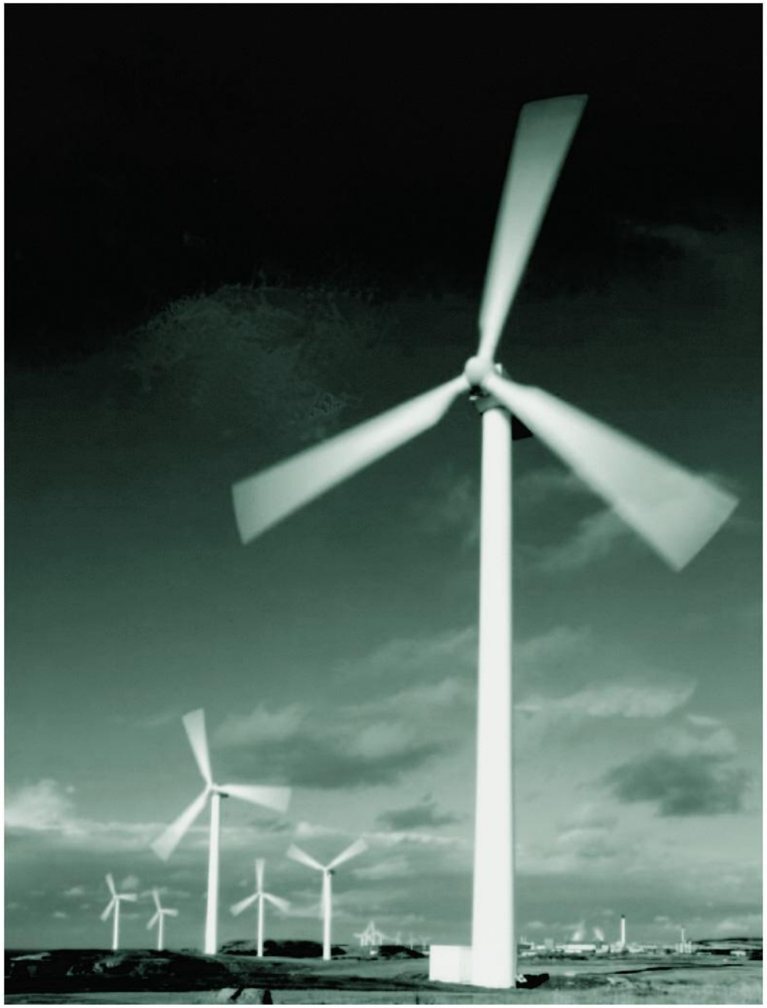
Richard Wolfson

2nd Edition

Rotational Motion

In this lecture you'll learn

- To describe the rotational motion of rigid bodies
  - You'll use an analogy between new quantities describing rotational motion and familiar quantities from one-dimensional linear motion
- To calculate the rotational inertias of objects made of discrete and continuous distributions of matter
  - Rotational inertia is the rotational analog of mass
- To handle problems involving both linear and rotational motion
- To describe rolling motion

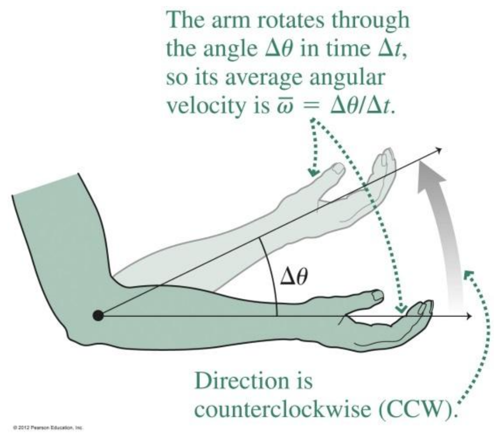


Angular Velocity

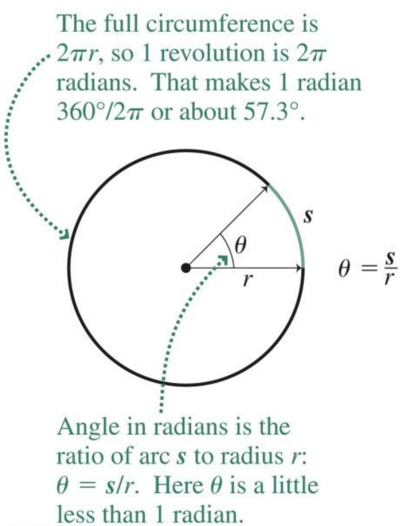
- Angular velocity ω is the rate of change of angular position.

