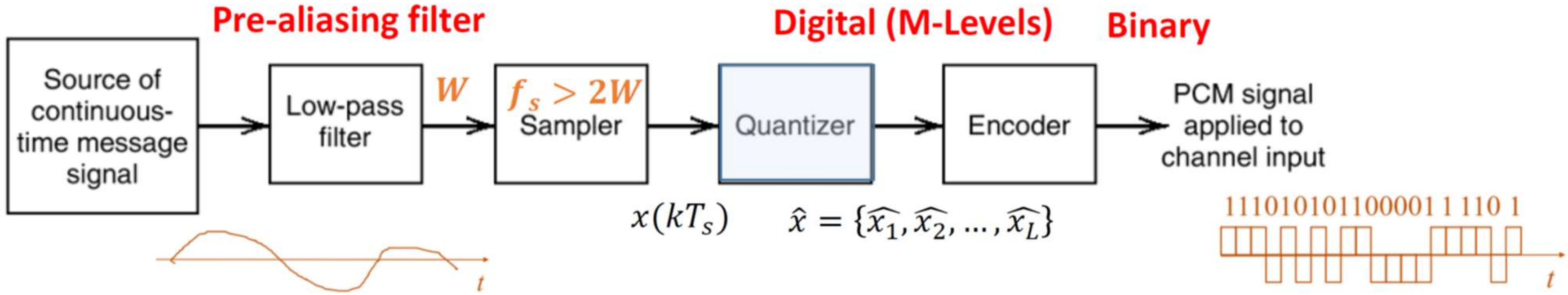
Pulse Modulation

Part 2

Pulse Code Modulation: Overview

- Sources are of two types: **analog and digital**.
- An analog source can be converted into digital via sampling, quantization, and binary encoding. This process is called **pulse code modulation**



➢ **Sampler:** If W is the highest frequency component in a signal, then the sampling rate required to reconstruct the message from its samples should follow the Nyquist rate where $\boldsymbol{f_s > 2W}$.

➢ **Three types of sampling were discussed in previous lectures; ideal, natural, and flat-topped**.

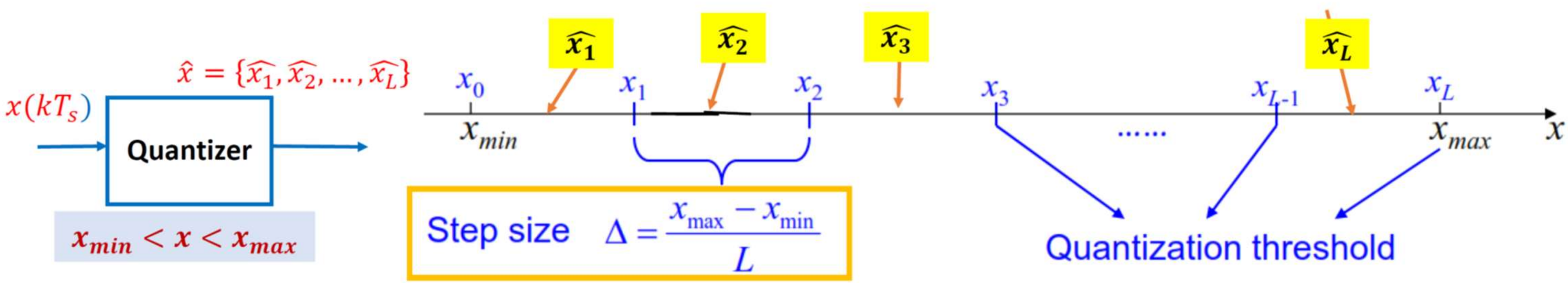
➢ The output of the sampler is a continuous amplitude discrete time signal.

➢ **Quantizer:** Converts the continuous amplitude samples $x(kT_s)$ into **discrete** level samples $\hat{x}(kT_s)$ taken from a finite set of L possible values $\hat{x} = \{\widehat{x_1}, \widehat{x_2}, \dots, \widehat{x_L}\}$.

➢ **Binary Encoder:** Each quantized level is represented by $\boldsymbol{r = \log_2 L}$ binary digits

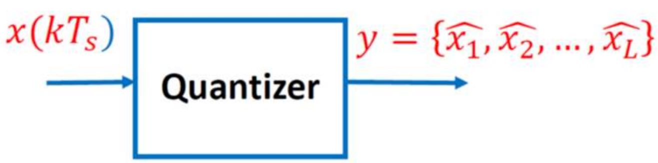
Quantization: Basic Definitions

- **Quantization**: is defined as the process of converting the continuous amplitude sample $x(kT_s)$ of a message signal into a discrete amplitude $\hat{x}(kT_s)$ taken from a finite and countable set of L possible values $\hat{x} = \{\widehat{x_1}, \widehat{x_2}, \dots, \widehat{x_L}\}$.
- The **dynamic range** of the quantizer is the range of values for which the quatizer is designed**,** $\boldsymbol{x_{min} < x < x_{max}}$
- This range is partitioned into L intervals such that if $x(kT_s) \in R_i$, the quantizer output will be a level $\widehat{x_i} = \{\widehat{x_1}, \widehat{x_2}, \dots, \widehat{x_L}\}$
- The qauntizer output is called a **representation or reconstruction level**
- The boundary points separating adjacent regions are called **decision or threshold levels**.
- The spacing between representation levels is called the **step size (Δ)**

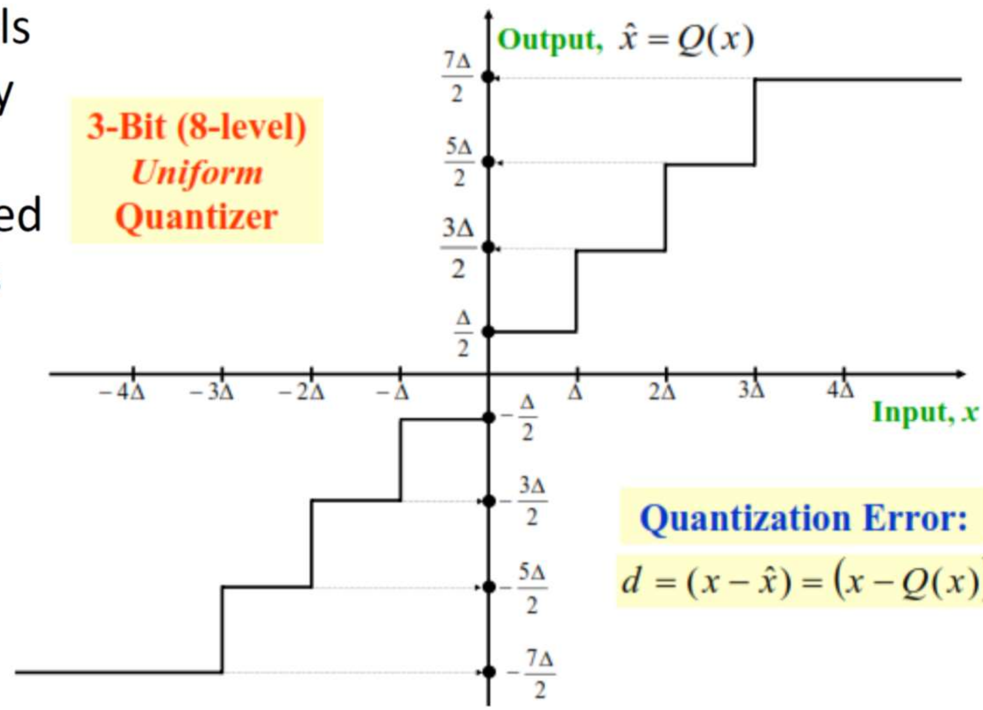


The Uniform Quantizer: Input-Output Characteristic

- A quantizer is called **uniform** when the L regions are of equal length Δ and the spacing between representation levels is uniform and equals to Δ.
- The input-output characteristic of a uniform quantizer (midrise type) is shown below for L=8.
- If the dynamic range of the quantizer varies between $x_{min} < x < x_{max}$, then $\boldsymbol{\Delta = \frac{x_{max} - x_{min}}{L}}$
- When the spacing between the adjacent levels is made small, $\hat{x}(kT_s)$ can be made practically indistinguishable from $x(kT_s)$.
- There is always a loss of information associated with the quantization process. Therefore, it is not possible to completely recover the sampled signal from the quantized signal.

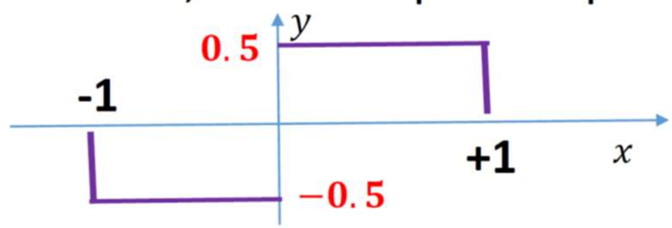


$$\boldsymbol{e = (x - y).}$$



Example: the one-bit quantizer

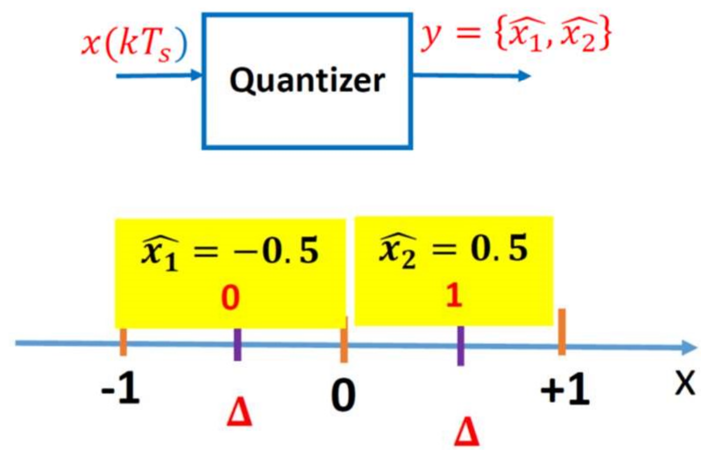
- **Example:** The signal $\mathrm{x(t)} = \cos(2\pi t)$ is uniformly sampled at a rate of 20 samples per second. The samples are applied to a sign detector, whose input-output characteristic is defined as:



