Problem 2.1

Evaluate the Fourier transform of the damped sinusoidal wave $g(t) = \exp(-t)\sin(2\pi f_c t)u(t)$ where $u(t)$ is the unit step function

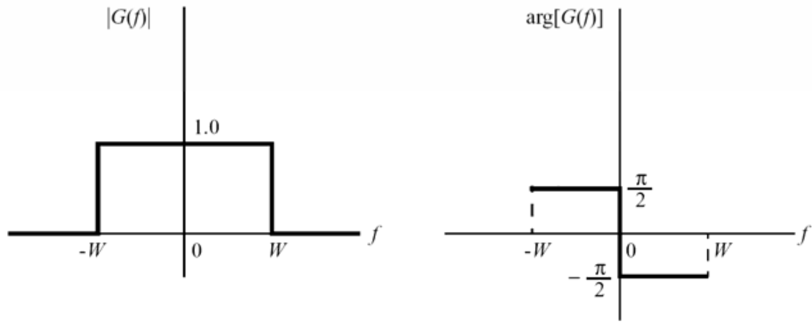
Solution

The Fourier transform of $g(t)$ is

$$
\begin{aligned}
G(f) &= \int_0^{\infty} \exp(-t)\sin(2\pi f_c t)\sin(-j2\pi f_c t)dt \\
&= \frac{1}{2j}\int_0^{\infty} \exp(-t)[\exp(j2\pi f_c t) - \exp(-j2\pi f_c t)]\exp(-j2\pi f t)dt \\
&= \frac{1}{2j}\int_0^{\infty} [\exp(j2\pi(f_c - f)t - t)]dt \\
&= \frac{1}{2j}\left[\frac{1}{j2\pi(f_c - f) - 1}\exp(j2\pi(f_c\text{-}f)t\text{-}t) + \frac{1}{j2\pi(f_c - f) + 1}\exp((-j2\pi(f_c + f)t)\text{-}t)\right]_{t=0}^{\infty} \\
&= \frac{1}{2j}\left(\frac{1}{j2\pi(f_c - f) - 1} + \frac{1}{j2\pi(f_c - f) + 1}\right) \\
&= \frac{1}{2j}\left(\frac{(j2\pi(f_c - f) + 1) + (j2\pi(f_c - f) - 1)}{1 + 4\pi^2(f_c - f)^2}\right) \\
&= \frac{2\pi f_c}{1 + 4\pi^2(f - f_c)^2}
\end{aligned}
$$

Problem 2.2

Determine the inverse Fourier transform of the frequency function $G(f)$ defined by the amplitude and phase spectra shown in Fig. 2.5.



Solution

$$g(t) = \int_{-W}^{0} e^{j\pi/2} \cdot e^{j2\pi ft} df + \int_{0}^{W} e^{-j\pi/2} e^{j2\pi ft} df$$

$$= \left[\frac{1}{j2\pi t} e^{j\left(\frac{\pi}{2} + 2\pi ft\right)}\right]_{f=-W}^{0} + \left[\frac{1}{j2\pi t} e^{j\left(-\frac{\pi}{2} + 2\pi ft\right)}\right]_{f=0}^{W}$$

$$= \frac{1}{j2\pi t}\left(e^{j\left(\frac{\pi}{2} - 2\pi Wt\right)} - e^{j\pi/2}\right) + \frac{1}{j2\pi t}\left(e^{-j\pi/2} - e^{j\left(-\frac{\pi}{2} - j2\pi Wt\right)}\right)$$

$$= \frac{1}{j2\pi t}(e^{-j\pi/2} - e^{j\pi/2}) + \frac{1}{j2\pi t} e^{-j2\pi Wt}(e^{j\pi/2} - e^{-j\pi/2})$$

$$= -\frac{1}{\pi t} + \frac{1}{\pi t} e^{-j2\pi Wt} = \frac{1}{\pi t}(e^{-j2\pi Wt} - 1)$$

Note: If we let $W \to \infty$, $G(f) \to j\,\mathrm{sgn}(t)$, the inverse of which $-\frac{1}{\pi t}$. This result agrees with the limiting value of the solution for $W = \infty$.

