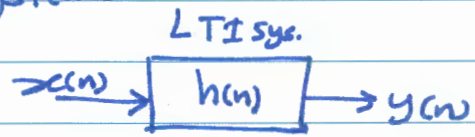


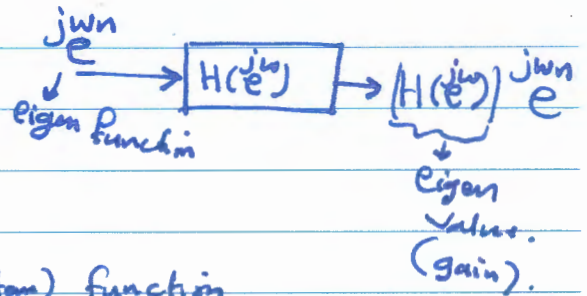
Chapter - 5 -

Transfer Analysis of LTI systems

$$y(n) = x(n) * h(n) \\ = \sum_{k=-\infty}^{\infty} x(k) h(n-k)$$



$$Y(e^{j\omega}) = X(e^{j\omega}) \cdot \underbrace{H(e^{j\omega})}_{\text{Freq. Response.}}$$



$$Y(z) = X(z) \cdot \underbrace{H(z)}_{\text{Transfer (system) function}}$$

Frequency Response of LTI systems:

$$|Y(e^{j\omega})| = |X(e^{j\omega})| \cdot |H(e^{j\omega})| \rightarrow \text{Magnitude Response or 'gain'}$$

$$\angle Y(e^{j\omega}) = \angle X(e^{j\omega}) + \angle H(e^{j\omega}) \rightarrow \text{Phase Response or phase shift.}$$

* Undesirable effects $\Rightarrow$ called magnitude or phase distortions.

Ideal Frequency Selective Filters

* Ideal Low-pass Filter:

$$H_{lp}(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| \leq \pi \end{cases}$$

$H_{lp}(e^{j\omega})$ is periodic with period of 2π .

$$\Rightarrow h_{lp}(n) = \frac{\sin \omega_c n}{\pi n}, \quad -\infty < n < \infty$$

* low-pass filter selects low freq. components and rejects high freq. comp.

High-pass Filter:

$$H_{hp}(e^{j\omega}) = \begin{cases} 0, & |\omega| < \omega_c \\ 1, & \omega_c < |\omega| \leq \pi \end{cases}$$

Since $H_{hp}(e^{j\omega}) = 1 - H_{lp}(e^{j\omega}) \Rightarrow h_{hp}(n) = \delta(n) - \frac{\sin \omega_c n}{\pi n}$

* Ideal high-pass filter selects high freq. components and rejects low freq. comp.

* The Ideal low-pass filter is noncausal, and its impulse response $h_{lp}(n)$ extends from $-\infty$ to $+\infty \Rightarrow$ Therefore, it is not possible to compute the output of either ideal low-pass or high-pass filters either recursively or nonrecursively, i.e. systems are not computationally realizable.

* Phase Response of ideal low-pass filter is specified to be zero. So, causal approximation to ideal frequency selective filters have nonzero phase response.

phase distortion and delay

Ideal delay system: $h_{id}(n) = \delta(n - n_d)$, $n_d \rightarrow \text{Int}$

$$H_{id}(j\omega) = e^{-j\omega n_d}$$

$$|H_{id}(j\omega)| = 1, \quad \angle H_{id}(j\omega) = -\omega n_d, \quad |\omega| < \pi.$$

* Design approximation to ideal filters and other LTI systems. We accept linear-phase response rather than zero phase response. $\Rightarrow$ so ideal lowpass filter with linear phase.

$$H_{lp}(j\omega) = \begin{cases} e^{-j\omega n_d} & , \quad |\omega| < \omega_c \\ 0 & , \quad \omega_c < |\omega| \leq \pi \end{cases}$$

$$\Rightarrow h_{lp}(n) = \frac{\sin(\omega_c(n - n_d))}{\pi(n - n_d)}, \quad -\infty < n < \infty$$

$$\angle H_{lp}(j\omega) \approx -\phi_0 - \omega n_d \quad (\text{linear approximation}).$$