$$\text{Average:} \quad \bar{\omega} = \frac{\Delta\theta}{\Delta t}$$

$$\text{Instantaneous:} \quad \omega = \frac{d\theta}{dt}$$

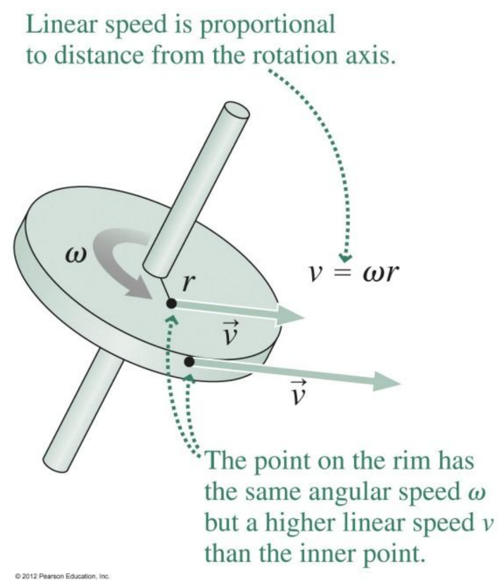


- Angular and linear velocity
  - The linear speed of a point on a rotating body is proportional to its distance from the rotation axis:



$$\omega = \frac{d\theta}{dt} = \frac{1}{r}\frac{ds}{dt} = \frac{v}{r}$$

$$v = \omega r$$



Angular Acceleration

- Angular acceleration α is the rate of change of angular velocity.

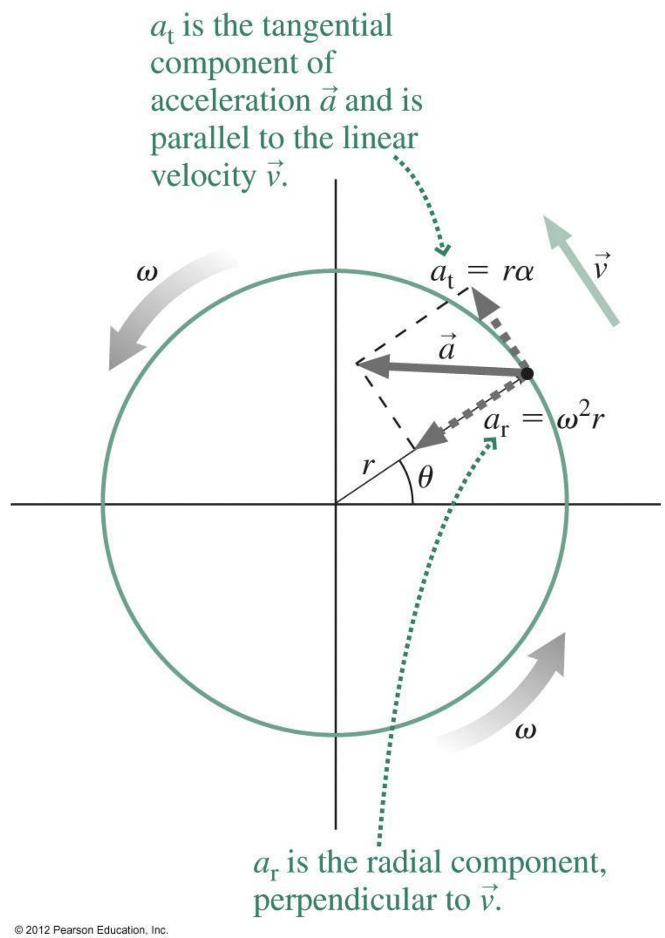
$$\text{Average:} \quad \bar{\alpha} = \frac{\Delta\omega}{\Delta t} \qquad \text{Instantaneous:} \quad \alpha = \frac{d\omega}{dt}$$