Problem 2.3

Suppose $g(t)$ is real valued with a complex-valued Fourier transform $G(f)$. Explain how the rule of Eq. (2.31) can be satisfied by such a signal.

Solution

With $G(f)$ being complex valued, we may express it as

$$G(f) = G_r(f) + jG_i(f)$$

where $G_r(f)$ is the real part of $G(f)$ and $G_i(f)$ is its imaginary part. Hence,

$$G(0) = G_r(0) + jG_i(0).$$

According to Eq. (2.31) in the text,

$$\int_{-\infty}^{\infty} g(t)dt = G_r(0) + jG_i(0)$$

With $g(t)$ being real valued, this condition can only be satisfied if the imaginary part $G_i(0)$ is zero.

Problem 2.4

Continuing with Problem 2.3, explain how the rule of Eq. (2.32) can be satisfied by the signal $g(t)$ described therein.

Solution

Since $g(t)$ is real valued, it follows that the integral $\int_{-\infty}^{\infty} G(f)df$ must likewise be real valued. For this condition to be satisfied, the imaginary part of $G(f)$ must be an odd function of f.

Problem 2.5
Develop the detailed steps that show that the modulation and convolution theorems are indeed the dual of each other.

Solution
The modulation theorem states that

$$g_1(t)g_2(t) \rightleftharpoons \int_{-\infty}^{\infty} G_1(\lambda)G_2(f-\lambda)d\lambda \qquad (1)$$

To apply the duality theorem, we say that if
$g_1(t)g_2(t) \rightleftharpoons X(f)$, then

$X(f) \rightleftharpoons g_1(-f)g_2(-f)$

For the problem at hand, we may therefore write

$$\int_{-\infty}^{\infty} G_1(\lambda)G_2(f-\lambda)d\lambda \rightleftharpoons g_1(-f)g_2(-f) \qquad (2)$$

Next, we apply Eq. (2.21), which states that if $g(t) \rightleftharpoons G(f)$ then $g(-t) \rightleftharpoons G(-f)$. Hence, applying this rule to Eq. (2), we may write

$$\int_{-\infty}^{\infty} G_1(\lambda)G_2(\lambda-t)d\lambda \rightleftharpoons g_1(f)g_2(f)$$

which is a statement of the convolution theorem, with $G_1(t) \rightleftharpoons g_1(f)$ and $G_2(t) \rightleftharpoons g_2(f)$.

Problem 2.6

Develop the detailed steps involved in deriving Eq. (2.53), starting from Eq. (2.51).

Solution

According to Eq. (2.51),

$$\int_{-\infty}^{\infty} g_1(\tau)g_2(t-\tau)d\tau \rightleftharpoons G_1(f)G_2(f)$$

According to Eq. (2.21), if $g(t) \rightleftharpoons G(f)$, then $g(-t) \rightleftharpoons G(-f)$. Hence, applying this rule to the problem at hand, we may write

$$\int_{-\infty}^{\infty} g_1(\tau)g_2(\tau-t)d\tau \rightleftharpoons G_1(f)G_2(-f)$$

Next, we note that if we complex conjugate the term $g_2(\tau - t)$, then the conjugation theorem of Eq. (2.22) teaches us that

$$\int_{-\infty}^{\infty} g_1(\tau)g_2^*(\tau-t)d\tau \rightleftharpoons G_1(f)G_2^*(-f)$$

which is the desired result, except for the fact that we have interchanged the roles of variables t and τ.