* Group delay: measure of the linearity of the phase defined as negative slope of phase at specific frequency.

$$\tau(\omega) = -\frac{d}{d\omega} \angle H(e^{j\omega})$$

Example: $h(n) = \delta(n-5)$ (ideal delay system).

$$H(z) = z^{-5}$$

$$H(e^{j\omega}) = H(z) \Big|_{z=e^{j\omega}} = 1 e^{-j5\omega}$$

$$|H(e^{j\omega})| = 1, \quad \angle H(e^{j\omega}) = -5\omega$$

$$\tau(\omega) = -\frac{d}{d\omega} (-5\omega) = 5 \text{ samples.}$$

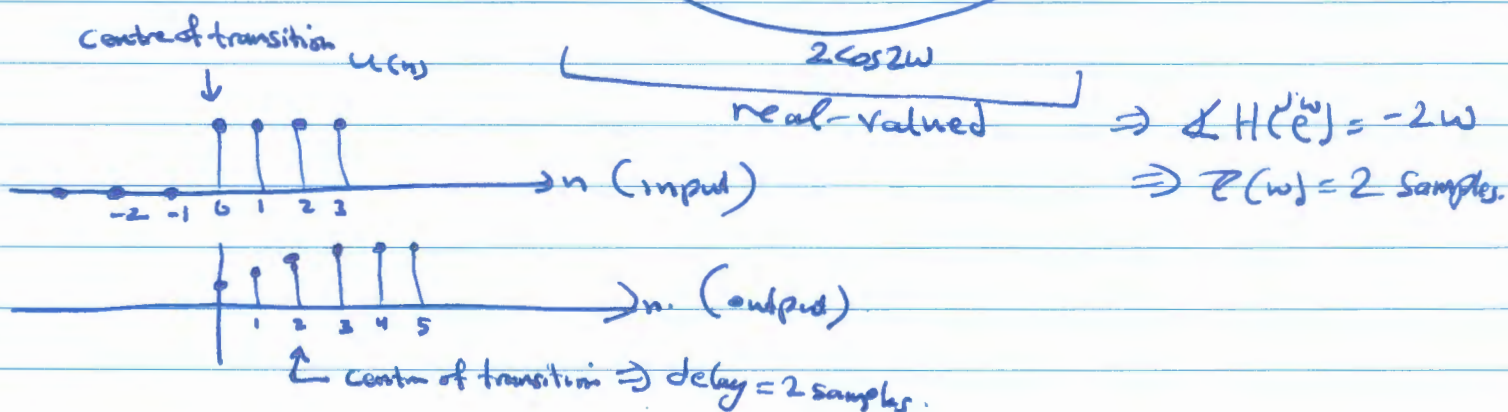
Example 2: $h(n) = \frac{1}{5} \{ \underset{\substack{\uparrow \\ n=0}}{1}, 1, 1, 1, 1 \}$ ← 5-point Moving Averager.

$$h(n) = \frac{1}{5} (\delta(n) + \delta(n-1) + \delta(n-2) + \delta(n-3) + \delta(n-4)).$$

$$H(z) = \frac{1}{5} (z^0 + z^{-1} + z^{-2} + z^{-3} + z^{-4})$$

$$H(e^{j\omega}) = \frac{1}{5} (e^{j0\omega} + e^{-j\omega} + e^{-j2\omega} + e^{-j3\omega} + e^{-j4\omega})$$

$$= \frac{1}{5} (e^{j2\omega} + e^{j\omega} + e^0 + e^{-j\omega} + e^{-j2\omega}) e^{-j2\omega}$$



* See Example 5.1 (effects of attenuation and group delay).

System Functions for systems characterized by LCCDE:

$$\sum_{k=0}^N a_k y(n-k) = \sum_{k=0}^M b_k x(n-k)$$

Another Assumption that system is Causal.

* LTI systems satisfy Linear Const-coeff. difference equation are best developed through Z-transform.

