CHAPTER 5

FINDING THE EFFICIENT SET

In the previous chapter we learned how individual securities combined to form portfolios. We were seeking the answer to the question, If we combine a group of securities together to form a portfolio, what will the portfolio look like in terms of its risk and expected return? In this chapter, we will direct our attention to answering the following question: Given an available group of securities, how do we determine the *best* way to combine the securities into portfolios? In answering this question, we will be seeking those portfolios that are expected to produce the maximum amount of return given the level of risk exposure.

THE MINIMUM VARIANCE AND EFFICIENT SETS

Assume that the points plotted in Figure 5.1 denote the positions of individual *stock* investments. To illustrate, the stock denoted by point *A* has an expected rate of return of 10 percent and a standard deviation of 15 percent. These individual stocks can be combined into portfolios. For example, we can invest in stocks *A* and *B* and attain positions anywhere along the broken combination line. As you might imagine, by taking positive positions in some of the stocks and short positions in others, we can form a wide variety of portfolios that would be positioned at various points on the graph.

All of these attainable portfolio positions represent the set of investment opportunities available to us. We would, of course, prefer some of these positions to others. Given the level of risk, or standard deviation, we prefer positions with higher expected rates of return; given the level of expected return, we prefer positions of lower risk. In any case, given the characteristics of the available population of stocks, the investment opportunity set has a perimeter that is represented by the bullet-shaped curve. From now on, we're going to refer to this *perimeter* as the ***minimum variance set.***

Each point on the minimum variance set represents a portfolio, with portfolio weights allocated to each of the stocks in the population. Each of the portfolios in the minimum variance set meets the following criterion: *Given a particular level of expected rate of return, the portfolio on the minimum variance set has the lowest standard deviation (or variance) achievable with the available population of stocks.* As befitting its shape, from time to time we shall refer to the minimum variance set as the *bullet.* We will refer to the bullet as the minimum variance set irrespective of whether standard deviation or variance is being measured along the horizontal axis. Since the portfolios

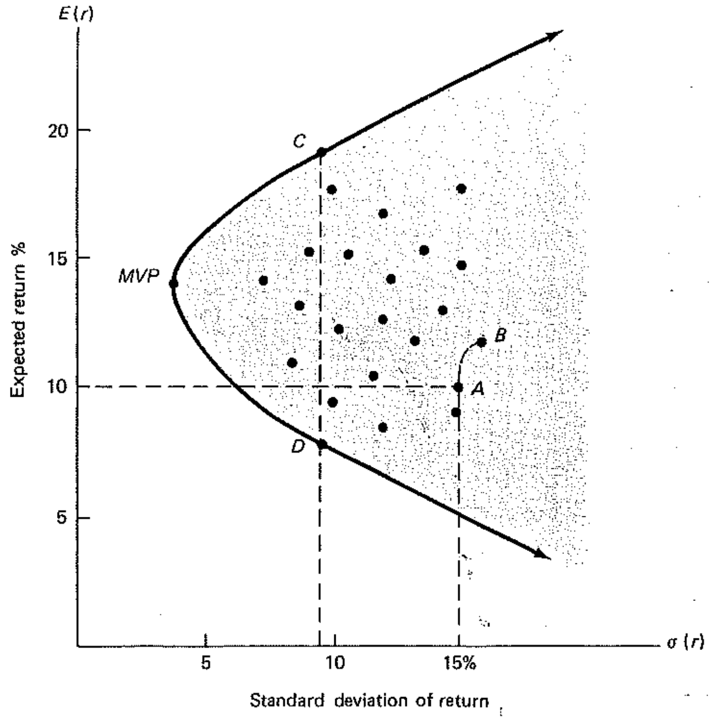


FIGURE 5.1 MINIMUM VARIANCE SET.

that minimize *variance* given expected return are identical to the portfolios that minimize *standard deviation,* the terms *minimum variance set* and *minimum standard deviation set* can be used interchangeably.

The minimum variance set can be divided into two halves, a top and a bottom. The halves are separated at the point *MVP.* This point represents the single portfolio with the lowest possible level of standard deviation, the ***global minimum variance portfolio.*** The most desirable portfolios for us to hold are those in the top half of the bullet. The most undesirable are those in the bottom half.

The top half of the bullet is called the ***efficient set.*** All the portfolios in the efficient set meet the following criterion: *Given a particular level of standard deviation, the portfolios in the efficient set have the highest attainable expected rate of return.* Thus while portfolios *C* and *D* both meet the criterion for the minimum variance set (lowest standard deviation, given expected return), only *C* meets the criterion for the efficient set (highest expected return, given standard deviation). Portfolio *D* actually has the lowest expected return, given its standard deviation level.

FINDING THE EFFICIENT SET WITH SHORT SELLING

In practice, you will find the minimum variance and efficient sets using a computer. To illustrate the process employed, we shall consider an example where we build portfolios from three available stocks: *A*, Acme Steel; *B*, Brown Drug; and *C*, Consolidated Electric. The three stocks have the following expected rates of return:

$$\text{Acme Steel: } E(r_A) = 5\%$$

$$\text{Brown Drug: } E(r_B) = 10\%$$

$$\text{Consolidated Electric: } E(r_C) = 15\%$$

The covariance matrix for the stocks is given by

	A	*B*	*C*
A	.25	.15	.17
B	.15	.21	.09
C	.17	.09	.28

By taking the square root of the variances going down the diagonal of the matrix, we can compute the standard deviations of the stocks as follows:

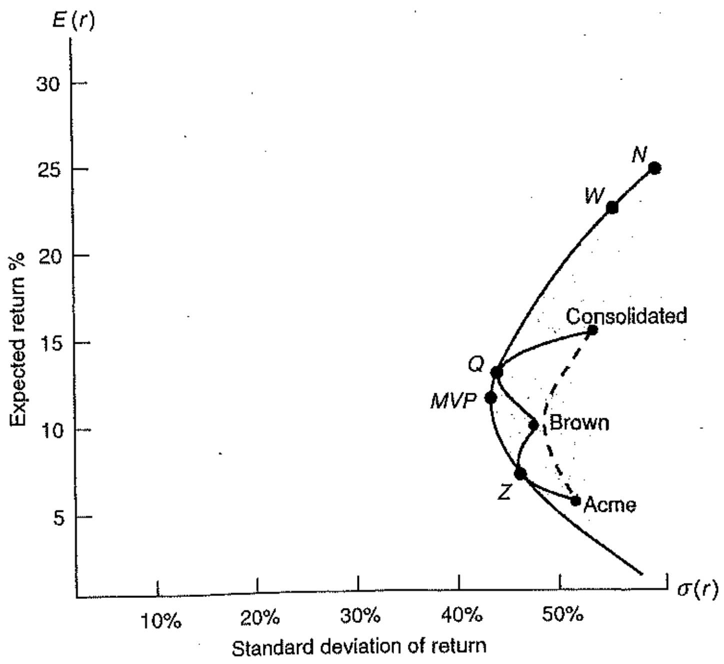
$$\text{Acme Steel: } \sigma(r_A) = .50$$

$$\text{Brown Drug: } \sigma(r_B) = .46$$

$$\text{Consolidated Electric: } \sigma(r_C) = .53$$

The expected returns and standard deviations of the three stocks are plotted in Figure 5.2. The minimum variance set of portfolios of the three stocks is plotted as the solid curve. We shall now examine the procedure a computer might follow in finding the portfolios in the minimum variance set.

FIGURE 5.2 MINIMUM VARIANCE SET FOR CONSOLIDATED, BROWN, AND ACME.

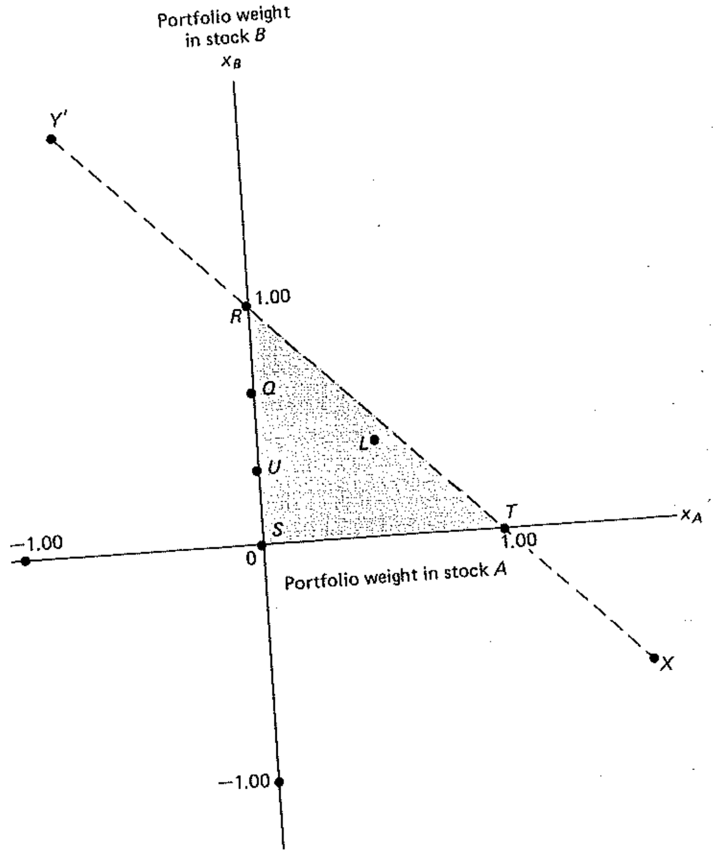


In Figure 5.3 we are plotting the portfolio weights in Acme Steel and Brown Drug. The portfolio weight in Consolidated Electric is not represented on the diagram, but it is implied by the values for x_A and x_B. For example, if we are at point *U*, the portfolio weight in Brown is equal to .30, and the portfolio weight in Acme is given by .00. Since there are only three stocks, the portfolio weight in Consolidated is always equal to $1 - x_A - x_B$ and is, therefore, .70. Each point in Figure 5.3 represents a portfolio with particular weights allocated to each of the three stocks.

Now consider the triangle drawn in Figure 5.3. The points of the triangle are given by *R*, *S*, and *T*. All positions inside the triangle represent portfolios where we have invested positive amounts of money in each of the three stocks. To illustrate, consider the point labeled *L*. This point represents a portfolio where we are investing 50 percent of our money in Acme, 45 percent of our money in Brown, and the remaining 5 percent of our money in Consolidated. For portfolios on the perimeter of the triangle, we are investing a combined total of 100 percent of our money in two of the stocks, and we aren't taking any position at all in the third. At point *Q* on perimeter *RS*, we're investing 60 percent in Brown, 40 percent in Consolidated, and nothing in Acme. On perimeter *RT* we are taking no position in Consolidated.

If we are outside the triangle, at any point to the northeast of the line labeled *Y′X*, we are selling Consolidated short. If we are positioned to the west of the figure's vertical axis, we are selling Acme short, and if we are anywhere to the south of the horizontal axis, we are selling Brown short.

FIGURE 5.3 PORTFOLIO WEIGHTS IN THE THREE-STOCK PORTFOLIO



zontal axis, we are selling Brown short. At point Y', we are selling Acme short and adding the proceeds to our equity to invest in Brown.

THE ISOEXPECTED RETURN LINES

Suppose we wish to find a set of portfolios all of which have the same expected rate of return. The weights for these portfolios are given by one of the ***isoexpected return lines.***[1] There is a family of such lines, each representing a given expected rate of portfolio return. The lines are drawn in $x_A x_B$ space, and any of them can be expressed by the following relationship with intercept a_0 and slope a_1:

$$x_B = a_0 + a_1 \cdot x_A$$

Given a value for x_A, we solve the preceding equation for a value for x_B (and an implied value for x_C), which will produce a portfolio with the desired expected rate of return. The values for a_0 and a_1 are determined by the relative expected rates of return on the three stocks. To compute a_0 and a_1, we start with the formula for the expected return on a three-stock portfolio:

$$E(r_P) = x_A E(r_A) + x_B E(r_B) + (1 - x_A - x_B)E(r_C)$$

Multiplying through by $E(r_C)$ and solving for x_B as a function of x_A, we get

$$x_B = \underbrace{\frac{E(r_C) - E(r_P)}{E(r_C) - E(r_B)}}_{a_0} + \underbrace{\left(\frac{E(r_A) - E(r_C)}{E(r_C) - E(r_B)}\right)}_{a_1} x_A$$

In the case of the three stocks of our example, the slope is computed as

$$a_1 = \frac{.05 - .15}{.15 - .10} = -2.00$$

The intercept is computed as

$$a_0 = \frac{.15 - E(r_P)}{.15 - .10}$$

Thus, the value for the intercept depends on the desired expected rate of return on the portfolio. Let's assume we want to find the set of portfolio weights that will give us an expected rate of return of 10 percent. In this case, the value for a_0 is given by 1.00:

$$a_0 = \frac{.15 - .10}{.15 - .10} = 1.00$$

The formula for the 10 percent isoexpected return line is, thus,

$$x_B = 1.00 - 2.00x_A$$

Suppose we invest 50 percent of our money in Acme. According to the preceding formula, we must invest 0 percent of our money in Brown and the remaining 50 percent in Consolidated to construct a portfolio with an expected return of 10 percent:

$$E(r_P) = x_A E(r_A) + x_B E(r_B) + (1 - x_A - x_B)E(r_C)$$
$$.10 = .50 \times .05 + .00 \times .10 + .50 \times .15$$

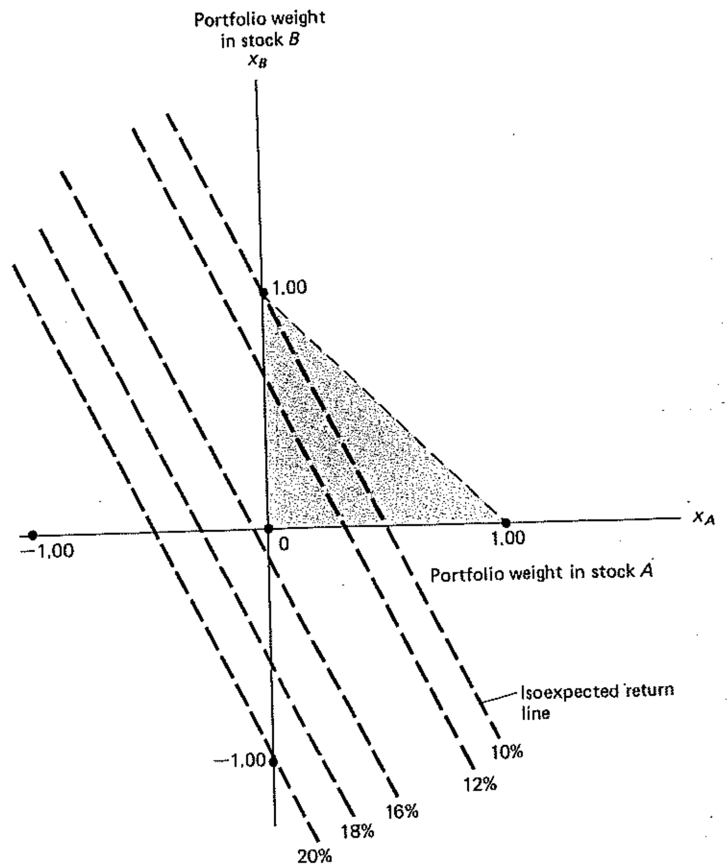
[1] The process of finding the minimum variance set described in this chapter was originally developed by Harry Markowitz (1952).

These, of course, are not the only weights that will produce a 10 percent expected rate of return. For example, the combination $x_A = 1.00$, $x_B = -1.00$, and $x_C = 1.00$ will also do the job. Think of all the combinations of weights in the three stocks that represent portfolios with 10 percent expected rates of return. These are plotted as the isoexpected return line labeled 10 percent in Figure 5.4.

If we want to see the combinations of weights that are consistent with a different value for the portfolio expected return, we must look to a different isoexpected return line. For example, all of the combinations of weights represented by the line labeled 12 percent are those that provide a 12 percent expected return on the portfolio.

In the case of this example, as we move to the northeast, we move to isoexpected return lines with lower and lower expected rates of return. The isoexpected return lines also have a negative slope. In general, however, the slope and relative position of the isoexpected return lines are dependent on the relative expected returns of the three stocks considered. To see this, suppose we switched the expected returns on Acme and Consolidated. In this case, while the lines would still have a negative slope, we now move to isoexpected return lines with higher and higher expected returns as we move in a northeasterly direction.

FIGURE 5.4 ISOEXPECTED RETURN LINES.



THE ISOVARIANCE ELLIPSES

Now suppose we want to find a set of portfolios all of which have the same *variance* of return. The portfolio weights for members of this set are given by one of the ***isovariance ellipses.*** As with the isoexpected return lines, there is a family of these ellipses, each representing a different level of portfolio variance.

To find the isovariance ellipse representing a given level of portfolio variance, we begin with the equation of the variance for a three-stock portfolio:

$$\sigma^2(r_P) = x_A^2\sigma^2(r_A) + x_B^2\sigma^2(r_B) + (1 - x_A - x_B)^2\sigma^2(r_C) + 2x_Ax_B \operatorname{Cov}(r_A, r_B) \\ + 2x_A(1 - x_A - x_B) \operatorname{Cov}(r_A, r_C) + 2x_B(1 - x_A - x_B) \operatorname{Cov}(r_B, r_C)$$

Suppose we want to find the ellipse of portfolio weights consistent with a 30 percent portfolio variance. To find two points on the ellipse, select an arbitrary value for x_A (say, .00) and substitute this and the values for the variances and covariances into the variance formula:

$$.30 = .00^2 \times .25 + x_B^2 \times .21 + (1 - .00 - x_B)^2 \times .28 + 2 \times .00 \times x_B \times .15 \\ + 2 \times .00 \times (1 - .00 - x_B) \times .17 + 2 \times x_B \times (1 - .00 - x_B) .09$$

or, after simplifying

$$.3 = .28 - .38x_B + .31x_B^2$$

The equation now has only one unknown value, that being x_B. This is a quadratic equation, so two values for x_B will make the right-hand side equal to the left-hand side. The two values are

$$x_B = 1.28$$

and

$$x_B = -.05$$

Two portfolios that both have a 30 percent variance are, thus,

$$x_A = .00, \quad x_B = 1.28, \quad \text{and} \quad x_C = -.28$$

and

$$x_A = .00, \quad x_B = -.05, \quad \text{and} \quad x_C = 1.05$$

These portfolios are plotted in Figure 5.5 at the points labeled *W* and *X*. These are two of the many portfolios on the 30 percent isovariance ellipse. To find two more, we merely select another arbitrary value for x_A, say, 1.00. We substitute this new value for x_A in the portfolio variance equation and solve once again for two values for x_B.

These solutions are plotted at the points labeled *C* and *B* in Figure 5.5. By repeating the process, we can produce as many points in the ellipse as desired. Suppose we find there are no solutions for x_B, given the arbitrarily chosen value for x_A. This means that given the characteristics for the stocks, it is impossible to construct a portfolio with a variance as low as 30 percent with x_A equal to the chosen value. The chosen value for x_A, therefore, must be outside the horizontal range of the ellipse.

