Discussion 11.1, 11.2, 11.4

Section 11.1

1, 2, $\boxed{4}$, $\boxed{5}$, $\boxed{6}$

$|f(x,y)-f(x,w)|<\varepsilon$

11.1.4. Suppose that $H = [a,b]\times[c,d]$ is a rectangle, that $f : H \to \mathbf{R}$ is continuous, and that $g : [a,b]\to\mathbf{R}$ is integrable. Prove that

$$\boxed{F(y)=\int_a^b g(x)f(x,y)\,dx}$$

is uniformly continuous on $[c,d]$.

$\Rightarrow$ g is bdd

$|g(x)|\le M,\ \forall x\in[a,b]$

proof. We want to show that $\forall\varepsilon>0,\ \exists$ a $\delta>0$ s.t if $|y-w|<\delta$, $y,w\in[c,d]\Rightarrow$

$$|F(y)-F(w)|<\varepsilon.$$

Let $\varepsilon>0$, Since f is uniformly cont. on H, $\exists\,\delta>0$ s.t $y,w\in[c,d]$, $|y-w|<\delta \Rightarrow |f(x,y)-f(x,w)|<\frac{\varepsilon}{M(b-a)}$

Therefore, $|F(y)-F(w)| = \left|\int_a^b f(x,y)\,g(x)\,dx - \int_a^b f(x,w)\,g(x)\,dx\right|$

$$=\left|\int_a^b g(x)\big[f(x,y)-f(x,w)\big]\,dx\right|$$

$$\le\int_a^b \underbrace{|g(x)|}_{\le M}\,|f(x,y)-f(x,w)|\,dx$$

Since g is integrable on [a,b] $\Rightarrow$ g is bdd on [a,b]

$$\le M\int_a^b \underbrace{|f(x,y)-f(x,w)|}\,dx$$

f is cont. on H "compact"
$\Rightarrow$ f is uniformly cont. on H.

$$\le M\int_a^b \frac{\varepsilon}{M(b-a)}\,dx=\varepsilon.$$

$\therefore$ F is uniformly cont on $[c,d]$.

11.1.5. Evaluate each of the following expressions.

a) $$\lim_{y\to 0}\int_0^1 e^{x^3y^2+x}\,dx$$

b) $$\frac{d}{dy}\int_0^1 \sin(e^x y - y^3 + \pi - e^x)\,dx \quad \text{at } y = 1$$

✓ c) $$\frac{\partial}{\partial x}\int_1^3 \underbrace{\sqrt{x^3+y^3+z^3-2}}_{f(x,y,z)}\,dz \quad \text{at } (x, y) = (\overset{x}{1}, \overset{y}{1})$$

$$H = [\underset{x}{1,3}] \times [\underset{y}{1,3}] \times [\underset{z}{1,3}]$$

$$f = \sqrt{x^3+y^3+z^3-2} \geq \sqrt{1+1+1-2} = 1 > 0$$

$$f_x = \frac{3x^2}{2\sqrt{x^3+y^3+z^3-2}}$$

$$f_y = \frac{3y^2}{2\sqrt{x^3+y^3+z^3-2}}, \quad f_z = \ldots\ldots$$

f_x, f_y, f_z exist + cont. on H

thm11.5 $\Longrightarrow$ At $(x,y) = (1,1)$

$$\frac{\partial}{\partial x}\int_1^3 f(x,y,z)\,dz = \int_1^3 \left(\frac{\partial f}{\partial x}\right) dz$$

$$= \int_1^3 \frac{3x^2}{2\sqrt{x^3+y^3+z^3-2}}\,dz \Bigg|_{(x,y)=(1,1)}$$

$$= \int_1^3 \frac{3}{2\sqrt{z^3}}\,dz$$

$$= \frac{3}{2}\int_1^3 z^{-\frac{3}{2}}\,dz = \ldots\ldots$$

11.1.6. Suppose that f is a continuous real function.

a) If $\int_0^1 f(x)\,dx = 1$, find the exact value of

$$\lim_{y\to 0}\int_0^2 f(|x-1|)e^{x^2y+xy^2}\,dx.$$

cont. on $\mathbb{R}$ & hence on $[0,2]$

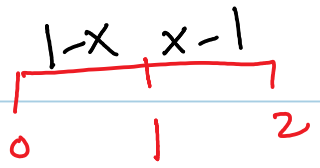
cont. on $\mathbb{R}^2$

$\therefore$ the integrand is cont. on $\mathbb{R}^2$

by thm 11.4

$$=\int_0^2 \lim_{y\to 0}\left[f(|x-1|)\,e^{x^2y+xy^2}\right]dx$$

$$=\int_0^2 f(|x-1|)\,dx$$



$$=\int_0^1 f(1-x)\,dx+\int_1^2 f(x-1)\,dx$$

$y=1-x$

$dy=-dx$

$w=x-1$

$dw=dx$

$$=\int_1^0 f(y)\,(-dy)+\int_0^1 f(w)\,dw$$

$$=\int_0^1 f(y)\,dy+\int_0^1 f(w)\,dw$$

$$=1+1=2.$$

Section 11.2 (1 – 10)

11.2.6. Prove that if $\alpha > 1/2$, then

$|h\cdot 0|^\alpha \log(h^2)$

$$f(x,y)=\begin{cases}|xy|^\alpha\log(x^2+y^2) & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$$

is differentiable at (0, 0).