- $y(t) = \begin{cases} 0.5, & 0 < x < 1 \\ -0.5, & -1 < x < 0 \end{cases}$
- The next figures depict the input samples to the sign detector, the quantized output, and the quantization error defined as $e = (x - y)$.
- **Here,** $\Delta = \frac{1-(-1)}{2} = 1$;

t=	0	0.0500	0.1000	0.1500	0.2000	0.2500
x(t)=	1.0000	0.9511	0.8090	0.5878	0.3090	0.0000
y =	0.5	0.5	0.5	0.5	0.5	0.5
e=	0.5	0.4511	0.3090	0.0878	-0.191	-0.5

Note that $-\frac{\Delta}{2} < e < \frac{\Delta}{2} \Rightarrow -0.5 < e < 0.5$

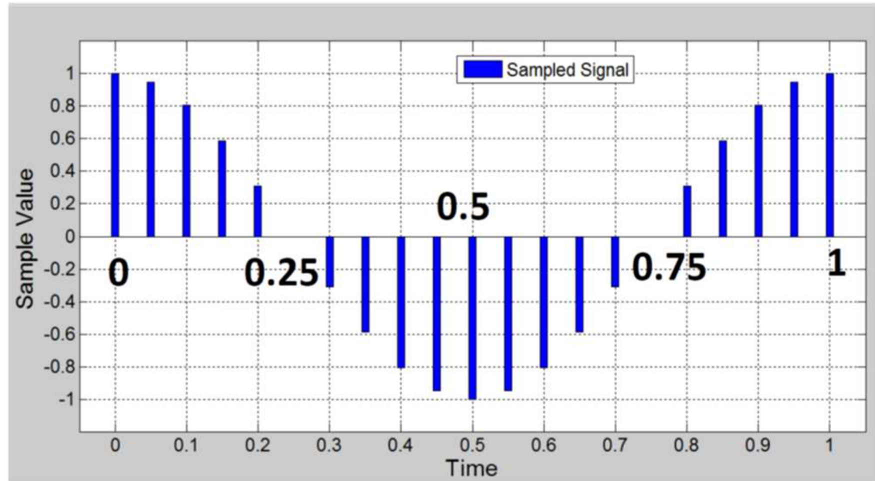


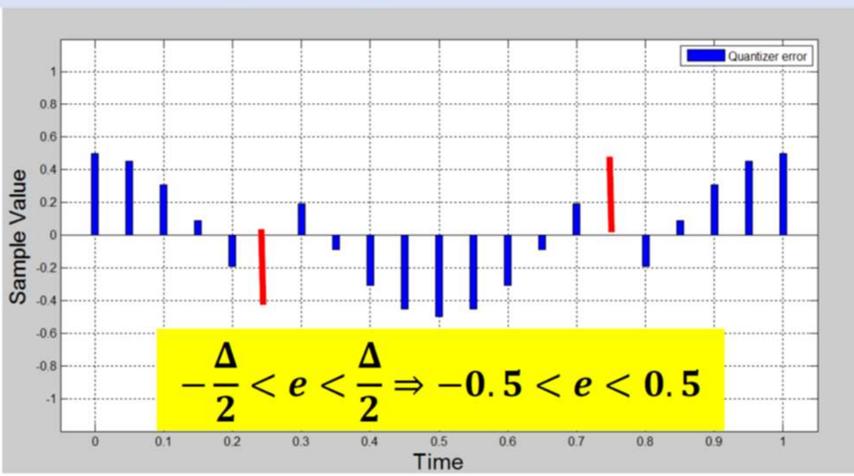
Quantization: the one-bit quantizer

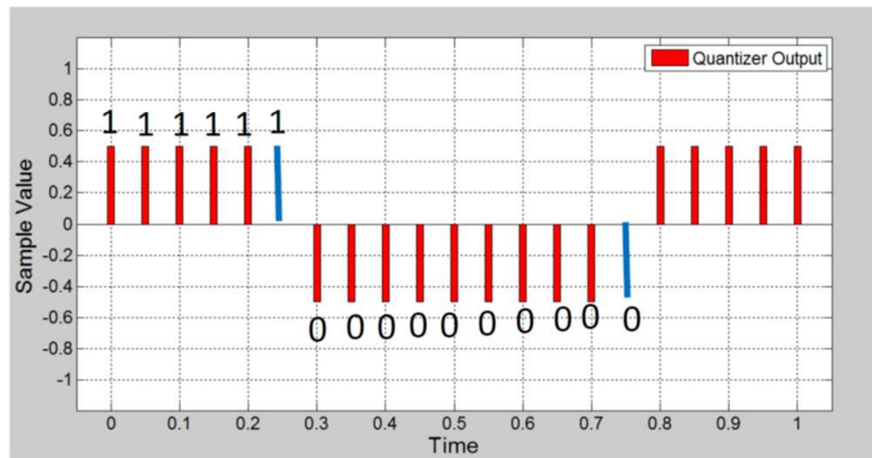
- **Example:** The signal $x(t) = \cos(2\pi t)$ is uniformly sampled at a rate of 20 samples per second.
- The samples are applied to a sign detector.
- Binary digits are assigned to the quantizer output.

t=	0	0.0500	0.1000	0.1500	0.2000	0.2500
x(t) =	1.0000	0.9511	0.8090	0.5878	0.3090	0.0000
y =	0.5	0.5	0.5	0.5	0.5	0.5
e=	0.5	0.4511	0.3090	0.0878	-0.191	-0.5

Note that $-\frac{\Delta}{2} < e < \frac{\Delta}{2} \Rightarrow -0.5 < e < 0.5$



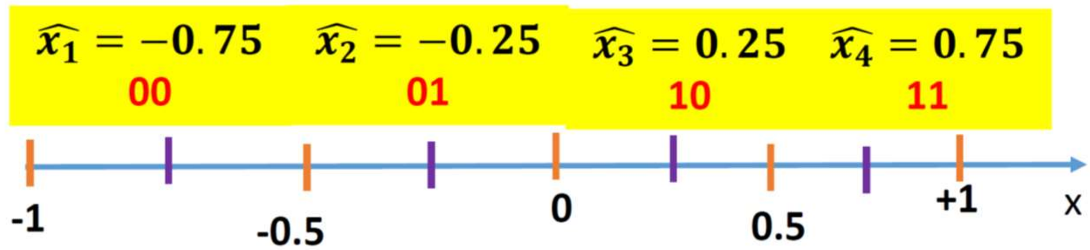




Quantization: the two-bit quantizer

- **Example**: The signal $x(t) = \cos(2\pi t)$ is sampled uniformly at a rate of 20 samples per second. The samples are applied to a four-level uniform quantizer with input-output characteristic

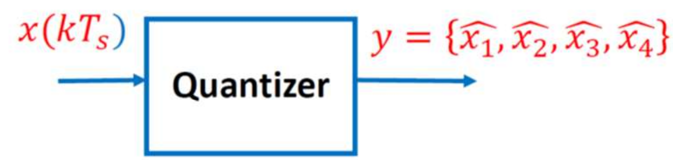
- $$y(t) = \begin{cases} 0.75, & 0.5 < x < 1 \\ 0.25, & 0 < x < 0.5 \\ -0.25, & -0.5 < x < 0 \\ -0.75, & -1 < x < -0.5 \end{cases}$$



- The next figures depict the quantizer operation, the input samples to the sign detector, the quantized output, and the quantization error defined as $e = (x - y)$.

t =	0	0.0500	0.1000	0.1500	0.2000	0.2500
x(t) =	1.0000	0.9511	0.8090	0.5878	0.3090	0.0000
y =	0.75	0.75	0.75	0.75	0.25	0.25
e =	0.25	0.2011	0.059	-0.1622	0.059	-0.25

Note that $-\frac{\Delta}{2} < e < \frac{\Delta}{2} \Rightarrow -0.25 < e < 0.25$