To produce another ellipse consistent with a different value for $\sigma^2(r_P)$, simply select another desired value, substitute it in the portfolio variance equation, and go through the same process just described. If a portfolio variance level of less than 30 percent is selected, the new ellipse will be found inside the 30 percent ellipse. Note that

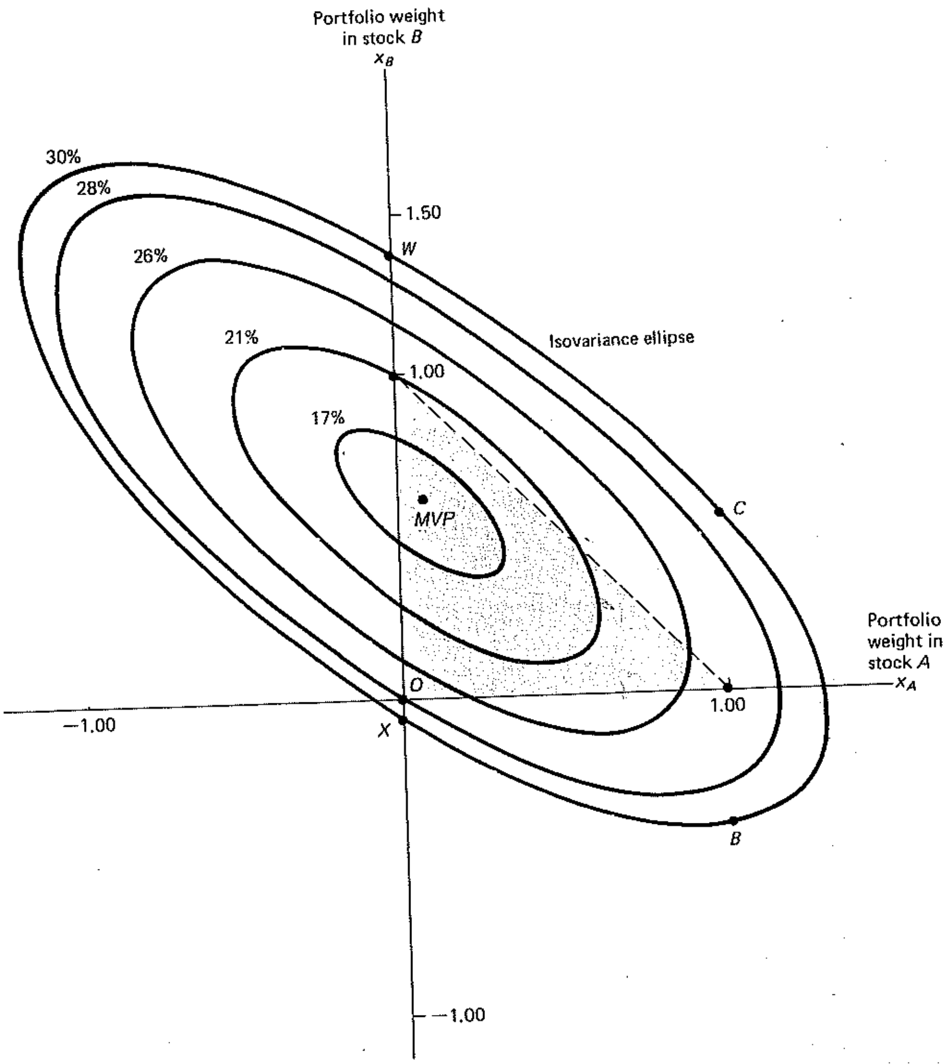


FIGURE 5.5 ISOVARIANCE ELLIPSES.

the ellipse for the 28 percent variance is inside the ellipse for the 30 percent varianc As smaller values for the portfolio variance are selected, the ellipses become small and smaller in size, converging on the point labeled *MVP.* This point represents tl lowest possible portfolio variance level achievable, given the covariance matrix for tl three stocks. If we were to attempt to construct an ellipse for a still lower portfolio va ance, we would find no solutions, no matter what values for x_A were arbitrarily chose

Note that the ellipses are all concentric about the point *MVP.* Aside from this, tl ellipses are in a way similar to lines denoting points of equal altitude on a topograpl map. Such a map is drawn in Figure 5.6. By studying the map, we can determine th the gray area is positioned at the top of a hill, at an altitude of approximately 500 fe above sea level. In Figure 5.5 the isovariance ellipses represent points of constant va ance, instead of altitude, with the *MVP* point positioned at the bottom of a vall rather than at the top of a hill.

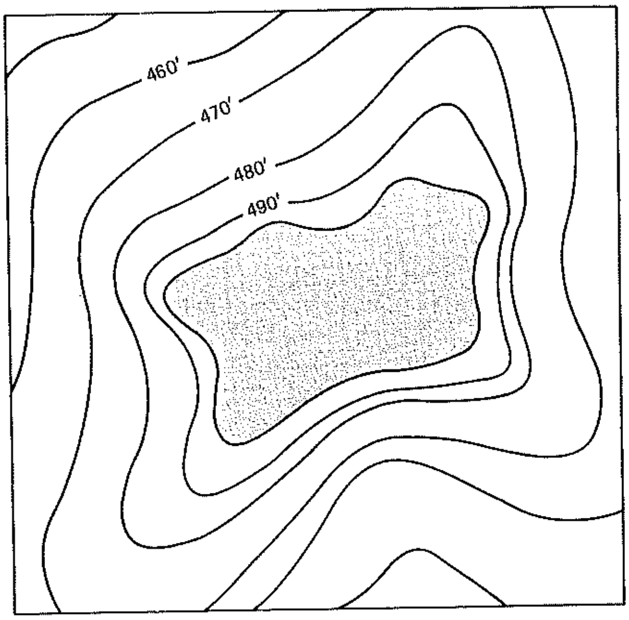


FIGURE 5.6 TOPOGRAPHIC MAP.

THE CRITICAL LINE

The isoexpected return lines are superimposed on the isovariance ellipses in Figure 5.7.

The ***critical line*** is drawn as line *NY* in Figure 5.7. It shows the portfolio weights for the portfolios in the minimum variance set. The critical line can be found by tracing out the points of tangency between the isoexpected return lines and the isovariance ellipses. We can say that finding the minimum variance set is tantamount to finding the location of the critical line.

To find the minimum variance set with a computer, you might provide the computer with the following set of instructions:

1. Find the portfolio weights that minimize portfolio variance, subject to the constraint that the expected rate of return on the portfolio is equal to some predetermined level.
2. For any given portfolio constructed, the sum of the portfolio weights for all stocks in the portfolio must be equal to 1.
3. The portfolio weight assigned to any one stock may take any value from plus to minus infinity. (This allows the computer to sell short in unlimited amounts.)
4. The expected rate of return to a portfolio is given by
$$E(r_P) = x_A E(r_A) + x_B E(r_B) + (1 - x_A - x_B)E(r_C)$$
5. The variance of return to the portfolio is given by
$$\sigma^2(r_P) = x_A^2\sigma^2(r_A) + x_B^2\sigma^2(r_B) + (1 - x_A - x_B)^2\sigma^2(r_C) + 2x_Ax_B \operatorname{Cov}(r_A, r_B) + 2x_A(1 - x_A - x_B)\operatorname{Cov}(r_A, r_C) + 2x_B(1 - x_A - x_B)\operatorname{Cov}(r_B, r_C)$$
6. You provide the computer with estimates of the expected rates of return on the three stocks as well as estimates of the numbers in the covariance matrix.

You now provide the computer with some target expected rate of return. The computer's task is to minimize the variance of the portfolio subject to having an expected

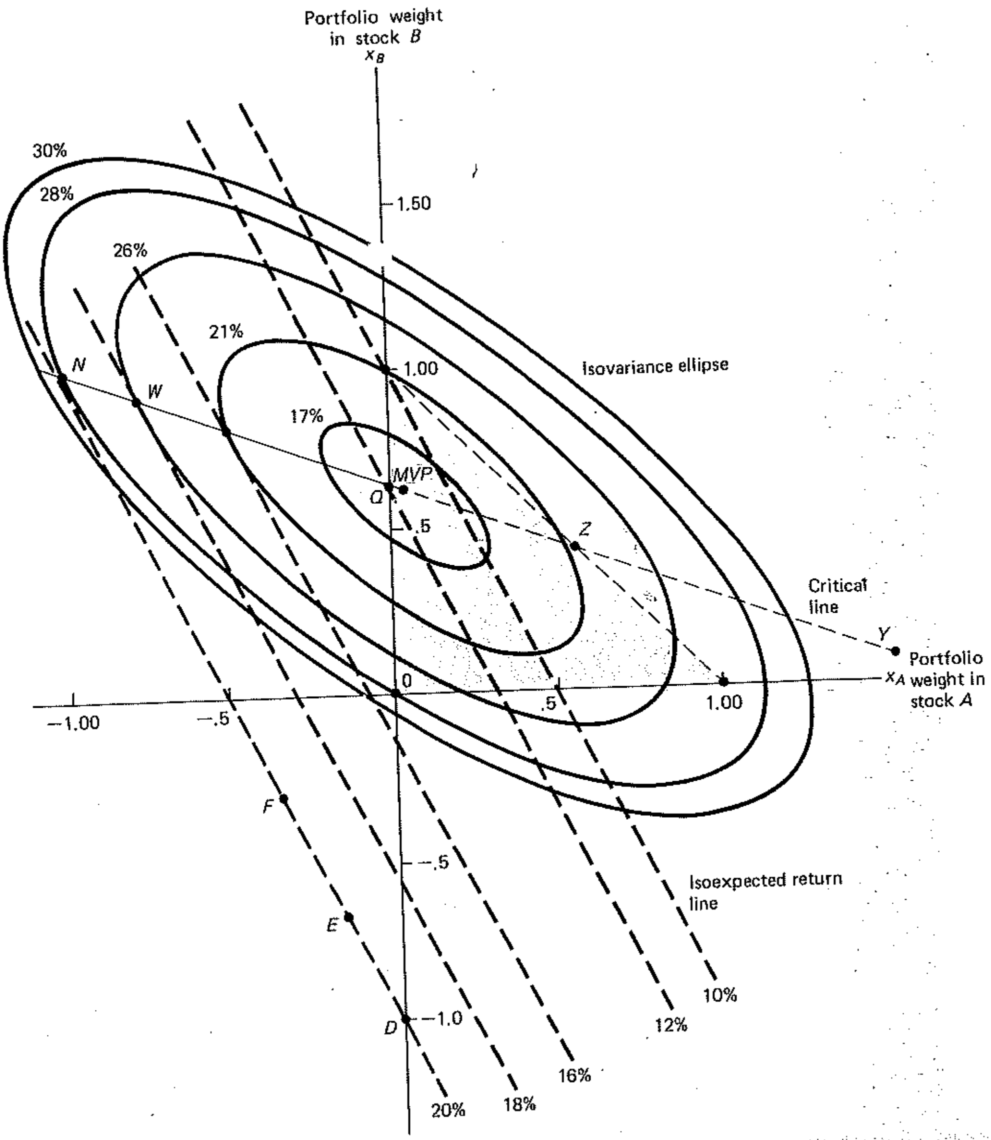


FIGURE 5.7 PORTFOLIO WEIGHTS IN THE MINIMUM VARIANCE SET.

rate of return on the portfolio equal to the target. Assume the first target expected rat of return is equal to 20 percent. The computer "knows" the portfolio weights that pro vide the solution to the problem lie somewhere on the 20 percent isoexpected retur line. The process of finding the solution begins at a position on the line correspondin to some arbitrary value for x_A. Let's assume this value for x_A is equal to .00, so we begi at the position labeled D on the 20 percent line in Figure 5.7.

The point D represents three portfolio weights for the three stocks. The compute substitutes these weights, along with the given variances and covariances from th covariance matrix, into the formula you have provided for portfolio variance. The por folio variance consistent with the weights of point D is computed. The computer no moves by some predetermined distance toward either the southeast or the northwe along the 20 percent isoexpected return line. Suppose it moves to the southeast and recomputes the portfolio variance for the new point. If you move to the southeast along the 20 percent isoexpected return line, you move out of the "valley" toward higher variance positions. Thus, the variance consistent with the portfolio weights of the new point will be greater than that of point D. The computer now knows it has moved in the wrong direction along the line. It retraces its steps and moves to a point toward the northwest of D, say, to point E. It again computes the variance and finds that it's lower than the variance of point D. It now "knows" it's moving in the right direction, so it takes another jump to point F. The computed variance is again smaller because we are continuing to move deeper into the valley.

The computer continues to move to the northwest until it takes a jump past point N. Beyond N the computed variance will begin to increase, because we have gone beyond the isoexpected return line's deepest penetration into the valley. The sizes of the jumps are now reduced, and the computer now reverses direction toward the southeast. The process is repeated until the computer gradually iterates to point N. This point represents the lowest variance position on the isoexpected return line, a portfolio variance of 28 percent. Point N is also the point where the 28 percent isovariance ellipse is tangent to the 20 percent isoexpected return line.

We have found one of the portfolios in the minimum variance set. Reading from the graph, we find that the portfolio weights for the portfolio that minimizes variance, given a 20 percent expected rate of return, are approximately

$$x_A = -1.00$$

$$x_B = 1.00$$

$$x_C = 1.00$$

This portfolio has a 28 percent variance and a 53 percent standard deviation. The portfolio is plotted in the $E(r_P)$, $\sigma(r_P)$ mapping of Figure 5.2 at point N.

To find another point on the minimum variance set, we select another target expected rate of return, perhaps 18 percent. The 18 percent isoexpected return line again can be found in Figure 5.7. In the same fashion described, the computer iterates to point W, the line's deepest penetration into the "valley." This is the point of tangency between the line and the 26 percent isovariance ellipse. Since a 26 percent variance is consistent with a 51 percent standard deviation, this minimum variance portfolio is plotted at point W in Figure 5.2.

This process is repeated to find as many points as desired on the minimum variance set. In actual practice, extremely efficient computer algorithms are employed so computers can find the solutions quickly and accurately. However, the description furnished here should give you a rough idea of what is being accomplished.

As said before, the line passing through the portfolio weights in the minimum variance set is called the critical line. In Figure 5.7, as we move from point W to point Q, we move along the minimum variance set of Figure 5.2 from corresponding points W to Q.

Note that at point Q we are on the western border of the triangle of Figure 5.7. Therefore, we are taking no position at all in Acme, and we are investing *positive* amounts of money in both Brown and Consolidated. This means portfolio Q must be on the combination line between Brown and Consolidated, somewhere between the positions of the two stocks. In Figure 5.2 we see that portfolio Q lies at the point of tangency between the minimum variance set and the combination line between Brown and Consolidated.

For all the portfolios between N and Q, we are short-selling Acme and investing positive amounts of money in Brown and Consolidated. As we move past point Q into the triangle, we begin taking positive positions in all three of the stocks. When we reach the point *MVP*, we have reached the lowest point in the "valley," the global minimum variance portfolio. This portfolio is also labeled *MVP* in Figure 5.2. As we move beyond *MVP* on the critical line, we move to the inferior positions on the bottom half of the minimum variance set.

At point *Z* in Figure 5.7, the critical line passes through the northeastern edge of the triangle. At this point we are taking no position at all in Consolidated, and we are investing positive amounts of money in Acme and Brown. Thus, portfolio *Z* must be positioned in Figure 5.2 at point *Z* where the combination line between Acme and Brown is tangent to the minimum variance set.

Note that the critical line doesn't pass through the southern edge of the triangle, so the combination line between Acme and Consolidated isn't tangent to the minimum variance set.

As we move past point *Z* on the critical line, we begin selling Consolidated short and investing positive amounts in Acme and Brown. Eventually, where the critical line passes through the horizontal axis, we begin to sell both Consolidated and Brown short and use the proceeds to invest in Acme.

It is important to recognize that the critical line represents all the portfolios in the minimum variance set. The portfolios we are interested in as investors are those in the efficient set. In Figure 5.7 the part of the critical line representing portfolios in the efficient set is represented by the solid portion of the line.

FINDING THE MINIMUM VARIANCE WITHOUT SHORT SELLING

Some financial institutions do not sell short as a matter of policy. Consequently, it may be of interest to discuss how the minimum variance set is determined when you are constrained not to sell short.

If you can't sell any stock short, the portfolio weight for each stock must be no less than 0 and no greater than 1. This means you are constrained to stay on, or within, the boundaries of the triangle of Figure 5.3. Consequently, those positions on the minimum variance set, which correspond to points on the critical line that are outside the triangle (such as point *N* in Figure 5.7), are no longer available to you.

However, since the critical line passes through the triangle, some of the portfolios that were available before will still be available, even though you can no longer sell short. These will be the portfolios for which you were investing either positive amounts in each stock or positive amounts in two of the stocks with no position at all in the third.

The triangle of Figure 5.3 is reproduced in Figure 5.8. Let's start with portfolio *Z* on the northeastern edge of the triangle. This portfolio is positioned at point *Z* in Figure 5.9 on the bottom half of the bullet. As we move from point *Z* to point *Q* on the critical line of Figure 5.8, we move from point *Z* to point *Q* in the minimum variance set of Figure 5.9. This section of the minimum variance set is identical to the case where short selling is allowed. Once we get to point *Q*, however, we can't continue toward the northwest on the critical line. Consider your options at this point. You can either (1) move up the western edge to point *R*, (2) move down the western edge to point *S*, or (3) move back inside the triangle somewhere.

Let's first consider option 1, moving up the western edge. As you move up the western edge, you are taking positive positions in Brown and Consolidated and no

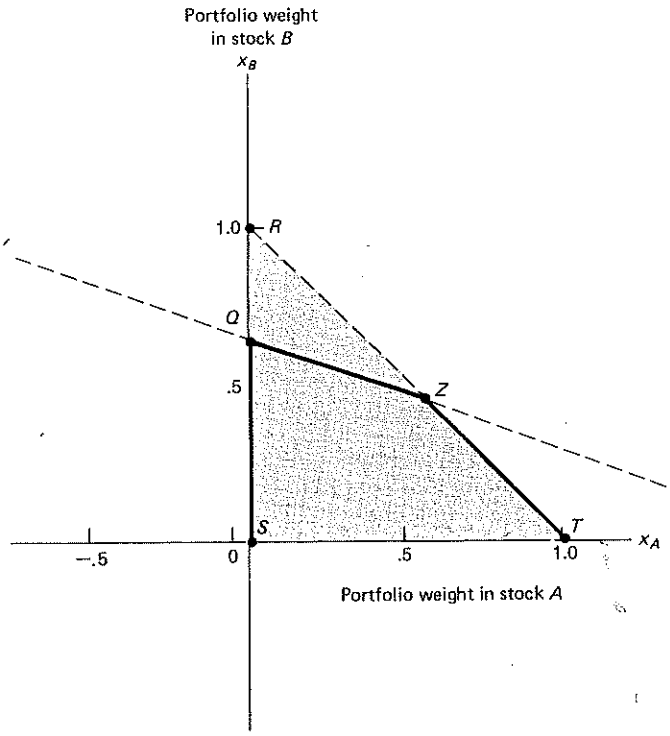


FIGURE 5.8 CRITICAL LINE WITH NO SHORT SELLING.

position at all in Acme. As you move closer to point R, your portfolio weight in Brown becomes larger and larger, finally reaching 1.00 when you reach point R. In Figure 5.9 you have been moving on the combination line between Brown and Consolidated from point Q to the position of Brown's stock at point B. The portfolios on this segment of the combination line aren't minimum variance portfolios. The portfolios between Q and Z have lower variances for the same expected return, and they are available, even though you can't sell short.

Now let's consider option 3, moving back inside the triangle. We already know, given any portfolio inside the triangle, other portfolios positioned on the same isoexpected return line but closer to the critical line have lower variance, given their expected return. Therefore, any portfolio inside the triangle, which is off the critical line, can't be minimum variance.

This leaves us with option 2, moving down the western edge toward point S. If we take this option, we are again taking positive positions only in Brown and Consolidated, but now we are moving up the combination line between the two stocks in Figure 5.9 toward the position of Consolidated at point C. Given our constraint not to sell short, these portfolios represent the lowest variance positions we can attain, given the expected returns, and they are part of the constrained minimum variance set. Given that we can't sell short, we can't move beyond point C, so the minimum variance set runs from point C, through point Q, to at least point Z. To see where it goes from Z, let's take a closer look at the portfolio position represented by that point.

