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MATH1321

Calculus 2

Chapter 10.9

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التشابتر، هاد تلخيص دكتور
عبدالرحيم موسى وضفت عليه
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 Convergence of Taylor Series

Th (Taylor's Theorem)

Assume $f, f', f'', \dots, f^{(n)}$ are continous on $[a,b]$ and $f^{(n)}$ is differentiable on (a,b), then there exists a number $c \in (a,b)$ such that :

تقريب لـ f(b)

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}.$$

Remainder

* Note that Taylor's Theorem is a generalization of the MVT. $f'(c) = \frac{f(b)-f(a)}{b-a}$
* If we change b by x, we get Taylor's formula:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(X)$$

$P_n(x)$

where $f(x) = P_n(x) + R_n(x)$ Taylor Formula

- $R_n(x)$ is the Remainder of order n "or the error term" results from approximating f by $P_n(x)$ on an open interval I contains a given by

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \quad \text{where } c \in (a, x)$$

center

* Convergence of Taylor Series:

If $\lim_{n\to\infty} R_n(x) = 0$ for all $x \in I$, then the Taylor Series generated by f at $x = a$ converges to f on I :

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

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Exp show that the Taylor series generated by $f(x) = e^x$ at $x = 0$ converges to $f(x)$ for every real value x. (78)

$f(x) = e^x$ is smooth on $\mathbb{R} = (-\infty, \infty)$. Thus, has all orders of all derivatives. Taylor's formula is

$c \in (0, x)$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + R_n(x) \quad \text{, where}$$

$R_n(x) = \frac{e^c}{(n+1)!} x^{n+1}$. Note that $\lim_{n\to\infty} R_n(x) = e^c \lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} = 0$

Thus, the series converges to e^x for every $x \Rightarrow$ Th 5 Sec. 10.1

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^k}{k!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

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* Note that $e = \sum_{k=0}^{\infty} \frac{1}{k!}$ as a series: $c \in (0, 1)$ ($a = 0$, $x = 1$), $0 < c < 1$, $e^0 < e^c < e < 3$

$$e = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} + R_n(1) \text{, where } R_n(1) = \frac{e^c}{(n+1)!} < \frac{3}{(n+1)!}$$

Th (The Remainder Estimation Theorem)

Given Taylor's formula: $f(x) = P_n(x) + R_n(x)$.

If $|f^{(n+1)}(t)| \le M$ (positive; upper bound for $f^{(n+1)}$) for all $t \in (a, x)$, then the Remainder

$$\boxed{|R_n(x)| \le \frac{M}{(n+1)!} |x - a|^{n+1}.} \; *$$

Further, if $*$ holds for every n, then the series converges to $f(x)$.

Then the series converges for all x.

Exp show that the Taylor series for $\sin x$ at $x = 0$ converges for all x.

show that Taylor's formula for $\sin x$ at $x = 0$ is $\quad c \in (a, x) = (0, x)$

Recall that Taylor's formula for $\sin x$ at $x=0$ is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + R_{2n+1}(x).$$

Now apply The Remainder Estimation Theorem:

$$|R_{2n+1}(x)| = \left| \frac{f^{(2n+2)}(c)}{(2n+2)!} x^{2n+2} \right| \le \frac{1 |x|^{2n+2}}{(2n+2)!}.$$

$M = 1$ (sin and cos ≤ 1)

But $\lim_{n\to\infty} |R_{2n+1}(x)| \le \lim_{n\to\infty} \frac{|x|^{2n+2}}{(2n+2)!} = 0 \quad \forall x$

$\Rightarrow$ The Maclaurin series for $\sin x$ converges to $\sin x$ $\forall x$.

$\Rightarrow \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$.

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Exp Show that Taylor series for $\cos x$ at $x=0$ converges to $\cos x$ for every value of x.

