14.5 Directional Derivatives and Gradient Vectors.

Directional Derivatives in the plane.

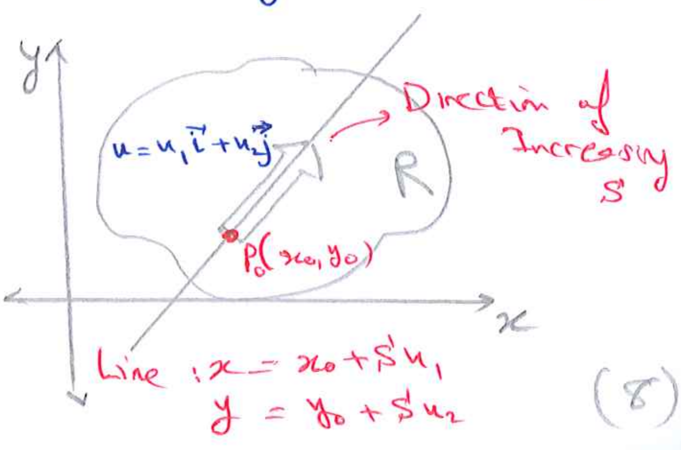
Def: The derivative of f at $P_0(x_0, y_0)$ in the direction of the unit vector $\vec{u} = u_1\vec{i} + u_2\vec{j}$ is the number :

$$\left(D_{\vec{u}} f\right)_{P_0} = \left(\frac{df}{ds}\right)_{\vec{u}, P_0} = \lim_{s \to 0} \frac{f(x_0 + s u_1, y_0 + s u_2) - f(x_0, y_0)}{s}$$

"We read it : The derivative of f at P_0 in the direction of $\vec{u}$ ".

Note : $f_x(x_0, y_0)$ is the directional derivative of f at P_0 in the $\vec{i}$ direction.

$f_y(x_0, y_0)$ is the directional derivatives of f at P_0 in the $\vec{j}$ direction.



Example: Using the definition, find the derivative of

$f(x,y) = x^2 + xy$ at $P_0(1,2)$ in the Direction

of the unit vector $\vec{u} = \frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}$

sol: $$\left(D_{\vec{u}} f\right)_{P_0(1,2)} = \lim_{s \to 0} \frac{f\left(1+\frac{1}{\sqrt{2}}s,\ 2+\frac{1}{\sqrt{2}}s\right) - f(1,2)}{s}$$

$$= \lim_{s \to 0} \frac{\left(1+\frac{1}{\sqrt{2}}s\right)^2 + \left(1+\frac{1}{\sqrt{2}}s\right)\left(2+\frac{1}{\sqrt{2}}s\right) - [1+2]}{s}$$

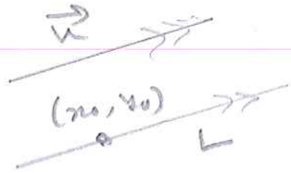
$$= \lim_{s \to 0} \frac{\left(1+\sqrt{2}s+\frac{1}{2}s^2\right) + \left(2+\frac{3}{\sqrt{2}}s+\frac{1}{2}s^2\right) - 3}{s}$$

$$= \lim_{s \to 0} \frac{\frac{5}{\sqrt{2}}s + s^2}{s}$$

$$= \lim_{s \to 0} \frac{5}{\sqrt{2}} + s = \frac{5}{\sqrt{2}}$$

(i-e) The rate of change of $f(x,y) = x^2 + xy$ at $P_0(1,2)$ in the Direction of $\vec{u}$ is $\frac{\sqrt{5}}{2}$

Calculation of Gradients:



Consider the line $L: x = x_0 + u_1 s \;, y = y_0 + u_2 s$

through $P_0(x_0, y_0)$, parametrized with the arc length

parameter s increasing in the direction of $\vec{u} = u_1 \vec{i} + u_2 \vec{j}$

Then, Using chain Rule:

$$\left(\frac{df}{ds}\right)_{\vec{u}, P_0} = \left(\frac{\partial f}{\partial x}\right)_{P_0} \cdot \frac{dx}{ds} + \left(\frac{\partial f}{\partial y}\right)_{P_0} \cdot \frac{dy}{ds}$$

$$= \left(\frac{\partial f}{\partial x}\right)_{P_0} \cdot u_1 + \left(\frac{\partial f}{\partial y}\right)_{P_0} \cdot u_2$$

$$= \underbrace{\left[\left(\frac{\partial f}{\partial x}\right)_{P_0} \vec{i} + \left(\frac{\partial f}{\partial y}\right)_{P_0} \vec{j}\right]}_{\text{Gradient of } f \text{ at } P_0} \cdot \underbrace{\left[u_1 \vec{i} + u_2 \vec{j}\right]}_{\text{Direction } \vec{u}}$$