At point Z we are investing positive amounts in Acme and Brown and nothing at all in Consolidated. Point Z, therefore, lies on the combination line between Acme and

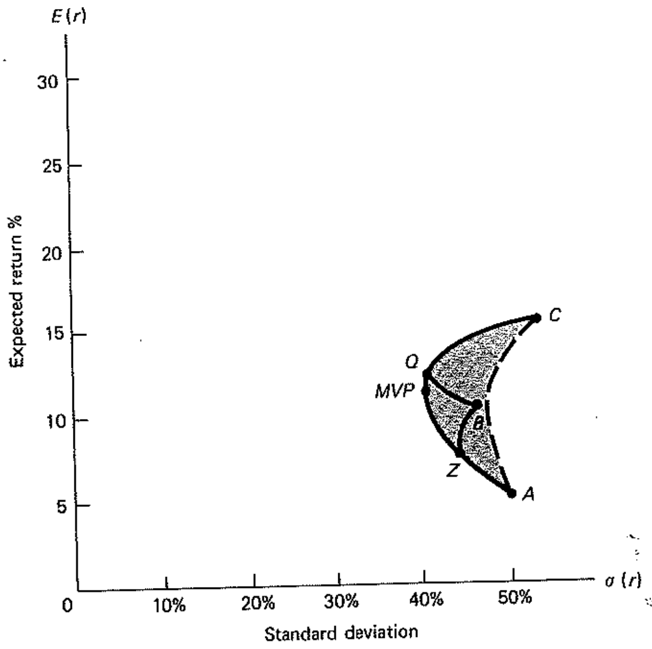


FIGURE 5.9 MINIMUM VARIANCE SET WITH NO SHORT SELLING.

Brown as drawn in Figure 5.9. If we move up the northeastern edge of the triangle toward point *R*, we move along the combination line toward the position of Brown's stock at point *B*. These aren't minimum variance portfolios, because some of the portfolios on the bullet between *Q* and *Z* have lower variance and the same expected return. However, if we move down the northwestern edge instead, toward point *T*, we move along the combination line of Figure 5.9 from *Z* to the position of Acme's stock at *A*. These are minimum variance portfolios, given our no-short-selling constraint.

Thus, the minimum variance set begins at point *C* and runs through points *Q* and *Z*, finally ending at point *A*. Note that the minimum variance set for the case of no short selling falls inside the bullet drawn on the basis of unrestricted short selling. That is because the unconstrained minimum variance set took full advantage of all possible strategies in terms of the weights allocated to the three stocks. When you rule out most of these strategies by disallowing short sales, you rule out many opportunities to further reduce variance, given the level of expected return, or to increase expected return, given the level of variance.

In this example, the minimum variance set without short selling partially coincides with the minimum variance set where short selling is allowed. That is because the critical line goes through the triangle. If the critical line doesn't pass through the triangle, as is the case in Figure 5.10, the minimum variance set without short selling will fall inside the unconstrained minimum variance set as in Figure 5.11 (see page 96).

To see some examples of security volatility and correlation, you can go to the Web site www.TheNewFinance.com. Once there, go to the area labeled Modern Investment Theory. Install PManager into your computer. Then go to the area labeled Sessions. Copy the session called No Short Selling into the directory in you computer called Optimize. You will find it on your C: drive.

Now go to the Programs section of Windows and run PManager. Go to Open under File, and select and open the file No Short Selling. Bring up the window Expected Return under Optimize. You will see the assumed expected returns for Acme, Brown, and Con-

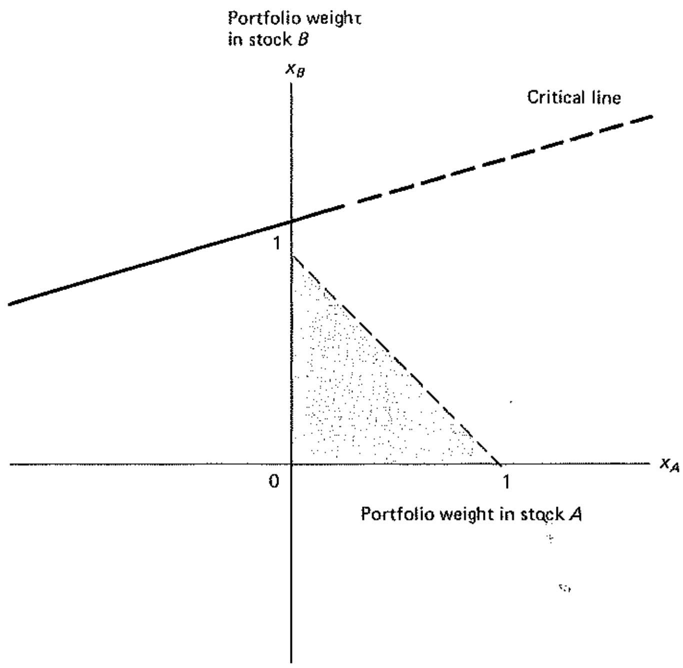


FIGURE 5.10 CASE WHERE CRITICAL LINE DOESN'T PASS THROUGH TRIANGLE.

solidated. Now bring up the window Correlation Matrix under Optimize. Correlations are in black and standard deviations are in red. These are the numbers we have assumed in this chapter. Next bring the window Exp. Ret/Asset Frontier under Optimize. You should see the bullet shown in Figure 5.9. Highlight Tile under window. The correction matrix, efficient frontier, and expected returns should be displayed side by side. You can now make changes in the assumptions regarding expected return, standard deviation, and correlation to see their impact on the frontier. To do this, point to and click on either the expected return or correlation window, and type in your new assumptions. Then click on the frontier window to see the result.

TWO IMPORTANT PROPERTIES OF THE MINIMUM VARIANCE SET

Property I If we combine two or more portfolios on the minimum variance set, we get another portfolio on the minimum variance set.

This important property follows directly from the fact that the critical line is a straight line. Recall that the critical line traces out the points of highest expected return on the isostandard deviation ellipses. The critical line is linear because the isostandard deviation ellipses are all symmetric about a common point (the minimum variance portfolio). As a result, when we trace out the points of highest expected return, we trace out a straight line.

To illustrate property I, consider portfolios 1 and 3 in Figure 5.12, where the portfolio weights for Acme and Brown are plotted on the horizontal and vertical axes,

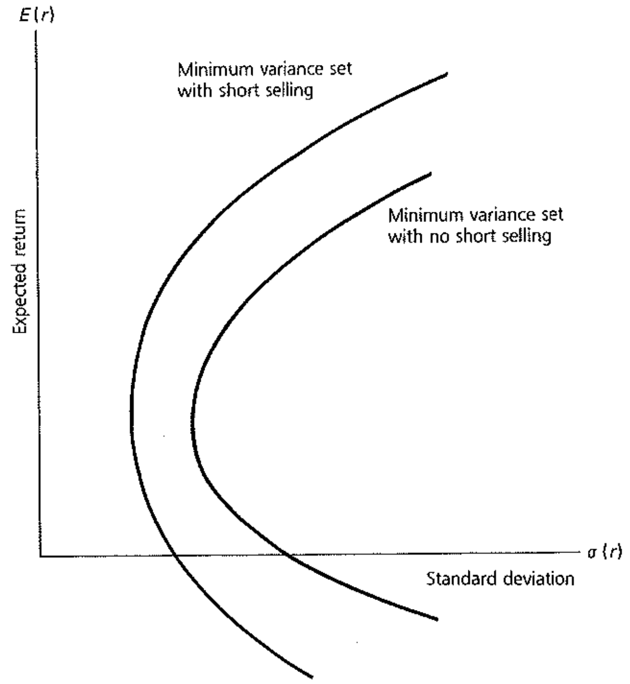


FIGURE 5.11 CORRESPONDING MINIMUM VARIANCE SETS.

respectively. Suppose we combine these two portfolios by investing $1,000 in each. The portfolio weights for the two portfolios are given by

	x_A	x_B	x_C
Portfolio 1	−1.50	1.20	1.30
Portfolio 3	.00	.70	.30

Given a $1,000 investment in each portfolio, these portfolio weights are consistent with the following dollar commitments:

	Acme	*Brown*	*Consolidated*
Portfolio 1	−$1,500	$1,200	$1,300
Portfolio 3	0	$ 700	$ 300
Combined portfolio 2	−$1,500	$1,900	$1,600

Because we are investing a total of $2,000 in the combined portfolio, the dollar positions in the three stocks are consistent with the following portfolio weights for the three stocks:

	A	*B*	*C*
Portfolio 2	−.75	.95	.80

If we plot the combined portfolio in Figure 5.12, it plots at point 2. Note that this point is on the critical line, so the combined portfolio is also a member of the minimum vari-

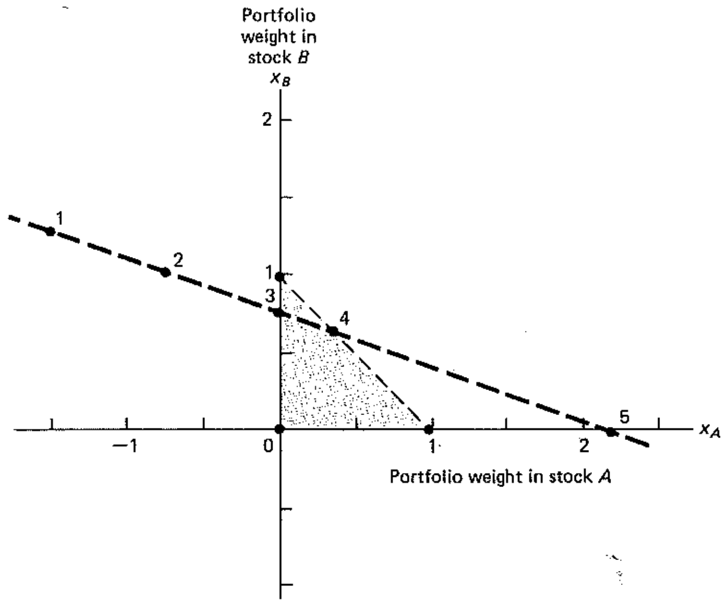


FIGURE 5.12 PORTFOLIO WEIGHTS IN ACME AND BROWN.

ance set. This same thing will happen no matter which, or how many, of the minimum variance portfolios are combined. We can, in fact, sell short some of the portfolios and use the proceeds to invest in others. As long as all the portfolios are in the minimum variance set, the combined portfolio will also be on the bullet.

As we will see in Chapter 7, the central prediction of the capital asset pricing model (CAPM) is that the market portfolio is positioned on the efficient set. The CAPM is a theory that assumes everyone can short sell without restriction and predicts the way securities would be priced if everyone used portfolio theory and invested in efficient portfolios. Keep in mind that the market portfolio is a combination of all the portfolios of every investor in the economy. Given property I, we know that if each investor holds an efficient portfolio, the combination of all of them will be efficient as well. In this sense, property I drives the central prediction of the CAPM.

Property II Given a population of securities, there will be a simple linear relationship between the beta factors of different securities and their expected (or average) returns if and only if the betas are computed using a minimum variance market index portfolio.[2]

The ***beta factor*** of a security describes the response of the security's returns to changes in the rates of return to the market portfolio, which is a portfolio composed of all risky (e.g., having positive standard deviations) investments in the economic system. To illustrate the concept of a beta factor, suppose we expect the return to the market portfolio to be 6 percent greater next month than it was last month. If this causes us to

[2] This property was originally discovered by Sharpe (1964). Its implications were not fully appreciated, however, until the publication of an important paper by Roll (1977).

revise upward our expectation for the rate of return on an individual stock by 12 percent, we can say the stock has a beta factor of 2.00. If, instead, our expectation for the stock increased by only 3 percent, the stock would have a beta of only .50.

Property II states that if we estimate betas by using a minimum variance portfolio as a proxy for the market portfolio, the relationship between our estimated betas for individual stocks and their average rates of return will be exactly linear. To see this, suppose we sample the returns to Acme, Brown, and Consolidated over a six-year period and find that the stocks produce the following rates of return:

Year	*Acme*	*Brown*	*Consolidated*
1	36%	35%	53%
2	−11	−8	−37
3	−18	−20	69
4	70	28	50
5	25	76	16
6	−72	−51	−61
Mean	5	10	15
Standard deviation	49.6	45.3	53.0

The sample covariance matrix for the three stocks for the six-year period can be computed from these returns as

Stock	*Acme*	*Brown*	*Consolidated*
Acme	.246	.179	.178
Brown	.179	.205	.112
Consolidated	.178	.112	.281

Based on these numbers we can now compute the minimum variance set. The minimum variance set and the positions of the three stocks are plotted on the left-hand side of Figure 5.13.

Now suppose we want to compute beta factors, with reference to some index portfolio, for the three stocks. Assume we select as such an index portfolio one of the portfolios in the minimum variance set, say, portfolio M in Figure 5.13.

Portfolio M is represented by a set of portfolio weights, one weight for each of the three stocks:

Portfolio Weights in Portfolio M	
Acme	−1.000
Brown	1.139
Consolidated	.861

Because we have six yearly returns for each of the three stocks, we can compute the corresponding six yearly returns to portfolio M. For each year we multiply the return to each stock by its portfolio weight and sum up the products. For example, the portfolio's return in the first year can be computed as

$$49.5\% = -1.00 \times 36\% + 1.139 \times 35\% + .861 \times 53\%$$

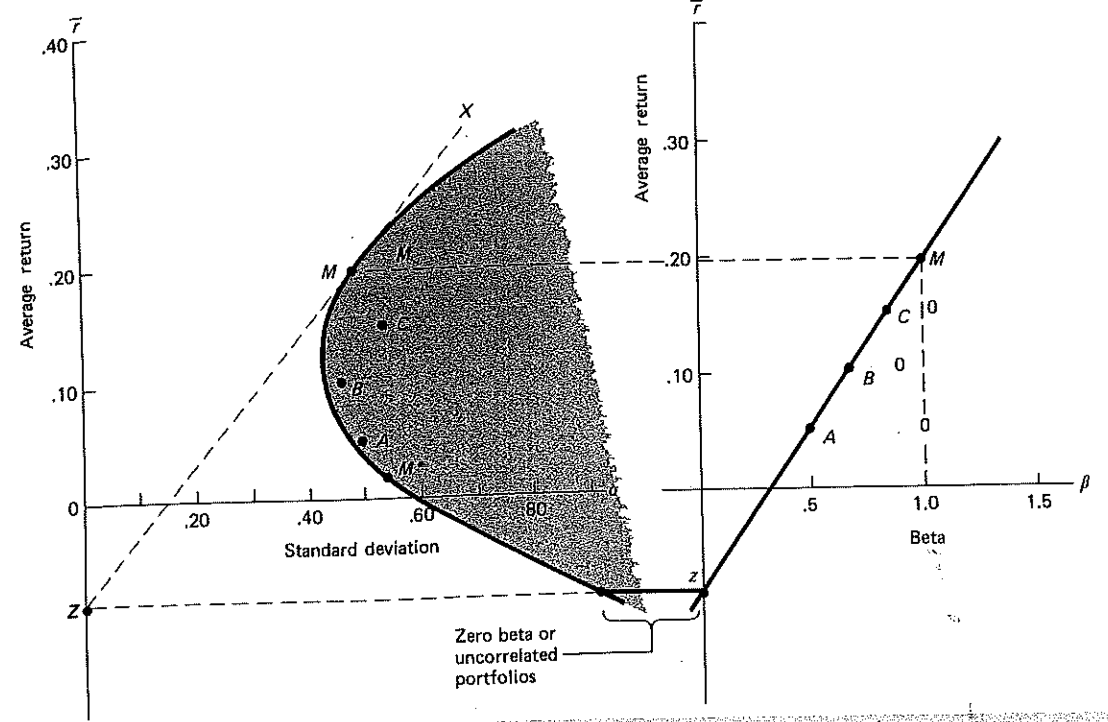


FIGURE 5.13 THE POSITIONING OF STOCKS IN AVERAGE RETURN–BETA SPACE.

In this way the six portfolio returns can be computed as

Year	Return to Portfolio M
1	49.5%
2	−29.9
3	54.6
4	4.9
5	75.3
6	−38.6
Mean	19.3
Standard deviation	47.3

Now we can compute the beta factor for each stock by relating the individual stock's returns to portfolio M's returns, as we do for Acme, Brown, and Consolidated in Figures 5.14(a), (b), and (c). The line in each figure is our estimate of the characteristic line for the stock. The slope of these lines is our estimate of the beta factor (computed as the ratio of each stock's sample covariance with portfolio M to portfolio M's sample variance). The betas are given by

	Beta Factor
Acme	.493
Brown	.670
Consolidated	.848

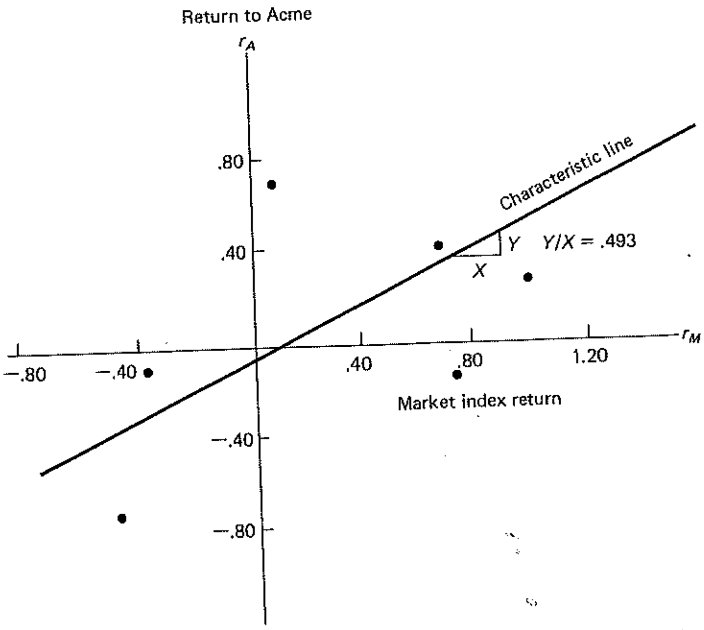
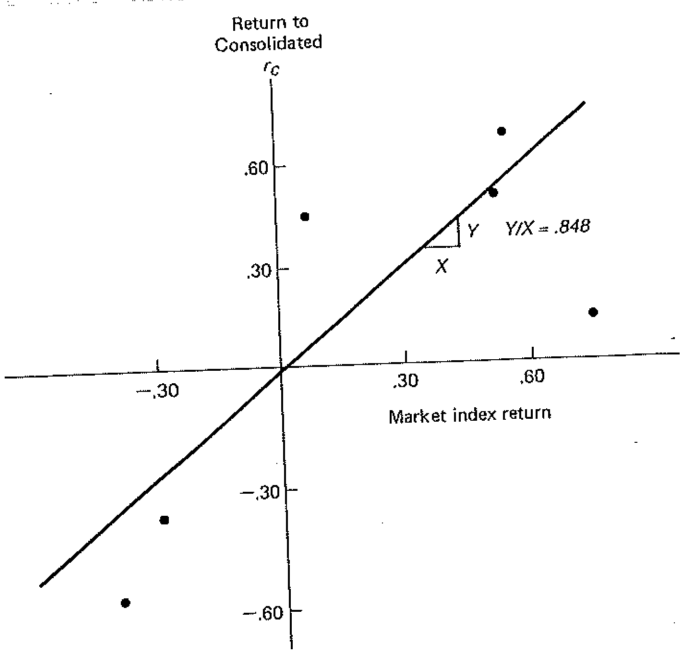


FIGURE 5.14(A) ESTIMATED CHARACTERISTIC LINE FOR ACME.

FIGURE 5.14(B) ESTIMATED CHARACTERISTIC LINE FOR CONSOLIDATED.