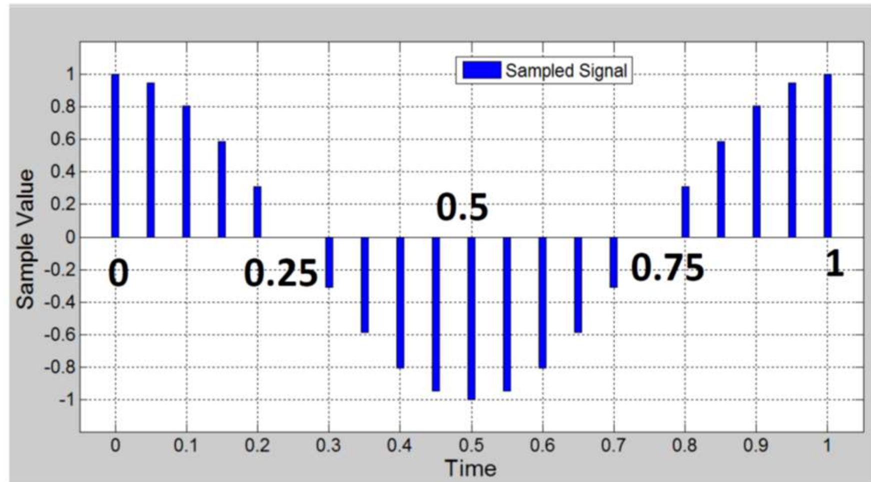
$$\Delta = \frac{1-(-1)}{4} = 0.5$$

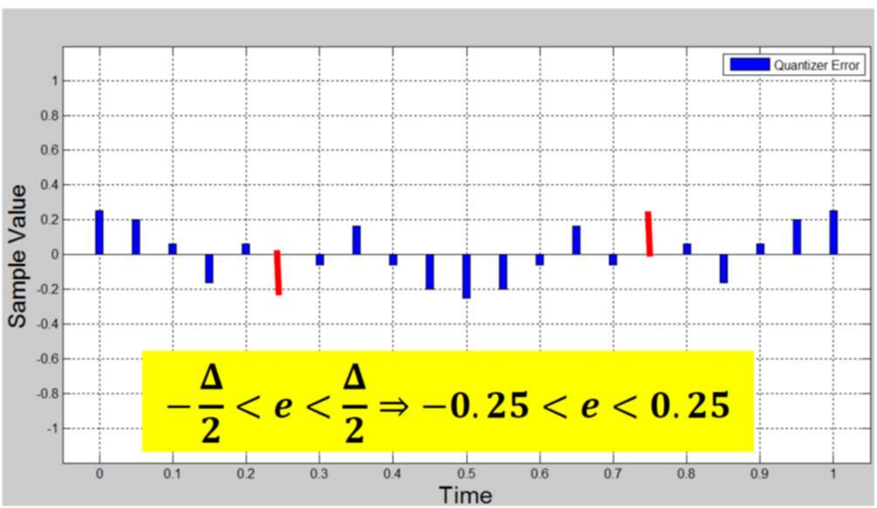
Quantization: the two-bit quantizer

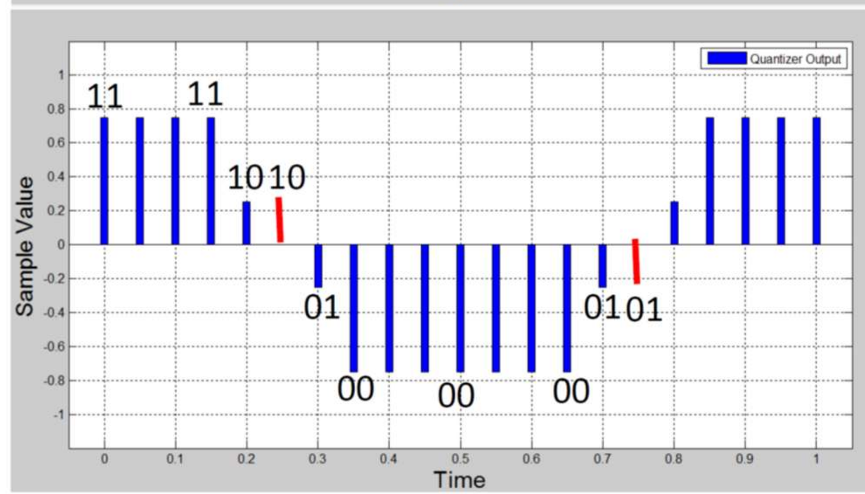
- **Example**: The signal $x(t) = \cos(2\pi t)$ is sampled uniformly at a rate of 20 samples per second. The samples are applied to a four-level uniform quantizer
- Binary digits are assigned to the quantizer output

t =	0	0.0500	0.1000	0.1500	0.2000	0.2500
x(t) =	1.0000	0.9511	0.8090	0.5878	0.3090	0.0000
y =	0.75	0.75	0.75	0.75	0.25	0.25
e =	0.25	0.2011	0.059	-0.1622	0.059	-0.25

Note that $-\frac{\Delta}{2} < e < \frac{\Delta}{2} \Rightarrow -0.25 < e < 0.25$

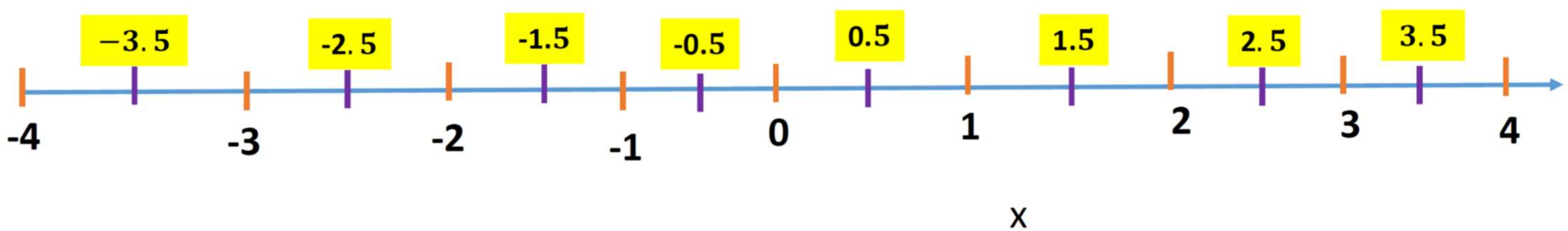






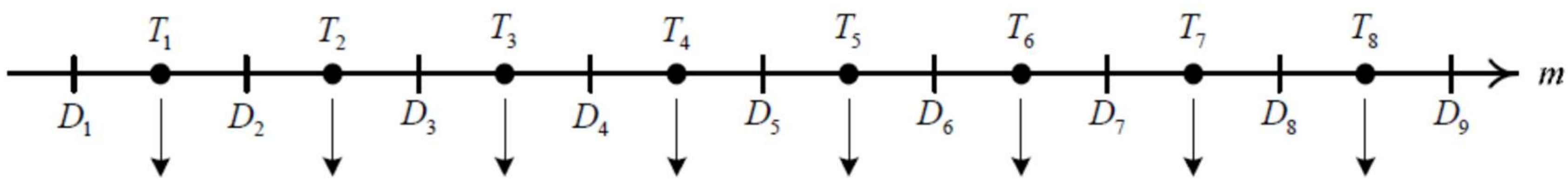
Example: Quantizer Design

- Design an 8-level uniform quantizer with a dynamic range of (-4, +4) V. Here, you need to specify the thresholds and the representation values.
- How many binary digits are needed to represent the samples? **(3 bits)**
- Find the representation value and the quantization error when a 1.64 V sample is applied to the quantizer.
- **Solution:** $\boldsymbol{\Delta = \frac{4-(-4)}{8} = 1}$
- **Since L = 8, then n=3.** $\mathbf{L = 2^{n}; n = \ log_2(L)}$
- **When x = 1.64, y = 1.5, and the error is:** $\boldsymbol{e = (x - \hat{x}) = (1.64 - 1.5) = 0.14}$.

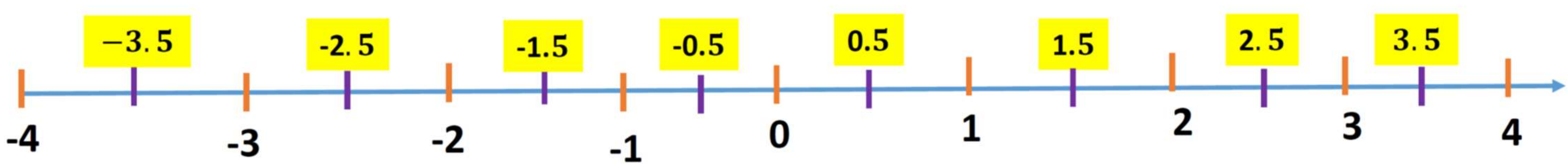


Example: continued

- Find the binary representation corresponding to the sample -2.1 V if natural binary encoding is employed **(-2.5; 001)**
- Are there other binary encoding formats?



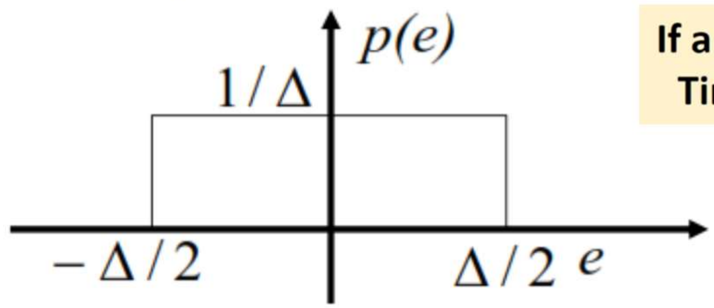
Ordinal:	0	1	2	3	4	5	6	7
NBC:	000	001	010	011	100	101	110	111
Gray:	000	001	011	010	110	111	101	100
FBC:	011	010	001	000	100	101	110	111



Quantization Noise

- The **quantization error** per sample is the difference between the input and output of the quantizer, i.e., $\boldsymbol{e = (x - \hat{x})}$
- The time average of the mean squared error $\boldsymbol{\frac{1}{T}\int (x - \hat{x})^2 dt}$ **; (Time Average)**
- The maximum error (also referred to as the resolution) = $|\frac{\Delta}{2}|$
- When Δ is small, the error, e, is assumed to be a uniform random variable over the interval $-\Delta|2 < e < \Delta|2$.
- The average quantization error (distortion) over all samples of the signal is
- $D = E(x - \hat{x})^2 = E(e^2) \Rightarrow D = \frac{1}{\Delta}\int_{-\Delta/2}^{+\Delta/2} (e)^2\, de \ \Rightarrow$ $\mathbf{D = \Delta^2|12}$ **(Statistical Average)**
- **Remark:** Note that D depends on the design of the quantizer and not on the signal applied to it, as we will see in the next two examples.

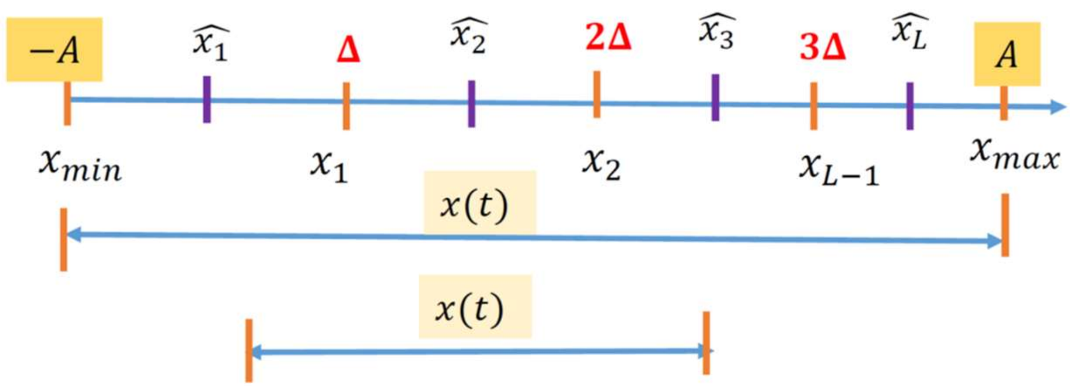
$$\left[-\frac{\Delta}{2}, \frac{\Delta}{2}\right]$$



If a process is assumed ergodic, then Time Average = Statistical Average

SQNR: Signal Matches Dynamic Range of a Uniform Quantizer

- **Example**: Let the sinusoidal signal $x(t) = Acos(2\pi f_0 t)$ be applied to a uniform quantizer with a dynamic range $(-A, A)$. We need to find the $SQNR$.
- **Solution:** The average power, P_x , in $x(t)$ is: $\boldsymbol{P_x = A^2|2}$
- The signal power to quantization noise ratio: $\boldsymbol{SQNR = \frac{A^2/2}{\Delta^2/12}}$
- Here, $\Delta = 2A/L$. If $L = 2^n$, then the $SQNR$ become: $\boldsymbol{SQNR = \frac{3}{2}L^2 = \frac{3}{2}2^{2n}}$
- In dB, the $SQNR$, becomes: $\boldsymbol{SQNR = 10log\frac{P_x}{D} = 6.02n + 1.76}$ (dB)
- $SQNR$ increases exponentially with the number n of bits per sample.
- There is a 6-dB improvement in $SQNR$ for each bit added to represent the sample values