So, by applying Z-transform to both sides and using linearity and time-shifting properties, we obtain:

$$\sum_{k=0}^N a_k z^{-k} Y(z) = \sum_{k=0}^M b_k z^{-k} X(z)$$
$$\Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^M b_k z^{-k}}{\sum_{k=0}^N a_k z^{-k}}$$

* $H(z)$ in factored form:

$$H(z) = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^M (1 - c_k z^{-1})}{\prod_{k=1}^N (1 - d_k z^{-1})}$$

* each of the factors $(1 - c_k z^{-1})$ in the numerator contributes a zero at $z = c_k$ and a pole at $z = 0$.

Similarly, $(1 - d_k z^{-1})$ in the denominator contributes a zero at $z = 0$ and a pole at $z = d_k$.

Z-domain	$b_k z^{-k}$	corresponds to	$b_k x(n-k)$	Time-domain
	$a_k z^{-k}$		$a_k y(n-k)$	

Example: (second order system)

$$H(z) = \frac{(1 + z^{-1})^2}{(1 - \frac{1}{2}z^{-1})(1 + \frac{3}{4}z^{-1})}$$

Find Diff. equation?

$$H(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 + \frac{1}{4}z^{-1} - \frac{3}{2}z^{-2}} = \frac{Y(z)}{X(z)}$$

Thus, $(1 + \frac{1}{4}\bar{z} - \frac{3}{8}\bar{z}^2)Y(z) = (1 + 2\bar{z} + \bar{z}^2)X(z)$

and Diff. eq. is:

$$y(n) + \frac{1}{4}y(n-1) - \frac{3}{8}y(n-2) = x(n) + 2x(n-1) + x(n-2)$$

Stability and Causality:

$$\sum_{k=0}^N a_k y(n-k) = \sum_{k=0}^M b_k x(n-k) \Rightarrow H(z) = \frac{\sum_{k=0}^M b_k \bar{z}^{-k}}{\sum_{k=0}^N a_k \bar{z}^{-k}}$$

Assumptions:

- linear

- Time-Invariant

but, No Assumption about stability and Causality.

So, from difference equation we can get system function $H(z)$ but not ROC.

* Many choices for ROC $\Rightarrow$ each choice of ROC leads to different impulse response, but all correspond to the same difference equation.

* If we assume system is Causal $\Rightarrow$ $h(n)$ must be right-sided therefore, ROC of $H(z)$ must be outside the outermost pole.

* If we assume system is Stable $\Rightarrow$ $h(n)$ must be absolutely summable i.e.

Identical $\rightarrow \sum_{n=-\infty}^{\infty} |h(n)| < \infty$ which is identical to:

$$\sum_{n=-\infty}^{\infty} |h(n) \bar{z}^{-n}| < \infty$$

* for $|z|=1$, condition for stability is equivalent to that ROC of $H(z)$ include unit circle.

Example: Determine ROC

$$y(n) - \frac{5}{2}y(n-1] + y(n-2) = x(n)$$

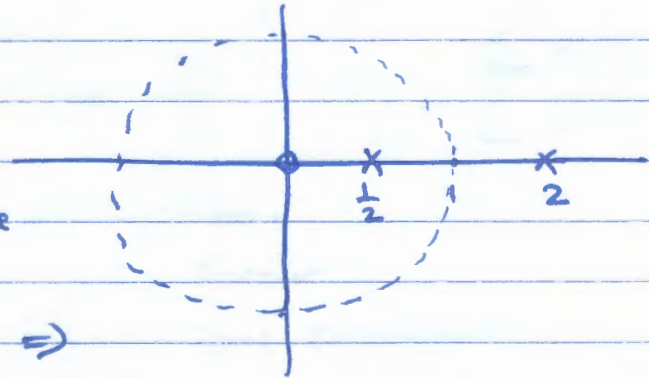
$$H(z) = \frac{1}{1 - \frac{5}{2}z^{-1} + z^{-2}} = \frac{1}{(1 - \frac{1}{2}z^{-1})(1 - 2z^{-1})}$$

* There are 3 choices for ROC:

- If system is assumed to be

Causal $\Rightarrow$ ROC $|z| > 2$

in this case, system is unstable



- If we assume system is stable $\Rightarrow$

$\Rightarrow$ ROC $\frac{1}{2} < |z| < 2$

- Third choice of ROC $|z| < \frac{1}{2} \rightarrow$ Not stable, non causal.