- Angular and tangential acceleration
  – The linear acceleration of a point on a rotating body is proportional to its distance from the rotation axis:

$$a_{t} = r\alpha$$

  – A point on a rotating object also has radial acceleration:

$$a_{r} = \frac{v^{2}}{r} = \omega^{2} r$$



Constant Angular Acceleration

- Problems with constant angular acceleration are exactly analogous to similar problems involving linear motion in one dimension.
  - The same equations apply, with the substitutions

$$x \to \theta, \quad v \to \omega, \quad a \to \alpha$$

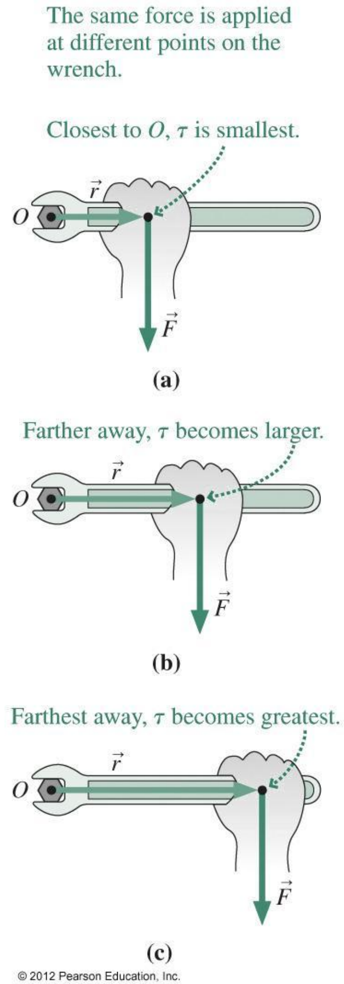
Table 10.1 Angular and Linear Position, Velocity, and Acceleration

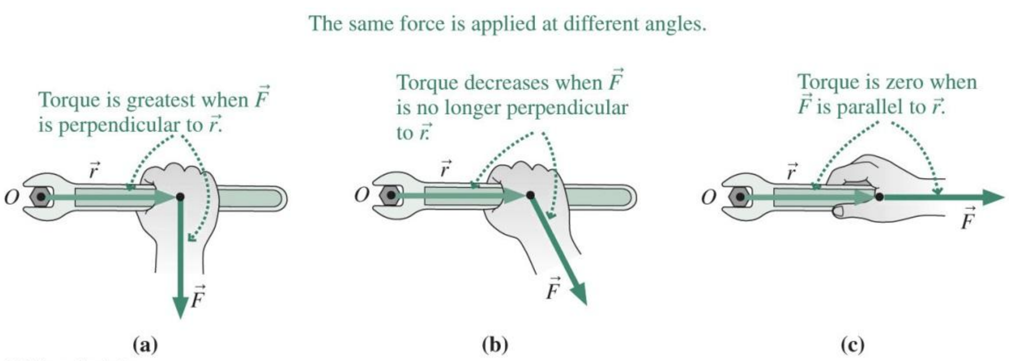
Linear Quantity	**Angular Quantity**
Position x	Angular position θ
Velocity $v = \dfrac{dx}{dt}$	Angular velocity $\omega = \dfrac{d\theta}{dt}$
Acceleration $a = \dfrac{dv}{dt} = \dfrac{d^2x}{dt^2}$	Angular acceleration $\alpha = \dfrac{d\omega}{dt} = \dfrac{d^2\theta}{dt^2}$