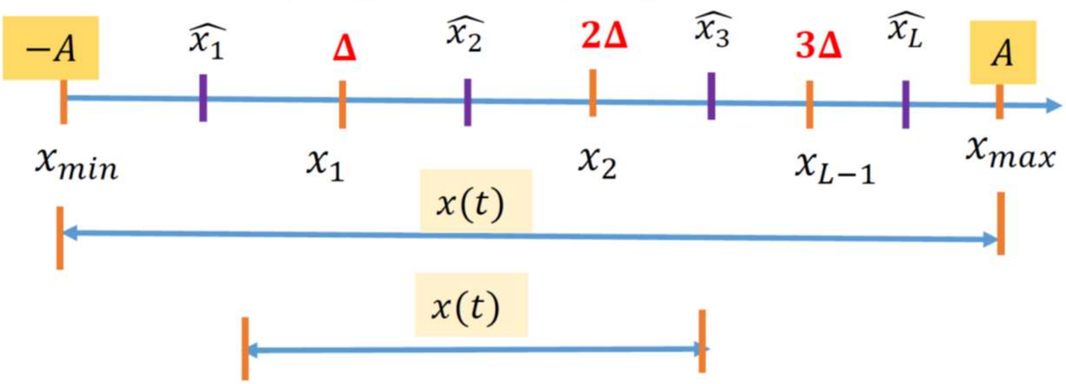
$$\Delta = \frac{x_{max} - x_{min}}{L}$$

Strong signal applied to a uniform quantizer

Weak signal applied to a uniform quantizer

Example: Weak Signal Applied to a Uniform Quantizer

- **Example**: Now, let the sinusoidal signal $x(t) = A/2cos(2\pi f_0 t)$ be applied to the same uniform quantizer of the previous example, with a dynamic range $(-A, A)$. We need to find the $SQNR$.
- **Solution:** $SQNR = \frac{(A/2)^2/2}{\Delta^2/12} = \frac{12}{32}L^2 = \frac{12}{32}2^{2n}$
- In dB, the $SQNR$, becomes: $SQNR = 10log\frac{P_x}{D} = 6.02n - 4.77$
- **Remark:** If the message signal $x(t)$ is a random signal, with an amplitude probability density function $f_X(x)$ and zero mean ($E(X) = 0$), then the SQNR is given as:
- $SQNR = \frac{E(X^2)}{E(x-\hat{x})^2} = \frac{\int_{-\infty}^{\infty} X^2 f_X(x)dx}{E(x-\hat{x})^2}$; Beyond the scope of this lecture.



$\Delta = \frac{x_{max} - x_{min}}{L}$

Strong signal applied to a uniform quantizer

Weak signal applied to a uniform quantizer

Example: Weak Signal Applied to a Uniform Quantizer

$x(t) = (1)cos(2\pi(10)t)$

Quantizer: (-10, 10);

n = 3 bits (8 levls)

$$\Delta = \frac{20}{8} = 2.5$$

Ideally sampled Signal

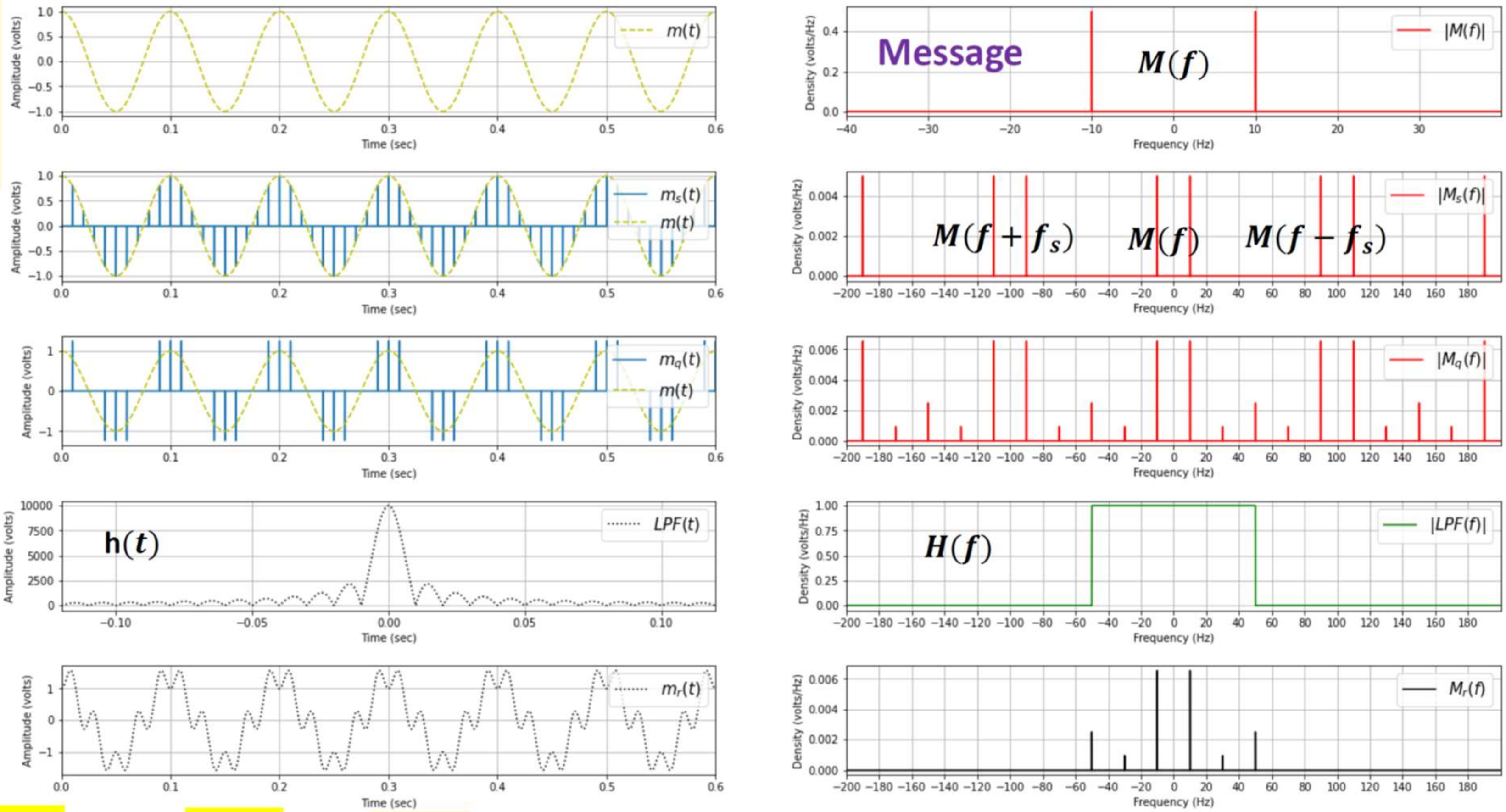
fs = 100

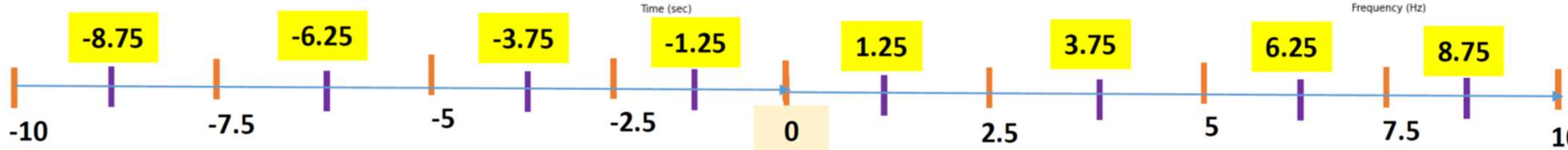
Quantized Signal

$-\frac{\Delta}{2}, +\frac{\Delta}{2}$, + 1.25, -1.25

Filter B.W = fs/2=50

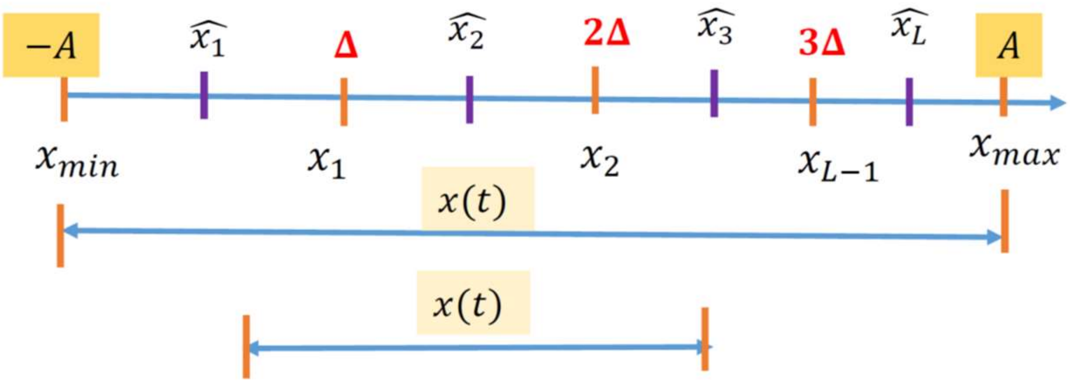
Recovered Signal





A Problem with the Uniform Quantizer

- **Problem**: Let the sinusoidal signal $x(t)$ be applied to a uniform quantizer with a dynamic range $(-A, A)$
- We have seen in the previous examples that the SQNR depends on the signal power. As the signal power decreases, the SQNR decreases (quantization noise is constant). **Quality deteriorates**
  - Case 1: $x(t) = A\cos(2\pi f_1 t)$, strong signal applied to the quantizer: $SQNR1 = \frac{3}{2}L^2$; with probability 0.2
  - Case 2: $x(t) = \frac{A}{2}\cos(2\pi f_2 t)$, weak signal applied to the quantizer: $SQNR2 = \frac{12}{32}L^2 = \frac{1}{4}SQNR1$ with probability 0.3
  - Case 3: $x(t) = \frac{A}{4}\cos(2\pi f_3 t)$, weak signal applied to the quantizer: $SQNR3 = \frac{4}{32}L^2 = \frac{1}{16}SQNR1$ with probability 0.5
- The average quantization noise is :
- $E(SQNR) = (0.2)\frac{3}{2}L^2 + (0.3)\frac{12}{32}L^2 + (0.5)\frac{1}{16}L^2 = 0.44375L^2$
- **Solution**: Use non-uniform quantization (the subject of the next lecture), the use of which will ensure an almost constant SQNR for strong as well as weak signal components.