* For Causal and stable system $\Rightarrow$ ROC outside outermost pole and include unit circle $\Rightarrow$ This requires all poles must be inside unit circle.

Inverse Systems: $H_i(z)$

$$h(n) * h_i(n) = \delta(n)$$

$$H(z) H_i(z) = 1$$

$$\Rightarrow H_i(z) = \frac{1}{H(z)} \quad \text{and} \quad H_i(e^{j\omega}) = \frac{1}{H(e^{j\omega})}$$

* Equivalently, log-magnitude and phase of the inverse system are negatives of the corresponding function for the original system

* Not all systems have an inverse, e.g. Ideal lowpass filter doesn't $\Rightarrow$ There is no way to recover frequency components above cutoff frequency (ω_c) that are set to zero by the filter.

* Rational systems functions have inverse:

$$H(z) = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^M (1 - c_k z^{-k})}{\prod_{k=1}^N (1 - d_k z^{-k})}$$

with zeros at $z = c_k$ and poles at $z = d_k$, in addition to possible zeros and poles at $z = 0$ and $z = \infty$, Then

$$H_i(z) = \left(\frac{a_0}{b_0}\right) \frac{\prod_{k=1}^N (1 - d_k z^{-k})}{\prod_{k=1}^M (1 - c_k z^{-k})}$$

* Poles of $H_i(z)$ are zeros for $H(z)$ and vice versa.

* What ROC associate with $H_i(z)$?

Answer from convolution theorem.

$$h(n) * h_i(n) = \delta(n)$$

So, $H(z)$ and $H_i(z)$ must overlap. If $H(z)$ is Causal, ROC is

$$|z| > \max_k |d_k|$$

* Thus, any ROC for $H_i(z)$ that overlaps with ROC of $H(z)$ is a valid ROC for $H_i(z)$.

Example: First-order System.

$$H(z) = \frac{1 - 0.5z^{-1}}{1 - 0.9z^{-1}}, \quad |z| > 0.9$$

$$H_i(z) = \frac{1 - 0.9z^{-1}}{1 - 0.5z^{-1}}$$

* Since $H_i(z)$ has only one pole, there are only two possibilities for ROC and the only choice that overlaps with $|z| > 0.9$ is $|z| > 0.5$
 $\Rightarrow$ impulse response of the inverse system is:

$$h_i(n) = (0.5)^n u(n) - 0.9 (0.5)^{n-1} u(n-1)$$

In this case, Inverse system is both stable and Causal.

Example 6

$$H(z) = \frac{z^{-1} - 0.5}{1 - 0.9z^{-1}}, \quad |z| > 0.9$$

$$H_i(z) = \frac{1 - 0.9z^{-1}}{z^{-1} - 0.5} = \frac{-z + 1.8z^{-1}}{1 - 2z^{-1}}$$

⇒ there are two possible ROC's : $|z| < 2$ and $|z| > 2$
 both ROC's overlap with $|z| > 0.9 \Rightarrow$ so, both are valid inverse system.

* For ROC $|z| < 2 \Rightarrow h_{i1}(n) = 2(2)^n u(-n-1) - 1.8(2)^{n-1} u(-n)$

* For ROC $|z| > 2 \Rightarrow h_{i2}(n) = -2(2)^n u(n) + 1.8(2)^{n-1} u(n-1)$

$h_{i1}(n) \rightarrow$ stable and noncausal.

$h_{i2}(n) \rightarrow$ Causal and nonstable.