Equations for Constant Linear Acceleration		**Equations for Constant Angular Acceleration**	
$\bar{v} = \frac{1}{2}(v_0 + v)$	(2.8)	$\bar{\omega} = \frac{1}{2}(\omega_0 + \omega)$	(10.6)
$v = v_0 + at$	(2.7)	$\omega = \omega_0 + \alpha t$	(10.7)
$x = x_0 + v_0 t + \frac{1}{2}at^2$	(2.10)	$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$	(10.8)
$v^2 = v_0^2 + 2a(x - x_0)$	(2.11)	$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$	(10.9)

Torque

- Torque τ is the rotational analog of force, and results from the application of one or more forces.
- Torque is relative to a chosen rotation axis.
- Torque depends on:
  - the distance from the rotation axis to the force application point.
  - the magnitude of the force $\vec{F}$.
  - the orientation of the force relative to the displacement $\vec{r}$ from axis to force application point: $\tau = rF\sin\theta$

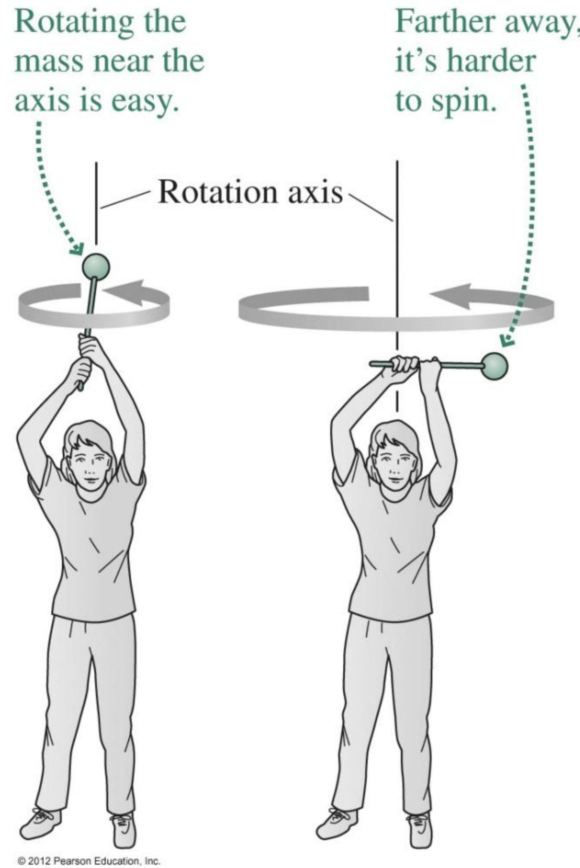




Rotational Inertia and the Analog of Newton's Law

- Rotational inertia I is the rotational analog of mass.
  - Rotational inertia depends on mass and its distance from the rotation axis.
- Rotational acceleration, torque, and rotational inertia combine to give the rotational analog of Newton's second law:

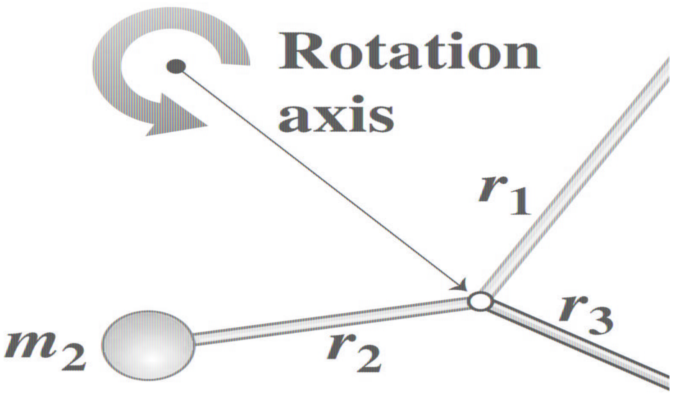
$$\tau = I\alpha$$



Finding Rotational Inertia

- For a single point mass *m*, rotational inertia is the product of mass with the square of the distance *R* from the rotation axis:

$$I = mR^2.$$

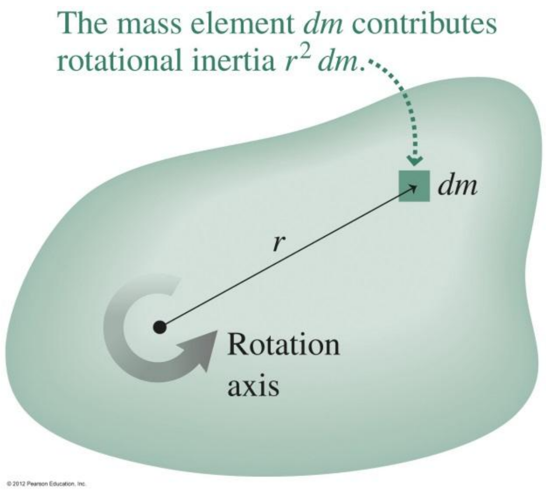


- For a system of discrete masses, the rotational inertia is the sum of the rotational inertias of the individual masses:

$$I = \sum m_i r_i^2$$

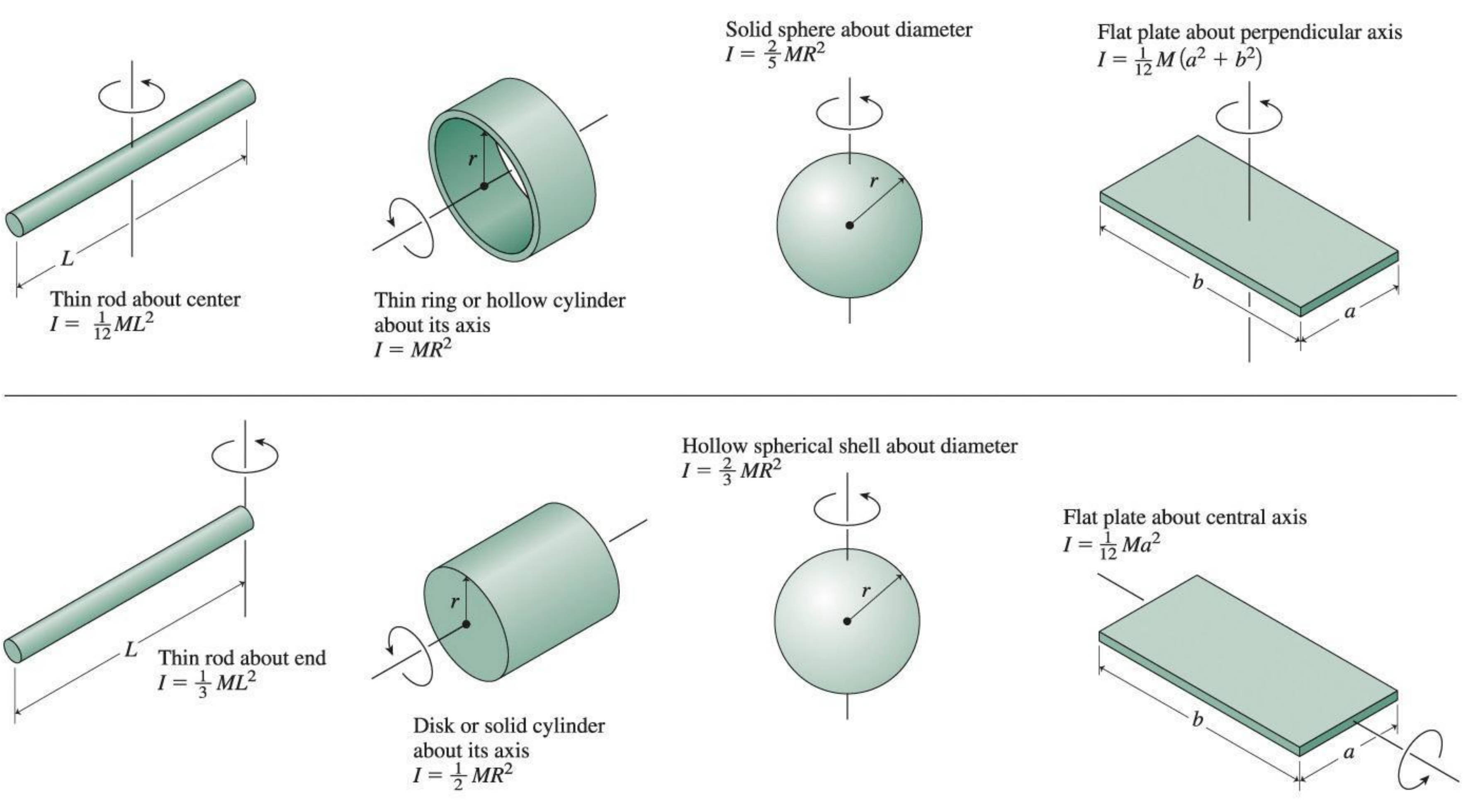
- For continuous matter, the rotational inertia is given by an integral over the distribution of matter:

$$I = \int r^2\, dm$$



Rotational Inertias of Simple Objects

Table 10.2 Rotational Inertias



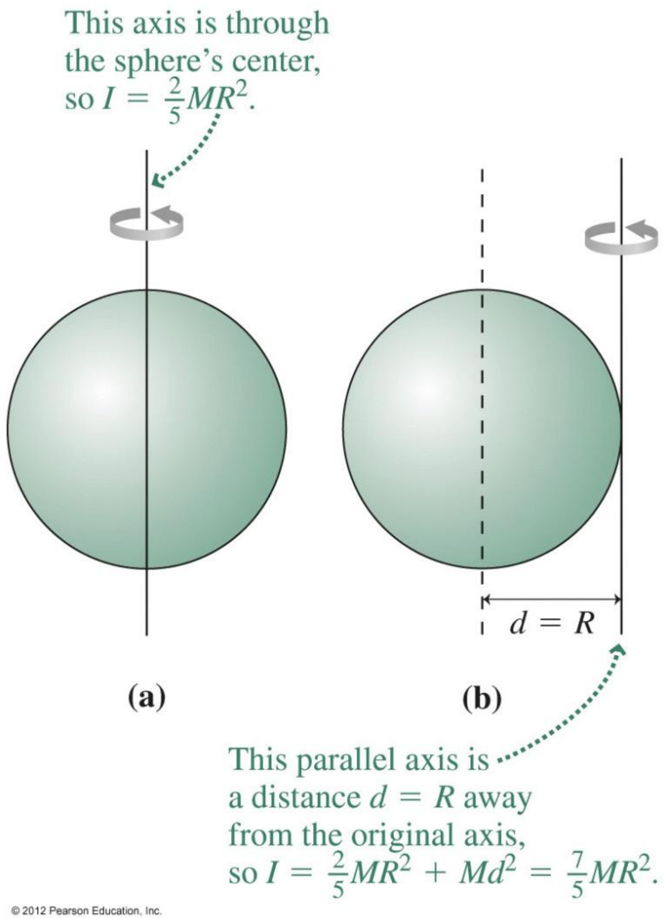
Parallel-Axis Theorem

- If we know the rotational inertia I_{cm} about an axis through the center of mass of a body, the **parallel-axis theorem** allows us to calculate the rotational inertia I through any parallel axis.

- The parallel-axis theorem states that

$$I = I_{\mathrm{cm}} + Md^2$$

where d is the distance from the center-of-mass axis to the parallel axis and M is the total mass of the object.