Problem 2.7
Prove the following properties of the convolution process:

(a) The commutative property:

$$g_1(t)\bigstar g_2(t) = g_2(t)\bigstar g_1(t)$$

Proof:

$$g_1(t)\bigstar g_2(t) = \int_{-\infty}^{\infty} g_1(\tau)g_2(t-\tau)d\tau$$

$$= \int_{-\infty}^{\infty} g_2(t-\tau)g_1(\tau)d\tau$$

Replace $t - \tau$ with λ. That is, $\tau = t - \lambda$. Hence

$$g_1(t)\bigstar g_2(t) = -\int_{+\infty}^{-\infty} g_2(\lambda)g_1(t-\lambda)d\lambda$$

$$= \int_{-\infty}^{\infty} g_2(\lambda)g_1(t-\lambda)d\lambda$$

$$= g_2(t)\bigstar g_1(t)$$

(b) The associative property:

$$g_1(t)\bigstar[g_2(t)\bigstar g_3(t)] = [g_1(t)\bigstar g_2(t)]\bigstar g_3(t)$$

Proof:

.

Let

$$x(t) = g_2(t)\bigstar g_3(t)$$

$$= \int_{-\infty}^{\infty} g_2(\tau)g_3(t-\tau)d\tau$$

Hence

$$I(t) = g_1(t)\bigstar \underbrace{[g_2(t)\bigstar g_3(t)]} = \int_{-\infty}^{\infty} g_1(\lambda)x(t-\lambda)d\lambda$$

$$= \int_{-\infty}^{\infty} g_1(\lambda)\int_{-\infty}^{\infty} g_2(\tau)g_3(t-\tau-\lambda)d\tau d\lambda \quad (1)$$

Replace $\tau + \lambda$ with μ; that is, $\tau = \mu - \lambda$. Hence, keeping λ fixed, we may write

$$I(t) = \int_{-\infty}^{\infty} g_1(\lambda)\int_{-\infty}^{\infty} g_2(\mu-\lambda)g_3(t-\mu)d\mu d\lambda \quad (2)$$

With μ fixed, the integral $\int_{-\infty}^{\infty} g_1(\lambda)g_2(\mu-\lambda)d\lambda$ is recognized as the convolution of $g_1(\mu)$ and $g_2(\mu)$, as shown by

$$g_{12}(\mu) = \int_{-\infty}^{\infty} g_1(\lambda)g_2(\mu-\lambda)d\lambda = g_1(\mu)\bigstar g_2(\mu)$$

We may therefore rewrite Eq. (1) as

Continued on next slide

Problem 2.7 continued

$$I(t) = \int_{-\infty}^{\infty} g_{12}(\mu)g_3(t-\mu)d\mu = g_{12}(t)\star g_3(t) = [g_1(t)\star g_2(t)]\star g_3(t)$$

(c) The distributive property:

$$g_1(t)\star[g_2(t)+g_3(t)] = g_1(t)\star g_2(t) + g_1(t)\star g_3(t)$$

Proof:

$$g_1(t)\star[g_2(t)+g_3(t)] = \int_{-\infty}^{\infty} g_1(\tau)[g_2(t-\tau)+g_3(t-\tau)]d\tau$$

$$= \int_{-\infty}^{\infty} g_1(\tau)g_2(t-\tau)d\tau + \int_{-\infty}^{\infty} g_1(\tau)g_3(t-\tau)d\tau$$

$$= g_1(t)\star g_2(t) + g_1(t)\star g_3(t)$$

Problem 2.8

Considering the sinc pulse sinc(t), show that

$$\int_{-\infty}^{\infty} \text{sinc}^2(t)dt = 1$$

Solution

This integral may be viewed as

$$I = \int_{-\infty}^{\infty} \text{sinc}(t) \cdot \text{sinc}(t)dt$$

which, in light of Rayleigh's energy theorem, may also be expressed as

$$I = \int_{-\infty}^{\infty} |\mathbf{F}[\text{sinc}(t)]|^2 df$$

From Eq. (2.25) in the text, we have

$$\mathbf{F}[\text{sinc}\,t] = \text{rect}(f)$$

Hence,

$$I = \int_{-\infty}^{\infty} \text{rect}^2(f)df$$

$$= \int_{-1/2}^{1/2} 1^2 df$$

$$= 1$$

Problem 2.9

Determine the Fourier transform of the squared sinusoidal signals:

(i) $g(t) = \cos^2(2\pi f_c t)$

(ii) $g(t) = \sin^2(2\pi f_c t)$

Solution

(i) Using the trigonometric identity

$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$$

we may express $g(t)$ as

$$g(t) = \frac{1}{2}(1 + \cos 4\pi f_c t)$$

Hence,

$$G(f) = \frac{1}{2}\delta(f) + \frac{1}{4}\delta(f - 2f_c) + \frac{1}{4}\delta(f + 2f_c)$$

(ii) Next, using the trigonometric identity

$$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$$

we may write

$$\sin^2(2\pi f_c t) = \frac{1}{2}(1 - \cos 4\pi f_c t)$$

Hence,

$$G(f) = \frac{1}{2}\delta(f) - \frac{1}{4}\delta(f - f_c) - \frac{1}{4}\delta(f + f_c)$$

Problem 2.10

Consider the function

$$g(t) = \delta\left(t+\frac{1}{2}\right) - \delta\left(t-\frac{1}{2}\right)$$

which consists of two delta functions at $t = \pm\frac{1}{2}$. The integration of $g(t)$ with respect to time t yields the unit rectangular function rect(t). Using Eq. (2.79), show that

$$\text{rect}(t) \rightleftharpoons \text{sinc}(f)$$

Solution

To begin, consider the transform pair

$$\delta(t) \rightleftharpoons 1$$

Hence, the Fourier transform of $g(t)$ is

$$G(f) = \exp(j\pi ft) - \exp(-j\pi ft)$$

from which we readily deduce that $G(0)$. Hence, applying Eq. (2.79) in the text yields

$$\mathbf{F}[\text{rect}(t)] = \frac{1}{j2\pi f}[\exp(j\pi f) - \exp(-j\pi f)]$$

$$= \frac{\sin(\pi f)}{\pi f} = \text{sinc}(f)$$

where we have used the identity

$$\sin(\pi f) = \frac{1}{2j}(e^{j\pi f} - e^{-j\pi f})$$

Problem 2.11

Using the Euler formula

$$\cos x = \frac{1}{2}\exp[(jx) + \exp(-jx)]$$

reformulate Eqs. (2.91) and (2.92) in terms of cosinusoidal functions.

Solution

$$\sum_{n=-\infty}^{\infty} \exp(j2\pi n f_0 t) = \sum_{n=1}^{\infty} \exp(j2\pi n f_0 t) + 1 + \sum_{n=-\infty}^{-1} \exp(j2\pi n f_0 t)$$

$$= 1 + \sum_{n=1}^{\infty} [\exp(j2\pi n f_0 t) + \exp(-j2\pi n f_0 t)]$$

$$= 1 + 2\sum_{n=1}^{\infty} \cos(2\pi n f_0 t)$$

We may therefore reformulate Eq. (2.91) as

$$\sum_{m=-\infty}^{\infty} \delta(t - mT_0) = f_0 + 2f_0\sum_{n=1}^{\infty} \cos(2\pi m f_0 t)$$

where $f_0 = 1/T$.

Similarly, we may write

$$\sum_{n=-\infty}^{\infty} \cos(j2\pi m f_0 t) = 1 + 2\sum_{m=1}^{\infty} \cos(2\pi m f_0 t)$$

Hence, we may reformulate Eq. (2.92) as

$$1 + 2\sum_{m=1}^{\infty} \cos(2\pi m f_0 t) = T_0 \sum_{n=-\infty}^{\infty} \delta(f - n f_0)$$

Problem 2.12

Discuss the following two issues, citing examples for your answers:

(a) Is it possible for a linear time-invariant system to be causal but unstable?

(b) Is it possible for such a system to be noncausal but stable?

