2.3 The Adjoint of a Matrix

* Let A be $n\times n$ matrix. The adjoint of A is defined by

$$\operatorname{adj} A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}^T = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix},$$

where A_{ij} is the cofactor of a_{ij}.

Th* $A(\operatorname{adj} A) = |A|\, I$

Proof
$$A(\operatorname{adj} A) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix} \begin{bmatrix} A_{11} & A & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & & A_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{j=1}^{n} a_{1j} A_{1j} & \sum_{j=1}^{n} a_{1j} A_{2j} & \cdots & \sum_{j=1}^{n} a_{1j} A_{nj} \\ \sum_{j=1}^{n} a_{2j} A_{1j} & \sum_{j=1}^{n} a_{2j} A_{2j} & \cdots & \sum_{j=1}^{n} a_{2j} A_{nj} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{j=1}^{n} a_{nj} A_{1j} & \sum_{j=1}^{n} a_{nj} A_{2j} & \cdots & \sum_{j=1}^{n} a_{nj} A_{nj} \end{bmatrix}$$

Using Lemma 2.1 : $a_{i1} A_{k1} + a_{i2} A_{k2} + \cdots + a_{in} A_{kn} = \sum_{j=1}^{n} a_{ij} A_{kj}$

$$= \begin{cases} |A| & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}$$

Hence,
$$A(\operatorname{adj} A) = \begin{bmatrix} |A| & 0 & \cdots & 0 \\ 0 & |A| & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & |A| \end{bmatrix} = |A|\, I$$

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Result If A is nonsingular, then $A^{-1} = \frac{1}{|A|} \operatorname{adj} A$

Proof: by Th* $\Rightarrow A\left(\frac{1}{|A|} \operatorname{adj} A\right) = I$

$$\Rightarrow \frac{1}{|A|} \operatorname{adj} A = A^{-1}$$

Exp Let A be 2×2 matrix.

$\boxed{1}$ Find adj A $\boxed{2}$ Find A^{-1}

$\boxed{1}$ $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow \operatorname{adj} A = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \begin{bmatrix} (-1)^2 a_{22} & (-1)^3 a_{12} \\ (-1)^3 a_{21} & (-1)^4 a_{11} \end{bmatrix} = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$

$\boxed{2}$
$$\boxed{A^{-1} = \frac{1}{|A|} \operatorname{adj} A = \frac{1}{a_{11}a_{22} - a_{21}a_{12}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}}$$

Exp Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$. Find A^{-1}

$\boxed{S_1}$ $(A|I) = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ \boxed{1} & 1 & 0 & 0 & 1 & 0 \\ \boxed{1} & 0 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \end{array}$ by Elimination easy

$= \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right)$ Hence $A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$

$\boxed{S_2}$ $A^{-1} = \frac{1}{|A|} \operatorname{adj} A = \frac{1}{1} \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$

$A_{ij} = (-1)^{i+j} |M_{ij}|$

Cramer's Rule

Th
- Let A be $n\times n$ nonsingular matrix. Let $\mathbf{b} \in \mathbb{R}^n$.
- Let A_i be the matrix obtained by replacing the i^{th} column of A by $\mathbf{b}$.
- If $\mathbf{x}$ is a unique solution of $A\mathbf{x}=\mathbf{b}$, then

$$x_i = \frac{|A_i|}{|A|} \quad \text{for } i=1,2,\cdots,n$$

Proof $A\mathbf{x}=\mathbf{b} \Leftrightarrow \mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{|A|}\,\text{adj}\,A\;\mathbf{b}$

Hence, $$x_i = \frac{b_1 A_{1i} + b_2 A_{2i} + \cdots + b_n A_{ni}}{|A|} = \frac{|A_i|}{|A|}$$

Exp Use Cramer's rule to solve

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 5 \\ 2x_1 + 2x_2 + x_3 &= 6 \\ x_1 + 2x_2 + 3x_3 &= 9 \end{aligned}$$

$Ax=b$

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 9 \end{bmatrix}, \quad |A| = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 0 & -2 & -1 \\ 0 & 0 & 2 \end{vmatrix} = -4$$

- $x_1 = \frac{|A_1|}{|A|} = \frac{\begin{vmatrix} \boxed{\begin{smallmatrix}5\\6\\9\end{smallmatrix}} & \begin{smallmatrix}2\\2\\2\end{smallmatrix} & \begin{smallmatrix}1\\1\\3\end{smallmatrix} \end{vmatrix}}{-4} = \frac{-4}{-4} = 1$
- $x_2 = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} \begin{smallmatrix}1\\2\\1\end{smallmatrix} & \boxed{\begin{smallmatrix}5\\6\\9\end{smallmatrix}} & \begin{smallmatrix}1\\1\\3\end{smallmatrix} \end{vmatrix}}{-4} = \frac{-4}{-4} = 1$
- $x_3 = \frac{|A_3|}{|A|} = \frac{\begin{vmatrix} \begin{smallmatrix}1\\2\\1\end{smallmatrix} & \begin{smallmatrix}2\\2\\2\end{smallmatrix} & \boxed{\begin{smallmatrix}5\\6\\9\end{smallmatrix}} \end{vmatrix}}{-4} = \frac{-8}{-4} = 2$

Hence, the solution is $(1, 1, 2)$.

Remark: To solve $n\times n$ system by Cramer's rule, we need to compute $n+1$ determinants. However, using Gaussian elimination involves much less computations.

Proof $x = A^{-1} b = \frac{1}{|A|} \operatorname{adj} A \; b$

$$= \frac{1}{|A|} \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$= \frac{1}{|A|} \begin{bmatrix} \sum_{k=1}^{n} b_k A_{k1} \\ \sum_{k=1}^{n} b_k A_{k2} \\ \vdots \\ \sum_{k=1}^{n} b_k A_{kn} \end{bmatrix}$$

Hence, $x_i = \frac{1}{|A|} \sum_{k=1}^{n} b_k A_{ki} = \frac{b_1 A_{1i} + b_2 A_{2i} + \cdots + b_n A_{ni}}{|A|}$

To see that this is $x_i = \frac{|A_i|}{|A|}$, recall that

$$A_i = \begin{bmatrix} a_{11} & a_{12} & \cdots & b_1 & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & b_2 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & b_n & \cdots & a_{nn} \end{bmatrix}$$

$|A_i|$ along the i^{th} column is

$$|A_i| = b_1 A_{1i} + b_2 A_{2i} + \cdots + b_n A_{ni}$$