$$\left(\frac{df}{ds}\right)_{\vec{u}, P_0} = (\nabla f)_{P_0} \cdot \vec{u}$$

∇f is read "grad f" or "gradient of f" or "del f" ∇

Def: The gradient vector (gradient) of $f(x,y)$ at a point $P_0(x_0, y_0)$ is the vector

$$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$$

Theorem: The Directional derivative is a dot product

$$\left(\frac{df}{ds}\right)_{\vec{u}, P_0} = (\nabla f)_{P_0} \cdot \vec{u}$$

Example: Find the derivative of $f(x,y) = xe^{y} + \cos(xy)$ at the point $(2,0)$ in the direction of $\vec{v} = 3\vec{i} - 4\vec{j}$.

Sol: $\vec{u} = \dfrac{\vec{v}}{|\vec{v}|} = \dfrac{3}{5}\vec{i} - \dfrac{4}{5}\vec{j}$, $|\vec{v}| = \sqrt{3^2 + 4^2} = 5$.

Now, $f_x\Big|_{(2,0)} = \left[e^{y} + -\sin(xy)[y]\right]\Big|_{(2,0)} = 1 - 0 = \boxed{1}$

$f_y\Big|_{(2,0)} = \left[xe^{y} - x\sin(xy)\right]\Big|_{(2,0)} = 2 - 0 = \boxed{2}$

$\therefore \nabla f\Big|_{(2,0)} = \vec{i} + 2\vec{j}$.

$\therefore (D_{\vec{u}} f)_{P_0} = (\nabla f)\Big|_{(2,0)} \cdot \vec{u} = \frac{3}{5}(1) - \left(\frac{4}{5}\right)(2) = \frac{-5}{5} = \boxed{-1}$

Recall: $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta$ $\quad$ θ between $\vec{u}$ & $\vec{v}$

$$\Rightarrow (D_{\vec{u}} f)_{P_0} = (\nabla f)_{P_0} \cdot \vec{u} = |\nabla f|_{P_0} |\vec{u}| \cos\theta$$

unit vector

$$= |\nabla f| \cos\theta$$

So, we have the following properties:

1) The function f increases most rapidly when $\cos\theta = 1$, or when $\theta = 0$ and $\vec{u}$ is the direction of ∇f. That is, at each point P in its domain, f increases most rapidly in the direction of the gradient vector ∇f at P. In this case

$$D_{\vec{u}} f = |\nabla f| \cos(0) = |\nabla f|$$

2) Similarly, f decreases most rapidly in the direction of $-\nabla f$. In this case: $D_{\vec{u}} f = |\nabla f| \cos(\pi) = -|\nabla f|$.

3) Any direction $\vec{u}$ orthogonal to $\nabla f \neq 0$ is a direction of zero change in f, because $\theta = \frac{\pi}{2}$

$$\Rightarrow D_{\vec{u}} f = |\nabla f| \cos\left(\frac{\pi}{2}\right) = 0$$

Example: Find the Directions in which

$$f(x,y) = \frac{x^2}{2} + \frac{y^2}{2}$$

(a) increases most rapidly at the point $(1,1)$.

(b) decrease most rapidly at the point $(1,1)$.

(c) What are the directions of zero change in f at $(1,1)$?

Sol:

$$f_x\Big|_{(1,1)} = x = 1 \quad , \quad f_y\Big|_{(1,1)} = y = 1$$

$$\therefore \quad \nabla f = \vec{i} + \vec{j} \quad , \quad |\nabla f| = \sqrt{2}.$$

(a) $\vec{u} = \dfrac{\nabla f}{|\nabla f|} = \dfrac{1}{\sqrt{2}}\vec{i} + \dfrac{1}{\sqrt{2}}\vec{j}$.

Note: $(D_{\vec{u}} f)_{(1,1)} = |\nabla f| = \sqrt{2}$.

(b) $\vec{u} = \dfrac{-\nabla f}{|\nabla f|} = -\dfrac{1}{\sqrt{2}}\vec{i} - \dfrac{1}{\sqrt{2}}\vec{j}$

(c) $(D_{\vec{u}} f)_{(1,1)} = (\nabla f)_{(1,1)} \cdot \vec{u} = 0$.

$$\Rightarrow \quad (\vec{i} + \vec{j}) \cdot (u_1\vec{i} + u_2\vec{j}) = 0$$

$\Rightarrow \quad u_1 + u_2 = 0$ and we know $\underbrace{u_1^2 + u_2^2 = 1}_{\text{unit vector}}$

$\Rightarrow \quad u_1 = -u_2$

$\Rightarrow \quad u_1^2 + u_1^2 = 1 \Rightarrow 2u_1^2 = 1 \Rightarrow u_1 = \pm \frac{1}{\sqrt{2}}$

If $u_1 = \frac{1}{\sqrt{2}}$, then $u_2 = -\frac{1}{\sqrt{2}} \Rightarrow \vec{u} = \frac{1}{\sqrt{2}} i - \frac{1}{\sqrt{2}} \vec{j}$

If $u_1 = \frac{-1}{\sqrt{2}}$, then $u_2 = \frac{1}{\sqrt{2}} \Rightarrow \vec{v} = \frac{-1}{\sqrt{2}} \vec{i} + \frac{1}{\sqrt{2}} \vec{j}$.