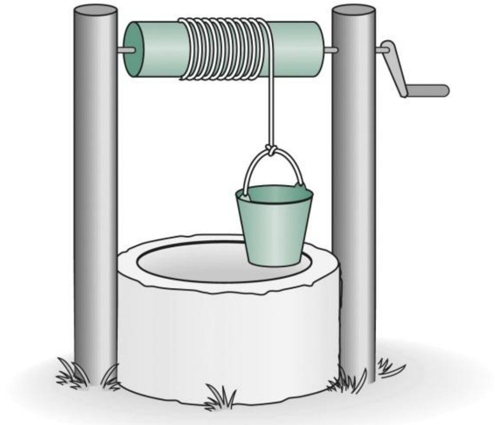


Combining Rotational and Linear Dynamics

- In problems involving both linear and rotational motion:
  - **IDENTIFY** the objects and forces or torques acting.
  - **DEVELOP** your solution with drawings and by writing Newton's law and its rotational analog. Note physical connections between the objects.
  - **EVALUATE** to find the solution.
  - **ASSESS** to be sure your answer makes sense.

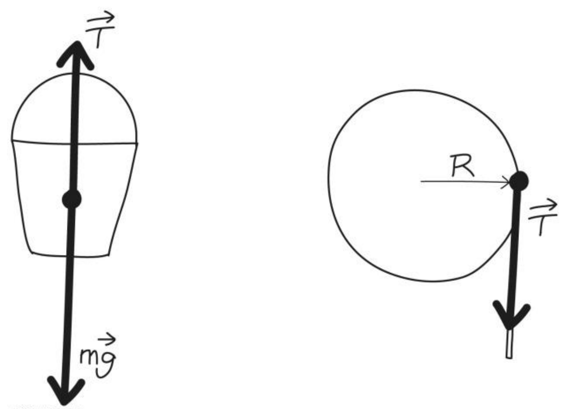
A bucket of mass *m* drops into a well, its rope unrolling from a cylinder of mass *M* and radius *R.*

What's its acceleration?



Free-body diagrams for bucket and cylinder

Rope tension $\vec{T}$ provides the connection



Newton's law, bucket:

$$F_{net} = mg - T = ma$$

Rotational analogy of Newton's law, cylinder:

$$RT = Ia/R$$

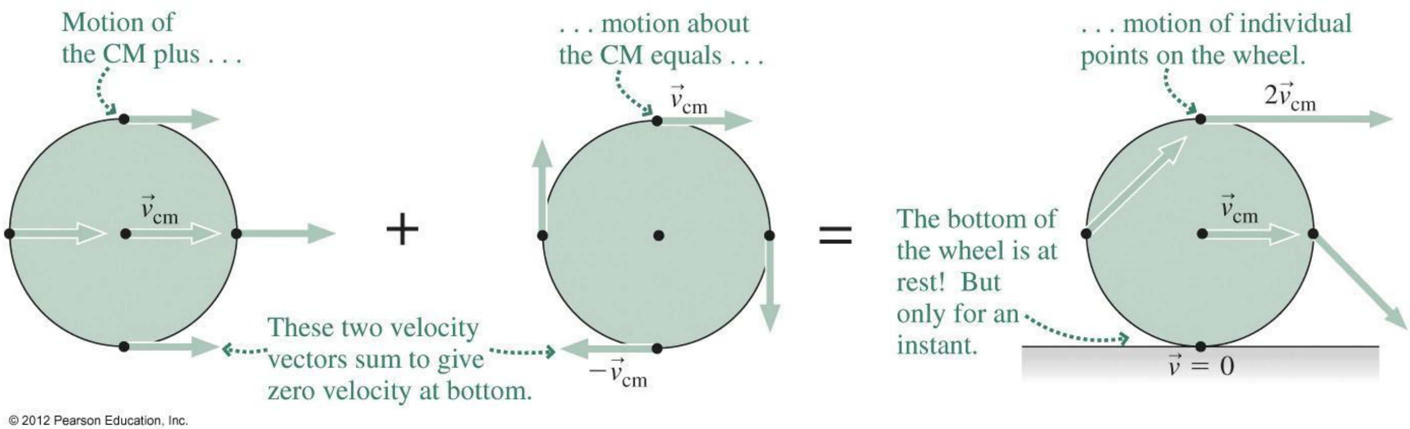
Here $I = \frac{1}{2}MR^2$

Solve the two equations to get

$$a = \frac{mg}{m + \frac{1}{2}M}$$

Rolling Motion

- Rolling motion combines translational (linear) motion and rotational motion.
  - The rolling object's center of mass undergoes translational motion.
  - The object itself rotates about the center of mass.
  - In true rolling motion, the object moves without slipping and its point of contact with the ground is instantaneously at rest.
  - Then the rotational speed ω and linear speed v are related by $v = \omega R$, where R is the object's radius.

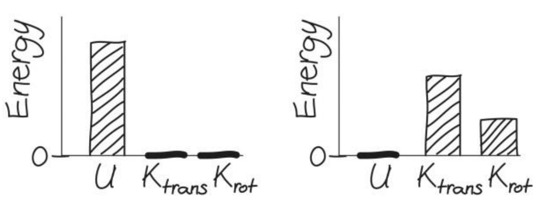


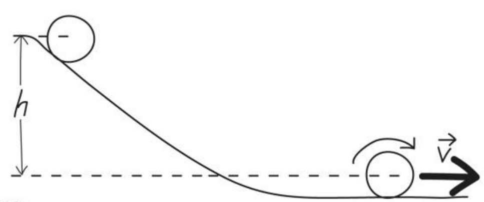
Rotational Energy

- A rotating object has kinetic energy $K_{\text{rot}} = \frac{1}{2} I\omega^2$ associated with its rotational motion alone.
  – It may also have translational kinetic energy: $K_{\text{trans}} = \frac{1}{2} Mv^2$.
- In problems involving energy conservation with rotating objects, both forms of kinetic energy must be considered.
  – For rolling objects, the two are related:
    - The relation depends on the rotational inertia.

Example: A solid ball rolls down a hill. How fast is it moving at the bottom?

Energy bar graphs





Equation for energy conservation

$$Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$$

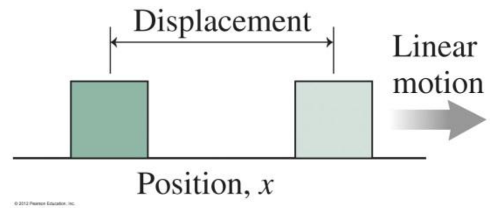
$$= \frac{1}{2}Mv^2 + \frac{1}{2}\left(\frac{2}{5}MR^2\right)\left(\frac{v}{R}\right)^2 = \frac{7}{10}Mv^2$$

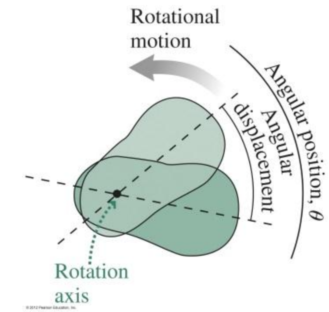
Solution:

$$v = \sqrt{\frac{10}{7}gh}$$

Summary

- Rotational motion in one dimension is exactly analogous to linear motion in one dimension.
  - Linear and angular motion:





  - Analogies between rotational and linear quantities:

Linear Quantity or Equation	Angular Quantity or Equation	Relation Between Linear and Angular Quantities
Position x	Angular position θ	
Speed $v = dx/dt$	Angular speed $\omega = d\theta/dt$	$v = \omega r$
Acceleration a	Angular acceleration α	$a_t = \alpha r$
Mass m	Rotational inertia I	$I = \int r^2\,dm$
Force F	Torque τ	$\tau = rF\sin\theta$
Kinetic energy $K_{\text{trans}} = \frac{1}{2}mv^2$	Kinetic energy $K_{\text{rot}} = \frac{1}{2}I\omega^2$	
Newton's second law (constant mass or rotational inertia):		
$F = ma$	$\tau = I\alpha$