Solution

(a) It is possible for a system to be causal but unstable. Causality means that the impulse response of the system $h(t)$ must be zero for negative t. Instability means that the BIBO criterion

$$\int_{-\infty}^{\infty} |h(t)|dt < \infty$$

is violated. Such a system could be represented by the impulse response

$$h(t) = \begin{cases} 0 & \text{for } t \le 0 \\ \exp(t) & \text{for } t > 0 \end{cases}$$

(b) By the same token, it is possible for the system to be stable but noncausal. In this second case, we may cite the impulse response

$$h(t) = \begin{cases} \exp(t) & \text{for } t \le 0 \\ 0 & \text{for } t > 0 \end{cases}$$

What does Problem 2.12 teach us?

The problem teaches us that the properties of stability and causality are independent.

Problem 2.13

The impulse response of a linear system is defined by the Gaussian function

$$h(t) = \exp\left(-\frac{t^2}{2\tau^2}\right)$$

where the parameter τ is used to adjust duration of the impulse response. Determine the frequency response of the system.

Solution

From Eq. (2.40) in the text, recall that

$$\exp(-\pi t^2) \rightleftharpoons \exp(-\pi f^2)$$

Next, from the dilation property of the Fourier transform described in Eq. (2.20), recall that if $h(t) \rightleftharpoons H(f)$, then

$$h(at) \rightleftharpoons \frac{1}{|a|} H\left(\frac{f}{a}\right)$$

where a is the dilation parameter. For the problem at hand, we have

$$a = \sqrt{\frac{1}{2\pi}}\ \frac{1}{\tau}$$

Accordingly, the frequency response of the system is

$$H(f) = \sqrt{2\pi}\ \tau \exp(-2\pi^2\tau^2 f^2)$$

Problem 2.14

A tapped-delay-line filter consists of N weights, where N is odd. It is symmetric with respect to the center tap, that is, the weights satisfy the condition

$$w_n = w_{N-1-n}, \qquad 0 \le n \le N-1$$

(a) Find the amplitude response of the filter.

(b) Show that this filter has a linear phase response. What is the implication of this property?

Solution

The impulse response of the filter is

$$h(t) = \sum_{n=0}^{N-1} w_n \delta(t - n\Delta\tau)$$

Hence, the frequency response of the filter is

$$H(f) = \sum_{n=0}^{N-1} w_n \exp(-j2\pi n f\Delta\tau)$$

To illustrate, consider the example of $N = 5$. Then

$$\begin{aligned} H(f) &= w_0 + w_1 \exp(-j2\pi f\Delta\tau) + w_2 \exp(-j4\pi f\Delta\tau) + w_3 \exp(-j6\pi f\Delta\tau) + w_4 \exp(-j8\pi f\Delta\tau) \\ &= \exp(-j4\pi f\Delta\tau)[w_0 \exp(j4\pi f\Delta\tau) + w_1 \exp(j2\pi f\Delta\tau) + w_2 + w_3 \exp(-j2\pi f\Delta\tau) \\ &\qquad + w_4 \exp(-j4\pi f\Delta\tau)] \qquad (1) \end{aligned}$$

For this example, the symmetry condition

$$w_n = w_{N-1-n} \qquad \text{for } 0 \le n \le N\text{-}1$$

reads as

$$w_n = w_{4-n} \qquad \text{for } 0 \le n \le 4$$

Hence, $w_0 = w_4$ and $w_1 = w_3$. Accordingly, we may rewrite Eq. (1) as

$$\begin{aligned} H(f) &= \exp(-j4\pi f\Delta\tau)[w_0 \exp(j4\pi f\Delta\tau) + \exp(-j4\pi f\Delta\tau) \\ &\qquad + w_1(\exp(j2\pi f\Delta\tau)) + \exp(-j2\pi f\Delta\tau) \\ &\qquad + w_2] \\ &= \exp(-j4\pi f\Delta\tau)[2w_0 \cos(4\pi f\Delta\tau) + 2w_1(2\pi f\Delta\tau) + w_2] \end{aligned}$$