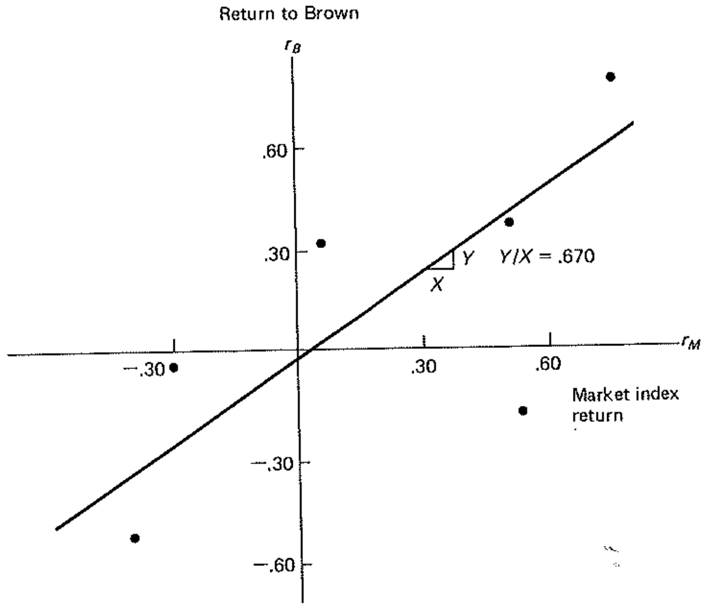


FIGURE 5.14(c) ESTIMATED CHARACTERISTIC LINE FOR BROWN.

At this point we plot the betas against the average rates of return for each stock. Given the position of the three stocks and portfolio *M*, as drawn in the left side of Figure 5.13, the plot *must* look like the right side of Figure 5.13. Note that we can draw a straight line through the positions of each of the three stocks on the right side. This will always be the case, no matter how many stocks we are dealing with, as long as the market index we select is in the minimum variance set for the stocks considered. In fact, we don't even have to go through the trouble of computing the betas for each of the stocks and then plotting them to find the line. The position of the line on a graph relating average or expected return to beta can be found directly.

To find the line relating average return to beta, first draw a line tangent to the bullet at the position of the index portfolio you have selected. The broken line *ZX* in Figure 5.13 is such a line. Now consider point *Z*, where the line of tangency intersects the vertical axis. Plot this same level of average return on the vertical axis of the right-hand graph of Figure 5.13 at *Z*. Now plot the index portfolio on the right-hand side. To do this, think of a plot like those in Figures 5.14(a) through 5.14(c) for the market index itself.

Because in this case, we are plotting the same returns on both the horizontal and vertical axes, all points will fall on a 45-degree line extending from the origin of the graph. The slope of this line, of course, would be equal to 1. The index portfolio, therefore, has a beta equal to 1 and is plotted at point *M* in the right side of Figure 5.13.

The relationship between beta and average return for all securities can now be found by drawing a straight line through points *Z* and *M*. *Every* security in the population considered will be positioned on this line. The position of each security on the line (and, therefore, the beta factor for each stock) is determined completely by the average return for the security in the time period observed.

Note that all securities with an average return equal to *Z* will have a beta equal to 0. Given that beta is equal to security covariance with the index portfolio divided by the index portfolio's variance, we know all securities positioned on the solid segment of the horizontal line passing through the bullet are completely uncorrelated with the index portfolio. One of these portfolios has the lowest variance and is, therefore, positioned on the bullet. We shall refer to this portfolio as the *minimum variance, zero beta portfolio.*

You should be able to see that if the index portfolio is positioned on the bullet above the minimum variance portfolio, the line on the right side of Figure 5.13 will be positively sloped. If it is positioned below the minimum variance portfolio, the line will be negatively sloped. With a market index like *M,* Consolidated has the largest beta, because it has the largest average return. On the other hand, if we selected an index portfolio like *M,* Acme would have the largest beta, because it has the smallest average return.

The relationship of property II stems from the fact that the combination lines between each of the individual securities and the index portfolio must be tangent to the bullet at the position of the index portfolio, as in Figure 5.15(a). If the combination lines didn't reflect *off* the bullet at this point but rather went *through* it, the bullet couldn't be efficient or minimum variance, as we have defined it.

In Figure 5.15(b) we are dealing with an index portfolio that is *inefficient.* Now the combination lines for the various stocks can move through the position of the portfolio at various angles relative to one another. It is no longer true that for changes in the weight assigned to each individual stock, the average return and variance of the portfolio change in the same proportion.

If the index portfolio is minimum variance, however, the combination line for all the securities must have the same slope at the position of the index portfolio on the bullet. This means that if we slightly change the portfolio weight assigned to *any* security, the standard deviation and average return of the index portfolio will change in the same proportions relative to one another for each and every security. Suppose, for example, we slightly change the weight in the index portfolio assigned to Acme, and we find that the change in the average return to the index portfolio is twice as great as the change in its standard deviation. We will find this is also the case when we change the weights assigned to Brown and Consolidated.

Consider, first, what determines the extent to which the portfolio's expected return changes as we change the portfolio weight. The magnitude of the change increases with the difference between the expected return to the stock and the index. In the case where they have the same expected return, there would be no change as we change the weight.

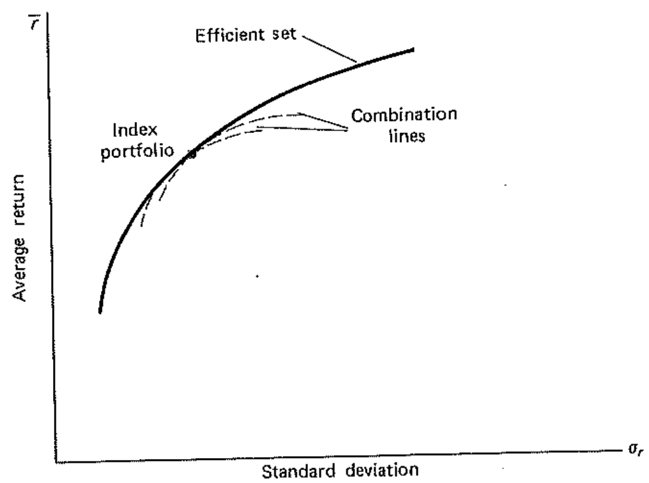


FIGURE 5.15(A) COMBINATION LINES FOR EFFICIENT INDEX PORTFOLIO.

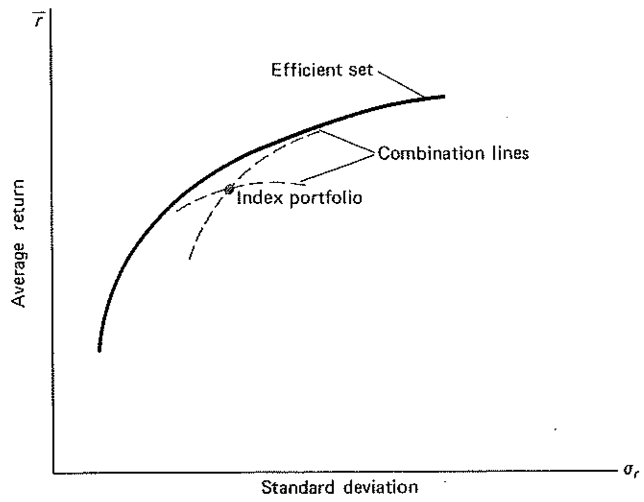


FIGURE 5.15(B) COMBINATION LINES FOR INEFFICIENT INDEX PORTFOLIO.

Now consider the effect of a change in the weight assignment on the index portfolio's *standard deviation.* In this case, the larger the covariance between the stock and the index portfolio, the larger will be the impact of a change in weight assignment on the portfolio's standard deviation. Because beta is computed as the ratio of this covariance to the variance of the market index, we can also say the effect on the portfolio's standard deviation is directly related to beta.

Thus, as we change the portfolio weight, the impact on portfolio expected return depends on the expected return to the stock (relative to the index, which is a constant across all the stocks). The impact on portfolio standard deviation increases with the beta of the stock. If the index is minimum variance, the change in expected return is in the same proportion to the change in standard deviation as we make slight changes in the portfolio weights assigned to each and every stock in the population. That is so, however, only if security betas are linearly related to security expected rates of return. This relationship exists under property II.

A proof of property II is given in Appendix 4 following this chapter. Additional properties of the minimum variance set are given in Appendix 10 following the last chapter in this book.

Property II is extremely important and will be referred to many times throughout this book. To appreciate the importance of this property, consider the fact that armed with properties I and II, we can get a sneak preview of the essential characteristics of the capital asset pricing model, which is more completely discussed in Chapter 7.

The capital asset pricing model describes the way expected returns on different securities will relate to their risks if everyone in the economy used portfolio theory, as we have described it, to determine his or her investment positions. In such an event, we all would take positions scattered along the efficient set. If I were more aggressive than you, my position would be higher on the bullet than yours, but we would both be positioned somewhere on the bullet. The market portfolio is a portfolio containing all the capital investments in the economic system. It is, therefore, the aggregate of everyone's portfolio. On the basis of property I, we know that combinations of efficient portfolios are also efficient. This means that when we aggregate the efficient portfolios of all investors to obtain the market portfolio, it too will be efficient. The market portfolio will be sitting on the skin of the bullet.

In the capital asset pricing model, beta is taken to be the appropriate measure of risk of an individual security or investment. Betas are obtained by relating individual security returns to the returns of the market portfolio. We know, on the basis of property II, that because the market portfolio is efficient, there will be a simple linear relationship between the beta of any security and its expected rate of return. In the context of the CAPM, this relationship is referred to as the security market line. Thus, if index portfolio M on the left-hand side of Figure 5.13 is the market portfolio, we have the CAPM, and the solid line on the right-hand side is the security market line.

❖ SUMMARY

Given a plot of portfolio investment opportunities in expected return–standard deviation space, the bullet-shaped minimum variance set represents those portfolios that have the lowest possible variance, given a particular level of expected return. The portfolio in the minimum variance set with the lowest variance, or standard deviation, is called the minimum variance portfolio. All portfolios in the minimum variance set that have expected returns equal to or greater than the minimum variance portfolio are in the efficient set. Portfolios in the efficient set have the highest possible expected return, given their level of standard deviation.

The critical line provides the portfolio weights for the portfolios in the minimum variance set. This line traces out the points of tangency between the isoexpected return lines and the isovariance ellipses. An isoexpected return line shows the combinations of portfolio weights, all of which provide for a particular portfolio expected rate of return. An isovariance ellipse shows the combinations of portfolio weights, all of which provide for a particular portfolio variance. The slope and relative positions of the isoexpected return lines depend on the relative expected returns of the stocks considered. The shapes of the isovariance ellipses depend on the covariances between the stocks considered.

In the next chapter we will examine some of the properties of the minimum variance set.

APPENDIX 2

A Three-Dimensional Approach to Finding the Efficient Set

THE EXPECTED RETURN PLANE

The expected rate of return to a portfolio is a simple weighted average of the expected rates of return to the securities we are putting in the portfolio. Thus, securities combine in a linear fashion in terms of their expected rates of return.

Consider Figure A.2.1. This is a two-dimensional diagram. Now think of adding a third dimension that comes directly out from the page. We will plot expected portfolio return on this dimension. We now have three axes. On the floor or base of the diagram (which is actually Figure A.2.1) we have our two horizontal axes, which show the portfolio weights in Acme and Brown. On the vertical axis we are plotting the expected portfolio return, which corresponds to each combination of portfolio weights plotted on the base.

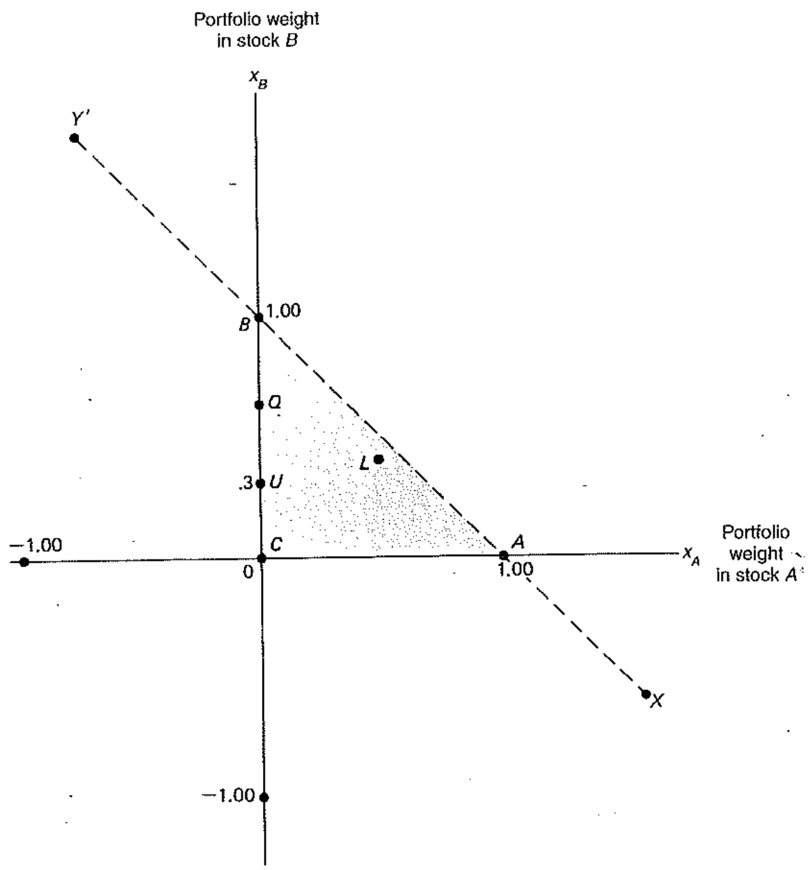


FIGURE A.2.1 PORTFOLIO WEIGHTS IN THE THREE-STOCK PORTFOLIO.

This three-dimensional diagram is depicted in Figure A.2.2. The plane depicted in the diagram is situated directly above the triangle covering positive positions in the three stocks. It is a flat surface sloping down toward you. The plane shows you the expected return to portfolios of Acme, Brown, and Consolidated, which are plotted on the base of the diagram.

Let's first consider the three points of the triangular plane. Consider point *A* on the base of the diagram. At *A*, you are investing all of your money in Acme and nothing in the other two stocks. Because you have formed a portfolio that is actually Acme and nothing else, it will have an expected return that is equal to 5 percent, the expected return for Acme. To find the expected portfolio return corresponding to point *A*, move directly up from the base at point *A*. You will hit the plane at a 5 percent rate of return relative to the vertical axis. Similarly, at point *C*, where you are investing everything in Consolidated, moving directly up from the base you hit the plane at a 15 percent expected return, the expected return for Consolidated. Moving up from *B*, where you are investing everything in Brown, you hit the plane at Brown's 10 percent expected return. Moving up from a point inside the triangle, where you are combining an investment in all three of the stocks, you find the portfolio's expected rate of return is a linear combination of the expected returns to the three stocks. Remember, the expected return to a portfolio is a simple weighted average of the expected returns to the stocks in the portfolio.

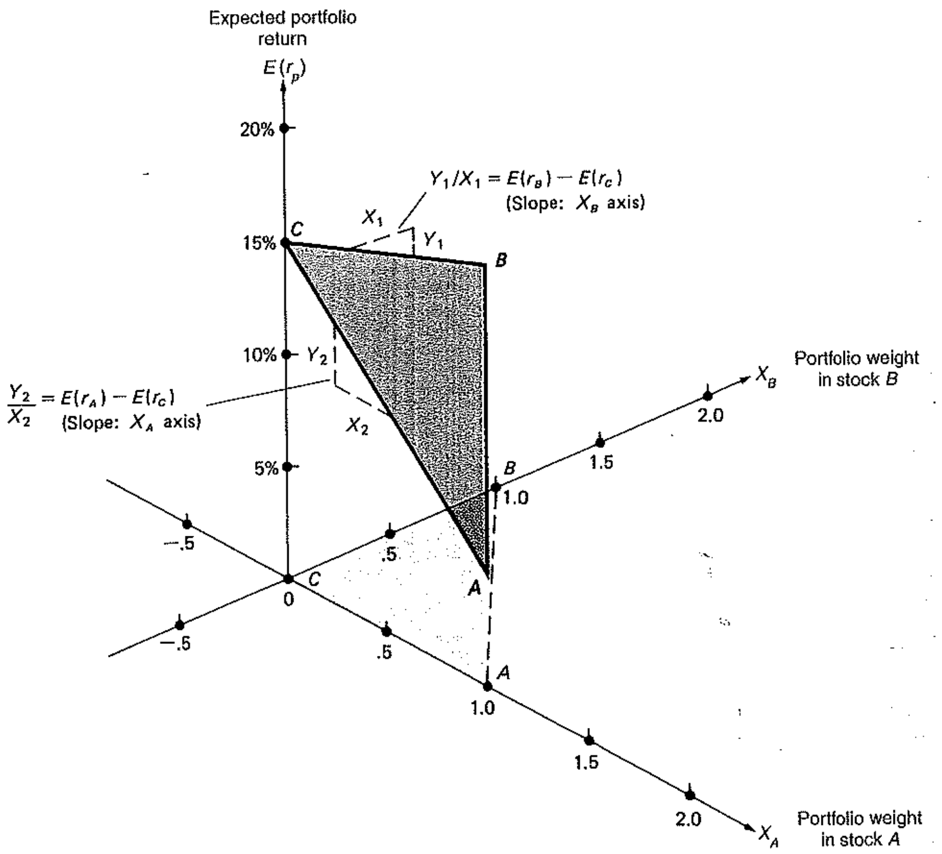


FIGURE A.2.2 THE EXPECTED RETURN PLANE.

You should be able to see that the plane is sloping down toward you only because Acme has the lowest expected rate of return. If Acme's expected return were, instead, the highest, the plane would then be sloping in an upward direction.

The plane actually extends indefinitely north, south, east, and west. For convenience, however, we have drawn only that segment positioned directly over the triangle, representing positive positions in each of the stocks.

ISOSTANDARD DEVIATION ELLIPSES

A similar diagram can be constructed showing the standard deviation of various portfolios of the three stocks. Figure A.2.3(a) is similar to Figure A.2.2, the only difference being that we are now plotting portfolio standard deviation on the vertical axis instead of expected return. For any given combination of portfolio weights, the standard deviation of the portfolio is computed by (1) taking each number in the covariance matrix and multiplying it by the portfolio weights for the two stocks associated with the covariance, (2) adding up the products, and (3) taking the square root of the sum. This is not, as in the case of the expected portfolio return, a simple linear process. Therefore, it shouldn't surprise us that the surface depicting the standard deviation of the portfolios represented on the base isn't flat.

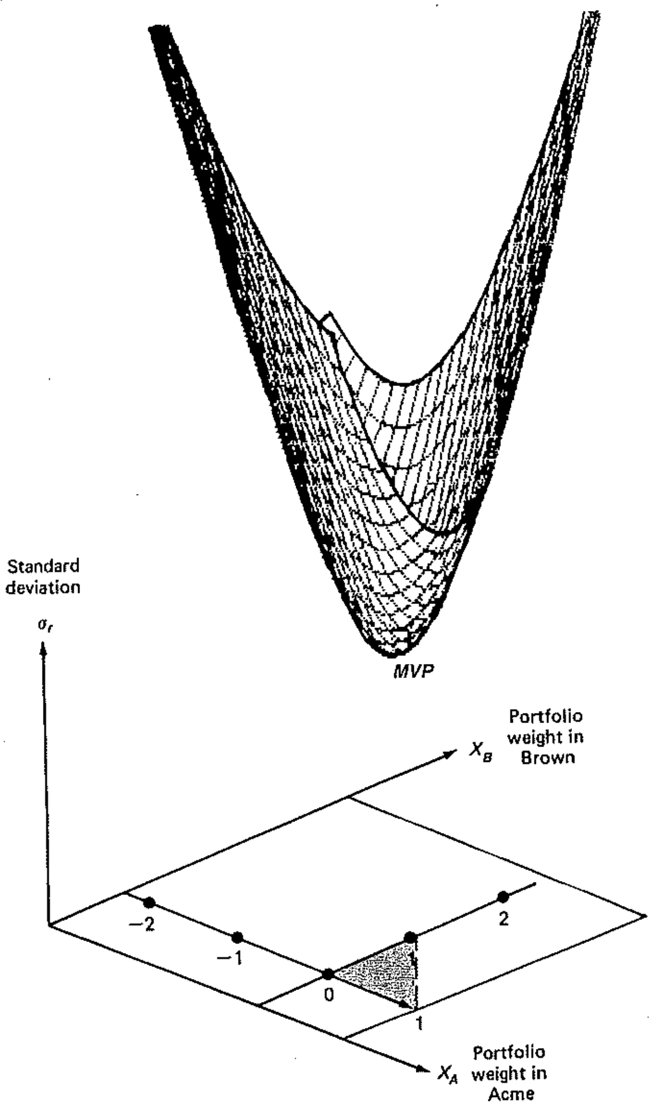


FIGURE A.2.3(A) PORTFOLIO STANDARD DEVIATION CORRESPONDING TO DIFFERENT PORTFOLIO WEIGHTS.