* In general, if $H(z)$ is Causal with zero's at $C_k, k=1, \dots, M$ then, its $H_i(z)$ will be Causal if and only if we associate ROC

$$|z| > \max_k |C_k| \quad \text{with } H_i(z).$$

⇒ if we also require that the inverse system be stable, then the ROC of $H_i(z)$ must include the unit circle ⇒ it must be true that

$$\max_k |C_k| < 1$$

i.e. all zeros of $H(z)$ must be inside unit circle.

⇒ Therefore LTI system is stable and causal and also has a stable and causal inverse system if and only if, both poles and zeros of $H(z)$ are inside unit circle. Such systems are referred to as minimum-phase systems.

Impulse Response for Rational System Functions:

* any rational function of z^{-1} with only first-order poles can be expressed in the form:

$$H(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^N \frac{A_k}{1 - d_k z^{-1}} \quad (*)$$

Where, the terms in the first sum would be obtained by long division of the denominator into numerator and would be present only if $M > N$. Coeff. A_k in the second sum are obtained using

$$A_k = (1 - d_k z^{-1}) H(z) \Big|_{z=d_k} \quad (\text{residues}).$$

* If $H(z)$ has multiple-order pole, its partial fraction expansion would have the form:

$$H(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{\substack{k=1 \\ k \neq i}}^N \frac{A_k}{1 - d_k z^{-1}} + \sum_{m=1}^S \frac{C_m}{(1 - d_i z^{-1})^m}$$

* If the system is assumed to be Causal $\Rightarrow$ ROC outside all poles of equation (*) above. and

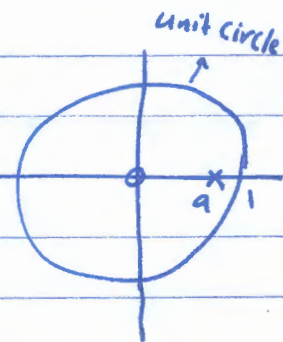
$$h(n) = \underbrace{\sum_{r=0}^{M-N} B_r \delta(n-r)}_{\text{only if } M > N} + \sum_{k=1}^N A_k d_k^n u(n)$$

Two Classes of LTI systems:

FIR: at least one nonzero pole of $H(z)$ is not cancelled by a zero. $\Rightarrow$ at least one term of $A_k (d_k)^n u(n)$, and so $h(n)$ will not be of finite length (i.e. will not be zero outside of finite interval), such systems are called Infinite Impulse Response (IIR) Systems.

Example 8 First-order IIR system

$$y(n] - a y[n-1] = x[n] \Rightarrow H(z) = \frac{1}{1 - a z^{-1}}$$



ROC $|z| > |a|$, condition for stability is $|a| < 1$
and Inverse z -transform of $H(z)$ is $h[n] = a^n u[n]$

2 FIR: $H(z)$ has no poles except at $z=0$, i.e. $N=0$

Thus, a partial fraction expansion is not possible. and $H(z)$ a polynomial in z^{-1}

$$H(z) = \sum_{k=0}^M b_k z^{-k} \quad (\text{assume } a_0=1)$$

$$\Rightarrow h[n] = \sum_{k=0}^M b_k \delta[n-k] = \begin{cases} b_n, & 0 \leq n \leq M \\ 0, & \text{otherwise} \end{cases}$$

In this case, the impulse response is finite in length, i.e. it is zero outside a finite interval ($h[n]=0, n \notin [0, M]$)

$\Rightarrow$ These systems are called Finite Impulse Response (FIR)

* For FIR systems, difference equation is identical to convolution sum, i.e.

$$y[n] = \sum_{k=0}^M b_k x[n-k]$$

Example:

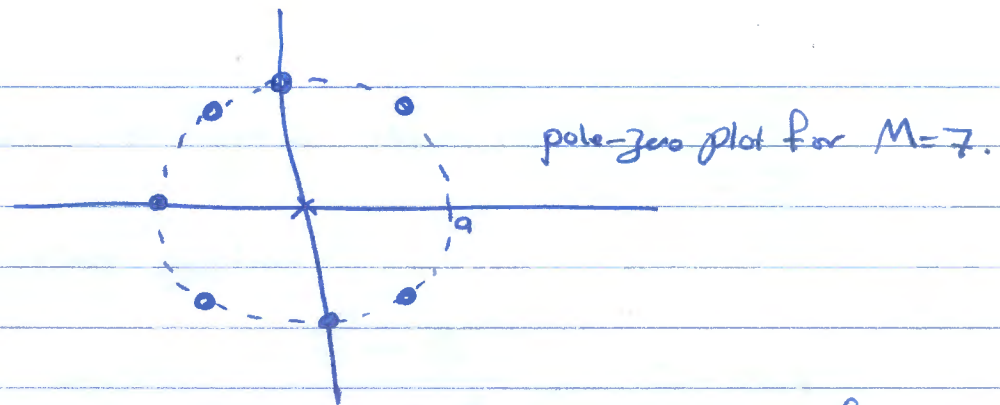
$$h[n] = \begin{cases} a^n, & 0 \leq n \leq M \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow H(z) = \sum_{n=0}^M a^n z^{-n} = \frac{1 - a^{M+1} z^{-(M+1)}}{1 - a z^{-1}} \quad \text{--- (*)}$$

Since zeros of the numerator are at:

$$z_k = a e^{j 2\pi k / (M+1)}, \quad k=0, 1, \dots, M$$

When, a is assumed real and positive, the pole at $z=a$ is cancelled by a zero at z_0 ($k=0$).



* The diff. equation satisfied by input and output of LTI system is discrete convolution

$$y(n) = \sum_{k=0}^M a_k x(n-k)$$

Also equation (*) above suggests I/O also satisfy the diff. eq.

$$y(n) - ay(n-1) = x(n) - a^{M+1}x(n-M-1)$$

Two diff. eq. result from two equivalent form of H(z) in eq. (*)

All-Pass Systems:

$$|H_{ap}(e^{j\omega})| = 1$$

All-Pass $\Leftrightarrow$ poles and zeros are in conjugate reciprocal pairs.

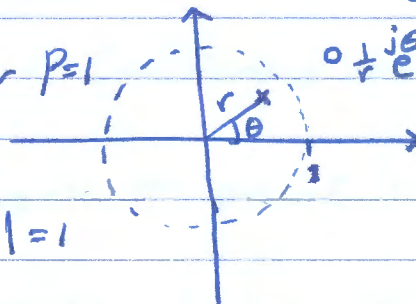
$$H_{ap}(z) = \prod_{i=1}^P \frac{z^{-1} - C_i^*}{1 - C_i z^{-1}}$$

poles: $C_i = r e^{j\theta}$
 $\frac{1}{C_i^*} = \frac{1}{r} e^{j\theta}$

* To show $|H_{ap}(e^{j\omega})| = 1$, consider $P=1$

$$|H_{ap}(e^{j\omega})| = \left| \frac{e^{-j\omega} - C^*}{1 - C e^{j\omega}} \right|$$

$$\xrightarrow{|e^{j\omega}|=1} \left| \frac{e^{-j\omega}(1 - C^* e^{j\omega})}{1 - C e^{j\omega}} \right|$$