$$\Delta = \frac{x_{max} - x_{min}}{L}$$

Strong signal applied to a uniform quantizer

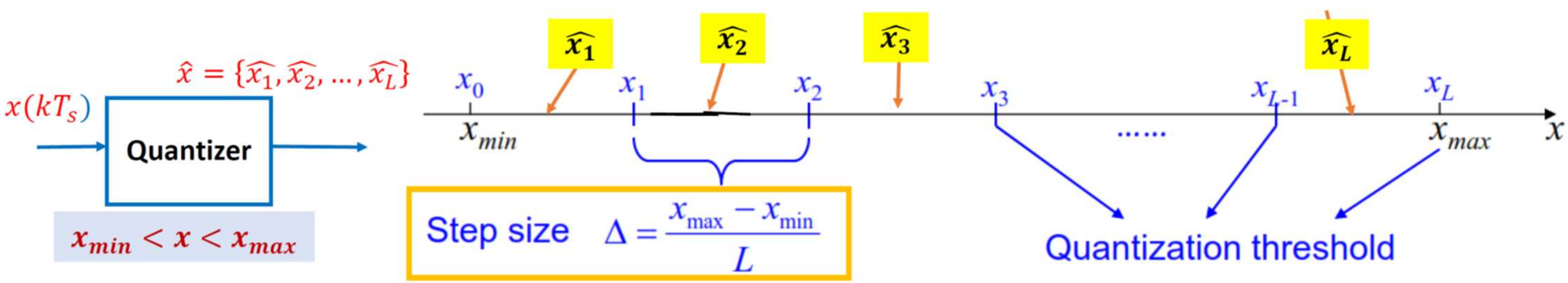
Weak signal applied to a uniform quantizer

Non-uniform Robust Quantization
Lecture Outline

- **In this lecture**
  - We define the non-uniform robust quantizer
  - Define the μ-law compressor and expander characteristics
  - Show the improvement in SQNR of non-uniform quantization over uniform quantization.
  - Present a number of illustrative examples

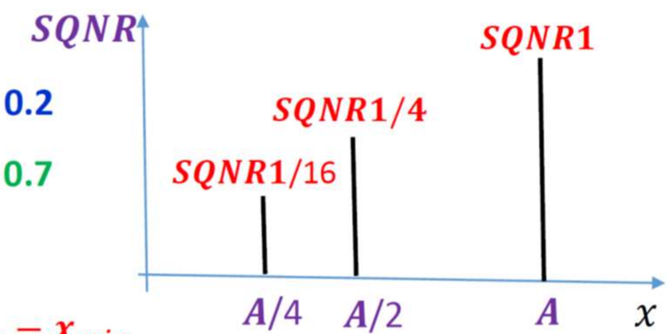
Quantization: Basic Definitions

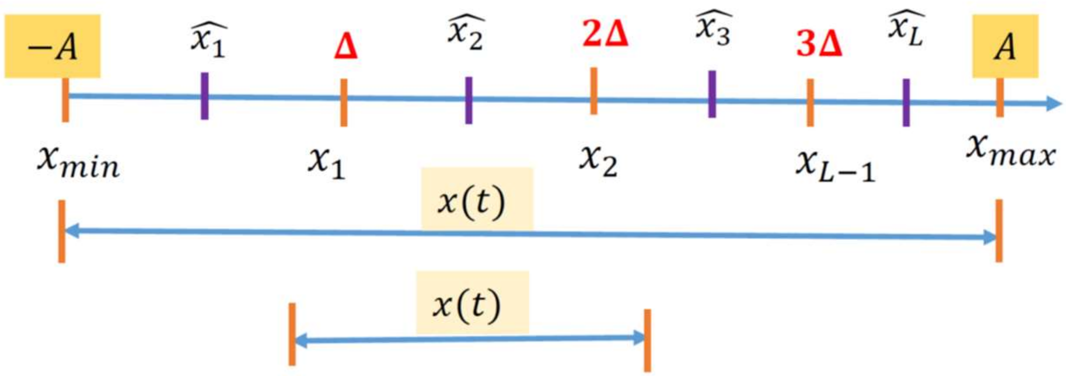
- **Quantization**: is defined as the process of converting the continuous amplitude sample $x(kT_s)$ of a message signal into a discrete amplitude $\hat{x}(kT_s)$ taken from a finite and countable set of L possible values $\{\widehat{x}|$.
- The **dynamic range** of the quantizer is the range of values for which the quatizer is designed, $x_{min} < x < x_{max}$
- This range is partitioned into L intervals such that if $x(kT_s) \in R_i$, the quantizer output will be a level $\widehat{x_i} = \{\widehat{x_1}, \widehat{x_2}, \dots, \widehat{x_L}\}$
- The qauntizer output is called a **representation or reconstruction level**
- The boundary points separating adjacent regions are called **decision or threshold levels**.
- The spacing between representation levels is called the **step size (Δ)**
- A quantizer is called **uniform** when the L regions are of equal length Δ and the spacing between representation levels is uniform and equals to Δ, where $\boldsymbol{\Delta = \frac{x_{max} - x_{min}}{L}}$.



Drawback of the Uniform Quantizer

- **The Drawback of the uniform quantizer**:
  - The uniform quantizer is easy to build; however, it is not optimal when the input signal is weak most of the time. Here, the signal does not use the entire set of available quantization levels.
  - Maximum SQNR is achieved when the signal strength matches the dynamic range of the quantizer.
- **Demonstration**: Let the sinusoidal signal $x(t)$ be applied to a uniform quantizer with a dynamic range $(-A, A)$
- We have seen in the previous lecture that the SQNR depends on the signal power. As the signal power decreases, the SQNR decreases, since the quantization noise is constant ($SQNR = \frac{P_x}{\Delta^2/12}$)
  - Let : $x(t) = A\cos(2\pi f_1 t)$, $SQNR1 = \frac{3}{2}L^2$; with probability 0.1
  - $x(t) = \frac{A}{2}\cos(2\pi f_2 t)$, $SQNR2 = \frac{12}{32}L^2 = \frac{1}{4}SQNR1$ with probability 0.2
  - $x(t) = \frac{A}{4}\cos(2\pi f_3 t)$, $SQNR3 = \frac{4}{32}L^2 = \frac{1}{16}SQNR1$ with probability 0.7





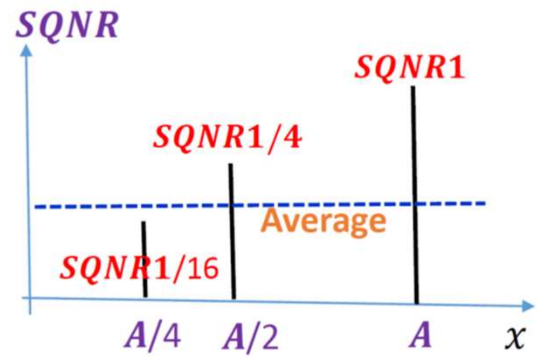
$$\Delta = \frac{x_{max} - x_{min}}{L}$$

Strong signal applied to a uniform quantizer

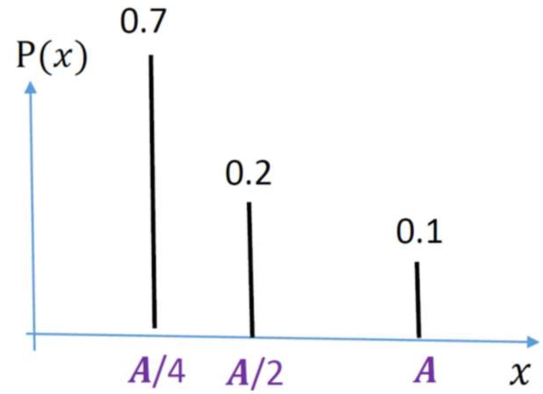
Weak signal applied to a uniform quantizer

Drawback of the Uniform Quantizer

- The average SQNR is:
- $E(SQNR) = (0.1)SQNR1 + \frac{(0.2)SQNR1}{4} + \frac{(0.7)SQNR1}{16} = (0.19375)SQNR1$
- The average SQNR is closer to that of the smallest value of the SQNR.
- The overall performance is not satisfactory .
- Small amplitudes will be subjected to more distortion than large amplitudes.
- The non-uniform quantization, the subject of this lecture, will ensure an almost constant SQNR for strong as well as weak signal components.

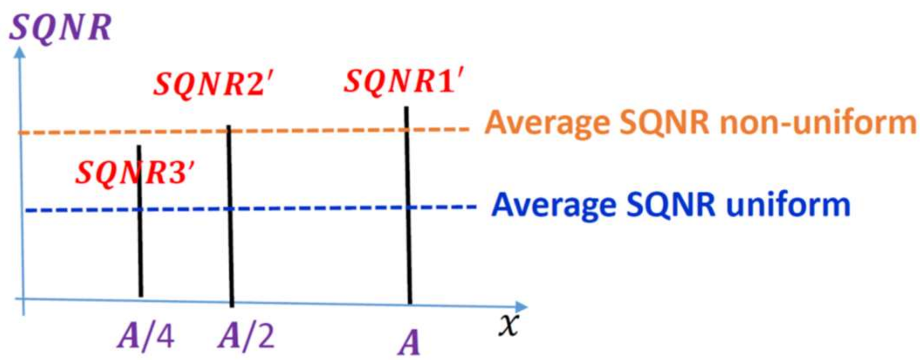


Uniform Quantizer



Two Remarks:

- **SQNR is higher than that of the uniform quanrtizer.**
- **The SQNR is almost the same for all signal levels**