If portfolio standard deviation is repeatedly calculated for the various combinations of portfolio weights on the base of the diagram, a plot of the resulting portfolio standard deviations would produce the three-dimensional surface of Figure A.2.3(a). This surface looks like a net holding a watermelon. The net has a somewhat elliptical shape.

If you pick a point on the base of the diagram representing a particular set of portfolio weights for the three stocks, the distance you would have to move directly upward to hit the three-dimensional net would correspond to the standard deviation of a portfolio with the selected weights.

In Figure A.2.3(b) we have sliced the net with a horizontal plane at a particular level of portfolio standard deviation (55 percent). The points of intersection between the net and the horizontal plane trace out as a portion of an ellipse. When the ellipse

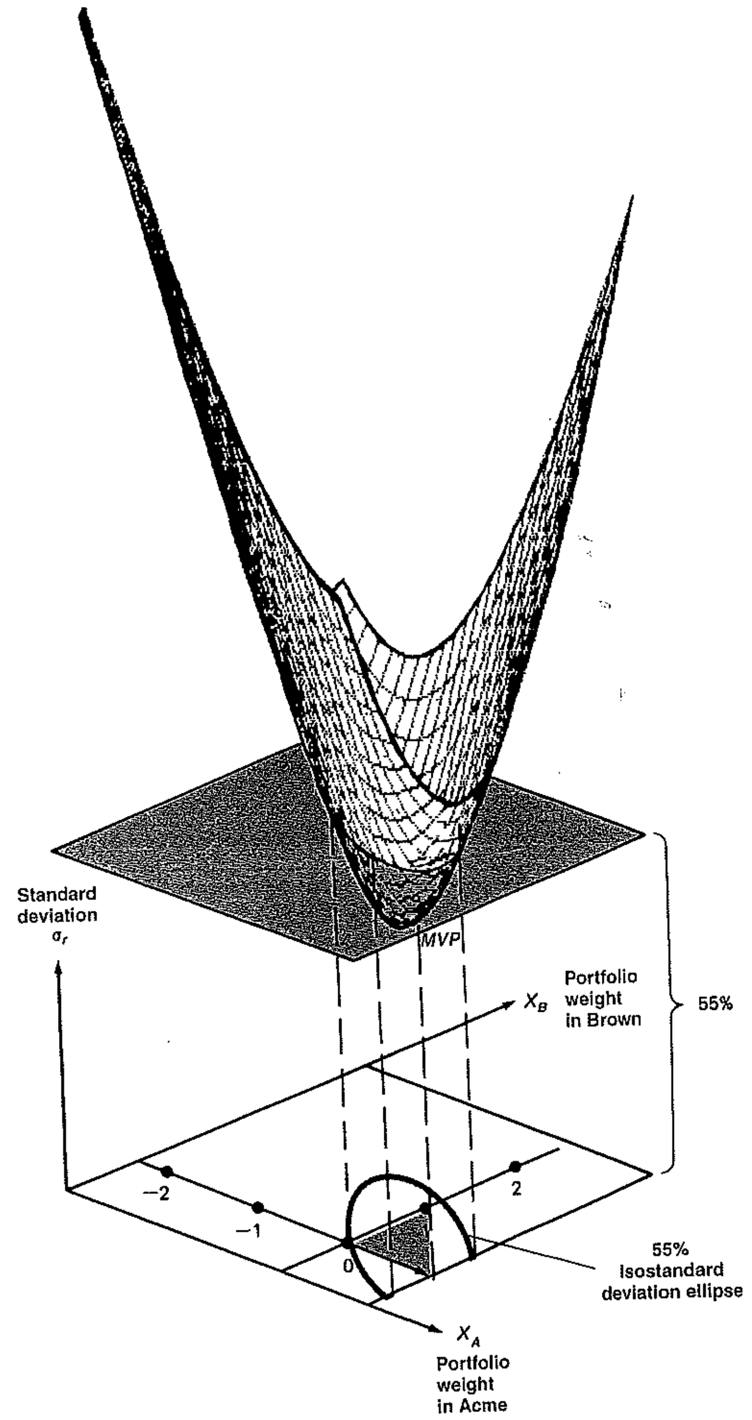


FIGURE A.2.3(B) DRAWING AN ISOSTANDARD DEVIATION ELLIPSE.

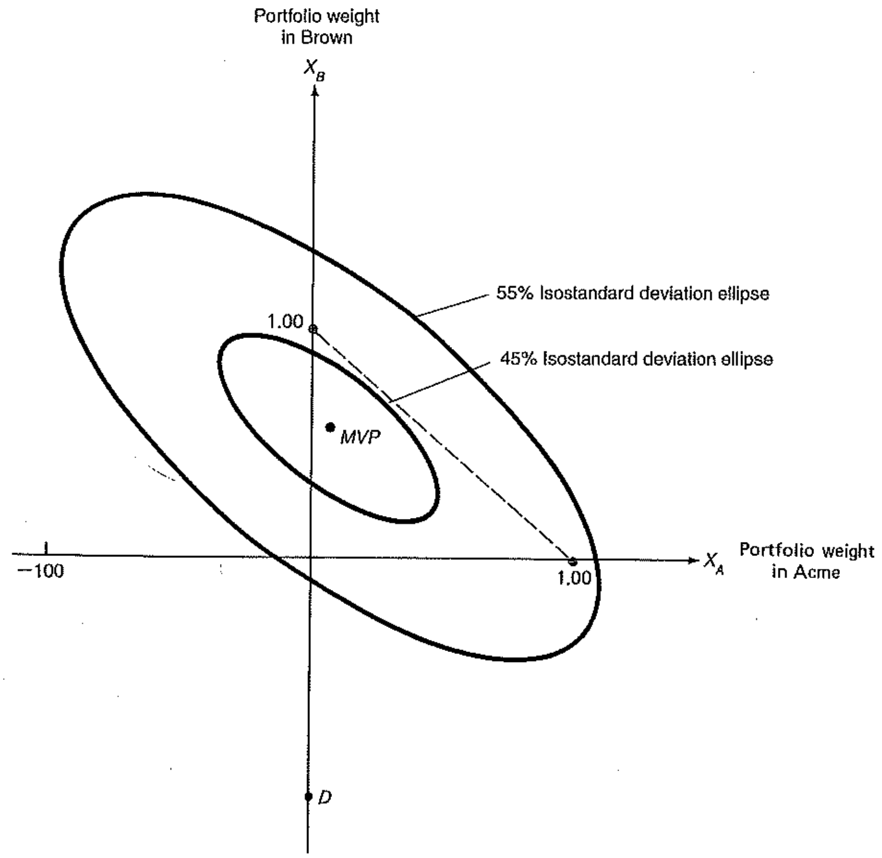
is superimposed on the base of the diagram, as in Figure A.2.3(b), it indicates those combinations of portfolio weights, all of which represent portfolios with 55 percent standard deviations. This ellipse is called an *isostandard deviation ellipse.* If we sliced the net of Figure A.2.3(b) with another horizontal plane at a different level of portfolio standard deviation, we would get another isostandard deviation ellipse representing a different portfolio standard deviation.

Two members of the family of isostandard deviation ellipses for Acme, Brown, and Consolidated are drawn in Figure A.2.4. The ellipses are centered about point *MVP.* The larger ellipse represents a larger portfolio standard deviation. Point *MVP* represents the one set of portfolio weights that produces the smallest possible portfolio standard deviation or variance. We have called this portfolio the *global minimum variance* portfolio. It is positioned at the lowest point of the net of Figure A.2.3(a) at point *MVP.* Drop a marble into the net, and it will settle at a position directly over *MVP.*

THE CRITICAL LINE

Now consider Figure A.2.5. In this figure we are again plotting expected portfolio return on the vertical axis, as in Figure A.2.2. In this figure, the isostandard deviation ellipses of Figure A.2.4 have been superimposed on the expected return plane of Figure A.2.2.

FIGURE A.2.4 TWO ISOSTANDARD DEVIATION ELLIPSES.



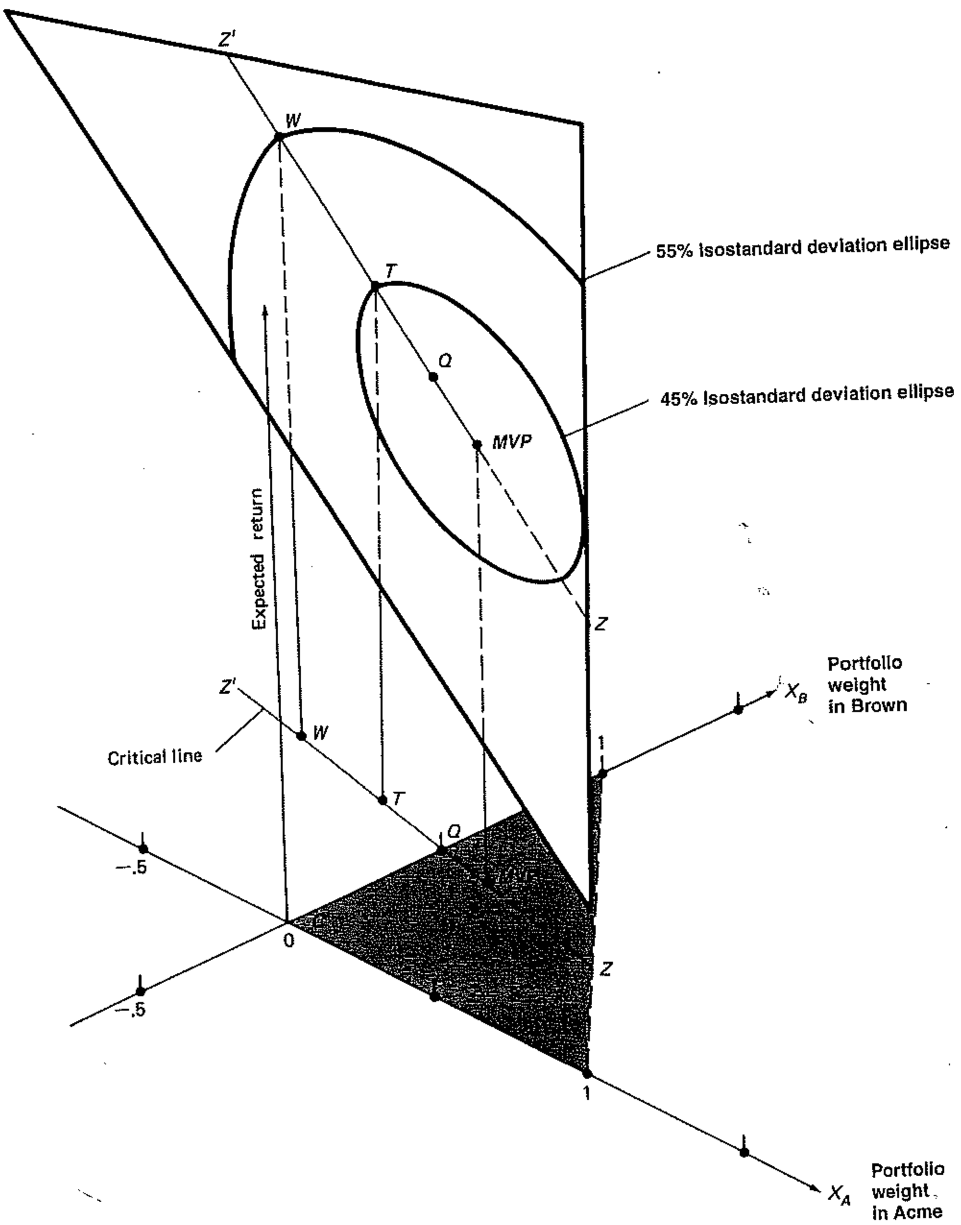


FIGURE A.2.5 SUPERIMPOSING THE ISOSTANDARD DEVIATION ELLIPSES ON THE EXPECTED RETURN PLANE.

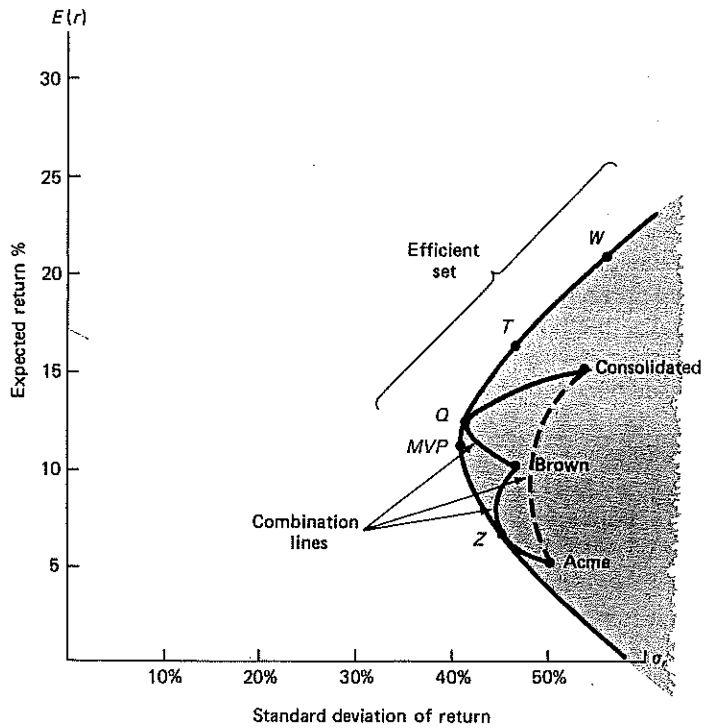
For reasons that will become obvious, the plane has now been drawn to extend farther out toward the northwest.

Our objective is to find the portfolio with the highest expected return, given the level of standard deviation. Suppose we want the standard deviation of our portfolio to be 45 percent. Given this constraint, we want the expected return on the portfolio to be as high as possible. If we want a 45 percent portfolio standard deviation, we must position ourselves somewhere on the 45 percent isostandard deviation ellipse. Given the location of the ellipse on the plane, we reach the highest possible point on the plane at point *T*. This is the portfolio with the highest possible expected return, given a 45 percent standard deviation. To construct this portfolio we must short-sell Acme and invest the proceeds in Brown and Consolidated. We know this is the case because point *T* is positioned over a point on the base of the diagram that represents a negative weight in Acme and a weight between 0.00 and 1.00 in Brown. The remaining portfolio position consists of a positive investment in Consolidated.

Because this portfolio provides the highest possible expected return, given a 45 percent standard deviation, it can be found on the efficient set of Figure A.2.6 at point *T*. If we want a portfolio with a larger standard deviation, say, 55 percent, we move to the 55 percent ellipse. Moving along the ellipse, we reach the highest point on the plane at point *W*. We are now selling additional amounts of Acme short, using the proceeds to increase our long position in both Brown and Consolidated. This portfolio can be found on the efficient set of Figure A.2.6 at point *W*.

Because the expected return plane is flat, and the isostandard deviation ellipses are concentric about point *MVP*, we can pass a straight line through the points on each ellipse representing the highest possible expected return. This is line $Z'Z$ in Figure A.2.5. This line, called the *critical line,* is superimposed on the base of the diagram and then plotted in two dimensions in Figure A.2.7. The portfolios in the efficient set (highest expected return, given standard deviation) are represented by the solid portion of the critical line. The broken portion of the line represents the remaining portfolios in the minimum variance set. These portfolios have the lowest possible expected return, given their standard deviation. Remember that each point in Figure A.2.7 represents the portfolio weights for a given portfolio. The points on the critical line represent the portfolio weights for all the portfolios in the minimum variance set.

FIGURE A.2.6 THE MINIMUM VARIANCE SET FOR CONSOLIDATED, BROWN, AND ACME.



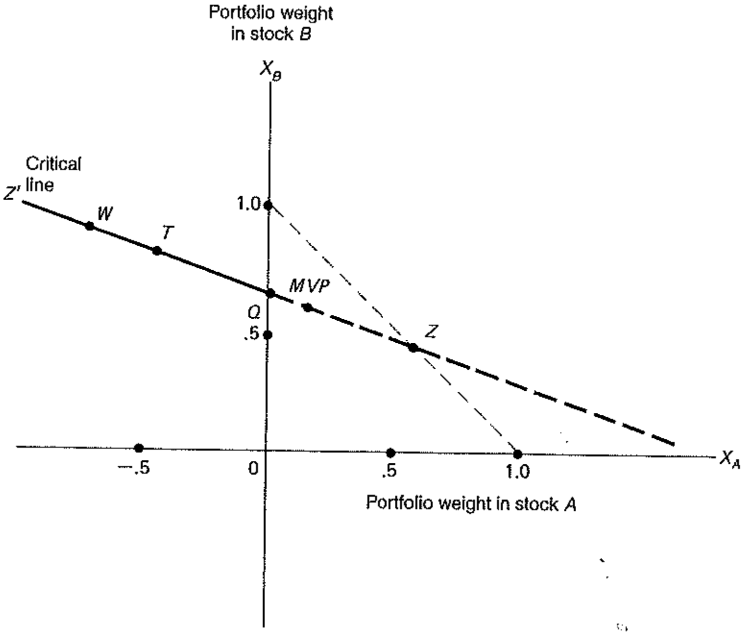


FIGURE A.2.7 THE CRITICAL LINE.

APPENDIX 3

Using Lagrangian Multipliers to Find the Minimum Variance Set

In Appendix 2, a graphic model of the procedure used to find the minimum variance set was presented. This model has considerable intuitive content but lacks mathematical precision. In this appendix, we present a numerical procedure for finding the minimum variance set for a three-stock portfolio. Once understood, the extension of this model to portfolios containing more than three stocks should be fairly straightforward.

