

7.5 Indeterminate Forms and L'Hopital Rule (16)

Indeterminate forms: $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty \cdot 0$, $\infty - \infty$, 0^0 , 1^∞ , ∞^0

Th (L'Hopital Rule): suppose f and g are differentiable on an open interval I s.t

$$f(a) = g(a) = 0$$

where $a \in I$

and $g'(x) \neq 0$ on I if $x \neq a$

$$\text{Then } \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

assuming $\xrightarrow{\text{this}} \lim$ exists.

Example ① $\lim_{x \rightarrow 2} \frac{x-2}{x^2-4} = \lim_{x \rightarrow 2} \frac{1}{2x} = \frac{1}{4} \quad \left(\frac{0}{0}\right)$

② $\lim_{x \rightarrow 0} \frac{x - \sin x}{x^3} = \lim_{x \rightarrow 0} \frac{1 - \cos x}{3x^2} = \lim_{x \rightarrow 0} \frac{\sin x}{6x} \quad \left(\frac{0}{0}\right)$

$$= \lim_{x \rightarrow 0} \frac{\cos x}{6} = \frac{1}{6}$$

③ $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x + x^2} = \lim_{x \rightarrow 0} \frac{\sin x}{1 + 2x} = \frac{0}{1} = 0 \quad \left(\frac{0}{0}\right)$

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④ $\lim_{x \rightarrow 0^+} \frac{\sin x}{x^2} = \lim_{x \rightarrow 0^+} \frac{\cos x}{2x} = \frac{1}{\text{very small}^+} = \infty$

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⑤ $\lim_{x \rightarrow 0^-} \frac{\sin x}{x^2} = \lim_{x \rightarrow 0^-} \frac{\cos x}{2x} = \frac{1}{\text{very small}^-} = -\infty$

$$\textcircled{6} \lim_{x \rightarrow \infty} \frac{\ln x}{2\sqrt{x}} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{\frac{1}{\sqrt{x}}} = \lim_{x \rightarrow \infty} \frac{1}{\sqrt{x}} = 0 \quad \textcircled{17} \left(\frac{\infty}{\infty}\right)$$

$$\textcircled{7} \lim_{x \rightarrow \infty} \frac{e^x}{x^2} = \lim_{x \rightarrow \infty} \frac{e^x}{2x} = \lim_{x \rightarrow \infty} \frac{e^x}{2} = \infty \quad \left(\frac{\infty}{\infty}\right)$$

$$\textcircled{8} \lim_{x \rightarrow \infty} \left(x \sin \frac{1}{x}\right) = \lim_{x \rightarrow \infty} \frac{\sin \frac{1}{x}}{\frac{1}{x}} = \lim_{h \rightarrow 0^+} \frac{\sin h}{h} = 1 \quad (\infty \cdot 0)$$

$$\begin{aligned} \textcircled{9} \lim_{x \rightarrow 0^+} \sqrt{x} \ln x &= \lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{\sqrt{x}}} = \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\frac{1}{2}x^{-\frac{3}{2}}} \quad (\infty \cdot 0) \\ &= \lim_{x \rightarrow 0^+} (-2\sqrt{x}) = 0 \end{aligned}$$

$$\begin{aligned} \textcircled{10} \lim_{x \rightarrow 1^+} \left(\frac{1}{x-1} - \frac{1}{\ln x}\right) &= \lim_{x \rightarrow 1^+} \frac{\ln x - x + 1}{(x-1)\ln x} \quad (\infty - \infty) \\ &= \lim_{x \rightarrow 1^+} \frac{\frac{1}{x} - 1}{\frac{x-1}{x} + \ln x} = \lim_{x \rightarrow 1^+} \frac{\frac{1}{x} - 1}{1 - \frac{1}{x} + \ln x} \\ &= \lim_{x \rightarrow 1^+} \frac{-\frac{1}{x^2}}{\frac{1}{x^2} + \frac{1}{x}} = \frac{-1}{1+1} = -\frac{1}{2} \end{aligned}$$

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} e^{\ln f(x)} = e^{\lim_{x \rightarrow a} \ln f(x)}$$

$$\begin{aligned} \text{Example } \textcircled{11} \lim_{x \rightarrow 0^+} (1+x)^{\frac{1}{x}} &= \lim_{x \rightarrow 0^+} e^{\ln (1+x)^{\frac{1}{x}}} \quad \left(\frac{\infty}{\infty}\right) \\ &= e^{\lim_{x \rightarrow 0^+} \frac{\ln(1+x)}{x}} = e^{\lim_{x \rightarrow 0^+} \frac{\frac{1}{1+x}}{1}} = e^1 = e \end{aligned}$$