Sol. We want

$$\lim_{(h,k)\to(0,0)} \underbrace{\frac{f(h,k)-f(0,0)-\nabla f(0,0)\cdot(h,k)}{\|(h,k)\|}}_{g(h,k)} = 0$$

$$f_x(0,0)=\lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{0-0}{h}$$

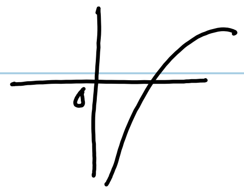
$$=\lim_{h\to 0} 0 = 0$$

Similarly, $f_y(0,0)=0$

$$\nabla f(0,0)=\left(f_x(0,0),\ f_y(0,0)\right)=(0,0)$$

$$=\lim_{(h,k)\to(0,0)}\frac{|hk|\log(h^2+k^2)-0-(0,0)\cdot(h,k)}{\sqrt{h^2+k^2}}$$

$$=\lim_{(h,k)\to(0,0)}\underbrace{\frac{|hk|\ \log(h^2+k^2)}{\sqrt{h^2+k^2}}}_{g(h,k)}$$



$$|g(h,k)| = \left| \frac{(\boxed{hk})^{\alpha} \log(h^2+k^2)}{\sqrt{h^2+k^2}} \right|$$

$|\log x|$
$= -\log x$

$$\le \frac{\left(\frac{h^2+k^2}{2}\right)^{\alpha}}{(h^2+k^2)^{\frac{1}{2}}} \left(-\log(h^2+k^2)\right)$$

$h^2+k^2-2hk \geq 0$

$h^2+k^2 \geq 2hk$

$\boxed{hk \le \frac{h^2+k^2}{2}}$

$\boxed{\alpha > \frac{1}{2}}$

$$= \frac{(h^2+k^2)^{\boxed{\alpha-\frac{1}{2}}}}{\boxed{2^{\alpha}}} \log\left(\frac{1}{h^2+k^2}\right) \longrightarrow 0$$

as $(h,k) \to (0,0)$.

Since $\boxed{\lim_{u\to 0} u^{\varepsilon} \log\left(\frac{1}{u}\right) = 0, \quad \forall \varepsilon > 0}$ (*)

$(0 \cdot \infty)$

(14.w) ✓

$\therefore \lim_{(h,k)\to(0,0)} g(h,k) = 0$ by Squeeze thm.

Section 11.4 All (1 – 11) .

11.4.4. Let $f, g : \mathbf{R} \to \mathbf{R}$ be twice differentiable. Prove that $\boxed{u(x, y) := f(xy)}$ satisfies

$$\boxed{x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = 0,}$$

and $\boxed{v(x, y) := f(x - y) + g(x + y)}$ satisfies the *wave equation*; that is,

$$\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial y^2} = 0.$$

$$V_x = f'(x-y)\cdot 1 + g'(x+y)\cdot 1$$

$$V_{xx} = f''(x-y) + g''(x+y)$$

Sol. $$\frac{\partial u}{\partial x} = f'(xy)\left[\frac{\partial}{\partial x}(xy)\right] = y\,f'(xy).$$

$$\frac{\partial u}{\partial y} = f'(xy)\ \frac{\partial}{\partial y}(xy) = x\,f'(xy)$$

$$\therefore\quad x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = xyf'(xy) - yxf'(xy) = 0.$$

11.4.7. Let

$$\boxed{u(x, t) = \frac{e^{-x^2/4t}}{\sqrt{4\pi t}},} \qquad t > 0,\ x \in \mathbf{R}.$$

a) Prove that u satisfies the *heat equation* (i.e., $\boxed{u_{xx} - u_t = 0}$ for all $t > 0$ and $x \in \mathbf{R}$).

$$u_x = \frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}} \cdot \frac{-2x}{4t} = \left(\frac{-x}{2t\sqrt{4\pi t}}\right)e^{-\frac{x^2}{4t}}$$

$$u_{xx} = \frac{-1}{2t\sqrt{4\pi t}}e^{-\frac{x^2}{4t}} - \frac{x}{2t\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}\left(\frac{-x}{2t}\right)$$

$$\boxed{u_{xx} = \frac{-1}{2t\sqrt{4\pi t}}e^{-\frac{x^2}{4t}} + \frac{x^2}{4t^2\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}}$$

$$u_t = \cdots\cdots$$

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$u \to L$ $\qquad x \geq a.$

b) If $a > 0$, prove that $u(x,t) \to 0$, as $t \to 0+$, uniformly for $x \in [a, \infty)$.

Sol.
$$u(x,t) = \frac{e^{-x^2/4t}}{\sqrt{4\pi t}},$$

$x \geq a \quad x^2 \geq a^2$

$-x^2 \leq -a^2$

$e^{-x^2} \leq e^{-a^2}$

$$|u(x,t) - \underset{L}{0}| = \frac{e^{\frac{-x^2}{4t}}}{\sqrt{4\pi t}} \leq \frac{e^{\frac{-a^2}{4t}}}{\sqrt{4\pi t}}$$

Now,
$$\boxed{\lim_{t\to 0^+} \frac{e^{\frac{-a^2}{4t}}}{\sqrt{4\pi t}} = 0} \quad \left(\tfrac{0}{0}\right)$$

by L'Hopital Rule (Exercise)

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$$|u(x,t) - 0| \leq \frac{e^{\frac{-a^2}{4t}}}{\sqrt{4\pi t}} \to 0 \text{ as } t \to 0^+$$

$$\Rightarrow \lim_{t\to 0^+} u(x,t) = 0 \quad \text{uniformly}$$

(indep. of x).