Finding the minimum variance portfolio for a given level of expected rate of return is a constrained optimization problem. In the case of a three-stock portfolio, our objective is to minimize the portfolio variance.

$$\begin{aligned}\text{Minimize } \sigma^2(r_p) = {} & x_A^2\sigma^2(r_A) + x_B^2\sigma^2(r_B) + x_C^2\sigma^2(r_C) \\ & + 2x_Ax_B\mathrm{Cov}(r_A, r_B) + 2x_Ax_C\mathrm{Cov}(r_A, r_C) \\ & + 2x_Bx_C\mathrm{Cov}(r_B, r_C)\end{aligned}$$

subject to a target expected return $E(r_p^*)$

$$E(r_p^*) = \sum_{J=1}^{3} x_J E(r_J)$$

so that the sum of the portfolio weights must be 1.00:

$$1.00 = \sum_{J=1}^{3} x_J$$

The first of these three equations is called the ***objective function*** and the last two equations the ***constraints.***

To begin to solve the problem we rewrite the objective function in Lagrangian form:

$$\begin{aligned}\text{Minimize } \sigma^2(r_p) = {} & x_A^2\sigma^2(r_A) + x_B^2\sigma^2(r_B) + (1 - x_A - x_B)^2\sigma^2(r_C) \\ & + 2x_Ax_B \operatorname{Cov}(r_A, r_B) + 2x_A(1 - x_A - x_B) \operatorname{Cov}(r_A, r_C) \\ & + 2x_B(1 - x_A - x_B) \operatorname{Cov}(r_B, r_C) \\ & + b[E(r_p) - x_AE(r_A) - x_BE(r_B) - (1 - x_A - x_B)E(r_C)]\end{aligned}$$

where

$$x_C = 1 - x_A - x_B$$
$$b = \text{the Lagrangian multiplier}$$

If we set the target expected return $E(r^*_p) = .15$ and substitute in the values for the variances, covariances, and expected returns given earlier, we have

$$\begin{aligned}\text{Minimize } \sigma^2(r_p) = {} & .25x_A^2 + .21x_B^2 + .28(1 - x_A - x_B)^2 \\ & + .30x_Ax_B + .34x_A(1 - x_A - x_B) \\ & + .18x_B(1 - x_A - x_B) \\ & + b[.10 - .05x_A - .10x_B - .15(1 - x_A - x_B)]\end{aligned}$$

Simplifying

$$\begin{aligned}\sigma^2(r_p) = {} & .19x_A^2 + .31x_B^2 - .22x_A - .38x_B + .34x_Ax_B \\ & + .28 + b(-.05 + .10x_A + .05x_B)\end{aligned}$$

Next, take the partial derivatives, set them equal to zero, and solve simultaneously:

$$\begin{aligned} .38x_A + .34x_B + .10b - .22 &= 0 \\ .34x_A + .62x_B + .05b - .38 &= 0 \\ .10x_A + .05x_B \qquad\quad - .05 &= 0 \end{aligned}$$

We get

$$\begin{aligned} x_A &= .24 \\ x_B &= .52 \\ x_C &= .24 \\ b &= -.48 \end{aligned}$$

If we plug the portfolio weights into the original objective function, we find

$$\sigma^2(r_p) = .1668$$

Technically, the Lagrangian multiplier, b, indicates the incremental change in the value of the objective function solution due to an infinitesimally small change in the constraint (in this instance, the target expected return). Because the objective function is nonlinear, its slope changes continuously and so should b.

APPENDIX 4

Proof of Property II

Given a population of stocks, the cross-sectional relationship between the beta factors of the stocks and their expected returns will be perfectly linear and deterministic as long as the betas are computed with reference to any portfolio in the minimum variance set for the population of stocks.

The minimum variance set of Figure A.4.1 is drawn on the basis of the assumption that short selling is allowed. We have selected an arbitrary portfolio in the minimum variance set, at point *M*. We have also selected an arbitrary stock *J* from the many in the population. Since *M* is on the minimum variance set and since short selling is allowed, the combination line between *M* and *J* must be tangent to the bullet at point *M*. The portfolio weight of stock *J* in portfolio *M* can be either positive or negative. In any case, for purposes of this proof, we are going to be interested in portfolios where we combine stock *J* with portfolio *M* in various proportions. However, the position of *J* in *M* is already nonzero. Because we want to distinguish portfolio *M* from stock *J*, we're going to redefine the portfolio weight of *J* in a portfolio allocated between portfolio *M* and stock *J*. We're going to define the weight to be the fraction of our money committed to *J* beyond what is already committed to *J* in portfolio *M*. In this sense, when we are at point *M* on the combination line, $x_J = .00$. When we are at a point such as M', the portfolio weight is negative. For points on the line between *M* and *J*, the portfolio weight is positive.

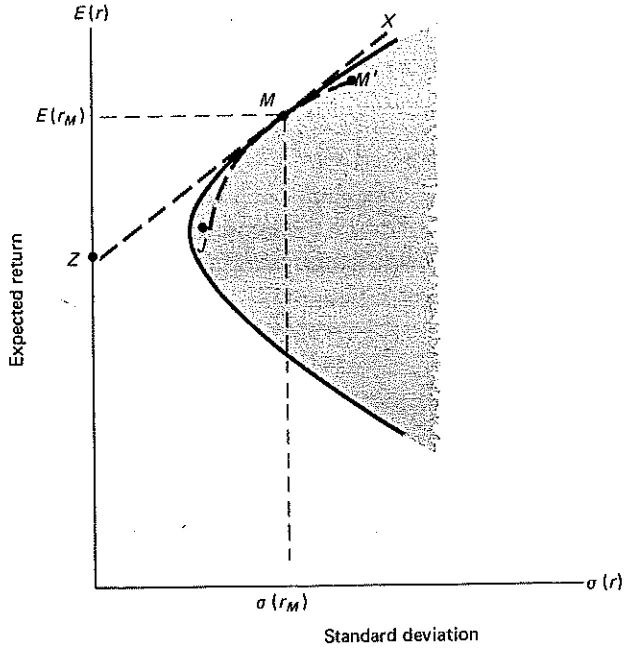


FIGURE A.4.1 COMBINATION LINE BETWEEN STOCK *J* AND PORTFOLIO *M*.

Given this definition of the portfolio weight, the expected rate of return to a portfolio in which we combine M and J is given by

$$E(r_p) = x_J E(r_J) + (1 - x_J)E(r_M)$$

and the standard deviation of the portfolio is given by

$$\sigma(r_P) = [x_J^2\sigma^2(r_J) + (1 - x_J)^2\sigma^2(r_M) + 2\,\mathrm{Cov}(r_J, r_M)x_J(1 - x_J)]^{1/2}$$

Note that $(1 - x_J)^2 = 1 + x_J^2 - 2x_J$, so that

$$\sigma(r_P) = [x_J^2\sigma^2(r_J) + \sigma^2(r_M) + x_J^2\sigma^2(r_M) - 2x_J\sigma^2(r_M) + 2\,\mathrm{Cov}(r_J, r_M)x_J - 2\,\mathrm{Cov}(r_J, r_M)x_J^2]^{1/2}$$

We know that the combination line between J and M' is tangent to the bullet at point M, because if it weren't tangent, it would penetrate the bullet, and this would be a violation of the definition of the bullet. The slope of the bullet at M is given by the slope of line ZM. Thus, the combination line JM' and line ZM have equal slopes at M. We will use the equality between the slopes to prove property II.

Our first step will be to get an expression for the slope of the combination line JM' at point M. We will then equate this expression to the slope of ZM. As you move along the combination line, both $\sigma(r_P)$ and $E(r_P)$ are changing in response to changes in x_J. Thus, we first derive expressions for the response of standard deviation and expected return to changes in the portfolio weight.

First, we take the derivative of $\sigma(r_P)$ with respect to x_J. The derivative of the nth power of a function is equal to the product of n, the $n - 1$ power of the function [in this case the function is $\sigma^2(r_P)$], and the derivative of the function. Thus, the derivative is the product of three terms. The first term is n:

$$n = \tfrac{1}{2}$$

The second term is the $n - 1$ power of the function.

$$[\sigma^2(r_P)]^{-1/2} = \frac{1}{\sigma(r_P)}$$

The third term is the derivative of the function $\sigma^2(r_P)$:

$$\frac{\partial\sigma^2(r_P)}{\partial x_J} = 2x_J\sigma^2(r_J) + 2x_J\sigma^2(r_M) - 2\sigma^2(r_M) + 2\,\mathrm{Cov}(r_J, r_M) - 4\,\mathrm{Cov}(r_J, r_M)x_J$$

We are interested in the derivative at point M in Figure A.4.1. At that point $\sigma^2(r_P) = \sigma^2(r_M)$ and $x_J = .00$, so the derivative of the function reduces to

$$\frac{\partial\sigma^2(r_P)}{\partial x_J} = -2\sigma^2(r_M) + 2\,\mathrm{Cov}(r_J, r_M)$$

Thus, the product of the three terms is given by

$$\frac{\partial\sigma(r_P)}{\partial x_J} = \frac{1}{2}\,\frac{1}{\sigma(r_P)}\,[-2\sigma^2(r_M) + 2\,\mathrm{Cov}(r_J, r_M)]$$

which reduces to

$$\frac{\partial\sigma(r_P)}{\partial x_J} = \frac{-1}{\sigma(r_M)}\,[\sigma^2(r_M) - \mathrm{Cov}(r_J, r_M)]$$

Now we know that $\beta_J = \text{Cov}(r_J, r_M)/\sigma^2(r_M)$, so it follows that $\text{Cov}(r_J, r_M) = \beta_J\sigma^2(r_M)$. Substituting this value for the covariance in the derivative and canceling, we get

$$\frac{\partial\sigma(r_P)}{\partial x_J} = -[\sigma(r_M) - \sigma(r_M)\beta_J]$$

Now we're going to take the derivative of the *expected value* of the portfolio with respect to x_J. By multiplying through by $E(r_M)$, we can write the expected return to the portfolio as

$$E(r_P) = x_J E(r_J) + E(r_M) - x_1 E(r_M)$$

Thus,

$$\frac{\partial E(r_P)}{\partial x_J} = E(r_J) - E(r_M) = -[E(r_M) - E(r_J)]$$

To find the change in portfolio expected return accompanying a change in portfolio risk along the combination line at point M, we take the derivative of $E(r_P)$ with respect to $\sigma(r_P)$ using the chain rule:

$$\frac{\partial E(r_P)}{\partial\sigma(r_P)} = \frac{\partial E(r_P)/\partial x_J}{\partial\sigma(r_P)/\partial x_J} = \frac{E(r_M) - E(r_J)}{\sigma(r_M) - \sigma(r_M)\beta_J}$$

Referring back to Figure A.4.1, we draw line ZM tangent to the minimum variance set at point M. The slope of this line is given by

$$\frac{E(r_M) - Z}{\sigma(r_M)}$$

Remember that at the point of tangency the slope of ZM is equal to the derivative of $E(r_P)$ with respect to $\sigma(r_P)$. Thus,

$$\frac{\partial E(r_P)}{\partial\sigma(r_P)} = \frac{E(r_M) - E(r_J)}{\sigma(r_M) - \beta_J\sigma(r_M)} = \frac{E(r_M) - Z}{\sigma(r_M)}$$

or by transposing

$$\frac{\sigma(r_M) - \beta_J\sigma(r_M)}{\sigma(r_M)} = \frac{E(r_M) - E(r_J)}{E(r_M) - Z}$$

Factoring $\sigma(r_M)$ from the left-hand side of the equation, canceling, and multiplying both sides by $E(r_M) - Z$, we get

$$[E(r_M) - Z](1 - \beta_J) = E(r_M) - E(r_J)$$

Solving for $E(r_J)$, we have property II:

$$E(r_J) = Z + [E(r_M) - Z]\beta_J$$

Thus, given that we are using as a market index a portfolio in the minimum variance set, the beta factor for any stock in the population from which we constructed the minimum variance set is deterministically related to its expected return in a linear fashion. In fact, by solving for β_J, we see that the beta of any stock can be computed as

$$\beta_J = \frac{E(r_J) - Z}{E(r_M) - Z}$$

APPENDIX 5

Utility and Risk Aversion[1]

Utility is a measure of well-being. The source of utility is consumption (of goods and services, natural beauty, and the like), and an important resource that can be transformed into consumption is dollar wealth. Because we are concerned with the utility derived from outcomes of investments, we will focus on the effect of these investments on our wealth level.

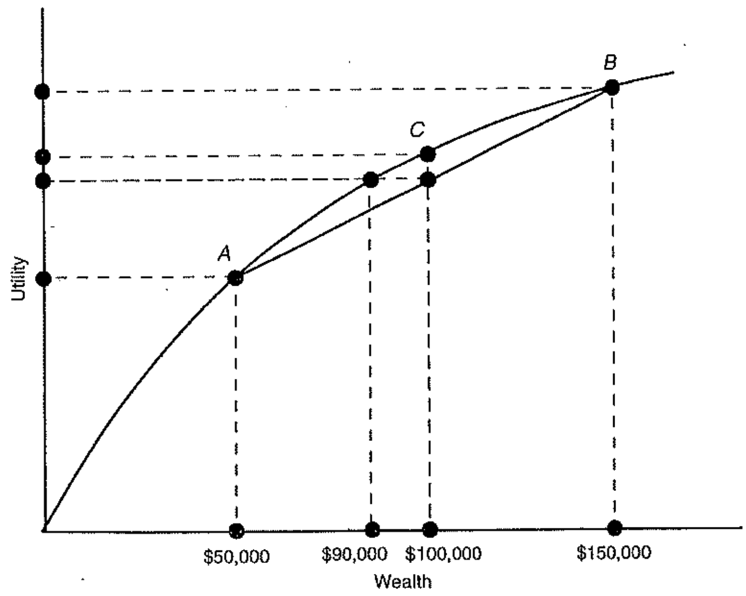
Figures A.5.1, A.5.2, and A.5.3 show the relationship between utility and wealth levels for three different investors.

For the investor of Figure A.5.1, utility increases with wealth (as it does with all three), but it increases at a diminishing rate. That is, as wealth goes from \$50,000 to \$100,000 to \$150,000, the gain from both increases is positive, but the gain from the second increase is less than the gain from the first. We can say that such an investor has *diminishing marginal utility.*

Consider how an investor with diminishing marginal utility reacts to a gamble versus a certain outcome. The gamble is an equal chance of ending up with a wealth level of \$50,000 or \$150,000. The expected dollar value of the gamble is \$100,000 ($.50 * \$50{,}000 + .50 * \$150{,}000$). The utility of each outcome is shown on the vertical axis. The expected utility of the outcomes ($.50 * U(\$50{,}000) + .50 * U(\$150{,}000)$) can be found as the midpoint of a straight line connecting the utilities at points A and B. Note that for this investor the expected utility of a gamble with an expected outcome of \$100,000 is less than the utility associated with a certain outcome of \$100,000.

We can also determine the dollar amount of a certain outcome at which this investor would see no difference between the gamble and the certain amount. We call this amount the *certainty equivalent,* and it can be found by projecting a horizontal line

FIGURE A.5.1 A RISK-AVERSE INVESTOR.



[1] I wish to thank Eli Talmor for his extensive contributions to this appendix.

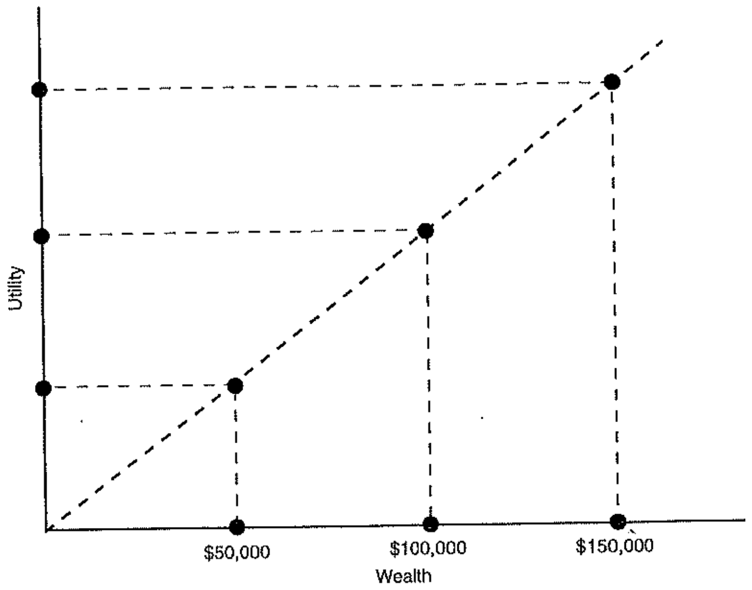


FIGURE A.5.2 A RISK-NEUTRAL INVESTOR.

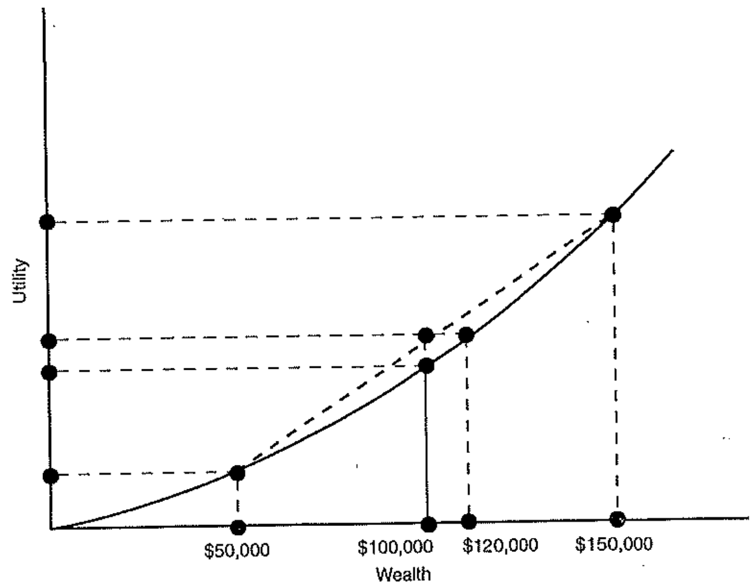


FIGURE A.5.3 A RISK-LOVING INVESTOR.

from the expected utility of the gamble to the utility curve at point *C* and then down to the horizontal axis of the graph. For investors like the one depicted in Figure A.5.1, there would be no difference between receiving a gamble with an expected outcome of $100,000 or receiving a certain sum of $90,000. For these investors, the certainty equivalent of this gamble is $90,000. Keep in mind that the value of the certainty equivalent in relation to the expected value of the gamble is related to the nature of the gamble as well as the nature of the investor's utility curve.

For investors with diminishing marginal utility, the certainty equivalents of gambles are less than their corresponding expected values. We describe such investors as *risk averse.*

The utility function of Figure A.5.2 is linear in wealth. The increase in utility is constant for each successive increase in wealth. Such investors have *constant marginal utility.* Note that for these investors, the certainty equivalent of the gamble is equal to its expected value. Moreover, that will be true no matter how spread out the possible outcomes are. These investors are indifferent to the risk of the gamble, and so we say that they are *risk neutral.*

The utility function of Figure A.5.3 increases at an increasing rate. Each successive increase in wealth brings an even greater increase in well-being than the one before. Investors are characterized by increasing marginal utility. They are willing to pay premiums for gambles. For them, the certainty equivalent of a gamble is even greater than its expected value. Note that the certainty equivalent for the risk-loving investor of the gamble discussed previously is \$120,000, \$20,000 more than its expected value. Moreover, the greater the risk, the greater the premium they are willing to pay. We describe such investors as *risk loving.*

MEASURES OF RISK AVERSION

As can be understood by looking at the three utility functions of Figures A.5.1 through A.5.3, investors' levels of risk aversion are determined by the curvature of their utility function. As we see in Figures A.5.4 and A.5.5, the more concave the function, the less the certainty equivalent relative to the expected value for the same gamble described previously and the greater the level of risk aversion. In the case of the investor depicted in Figure A.5.4, the certainty equivalent is \$90,000, while it is \$80,000 for the more risk-averse investor depicted in Figure A.5.5. Given that the expected payoff for the gamble is \$100,000, we can calculate the expected return to an "investment" in the gamble required by the two investors. The more risk-averse investor is willing to invest in the gamble at a price of \$80,000—that is, she would be willing to give up a certain sum of \$80,000 in exchange for the gamble. This represents a rate of return of 25 percent.

$$E(r) = \frac{\text{Expected payoff}}{\text{Investment}} - 1 = \frac{\$100{,}000}{\$80{,}000} - 1 = .25$$

Thus, the more risk-averse investor requires a 25 percent *risk premium* in order to invest in the gamble. The other investor, with the less concave utility function, requires a risk premium of only 11 percent.

$$E(r) = \frac{\$100{,}000}{\$90{,}000} - 1 = .11$$

Clearly, investors with more concave utility functions are more risk averse. To obtain a measure of the degree of risk aversion, we need a measure of the concavity of the utility function.