(18) $\begin{pmatrix} 0 \\ \infty \end{pmatrix}$

Th (Cauchy's Mean Value Theorem) CMVT

suppose • the functions f and g are continuous on $[a, b]$

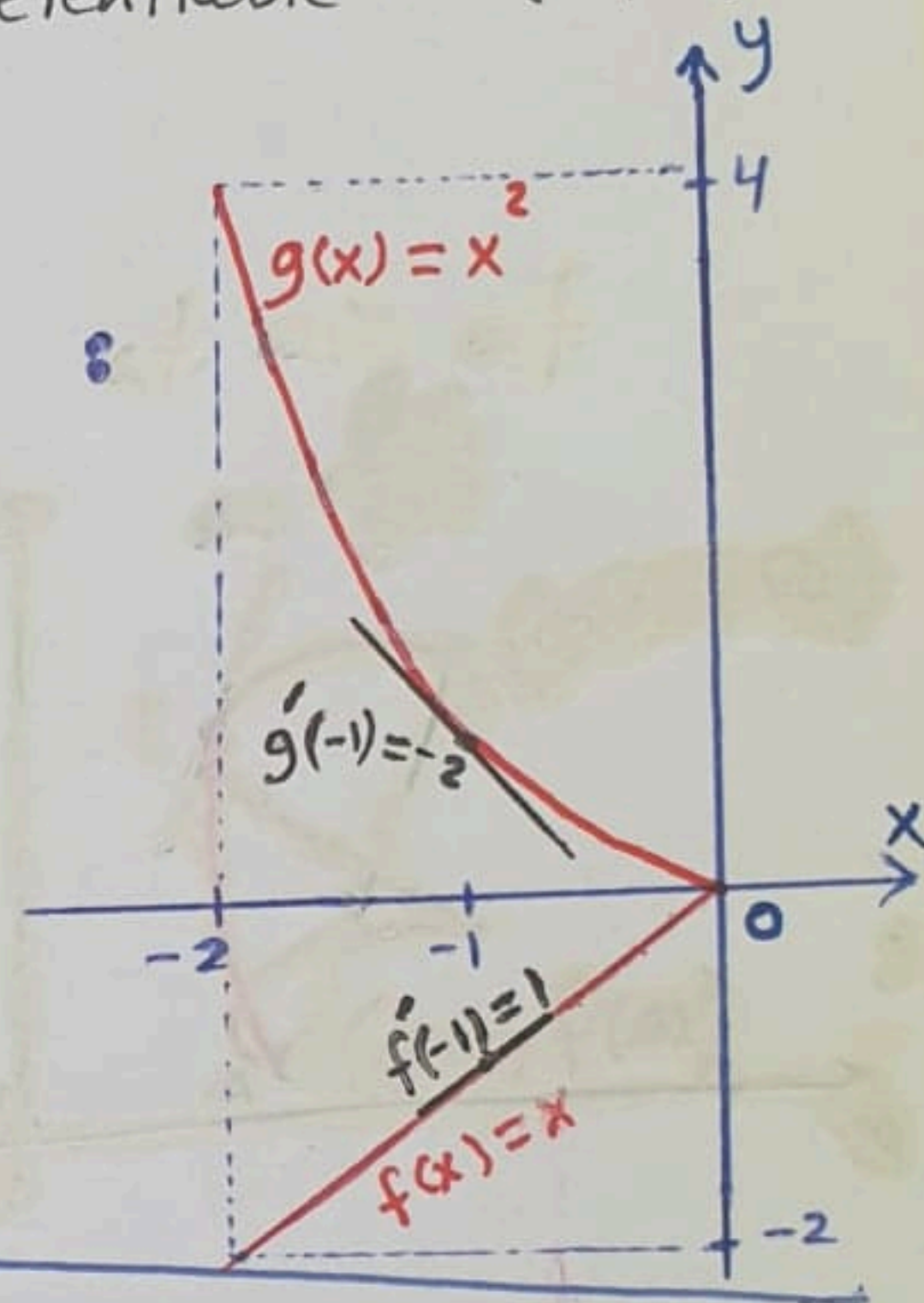
• f is differentiable on (a, b)

- $g'(x) \neq 0$ on (a, b)

Then there exists a number $c \in (a, b)$ 

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

slope of tangent = slope of secant



Example : Find values of c that satisfy **CMVT** for the function $f(x) = x$, $g(x) = x^2$ on $(-2, 0)$

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

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$$\frac{1}{2c} = \frac{f(0) - f(-2)}{g(0) - g(-2)}$$

$$\frac{1}{2c} = \frac{0 - -2}{0 - 4}$$

$$2C = \frac{-4}{2} = -2$$

$$c = -1$$

$$f(0) = 0$$
$$f(-2) = -2$$

$$g(0) = 0$$

$$g(-2) = 4$$

$$f'(x) = 1$$
$$g'(x) = 2x$$