The Taylor's formula for $\cos x$ at $x=0$ is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + R_{2n}(x).$$

Since $|\cos x|$ and $|$it's all derivative$| < 1$ we apply the Remainder Estimation Theorem with $\underline{\underline{M=1}}$

$0 \leq |R_{2n}(x)| \leq \frac{|x|^{2n+1}}{(2n+1)!}$. Now for every x we have $R_{2n}(x) \to 0$ as $n \to \infty$ by Th5 sech 10.1

Therefore, the series converges to $\cos x$ for every x. Thus,

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

Exp Find the first four nonzero terms in the Maclaurin series for the functions:

1) $\frac{1}{3}(2x + x\cos x) = \frac{2x}{3} + \frac{x}{3}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)$

$$= \frac{2x}{3} + \frac{x}{3} - \frac{x^3}{2!\,3} + \frac{x^5}{3(4!)} - \frac{x^7}{3(6!)} + \cdots$$

$$= x - \frac{x^3}{6} + \frac{x^5}{72} - \frac{x^7}{2160} + \cdots$$

2) $e^x \cos x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)$

$$= \left(\underline{\underline{1 + x}} + \cancel{\frac{x^2}{2!}} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) - \left(\cancel{\frac{x^2}{2!}} + \frac{x^3}{2!} + \frac{x^4}{2!\,2!} + \frac{x^5}{2!\,3!} + \cdots\right)$$

$$+ \left(\frac{x^4}{4!} + \frac{x^5}{4!} + \frac{x^6}{2!\,4!} + \cdots\right)$$

$$= \underline{\underline{1 + x}} - \frac{x^3}{3} - \frac{x^4}{6} + \cdots$$

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$\boxed{3}$ $\cos 2x$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\cos 2x = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} x^{2n}}{(2n)!} = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots$$

$$= 1 - \frac{2^2 x^2}{2!} + \frac{2^4 x^4}{4!} - \frac{2^6 x^6}{6!} + \cdots$$

Exp For what values of x can we replace $\sin x$ by $x - \frac{x^3}{3!}$ with an error of magnitude no more than 3×10^{-4}?

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

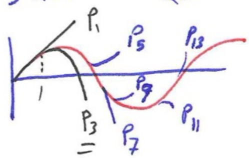
Using the Alternating Series Estimation Theorem:

The error after $\frac{x^3}{3!} < \left|\frac{x^5}{5!}\right|$. Therefore the error

$u_{n+1} = |a_{n+1}|$

will be less than 3×10^{-4} $\Leftrightarrow$ $\frac{|x|^5}{120} < 3 \times 10^{-4}$ $\Leftrightarrow$

$$|x| < \sqrt[5]{360 \times 10^{-4}} \simeq 0.514.$$



Exp Estimate the error if $P_3(x) = x - \frac{x^3}{6}$ is used to estimate the value of $\sin x$ at $x = 0.1$

$f(x) = \sin x = P_3(x) + R_3(x)$, where $R_3(x) = \frac{f^{(4)}(c)}{4!}(x-0)^4 = \frac{f^{(4)}(c)}{4!} x^4$

Apply The Remainder Estimation Th. with $M = 1$

$M \leq |f^{(4)}(c)| \leq 1$

$$\text{Error} = |R_3(x)| \leq \frac{|x^4|}{4!} = \frac{(0.1)^4}{24} < 4.2 \times 10^{-6}$$

Extra:

40. The estimate $\sqrt{1+x} = 1 + (x/2)$ is used when x is small. Estimate the error when $|x| < 0.01$.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}}$

$f''(x) = \frac{-1}{4}(1+x)^{-\frac{3}{2}}$

$f'''(x) = \frac{3}{8}(1+x)^{-\frac{5}{2}}$

$= 1 + \frac{x}{2} - \frac{x^2}{8} + \cdots$

By AST Theorem

Error $< \left|-\frac{x}{8}\right|$

$< \frac{x^2}{8} < \frac{0.01^2}{8}$

Error $< 1.25 \times 10^{-5}$

$-0.01 < x < 0.01$

$\frac{x^2}{8} < \frac{0.01^2}{8}$

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