Absolute risk aversion is a measure of investor reaction to uncertainty relating to *dollar changes* in their wealth. For any particular level of wealth, or point on the utility curve, we measure absolute risk aversion by the *relative change that is occurring in the slope of the function at the point.* The relative change in the slope is given by the change that occurs in the slope as we move away from the point divided by the slope of the utility function at the point.

The slope itself can be found by taking the first derivative of the utility function at the point. The change in the slope can be found by taking the second derivative. Thus, the relative change in the slope is found by dividing the second derivative (change in

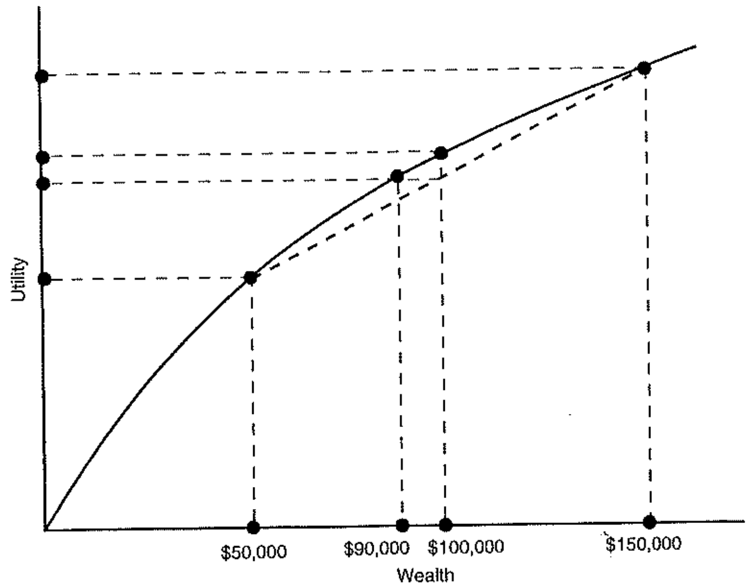


FIGURE A.5.4 A LESS RISK-AVERSE INVESTOR.

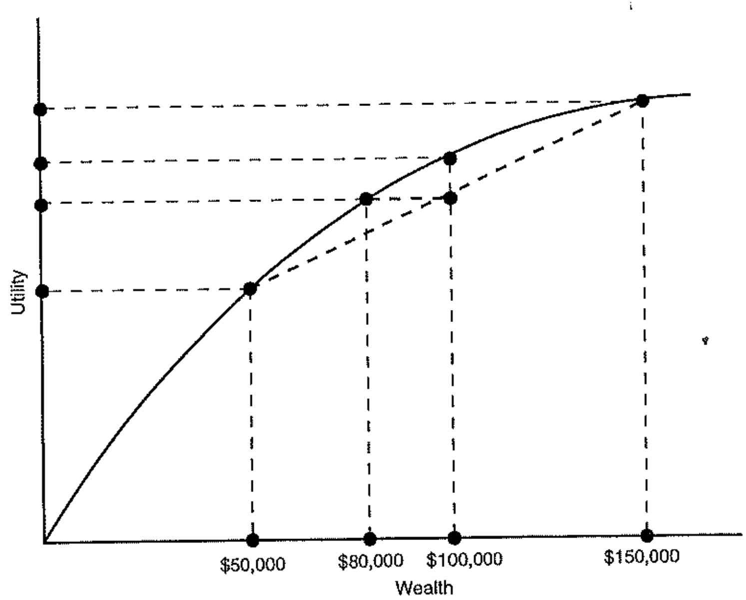


FIGURE A.5.5 A MORE RISK-AVERSE INVESTOR.

slope) by the first derivative (the slope itself). Concavity (and risk aversion) in the utility function increases with the *absolute value* of the relative change in the slope. Because the second derivative is negative (the slope becomes smaller and smaller as wealth increases), we measure the level of absolute risk aversion by the *negative* of the relative change in the slope.

$$\text{Absolute risk aversion} = -\frac{\text{Change in slope}}{\text{Slope}} = -\frac{\text{Second derivative}}{\text{First derivative}} = -\frac{U''\,(\text{wealth})}{U'\,(\text{wealth})}$$

Relative risk aversion is a measure of investor reaction to uncertainty relating to *percentage* changes in their wealth. It is similar to the absolute risk-aversion measure, except that the measure is scaled to reflect the investors' current wealth level.

$$\text{Relative risk aversion} = -\text{Wealth} * \frac{U''(\text{wealth})}{U'(\text{wealth})}$$

If investors are characterized by diminishing absolute risk aversion, you can expect them to commit more *dollars* to risky investments as their wealth increases. If they are characterized by diminishing relative risk aversion, you can expect them to commit a larger *fraction* of their wealth to risky investments as their wealth increases. Constant relative risk aversion, of course, implies diminishing absolute risk aversion. Casual observation of human behavior seems consistent with the notion that investors are characterized by diminishing absolute risk aversion and also by at least constant and perhaps diminishing relative risk aversion.

TYPES OF RISK-AVERSE UTILITY FUNCTIONS

It's common to observe people behaving in a risk-loving manner with relatively small amounts of money. (The price of a lottery ticket—its certainty equivalent—is greater than the expected value of its payoff.) Investors, however, typically behave in a risk-averse manner when they commit relatively large amounts to investment portfolios. Here we will focus on different types of risk-averse utility functions.

Under the *logarithmic utility function,* utility increases linearly with the natural log of wealth. If a is taken to be the level of utility associated with no wealth and b is taken to be the rate at which utility increases with wealth, the log utility function can be expressed as:

$U(\text{wealth}) = a + b\ln(\text{wealth})$
$U'(\text{wealth}) = b/\text{wealth}$
$U''(\text{wealth}) = -b/(\text{wealth})^2$
Absolute risk aversion = 1/wealth (decreasing with increasing wealth)
Relative risk aversion = 1 (constant with increasing wealth)

On the other hand, an *exponential utility function* takes the following form:

$U(\text{wealth}) = (1/a)e^{-a*\text{Wealth}}$
$U'(\text{wealth}) = e^{-a*\text{Wealth}}$
$U''(\text{wealth}) = -a * e^{-a*\text{Wealth}}$
Absolute risk aversion = a (constant with increasing wealth)
Relative risk aversion = $W * a$ (increasing with increasing wealth)

Thus, investors with exponential utility invest constant dollar amounts in risky investments as their wealth increases but decreasing *proportional* amounts.

Power utility functions can be expressed as follows:

$U(\text{wealth}) = (\text{wealth})^b/b$
$U'(\text{wealth}) = (\text{wealth})^{b-1}$
$U''(\text{wealth}) = (b-1)(\text{wealth})^{b-2}$
Absolute risk aversion = $(1 - b)$/wealth (decreasing with increasing wealth)
Relative risk aversion = $1 - b$ (constant with increasing wealth)

Finally, *quadratic utility functions* take the following form:

$U(\text{wealth}) = a - b * (\text{wealth})^2$
$U'(\text{wealth}) = 2b * (\text{wealth})$
$U''(\text{wealth}) = -2b$
Absolute risk aversion $= 2b/(1 - 2b * \text{wealth})$ (increasing with increasing wealth)
Relative risk aversion $= (2b * \text{wealth})/(1 - 2b * \text{wealth})$ (increasing with increasing wealth)

All four functions are concave in wealth. However, of the four, the quadratic is least appealing because of its presumption that both absolute risk aversion and relative risk aversion increase with wealth levels.

❖ QUESTION SET 1

1. What criterion must a portfolio meet to be in the minimum variance set?
2. Contrast the minimum variance set with the efficient set.
3. Referring to Figure 5.7,
   a. What part of the figure corresponds to portfolios having negative weight for stock C?
   b. What are the portfolio weights corresponding to point D?
   c. Could a portfolio variance of 30 percent be achieved if short selling were not permitted?
4. Suppose the expected returns on three stocks are as follows:

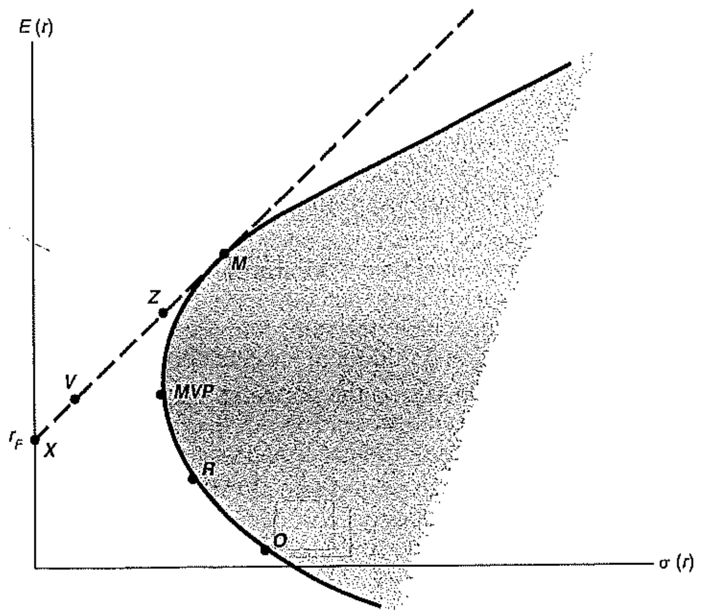
	X	Y	Z
$E(r)$	.07	.11	.16

   a. Find the equation of the isoexpected return line that corresponds to a portfolio expected return of .15 for these three stocks. (The line is to be expressed in terms of the weights on X and Y.)
   b. If the weight on stock Y were restricted to zero, what weights for stocks X and Z would result in a portfolio expected return of .15?
5. What criterion must a portfolio meet to be located on the critical line?
6. Refer to Figure 5.7. What would the critical line for stocks A, B, and C look like if you were restricted from selling A and B short but were permitted to sell C short?
7. Refer to Figure 5.7 and the data that were the basis for that figure.
   a. Find the portfolios that have variance of 26 percent and zero weight for stock A.
   b. Which, if any, of the portfolios you find in part (a) would constitute possible investments if short selling were not allowed?
8. Referring to Figure 5.7 and the relevant data for that figure, write out the equation that corresponds to the *particular* isovariance ellipse associated with a variance of 21 percent.
9. Referring to Figure 5.9, why does the minimum variance set "stop" at points A and C?
10. Which points on the minimum variance set depicted in Figure 5.9 correspond to portfolios in which all three of the stocks are used? Also, which points on the critical line depicted in Figure 5.8 correspond to these same portfolios?

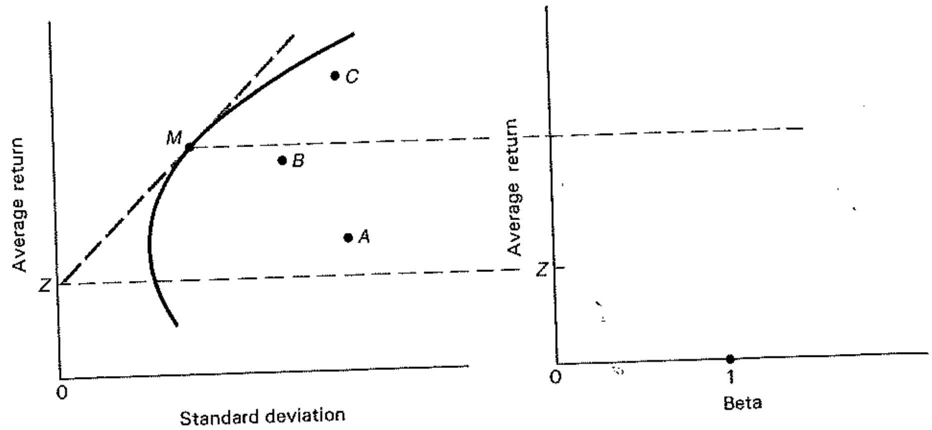
11. Consider the following statement: The geometry of the critical line is such that points on the line are always points of tangency between isoexpected return lines and isovariance ellipses. Is this statement true or false? Explain.
12. For an arbitrary set of stocks, suppose someone had constructed two minimum variance sets—one with short selling permitted and one with short selling not permitted. Even without knowing the details of the stocks' characteristics, what general statement could you make about the difference between the two minimum variance sets?
13. In finding the portfolios that make up the minimum variance set, what is the general approach that you need to employ?

❖ QUESTION SET 2

1. What can be said of a portfolio in the minimum variance set?
2. What criterion defines the global minimum variance portfolio?
3. How does the efficient set differ from the minimum variance set? If you could not obtain a portfolio on the efficient set, would you prefer à portfolio of the same expected standard deviation on the remaining minimum variance set or inside the bullet?
4. What is an isoexpected return line? An isovariance ellipse?
5. Define the critical line. How is it found?
6. Intuitively, why should it be true that when short selling is allowed, most securities will have either a positive or a negative weight?
7. How can we separate the decision on how much variance to assume from the decision on what stocks to purchase?
8. Given that a riskless asset is available, your client instructs you to get the highest rate of return with the minimum variance available. What point on the graph do you choose?



9. a. Given the following graphs, what points are necessary to draw the line showing the risk-reward trade-off for the individual stocks *A*, *B*, and *C*?
   b. What can we say about the relationships between the betas of stocks *A*, *B*, and *C*?
   c. What does the beta of *C* tell us about stock *C*? Is *C* more likely to be a utility or an emerging technology company?



10. Why do you think most of the stocks in a portfolio would have zero portfolio weights if they were on the minimum variance set with no short selling allowed?

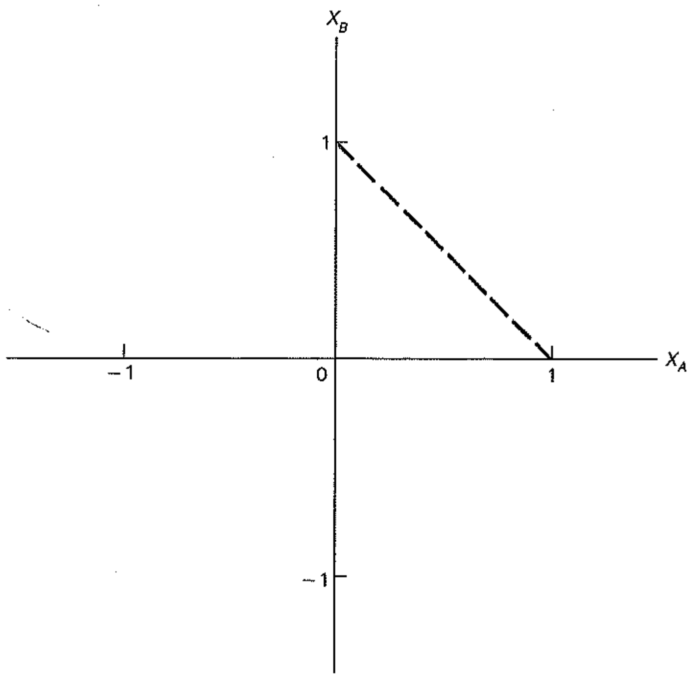
❖ Answers to Question Set 2

1. A portfolio in the minimum variance set has the least possible variance for a given level of expected return.
2. The global minimum variance portfolio has the least possible variance given the universe of securities available.
3. The efficient set includes only those portfolios on the top half of the "bullet," having the highest possible expected return given a level of portfolio variance. Given that you cannot have a portfolio on the efficient set, you would prefer a portfolio inside the bullet rather than on the lower half of the minimum variance set, since these represent the least desirable portfolios.
4. An isoexpected return line shows the set of portfolios all of which have the same expected rate of return. Different lines, showing different expected rates of return, are parallel to each other. An isovariance ellipse shows the set of portfolios all of which have the same variance of return. These ellipses are concentric around the global minimum variance portfolio.
5. The critical line shows the portfolio weights for portfolios in the minimum variance set. It is found by tracing out the points of tangency between the isoexpected return lines and the isovariance ellipses.
6. Given a definite expected rate of return with a definite variance, most securities will fall into the categories of desirable investments or undesirable investments. The undesirable investments are sold short to place more funds in the desirable investments.

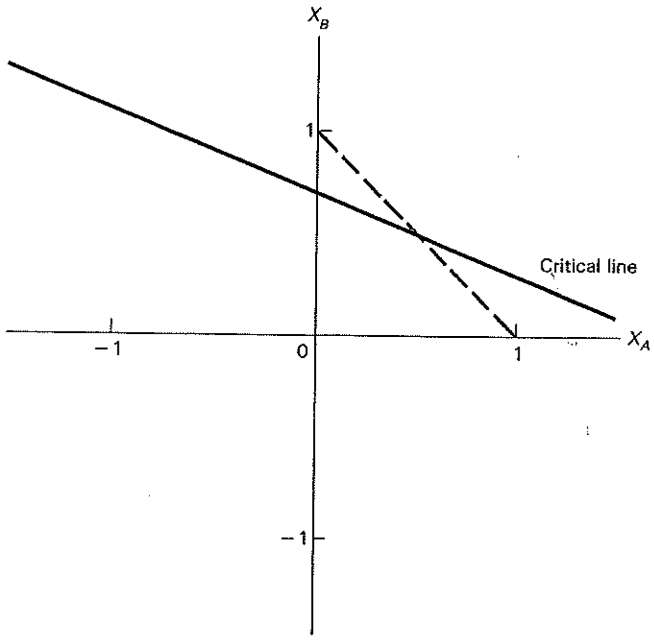
7. Given a population of risky assets and a riskless asset that we can either buy or sell, there exists a unique portfolio on the minimum variance set that is the one best portfolio of risky assets for the investor to hold. We can separate out the variance factor by taking a position in a risk-free asset. The more risk averse we are, the more of our portfolio we invest in the risk-free asset. All investors, however, if they hold any risky assets, still would prefer to invest a proportion of their assets in the unique portfolio on the minimum variance set.
8. You would choose point *X*, or the point where you get the highest rate of return for the lowest variance, in this case, *no* variance. This is a point on the new minimum variance set, which has been created by the addition of the possibility of investing in a riskless asset.
9. a. *M* and *Z* are necessary to tell us about the linear relationship of betas in the portfolio. *M* is the index portfolio, and a tangent to it intersects the vertical axis at point *Z*.
   b. The relationship between the betas is perfectly linear and deterministic.
   c. *C* is a stock with a higher beta, or risk factor. It is more likely to be an emerging technology company (i.e., a growth company) than it is to be a utility company (probably a stable stock with small, but certain, growth and dividends).
10. Unlike the case with short selling allowed, the portfolios with no short selling allowed do not allow the advantage of selling the less desirable securities to invest in the more desirable ones.

❖ PROBLEM SET

1. The following figure depicts in X_A, X_B space the possible portfolio weights in a three-stock portfolio. Indicate the areas of positive, negative, and zero portfolio weights for each of the three securities.



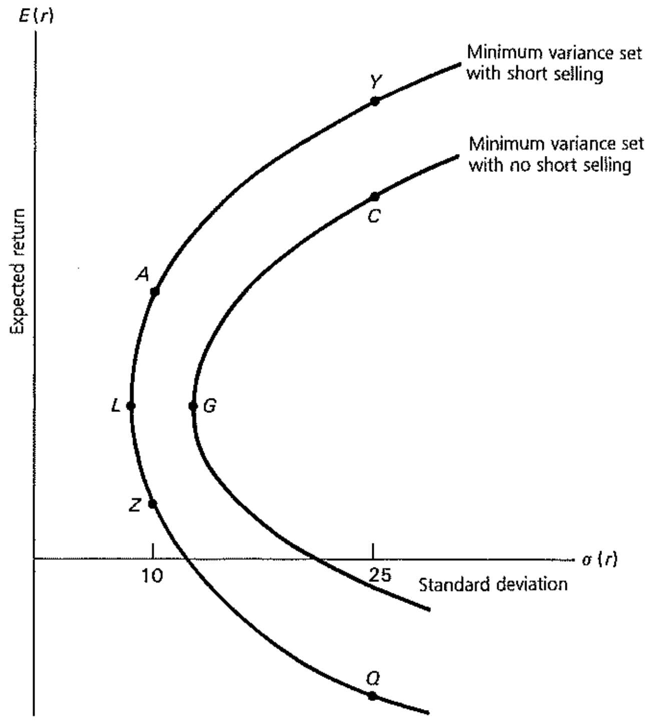
2. The following figure shows the critical line for a portfolio containing stocks A, B and C when there are no restrictions on short selling. What would the critical line look like in each of the following cases?
   a. No short selling allowed.
   b. Short selling not allowed in stock A.
   c. Short selling not allowed in stocks A and C.
   d. Short selling not allowed in stocks B and C.