Non-uniform Quantizer

Example: Signal Matches Dynamic Range of a Uniform Quantizer

$x(t) = (10)cos(2\pi(10)t)$

Quantizer: (-10, 10);

n = 3 bits (8 levls)

$$\Delta = \frac{20}{8} = 2.5$$

Ideally sampled Signal

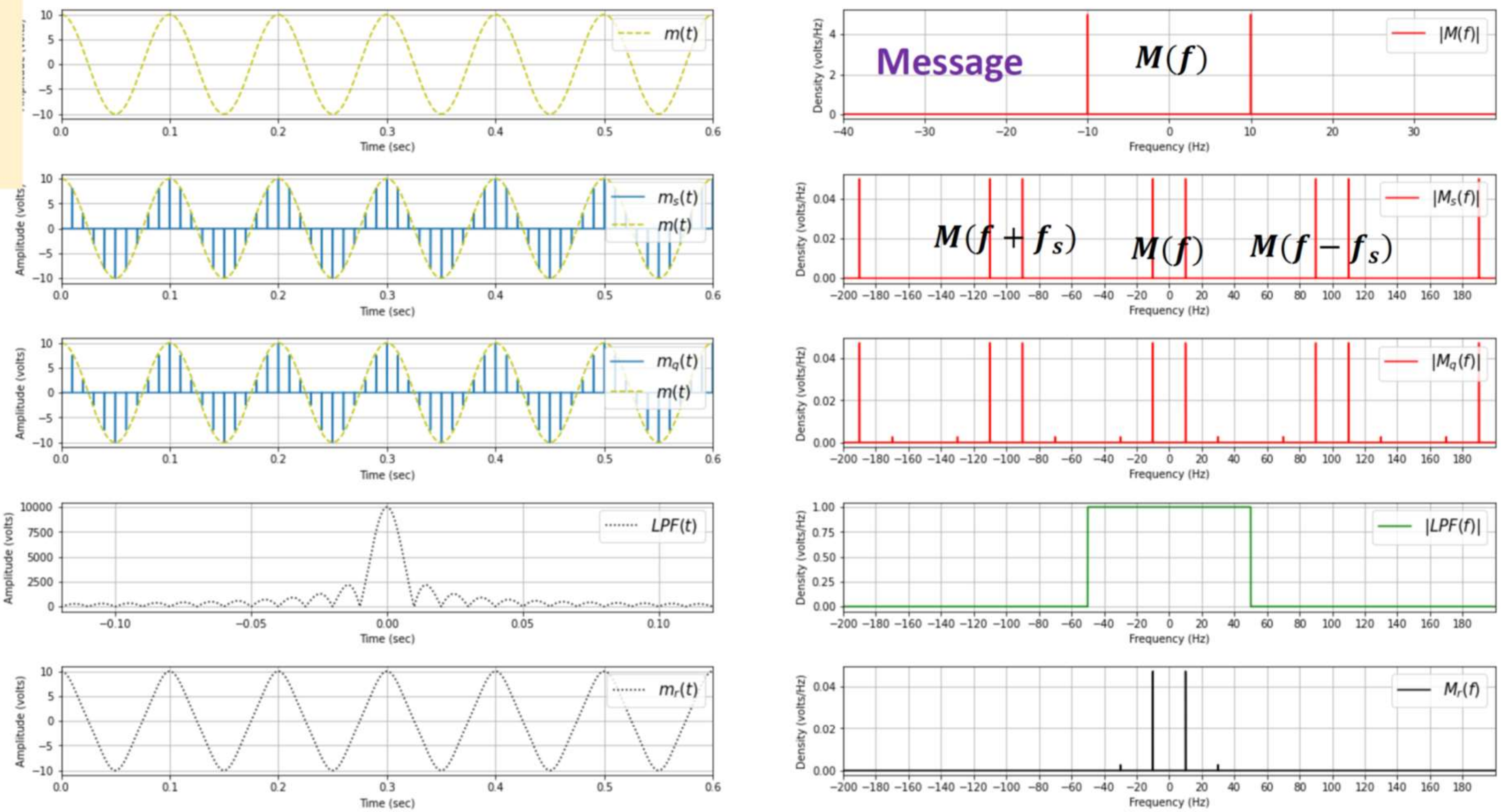
fs = 100

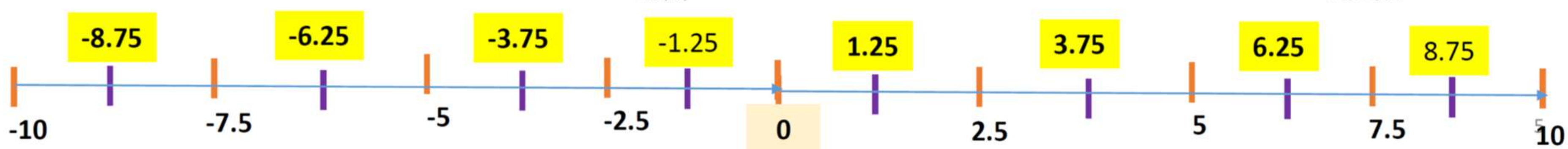
Quantized Signal

Filter B.W = fs/2=50

Recovered Signal

All levels are utilized





Example: Weak Signal Applied to a Uniform Quantizer

$x(t) = (1)cos(2\pi(10)t)$
Quantizer: (-10, 10);
n = 3 bits (8 levls)

$$\Delta = \frac{20}{8} = 2.5$$

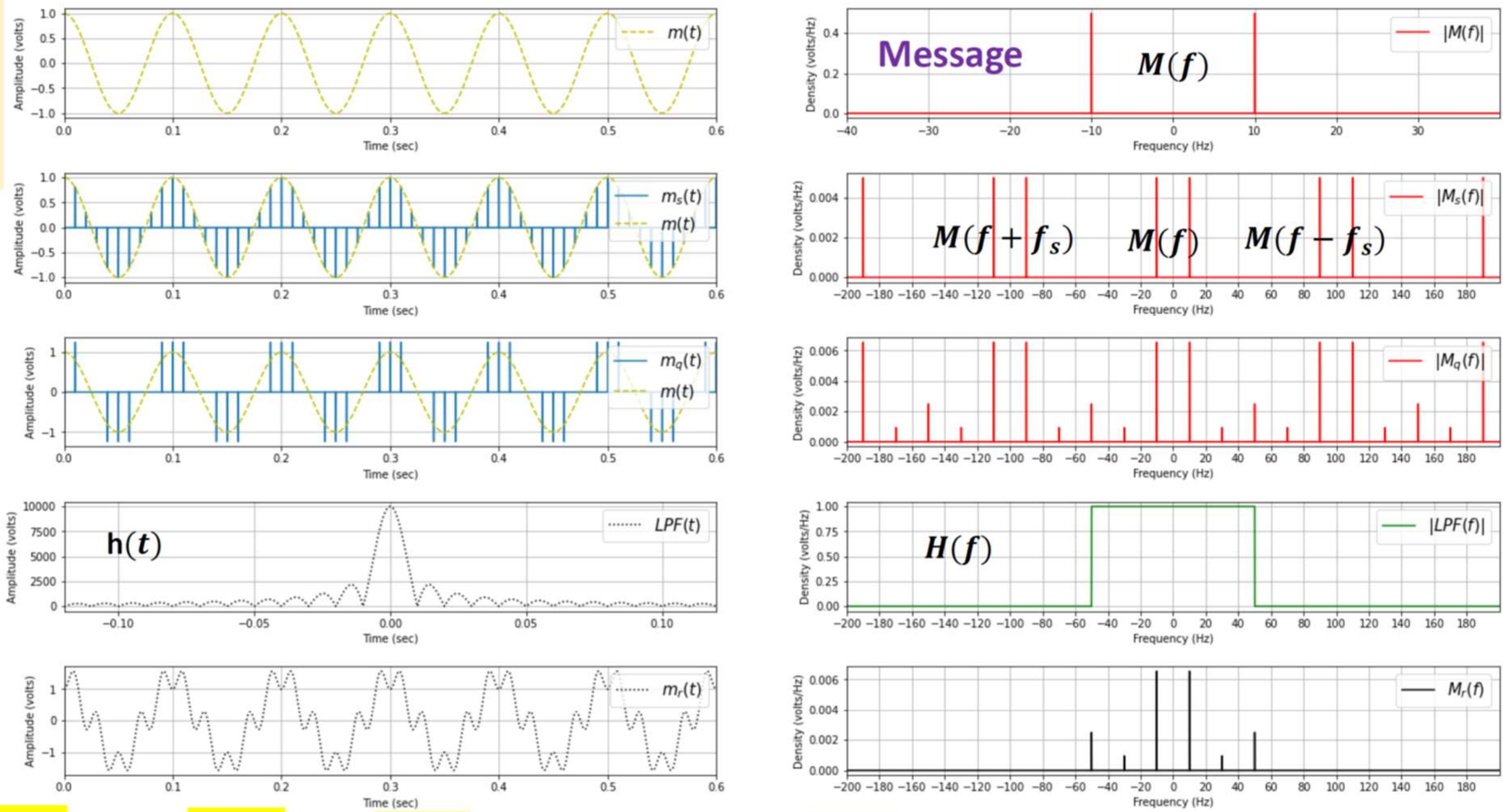
Ideally sampled Signal
fs = 100

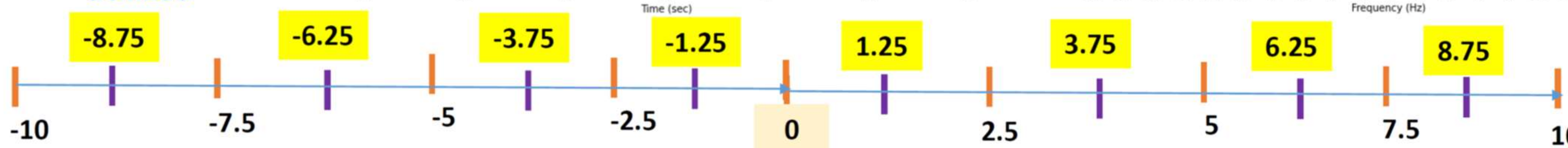
Quantized Signal
$-\frac{\Delta}{2}, +\frac{\Delta}{2}$, + 1.25, -1.25