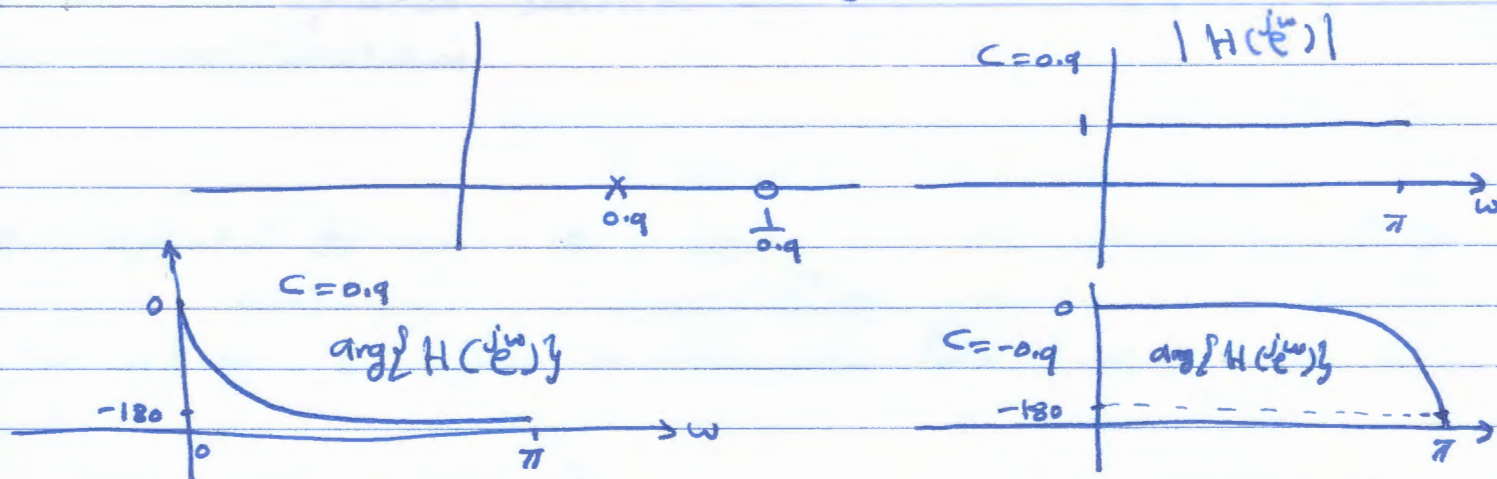


$$= \left| \frac{1 - C^* e^{j\omega}}{\underbrace{1 - C e^{j\omega}}_b} \right| = \frac{|b^*|}{|b|} = 1$$

* All-pass systems $\rightarrow$ form as product of $\frac{\bar{z}^{-1} - c^*}{1 - c \bar{z}^{-1}}$

Example: All-pass system

~~$H(z)$~~ $H(z) = \frac{\bar{z}^{-1} - c^*}{1 - c \bar{z}^{-1}}$, $c = 0.9$



* Any arbitrary system function can be factorized as a cascade of minimum-phase and all-pass sub-systems unless it hasn't poles at unit circle ($|z| = 1$).

Minimum-phase and All-pass decomposition

To Factor $H(z) = H_{\min}(z) \cdot H_{\text{ap}}(z)$

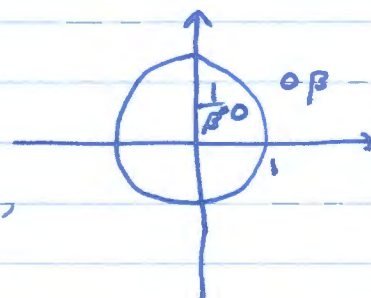
- ① Take zeros that lie outside $|z| = 1$ (unit-circle) and move to $H_{\text{ap}}(z)$.
- ② Add poles to $H_{\text{ap}}(z)$ in conjugate reciprocal locations of zeros.
- ③ Put zeros $H_{\min}(z)$ to cancel poles added to $H_{\text{ap}}(z)$.

Example: Suppose $H(z) = H_1(z)(1 - \beta z^{-1})$; $|\beta| > 1$, $H_1(z)$ min-phase

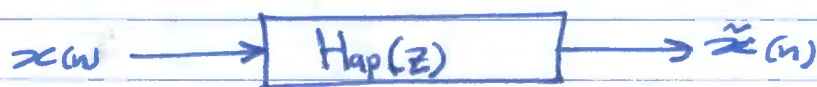
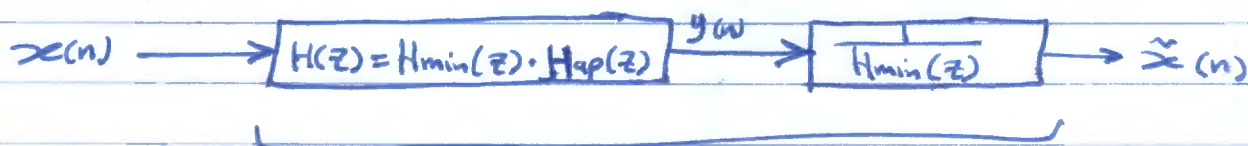
① $H(z) = H_1(z)(-\beta)(z^{-1} - \frac{1}{\beta})$

zero $z = \beta$ in $H(z)$
reflected inside $|z| = 1$.