3. Utilizing the diagram at the top of page 127, what portfolios, by letter, would you choose under the following constraints from your customer?
   a. You are allowed to short-sell stocks, but you can't have a standard deviation of more than 10.
   b. Your client wants the highest expected return, with a variance of no more than 25.
   c. Your client does not allow short selling and simply wants the best expected return along with the lowest variance.
4. Suppose that we have two portfolios known to be on the minimum variance set for a population of three stocks, A, B, and C. There are no restrictions on short sales. The weights for each of the two portfolios are as follows:

	X_A	X_B	X_C
Portfolio 1	.24	.52	.24
Portfolio 2	−.36	.72	.64

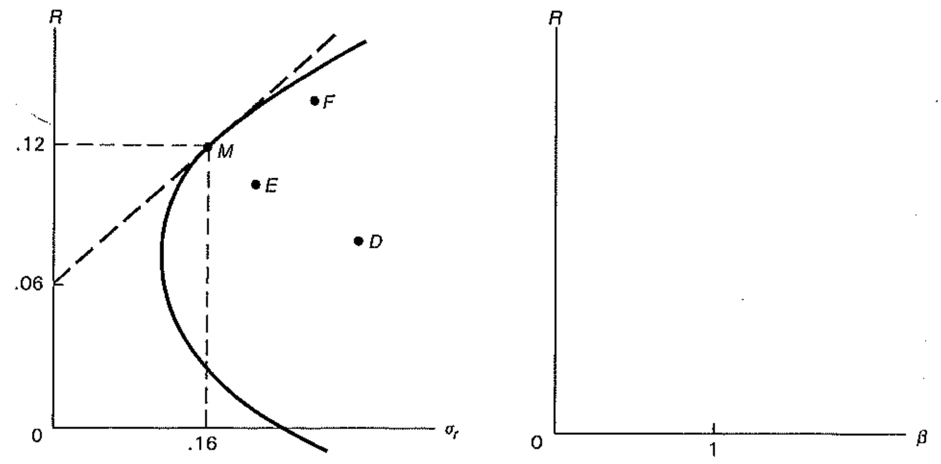
   a. What would the stock weights be for a portfolio constructed by investing \$2,000 in portfolio 1 and \$1,000 in portfolio 2?



b. Plot portfolios 1 and 2 and the combined portfolio in X_A, X_B space. Is the combined portfolio on the critical line?

c. Suppose you invest \$1,500 of the \$3,000 in stock A. How will you allocate the remaining \$1,500 between stocks A and B to ensure that your portfolio is on the minimum variance set?

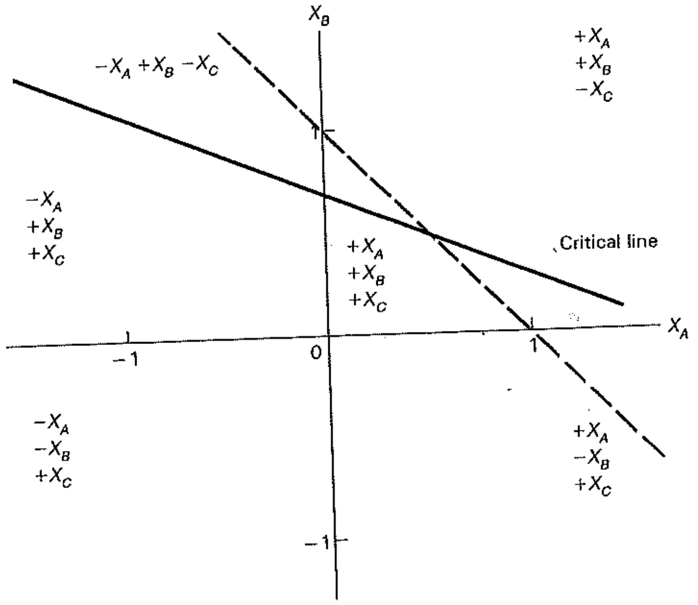
Refer to the following diagrams for Problems 5 and 6. The bullet represents the minimum variance set for a population of stocks. A line drawn tangent to the bullet at M intersects the mean return axis at .06.



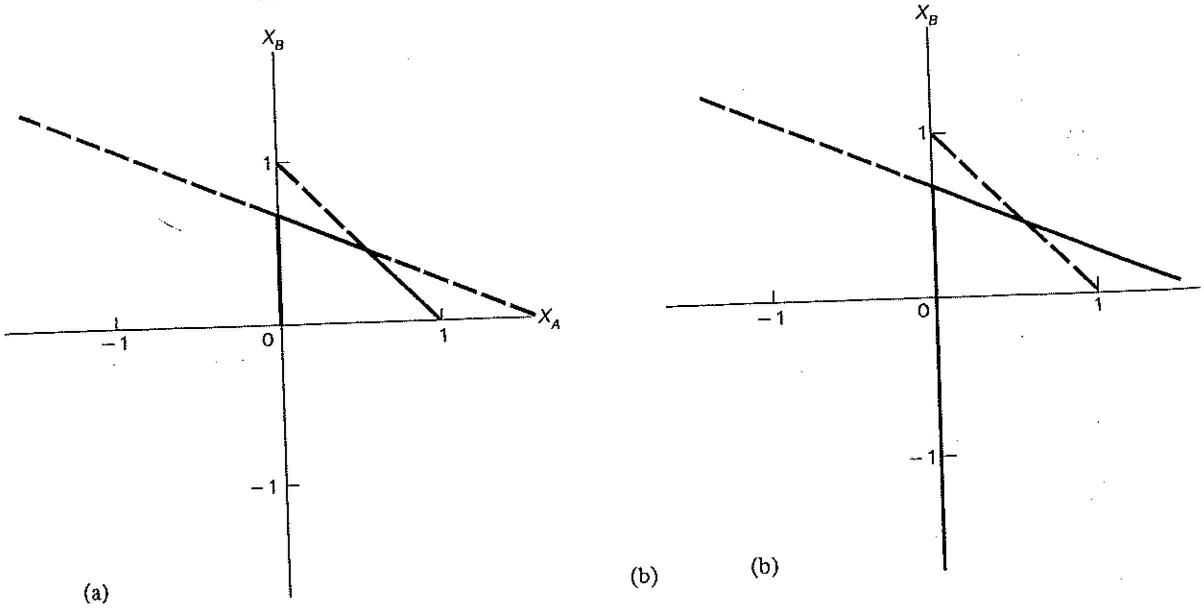
5. Assuming that short selling is allowed and that all betas of stocks and portfolios are computed with reference to M, draw the implied relationship between beta values and mean returns.
6. Suppose the mean returns for stocks D, E, and F are .08, .10, and .14, respectively. What would be the implied betas for these three stocks?

❖ Answers to Problem Set

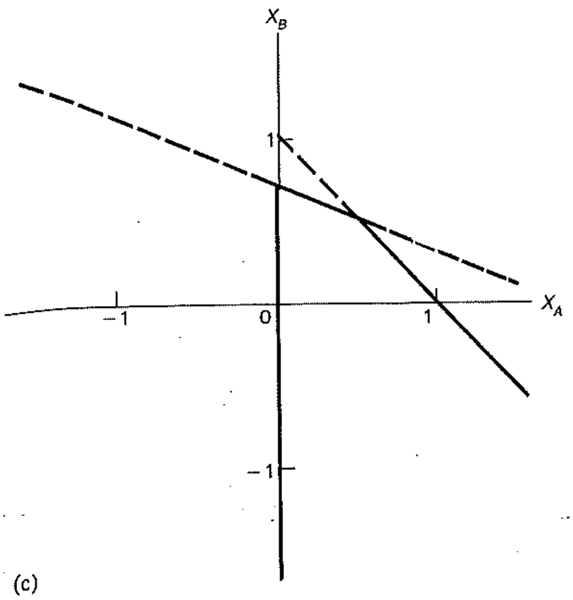
1.



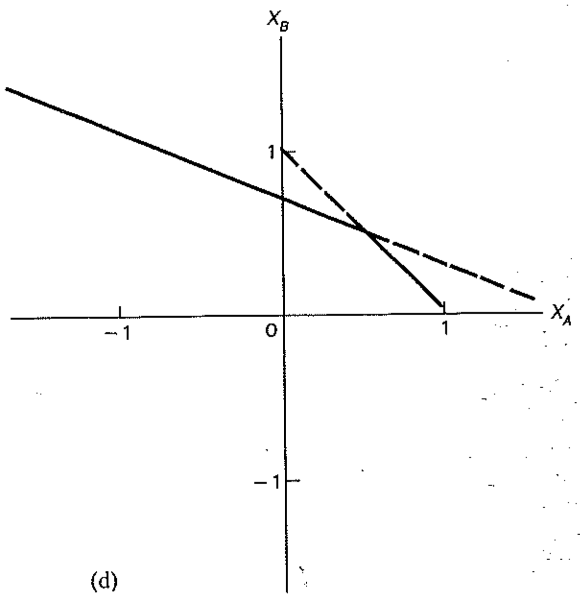
2.



(a) (b) (b)



(c)



(d)

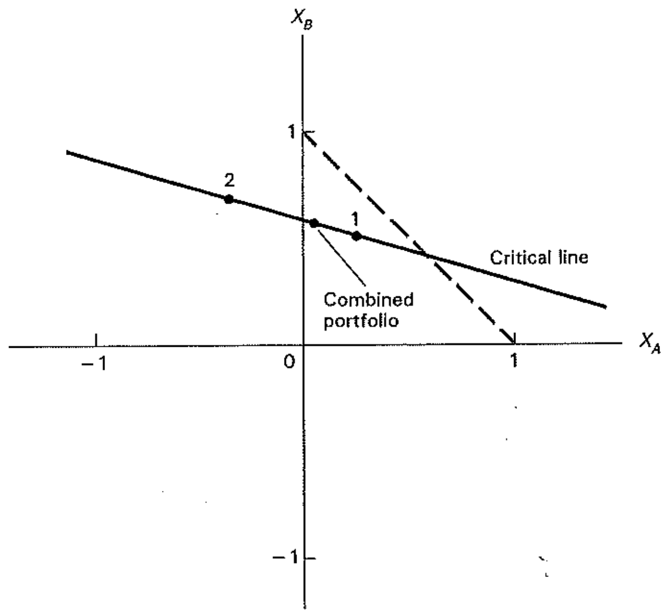
3. a. Point A represents the optimal portfolio in which you could invest given the constraint of a standard deviation of no more than 10.
   b. To get the highest rate of return with a standard deviation allowed of 25, you would pick point Y, which represents the portfolio on the minimum variance set that is also a part of the efficient set.
   c. In this case, the no-short-selling constraint does not allow you to be on the minimum variance set, and you would choose point G on the inside curve, which represents the smallest variance given the short-selling constraint.
4. a. Given a \$2,000 investment in portfolio 1 and a \$1,000 investment in portfolio 2, the dollars committed to each stock would be

	A	*B*	*C*	*Total*
Portfolio 1	\$480	\$1,040	\$480	\$2,000
Portfolio 2	−360	720	640	1,000
Combined portfolio	\$120	\$1,760	\$1,120	\$3,000

Since we are investing a total of \$3,000 in the combined portfolio, the dollar positions in the three stocks are consistent with the following portfolio weights:

	X_A	X_B	X_C
Combined portfolio	.04	.59	.37

b.



The combined portfolio is on the critical line.

c. Recall from Chapter 5 that if a portfolio is on the minimum variance set, it is also, by definition, on the critical line. In part (b) we plotted the critical line for our population of stocks in x_A, x_B space. The equation for the critical line takes the following form:

$$x_B = a + bx_A$$

Substituting in the values for x_A and x_B from portfolios 1 and 2, we get

$$.52 = a + .24b$$

$$.72 = a + -.36b$$

We can solve these equations simultaneously to obtain the slope and the intercept of the critical line

$$x_B = .6 - \tfrac{1}{3}x_A$$

Using this equation, we can find x_B for any given value x_A if we invest half our funds in stock A ($x_A = .5$), then

$$x_B = .6 - \tfrac{1}{3}(.5) = .43$$

Since $x_A + x_B + x_C = 1$, we know $x_C = 1 - x_A - x_B$. Substituting in our values for x_A and x_B, we find

$$x_C = 1 - .5 - .43 = .07$$

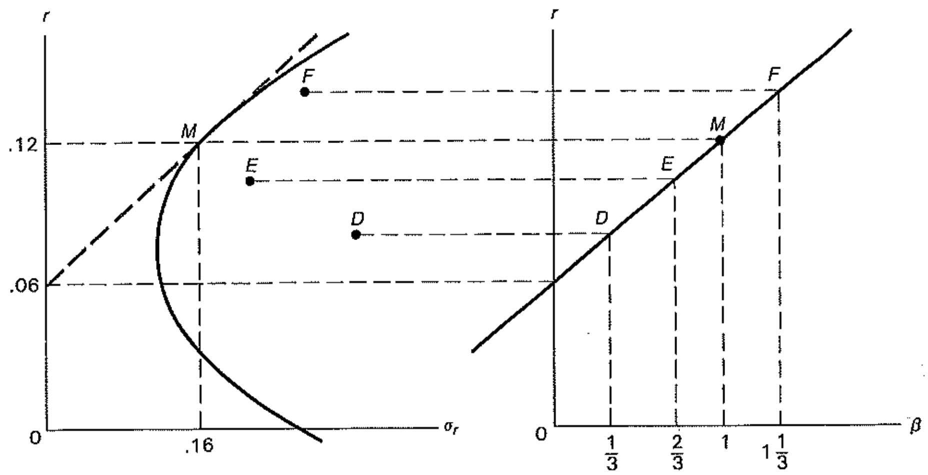
The minimum variance portfolio is

$$x_A = .50$$

$$x_B = .43$$

$$x_C = .07$$

5.



6. From property II we know that the relationship between beta factors of the stocks and their mean returns will be linear, since the betas were computed with reference to an index portfolio on the minimum variance set. The equation for a straight line in $\beta, \bar{r}$ space can be written

$$\bar{r}_J = .06 + (.12 - .06)\beta_J$$

Solving for β_J

$$\beta_J = \frac{(\bar{r}_J - .06)}{(.12 - .06)} = \frac{\bar{r}_J - .06}{.06}$$

By substituting for $\bar{r}_J$ in the mean returns of the three stocks, we can find their implied betas

$$\beta = \frac{.08 - .06}{.06} = .333$$

$$\beta_E = \frac{.10 - .06}{.06} = .666$$

$$\beta_F = \frac{.14 - .06}{.06} = 1.333$$

❖ Computer Problem Set

For problems 1 and 2, please refer to the file noshortselling.ses; for problems 3 and 4, please refer to the file Chapter5.ses.

1. You received $10,000 from a rich uncle and have decided to invest the money in the stock market. Your broker recommends you buy stock in three companies: *A*, *B*, and *C*. He tells you that at current market prices the stocks have an expected annual return of 20.4 percent, 24.5 percent, and 19.7 percent, respectively. On the basis of the stock's annual rates of return over the past 10 years, you construct the following covariance matrix:

	A	*B*	*C*
A	.0301	.0160	.0112
B	.0160	.0139	.0101
C	.0112	.0101	.0171

a. Using the data in the matrix, find the standard deviation and calculate the c relation coefficient for each pair of stocks. (The formula is given in Chapter
b. Using PManager, generate a series of efficient portfolios. Plot the portfolic mean–standard deviation space.

2. Just as you are about to decide how to allocate your uncle's money among the three stocks, your broker telephones. His firm's research department has just upgraded its estimate of the expected return on stock *A* to 30.0 percent from : percent. How will this new piece of information affect your investment decisic Plot the new efficient set for the three-stock portfolio and interpret the result:

3. An asset manager uses six types of assets to construct his portfolio. He choos certain market index for each asset. The assets and the relevant indices are

Asset Type	*Index Name*
U.S. large equity	S&P 500
U.S. small equity	Russell 2000
International equity	MSCI EAFE
U.S. fixed income	Long-term corporate bond index
Cash equivalents	90-day T-bills
Inflation	Consumer Price Index

Based on the assets' performance over the past 10 to 30 years, the expected return and standard deviation of each asset is estimated as follows:

Index Name	*Expected Return (annual percentage)*	*Standard Devi (annual percen*
S&P 500	9.10	13.30
Russell 2000	10.10	19.10
MSCI EAFE	9.60	18.50
Long-term corporate bond index	6.60	5.50
90-day T-bills	4.80	1.00
Consumer Price Index	3.40	0.95

Correlation between asset classes is already input in the PManager software package. Note that the numbers going down the diagonal of the matrix are tl standard deviations of the asset classes.

Correlation & Standard Deviation

	S&P 500	Russell	MSCI EAF	long cor	90 day T	CF
S&P 500	13.3000	0.8000	0.5200	0.3800	0.0000	-0.1
Russell 2000	0.8000	19.1000	0.4300	0.3100	-0.0500	-0.1
MSCI EAFE	0.5200	0.4300	18.5000	0.3600	-0.2000	-0.2
long corp bonds	0.3800	0.3100	0.3600	5.5000	0.1800	-0.1
90 day T-bills	0.0000	-0.0500	-0.2000	0.1800	1.0000	0.2
CPI-U	-0.1500	-0.1300	-0.2300	-0.1700	0.2400	0.5

a. In the PManager software package, open the file for Chapter 5. Find the e cient frontier.
b. What is the range of expected returns on the efficient frontier?
c. Suppose the asset manager requires at least 8% expected return for his p folio. Construct a portfolio for him with the lowest risk and justify your answer. Identify the position on the efficient frontier.

d. Without any calculation, rank the following asset allocations in terms of their risk (from low to high). Then, use the software package to find the expected return and risk of each. Plot their positions relative to the efficient frontier. What do you find about the relation between risk and return?

Asset Allocation	*Portfolio 1*	*Portfolio 2*	*Portfolio 3*	*Portfolio 4*
S&P 500	10	30	25	0
Russell 2000	10	30	50	0
MSCI EAFE	10	20	25	0
Long-term corporate bond index	30	20	0	30
90-day T-bills	20	0	0	40
Consumer Price Index	20	0	0	30

(The number indicates the percentage of asset invested in each portfolio.)

e. This asset manager now asks you to run a back test on your portfolio in part c and on the actual portfolio he used for the period from 1993.01 to 1997.12. The actual portfolio has 20% invested in S&P 500 and 16% in each of the other five assets. What is the expected return and risk of the actual portfolio? Plot the graph of their cumulative return for that period and compare their performance.

4. Suppose you are a member of the research team for the asset manager in question 1. Based on his previous request for an 8% expected return, what would be the impact on asset allocation for each of the following changes? (In each case, compare with the original asset allocation information given in question 1.)
   a. The standard deviations of Russell 2000 and MSCI EAFE were just upgraded to 15.5% and 14.6%, respectively.
   b. The newest information indicates that correlation (Russell 2000, Long-term corporate bond index) = 0.50.
   c. Correlation between S&P 500 and MSCI EAFE was updated to be 0.75.
   d. The most recent data show: correlation (S&P 500, 90-day T-bills) = 0.25.
   e. Volatility of long-term corporate bond index has increased to 10% and expected inflation rose to 5%.

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