Filter B.W = fs/2=50

Recovered Signal

Only two levels are utilized

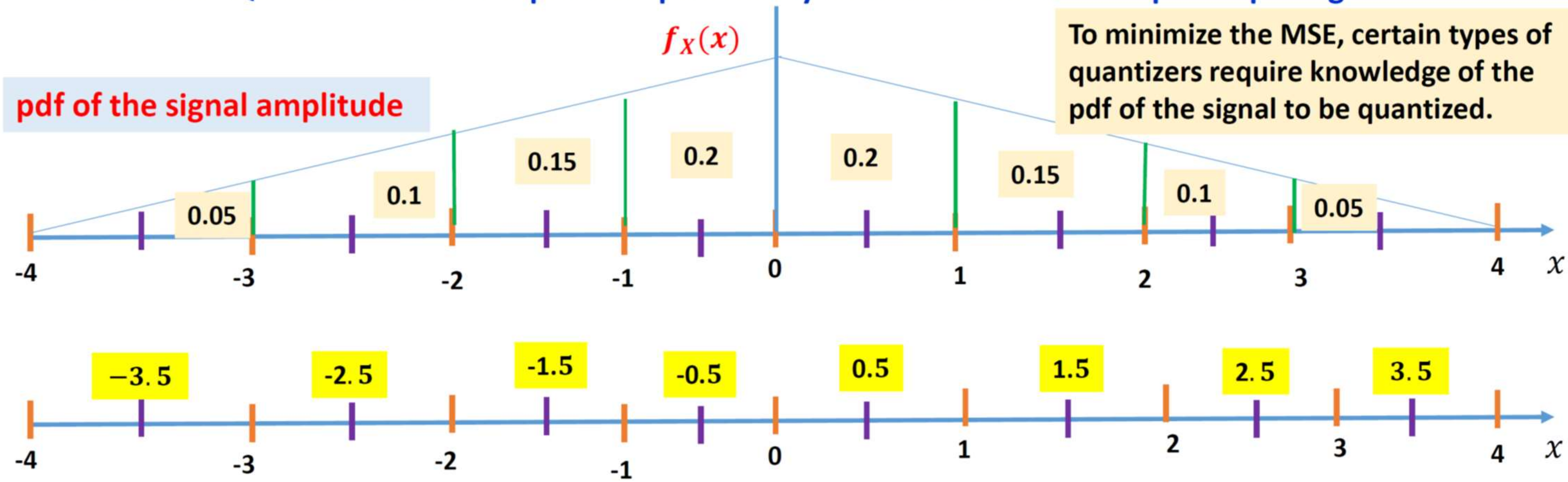




Uniform and Non-uniform Robust Quantization

- In practical applications, especially speech signals, small signal amplitudes occur more often than large signal amplitudes. This means that for the same signal, small amplitudes will be subject to distortion more than large amplitudes. That is, larger amounts of distortion have a higher probability of occurrence.
- The non-uniform robust quantizer, which employs the μ-law can
  - **Provide a SQNR that is somewhat constant and independent of the signal strength**
  - **Provide a SQNR that is also independent probability distribution of the sampled input signal**

To minimize the MSE, certain types of quantizers require knowledge of the pdf of the signal to be quantized.

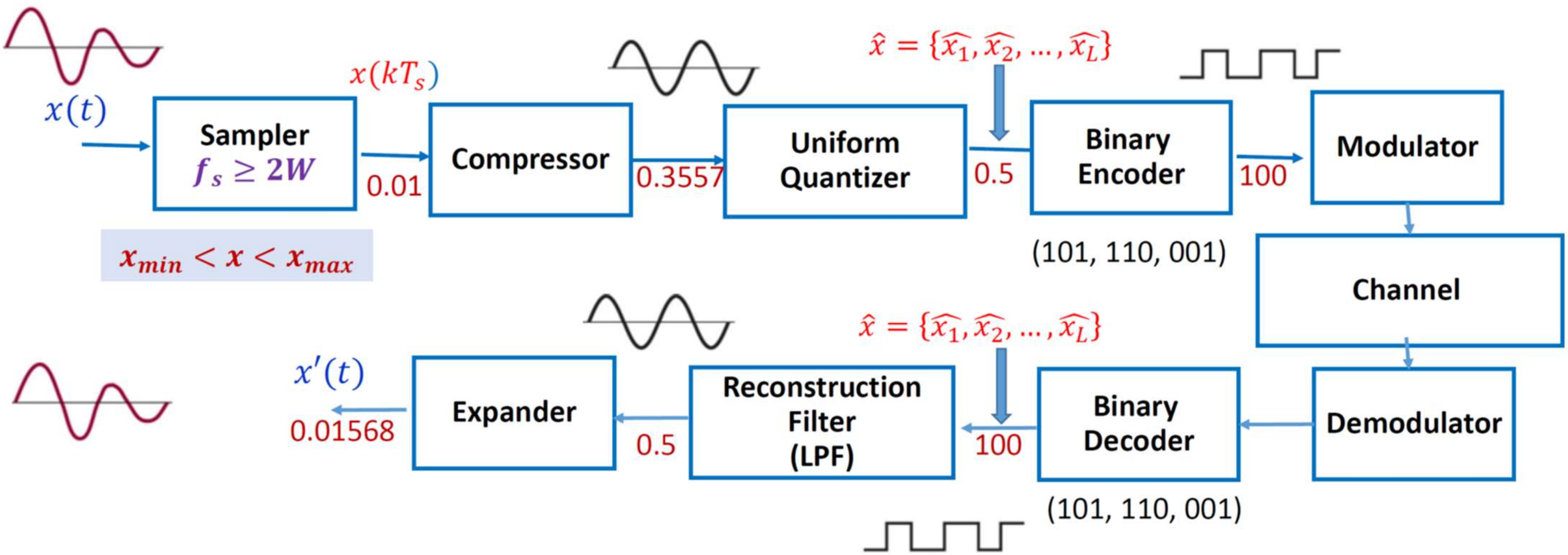


Non-uniform Robust Quantization

- We will use a type of non-uniform quantizers, called **companding**, **t**hat **does not require knowledge of the pdf of the signal to be quantized, and yields an almost uniform SQNR over a wide range of signal variations.**
- The process of pre-distorting the signal at the transmitter is known as (signal) **compression**. At the receiver, this process is reversed to remove distortion and is known as (signal) **expansion.** The two operations together, are typically, referred to as **companding** (or compansion).
- The compressor is a nonlinear operation that amplifies weak signal values more than it amplifies large signal values, thus stretching the signal over more representation levels. This will enhance weak signal levels and improve their SQNR. Large signal levels will also suffer from distortion, but the overall effect on the signal is an improvement in SQNR.
- Since the probability of smaller amplitudes is higher than the larger amplitudes, the overall result is an improvement.
- In North America, μ -law companding (with $\mu = 255$) is the standard.

Robust (Non-uniform) Quantization

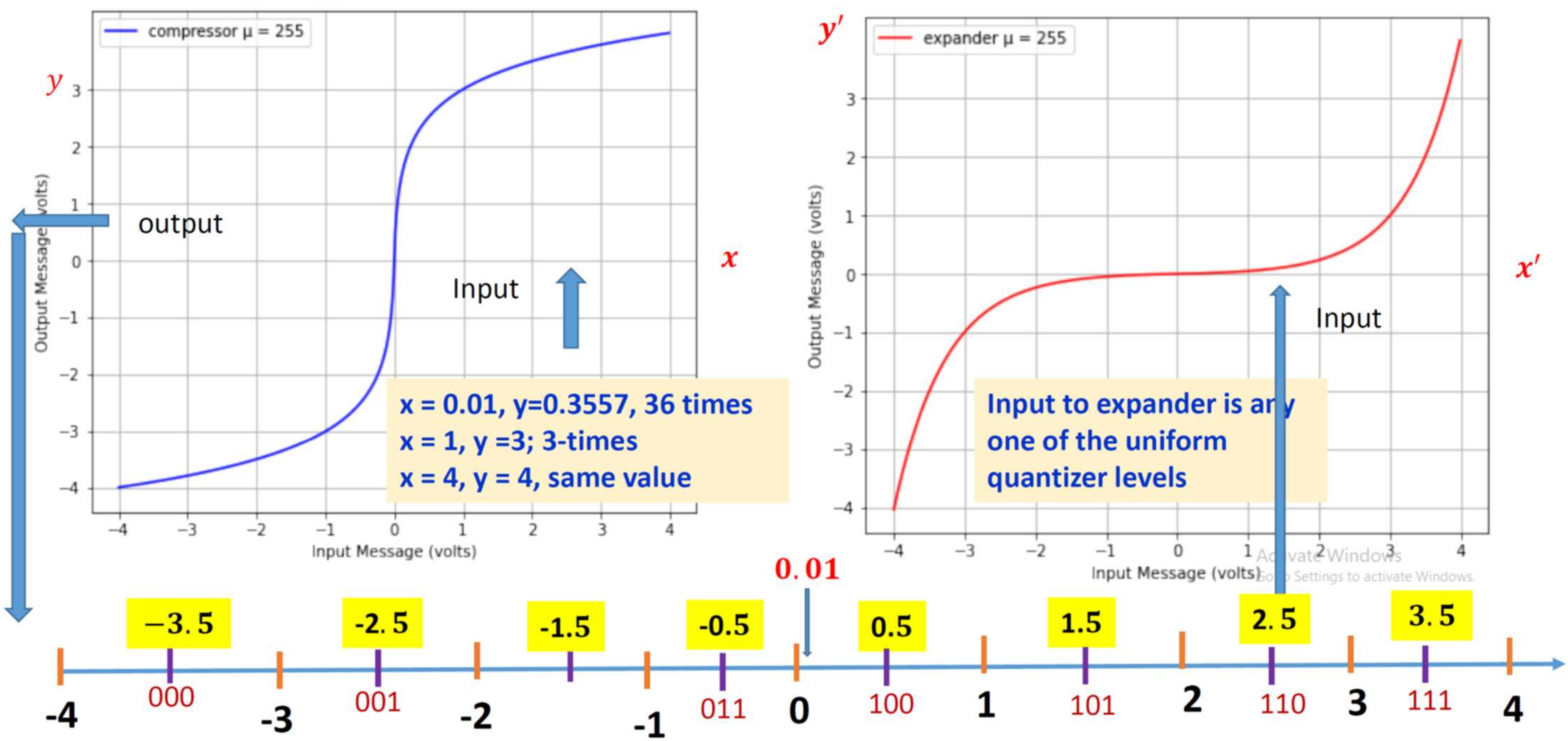
- In summary, companding is performed as follows:
  - Compress the signal using the μ-law. The output is approximately uniformly distributed.
  - Apply the compressed sample to a uniform quantizer
  - Transmit the quantized sample to the receiver.
  - Apply the received sample to the expander. The output is the desired signal value.



