7.5 Indeterminate Form and L'Hopital Rule

$\boxed{1}$ If $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = \frac{0}{0}$ OR $\frac{\infty}{\infty}$, Then $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = \lim\limits_{x\to a} \frac{f'(x)}{g'(x)}$

Example Find the limit of the following

$\boxed{1}$ $\lim\limits_{x\to 0} \frac{x\,2^x}{2^x - 1} = \frac{0}{0}$

$\lim\limits_{x\to 0} \frac{x\,2^x \ln 2 + 2^x \cdot 1}{2^x \ln 2} = \frac{1}{\ln 2}$

$\boxed{2}$ $\lim\limits_{x\to 0} \frac{\sin x - x}{x^3}$

$\lim\limits_{x\to 0} \frac{\cos x - 1}{3x^2} = \lim\limits_{x\to 0} \frac{-\sin x}{6x} = \lim\limits_{x\to 0} \frac{-\cos x}{6} = \frac{-1}{6}$

$\boxed{3}$ $\lim\limits_{x\to 0} \frac{(e^x - 1)^2}{x \sin x}$

$\lim\limits_{x\to 0} \frac{2(e^x - 1)\, e^x}{x \cos x + \sin x} = \lim\limits_{x\to 0} \frac{2(e^x - 1) e^x + 2e^x\, e^x}{x(-\sin x) + \cos x + \cos x}$

$= \frac{2}{2} = 1$

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$$\lim_{x\to\infty} \frac{\log_2 x}{\log_3 (x+3)}$$

$$\lim_{x\to\infty} \frac{\frac{1}{(\ln 2)\,x}}{\frac{1}{(\ln 3)(x+3)}} = \lim_{x\to\infty} \frac{(\ln 3)\,(x+3)}{(\ln 2)\,(x)}$$

$$= \frac{\ln 3}{\ln 2} \lim_{x\to\infty} \frac{x+3}{x}$$

$$= \frac{\ln 3}{\ln 2}$$

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$$\lim_{x\to\infty} \frac{3^x + 2x}{x^2+1}$$

$$\lim_{x\to\infty} \frac{3^x \ln 3 + 2}{2x} = \lim_{x\to\infty} \frac{(\ln 3)\cdot(\ln 3)\, 3^x}{2} =$$

$$= (\ln 3) \lim_{x\to\infty} 3^x = \infty$$

7.5 Indeterminate Form and L'Hopital Rule

$\lim_{x \to a} f(x)$

→ Indeterminate form $\frac{0}{0}$, $\frac{\infty}{\infty}$

limit exist by using l'Hopital Rule

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

→ Indeterminate form $0 \cdot \infty$, $\infty - \infty$

Convert to $\lim \frac{\text{Numerator}}{\text{Denominator}}$

$\frac{f(x)}{1/g(x)}$, Take out common factor, Rationalize Denominator

→ Indeterminate form 1^{∞}, 0^{0}, ∞^{0}

$$f(x)^{g(x)} = e^{g(x) \ln(f(x))}$$

[2] Indeterminate Product and difference. $0 \cdot \infty$, $\infty - \infty$

Exp:- Find the following limit

[1] $\lim_{x \to 0^+} x(\ln x)^2 = 0 \cdot \infty$

$$\lim_{x \to 0^+} \frac{(\ln x)^2}{\frac{1}{x}} = \frac{\infty}{\infty}$$

l'Hopital Rule

$$= \lim_{x \to 0^+} \frac{2 \ln x \frac{1}{x}}{\frac{-1}{x^2}}$$

$$= \lim_{x \to 0^+} \frac{2 \ln x}{\frac{-1}{x}} = \frac{-\infty}{-\infty} = \frac{\infty}{\infty}$$

$$\lim_{x \to 0^+} \frac{\frac{2}{x}}{\frac{1}{x^2}} = \lim_{x \to 0^+} 2x = 0$$

[2] $\lim_{x \to \frac{\pi}{2}^-} \left(\frac{\pi}{2} - x\right) \tan x = 0 \cdot \infty$

$$\lim_{x \to \frac{\pi}{2}^-} \frac{\frac{\pi}{2} - x}{\cot x} = \frac{0}{0}$$

$$\lim_{x \to \frac{\pi}{2}^-} \frac{-1}{-\csc^2 x} = \frac{1}{\csc^2(\frac{\pi}{2})} = \frac{1}{1}$$

$\boxed{131}$

(3) $\lim\limits_{x \to 0^+} \ln x - \ln \sin x = -\infty + \infty$

$$\lim_{x \to 0^+} \ln\left(\frac{x}{\sin x}\right)$$

$$\ln\left[\lim_{x \to 0^+}\left(\frac{x}{\sin x}\right)\right]$$

$$\ln\left[\lim_{x \to 0^+} \frac{1}{\cos x}\right] = \ln 1 = 0$$

$\boxed{4}$ $\lim\limits_{x \to 1^+}\left[\frac{1}{x-1} - \frac{1}{\ln x}\right]$

$$\left[\frac{1}{\text{Small}+} - \frac{1}{\text{Small}+}\right] = \infty - \infty$$

$$\lim_{x \to 1^+}\left[\frac{\ln x - x + 1}{(x-1)\ln x}\right]$$

$$\lim_{x \to 1^+}\left[\frac{\frac{1}{x} - 1}{(x-1)\frac{1}{x} + \ln x}\right] = \lim_{x \to 1^+} \frac{\frac{1}{x} - 1}{1 - \frac{1}{x} + \ln x}$$

$$= \lim_{x \to 1^+} \frac{\frac{-1}{x^2}}{+\frac{1}{x^2} + \frac{1}{x}} = \frac{-1}{1+1} = \frac{-1}{2}$$