(check : $(\nabla f)\big|_{(1,1)} \cdot \vec{u} = 0$ & $(\nabla f)\big|_{(1,1)} \cdot \vec{v} = 0$).

Gradients and Tangents to Level Curves:

If $f(x,y)$ has a Constant value C along a Smooth curve $\vec{r} = g(t)\vec{i} + h(t)\vec{j}$, (making the curve a level curve of f), then:

$$f(g(t), h(t)) = C .$$

$$\frac{d f(g(t),h(t))}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} = 0$$

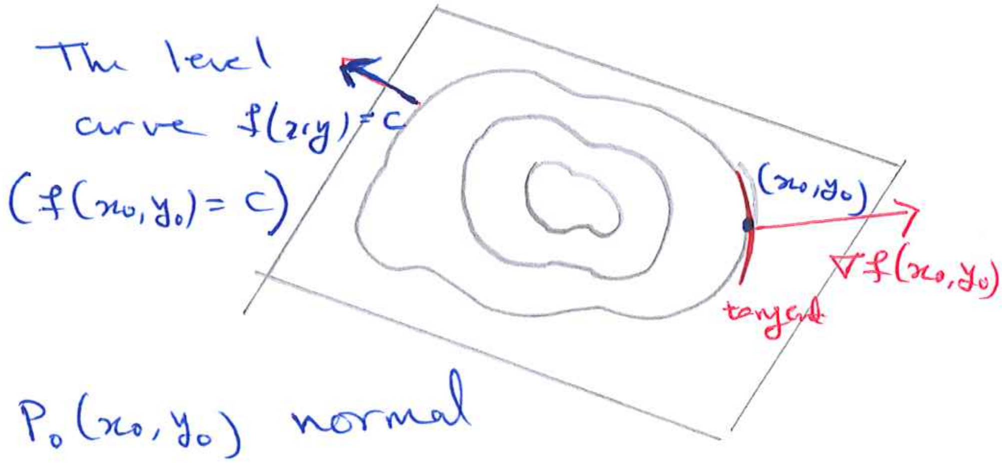
$$\Rightarrow \quad \frac{\partial f}{\partial x} \cdot g'(t) + \frac{\partial f}{\partial y} \cdot h'(t) = 0$$

$$\Rightarrow \left(\frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}\right)\cdot\left(g'(t)\vec{i} + h'(t)\vec{j}\right) = 0$$

$$\nabla f \cdot \frac{d\vec{r}}{dt} = 0 .$$

which means that ∇f is normal to the tangent vector $\frac{d\vec{r}}{dt}$, so ∇f is normal to the curve.

Result: At every point (x_0, y_0) in the domain of a differentiable function $f(x,y)$, the gradient of f is Normal to the Level curve through (x_0, y_0).



The line through $P_0(x_0, y_0)$ normal to the vector $\vec{N} = A\vec{i} + B\vec{j}$ has the equation $A(x-x_0) + B(y-y_0) = 0$

If $\vec{N} = (\nabla f)_{(x_0,y_0)} = f_x(x_0,y_0)\vec{i} + f_y(x_0,y_0)\vec{j}$.

then the equation is the tangent line given by

$$f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) = 0$$

Equation of the Tangent Line:

$$y - y_0 = m\,(x - x_0) \quad , \quad m = \left.\frac{dy}{dx}\right|_{(x_0, y_0)}$$

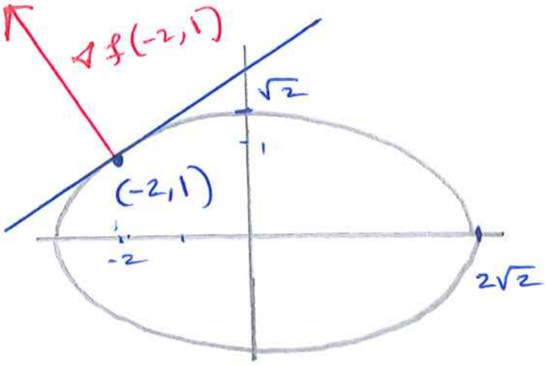
$$\Rightarrow \quad y - y_0 = \left.\frac{dy}{dx}\right|_{(x_0, y_0)} (x - x_0)$$

$$\Rightarrow \quad y - y_0 = \frac{-f_x(x_0, y_0)}{f_y(x_0, y_0)} (x - x_0)$$

$$\Rightarrow \quad (y - y_0)\, f_y(x_0, y_0) = -f_x(x_0, y_0)(x - x_0)$$

$$\Rightarrow \quad f_x(x_0, y_0)\,x + f_y(x_0, y_0)\,y = f_x(x_0, y_0)\,x_0 + f_y(x_0, y_0)$$