Non-uniform Robust Quantization

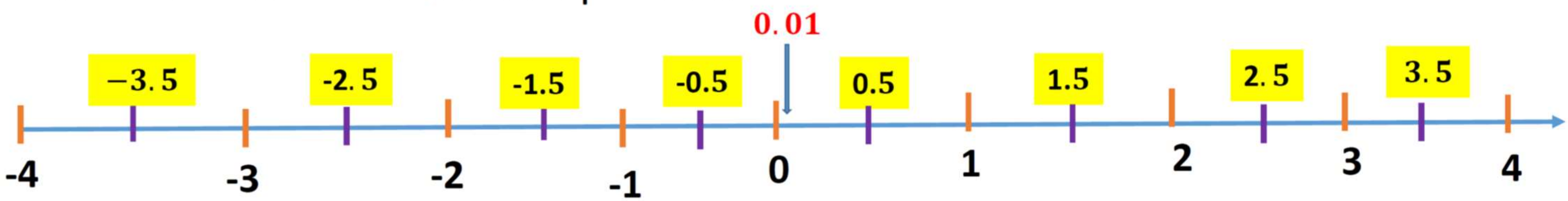
$$y = y_{max}\frac{ln\left[1+\mu\left(\frac{|x|}{x_{max}}\right)\right]}{ln(1+\mu)}sgn(x)$$

$$\frac{y'}{y'_{max}} = \left[\exp\left(\frac{x'}{x'_{max}}(\ln(1+\mu))\right)-1\right]/\mu\ sgn(x')$$



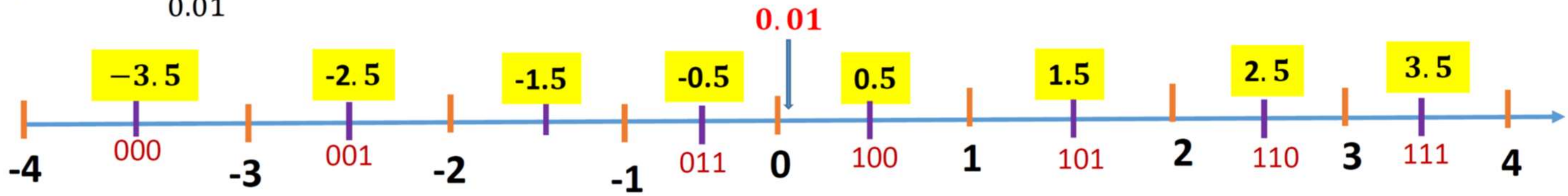
Non-uniform Robust Quantization

- **Example:** If a sample of magnitude 0.01 is applied to a 3-bit uniform quantizer with a dynamic range (-4, 4) V. Find the quantizer output, the quantization error, and the value of the sample at the receiving side.
- If the sample is applied to quantizer that employs companding, find the value of the received sample, and the quantization error.
- **Uniform Quantization**: when x = 0.01, $\hat{x} = 0.5$; This is the same voltage that is supposed to be received, assuming no transmission error.
- The error is: $|\mathbf{x} - \hat{\boldsymbol{x}}| = |\mathbf{0.01} - \mathbf{0.5}| = \mathbf{0.49}$
- The % of error is: $\hat{\boldsymbol{x}} = \frac{\mathbf{0.5 - 0.01}}{\mathbf{0.01}} \mathbf{100\%} = \mathbf{4900\%}$.
- The error is 49 times the sample value.



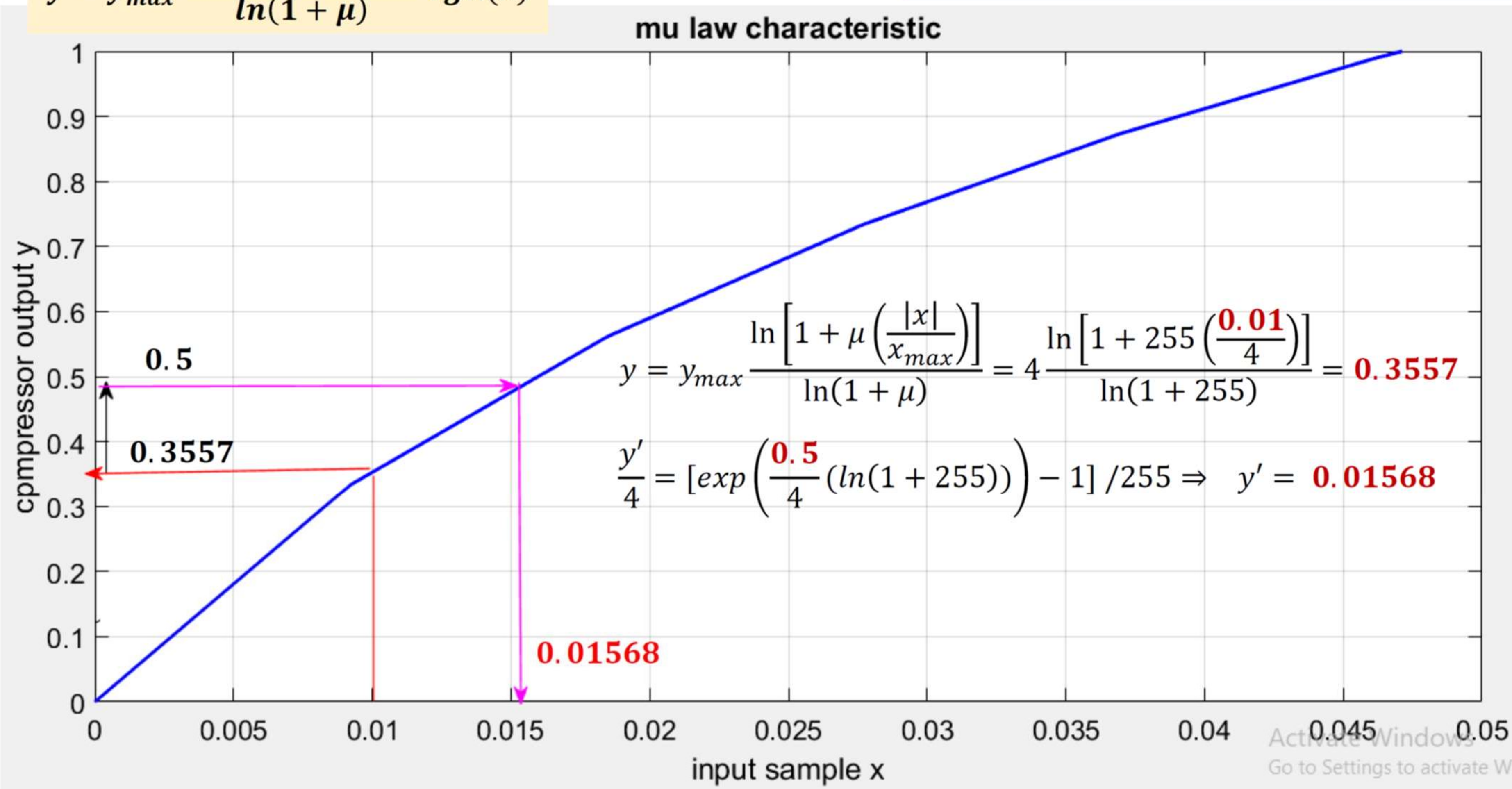
Non-uniform Robust Quantization

- **Non-uniform Quantization**: when x = 0.01 is applied to the μ-law,
- $\boldsymbol{y} = \boldsymbol{y_{max}} \frac{\ln\left[1+\mu\left(\frac{|x|}{x_{max}}\right)\right]}{\ln(1+\mu)} = 4\frac{\ln\left[1+255\left(\frac{0.01}{4}\right)\right]}{\ln(1+255)} = \mathbf{0.3557}$
- The 0.3557 is applied to the uniform quantizer, whose output is **0.5.**
- The 0.5 V is encoded as 100 and transmitted. Assume no transmission errors.
- At the receiver, the 0.5 is applied to the expander (the inverse μ-law characteristic)
- $\frac{\boldsymbol{y\prime}}{\mathbf{4}} = \left[\exp\left(\frac{\mathbf{0.5}}{\mathbf{4}}(\mathbf{ln}(1 + \mathbf{255}))\right) - 1\right]/255 \Rightarrow \boldsymbol{y}' = \mathbf{0.01568}$
- Received sample $\boldsymbol{y}' = \mathbf{0.01568}$
- The error is: $|\mathbf{x} - \boldsymbol{y}'| = |\mathbf{0.01} - \mathbf{0.0157}| = \mathbf{0.0057}$
- Error: $\frac{0.01-0.0157}{0.01}100\% = \mathbf{57}\%$



$$y = y_{max}\frac{ln\left[1+\mu\left(\frac{|x|}{x_{max}}\right)\right]}{ln(1+\mu)}sgn(x)$$

$$\frac{y'}{y'_{max}} = \left[exp\left(\frac{x'}{x'_{max}}(ln(1+\mu))\right)-1\right]/\mu$$



Uniform and Non-uniform Robust Quantization: More Values

Sample Value	Uniform Quantizer		Non-uniform Quantizer	
	Output	% Error	Output	% Error
0.01	0.5	$\frac{0.01-0.5}{0.01}100\% = \mathbf{4900}\%$	0.0157	$\frac{0.01-0.0157}{0.01}100\% = \mathbf{57}\%$
0.07	0.5	$\frac{0.07-0.5}{0.07}100\% = \mathbf{707}\%$	0.1098	$\frac{0.07-0.1098}{0.07}100\% = \mathbf{56.8}\%$
1.7	1.5	$\frac{1.7-1.5}{1.7}100\% = \mathbf{11.76}\%$	1.9922	$\frac{1.7-1.9922}{1.7}100\% = \mathbf{17.18}\%$
2.9	2.5	$\frac{2.9-2.5}{2.9}100\% = \mathbf{13.79}\%$	1.9922	$\frac{2.9-1.9922}{2.9}100\% = \mathbf{31.3}\%$
3.8	3.5	$\frac{3.8-3.5}{3.8}100\% = \mathbf{7.89}\%$	1.9922	$\frac{3.8-1.9922}{3.8}100\% = \mathbf{47.5}\%$

Uniform and Non-uniform Robust Quantization

- This figure shows a weak and a strong signal applied to a uniform and a non-uniform quantizer. As we can see, the non-uniform quantizer represents both signals quite adequately.