② $H(z) = \underbrace{H_1(z)(-\beta)}_{H_{\min}(z)} \underbrace{(1 - \frac{1}{\beta^*} z^{-1})}_{H_{\text{ap}}(z)} \cdot \frac{z^{-1} - \frac{1}{\beta}}{1 - \frac{1}{\beta^*} z^{-1}}$



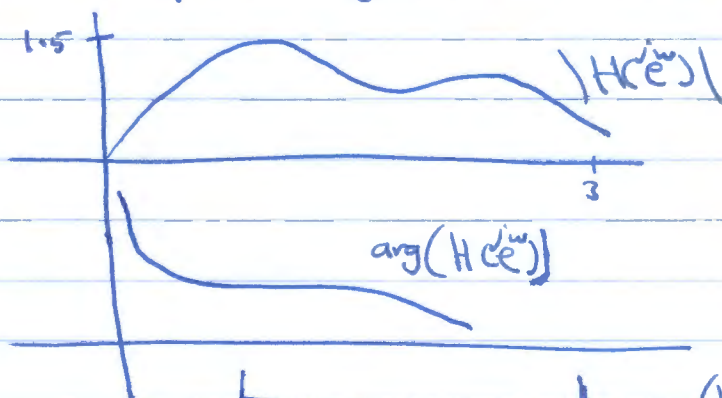
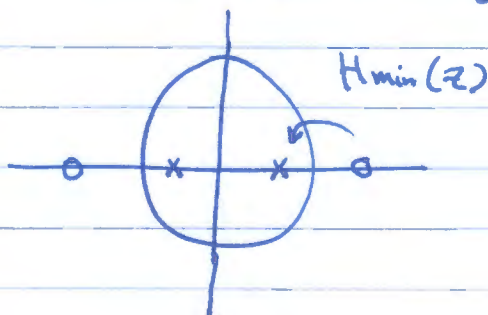
* The min-phase portion of any system has a stable, causal inverse system.



- No Magnitude distortion
- only phase distortion

Example: decompose $H(z)$ into min-phase and All-pass

$$H(z) = (1 - 0.9z^{-1})(1 + 0.9z^{-1})(1 - j0.7z^{-1})(1 + j0.7z^{-1})$$



$$H_{\text{ap}}(z) = \frac{(z^{-1} - 0.9)(z^{-1} + 0.9)}{(1 - 0.9z^{-1})(1 + 0.9z^{-1})}$$



$$H_{\min}(z) = \frac{1}{-0.8} (1 - 0.9z^{-1})(1 + 0.9z^{-1})(1 - j0.7z^{-1})(1 + j0.7z^{-1})$$

Example: $H(z) = \frac{1+5z^{-1}}{1+\frac{1}{2}z^{-1}}$, decompose $H(z)$ into H_{min} and H_{ap} ?

$$\begin{aligned}
 H(z) &= 5 \frac{z^{-1} + \frac{1}{5}}{1 + \frac{1}{2}z^{-1}} \\
 &= 5 \frac{z^{-1} + \frac{1}{5}}{1 + \frac{1}{2}z^{-1}} \cdot \frac{1 + \frac{1}{5}z^{-1}}{1 + \frac{1}{5}z^{-1}} \\
 &= 5 \underbrace{\frac{1 + \frac{1}{5}z^{-1}}{1 + \frac{1}{2}z^{-1}}}_{H_{min}(z)} \cdot \underbrace{\frac{z^{-1} + \frac{1}{5}}{1 + \frac{1}{5}z^{-1}}}_{H_{ap}(z)}.
 \end{aligned}$$