3 Indeterminate form : 1^{∞}, 0^{0}, ∞^{0}

Remark. If $\lim_{x \to a} \ln f(x) = L$, then $\ln \lim_{x \to a} f(x) = L$

$$e^{\ln \lim_{x \to a} f(x)} = e^{L}$$

$$\lim_{x \to a} f(x) = e^{L}$$

Exp: Find the following limit

① $\lim_{x \to \infty} \left(1 + \frac{3}{x}\right)^{x}$

In general

$$\lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^{x} = e^{a}$$

Assume $f(x) = \left(1 + \frac{3}{x}\right)^{x}$

$$\ln f(x) = x \ln\left(1 + \frac{3}{x}\right)$$

$$\lim_{x \to \infty} \ln f(x) = \lim_{x \to \infty} x \ln\left(1 + \frac{3}{x}\right) = \infty \cdot 0$$

$$\lim_{x \to \infty} \frac{\ln\left(1 + \frac{3}{x}\right)}{\frac{1}{x}} \quad \frac{0}{0}$$

$$\lim_{x \to \infty} \frac{\frac{1}{1 + \frac{3}{x}} \cdot \frac{-3}{x^2}}{\frac{-1}{x^2}} = \lim_{x \to \infty} \frac{3}{1 + \frac{3}{x}}$$

$$= 3$$

$$\lim_{x \to \infty} f(x) = e^{3}$$

Q61

(2) $\lim_{x\to\infty}\left(\frac{x+2}{x-1}\right)^{x}$

$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{x}=e^{a}$

$\lim_{y\to 0}\left(1+ay\right)^{1/y}=e^{a}$

$\lim_{x\to\infty}\left(\frac{x\left[1+\frac{2}{x}\right]}{x\left[1-\frac{1}{x}\right]}\right)^{x}$

$\lim_{x\to\infty}\frac{\left(1+\frac{2}{x}\right)^{x}}{\left(1+\frac{-1}{x}\right)^{x}}=\frac{\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^{x}}{\lim_{x\to\infty}\left(1+\frac{-1}{x}\right)^{x}}=\frac{e^{2}}{e^{-1}}=e^{3}$

Q58

(3) $\lim_{x\to 0}\left(e^{x}+x\right)^{1/x}=1^{\infty}$

$f(x)=\left(e^{x}+x\right)^{1/x}$

$\ln f(x)=\frac{1}{x}\ln\left(e^{x}+x\right)$

$\lim_{x\to 0}\ln f(x)=\lim_{x\to 0}\frac{\ln\left(e^{x}+x\right)}{x}=\frac{0}{0}$

$\lim_{x\to 0}\frac{1}{e^{x}+x}\,e^{x}+1=\frac{2}{1}=2$

$\lim_{x\to 0}f(x)=\lim_{x\to 0}\left(e^{x}+x\right)^{1/x}=e^{2}$

④ $\lim_{x \to 0^+} x^{\sin x}$

$$f(x) = x^{\sin x}$$

$$\ln f(x) = \ln x^{\sin x} = \sin x \ln x$$

$$\lim_{x \to 0^+} \ln f(x) = \lim_{x \to 0^+} \sin x \ln x = 0 \cdot -\infty$$

$$= \lim_{x \to 0^+} \frac{\ln x}{\csc x} = \frac{-\infty}{\infty}$$

l'Hopital Rule

$$= \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\csc x \cot x} = \lim_{x \to 0^+} - \frac{\sin x \tan x}{x} = \frac{0}{0}$$

l'Hopital Rule

$$= - \lim_{x \to 0^+} \frac{\tan x}{1} \cdot \lim_{x \to 0^+} \frac{\sin x}{x}$$

$$= - 0 \cdot 1 = 0$$

$$\lim_{x \to 0^+} f(x) = e^0 = 1$$

Outline Exercises

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$$\lim_{x\to 0^+} \frac{\sqrt{x}}{\sqrt{\sin x}}$$

$$\lim_{x\to 0^+} \sqrt{\frac{x}{\sin x}}$$

$$\sqrt{\lim_{x\to 0^+} \frac{x}{\sin x}} = \sqrt{\left(\lim_{x\to 0^+} \frac{\sin x}{x}\right)^{-1}} = \sqrt{1} = 1$$

Q62 $$\lim_{x\to\infty} \left(\frac{x^2+1}{x+2}\right)^{1/x}$$

let $f(x) = \left(\frac{x^2+1}{x+2}\right)^{1/x}$

$$\ln f(x) = \frac{\ln(x^2+1) - \ln(x+2)}{x}$$

$$\lim_{x\to\infty} \ln f(x) = \lim_{x\to\infty} \frac{\ln(x^2+1) - \ln(x+2)}{x}$$

$$= \lim_{x\to\infty} \frac{\frac{2x}{x^2+1} - \frac{1}{x+2}}{1}$$

$$= \lim_{x\to\infty} \frac{2x}{x^2+1} - \lim_{x\to\infty} \frac{1}{x+2} = 0$$

Solution $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \left(\frac{x^2+1}{x+2}\right)^{1/x} = e^0 = 1$