Example: Find an equation for the tangent to the ellipse : $\frac{x^2}{4} + y^2 = 2$ at $(-2,1)$



Sol: $f(x,y) = \frac{x^2}{4} + y^2 = 2$

$\therefore$ Ellipse is a level of the function

$$f(x,y) = \frac{x^2}{4} + y^2 .$$

$$f_x \Big|_{(-2,1)} = \frac{x}{2}\Big|_{(-2,1)} = -1 \quad , \quad f_y \Big|_{(-2,1)} = 2y \Big|_{(-2,1)} = 2$$

$$\therefore \vec{N} = (\nabla f)_{(x_0,y_0)} = -\vec{i} + 2\vec{j} .$$

$\therefore$ The tangent line is

$$f_x(-2,1)(x - -2) + f_y(-2,1)(y-1) = 0$$

$$\Rightarrow \quad (-1)(x+2) + 2(y-1) = 0$$

$$\Rightarrow \quad x - 2y = -4$$

$$\Rightarrow \quad y = \frac{x+4}{2}$$

Algebra Rules for Gradients:

1. $\nabla(f+g) = \nabla f + \nabla g$

2) $\nabla(f-g) = \nabla f - \nabla g$

3) $\nabla(kf) = k\nabla f$

4) $\nabla(fg) = f\nabla g + g\nabla f.$

5) $\nabla\left(\frac{f}{g}\right) = \frac{(g\nabla f - f\nabla g)}{g^2}$

Example: If $f(x,y) = x-y$, $g(x,y) = 3y$

then $\nabla(fg) = \nabla\left((x-y).(3y)\right) = \nabla(3xy - 3y^2)$

$$= 3y\vec{i} - (6y - 3x)\vec{j}$$

$$= 3y\vec{i} + 3x\vec{j} - 6y\vec{j}.$$

$=$

Now: $f\nabla g + g\nabla f = (x-y)(3\vec{j}) + (3y)(\vec{i} - \vec{j})$

$$= 3x\vec{j} - 3y\vec{j} + 3y\vec{i} - 3y\vec{j}$$

$$= 3y\vec{i} + 3x\vec{j} - 6y\vec{j}$$

Functions of three Variables:

For a differentiable function $f(x,y,z)$ and a unit vector $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$ in space

$$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$

and $$D_{\vec{u}} f = \nabla f \cdot \vec{u} = \frac{\partial f}{\partial x}u_1 + \frac{\partial f}{\partial y}u_2 + \frac{\partial f}{\partial z}u_3.$$

Example: (a) Find the derivative of $f(x,y,z) = x^3 - xy^2 - z$ at $P_0(1,1,0)$ in the direction of $\vec{v} = 2\vec{i} - 3\vec{j} + 6\vec{k}$.

(b) In what directions does f change most rapidly at P_0, and what are the rates of change in these directions.

Sol: $$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{2}{7}\vec{i} - \frac{3}{7}\vec{j} + \frac{6}{7}\vec{k} \quad , |\vec{v}| = 7$$

$$\Rightarrow \quad f_x(1,1,0) = (3x^2 - y^2)\Big|_{(1,1,0)} = 2$$

$$f_y(1,1,0) = -2xy\Big|_{(1,1,0)} = -2$$

$$f_z(1,1,0) = -1$$

$\therefore \nabla f \Big|_{P_0} = 2\vec{i} - 2\vec{j} - \vec{k}.$

$\therefore (D_{\vec{u}} f)\Big|_{P_0} = \nabla f \Big|_{P_0} \cdot \vec{u} = (2\vec{i} - 2\vec{j} - \vec{k}) \cdot \left(\frac{2}{7}\vec{i} - \frac{3}{7}\vec{j} + \frac{6}{7}\vec{k}\right)$

$= \frac{4}{7} + \frac{6}{7} - \frac{6}{7} = \frac{4}{7}$

(b) f increases most rapidly in the direction of ∇f.

$\therefore$ rate of change is $|\nabla f| = \sqrt{9} = 3$

f decreases most rapidly in the direction of $-\nabla f$.

$\therefore$ rate of change $= -|\nabla f| = -3$.