We may therefore generalize this result as

$$H(f) = \exp\left(-j2\pi\left(\frac{N\text{-}1}{2}\right)f\Delta\tau\right)\left[w_{\frac{N-1}{2}} + 2\sum_{n=0}^{\frac{N-1}{2}-1} w_n \cos(2\pi n f\Delta\tau)\right]$$

(a) The amplitude response of the filter is therefore

$$|H(f)| = w_{\frac{N-1}{2}} + 2\sum_{n=0}^{\frac{N-1}{2}-1} w_n \cos(2\pi n f\Delta\tau)$$

(b) The phase response of the filter is therefore

$$\arg(H(f)) = \exp\left(-j2\pi\left(\frac{N\text{-}1}{2}\right)f\Delta\tau\right)$$

which is linear with respect to the frequency f. The implication of this condition is that except for a delay, there is no phase distortion produced by the filter.

Problem 2.15

Derive the relationship of Eq. (2.142) between the two cross-correlation factors $R_{xy}(\tau)$ and $R_{yx}(\tau)$.

Solution

By definition

$$R_{yx}(\tau) = \int_{-\infty}^{\infty} y(t)x^*(t-\tau)dt$$

Complex conjugate both sides of the equation:

$$R_{yx}^*(\tau) = \int_{-\infty}^{\infty} x(t-\tau)y^*(t)dt$$

Next, replace τ with $-\tau$:

$$R_{yx}^*(-\tau) = \int_{-\infty}^{\infty} x(t+\tau)y^*(t)dt$$

Finally, replace $t + \tau$ with t, which is equivalent to replacing t with $t - \tau$; we therefore (since dt remains unchanged)

$$R_{yx}^*(-\tau) = \int_{-\infty}^{\infty} x(t)y^*(t-\tau)dt = R_{xy}(\tau)$$

Problem 2.16

Consider the decaying exponential pulse

$$g(t) = \begin{cases} \exp(-at) & t > 0 \\ 1, & t = 0 \\ 0, & t < 0 \end{cases}$$

Determine the energy spectral density of the pulse $g(t)$.

Solution

The Fourier transform of $g(t)$ is (see Eq. (2.12) in the text

$$G(f) = \frac{1}{a + j2\pi f}$$

The energy spectral density of the pulse is therefore

$$E_g(f) = |G(f)|^2$$

$$= \frac{1}{a^2 + 4\pi^2 f^2}, \qquad -\infty < f < \infty$$

Problem 2.17

Repeat Problem 2.16 for the double exponential pulse

$$g(t) = \begin{cases} \exp(-at), & t > 0 \\ 1, & t = 0 \\ \exp(at), & t < 0 \end{cases}$$

Solution

The Fourier transform of $g(t)$ is (see Eq. (2.16))

$$G(f) = \frac{2a}{a^2 + 4\pi^2 f^2}$$

The energy spectral density of the double exponential pulse is

$$E_g(f) = \frac{4a^2}{(a^2 + 4\pi^2 f^2)^2}, \qquad -\infty < f < \infty$$

Problem 2.18

In an implicit sense, Eq. (2.153) embodies *Parseval's power theorem*, which states that for a *periodic signal* $x(t)$ we have

$$\frac{1}{T}\int_{-T/2}^{T/2}|x(t)|^2 dt = \sum_{n=-\infty}^{\infty}|X(nf_0)|^2$$

where T is the period of the signal, f_0 is the fundamental frequency, and $X(nf_0)$ is the Fourier transform of $x(t)$ evaluated at the frequency nf_0. Prove this theorem.

Solution

Adapting Eq. (2.86) to the problem at hand, we may write

$$x_T(t) = f_0 \sum_{n=-\infty}^{\infty} X(nf_0)\exp(j2\pi n f_0 t) \quad (1)$$

where

$$x_T(t) = \begin{cases} x(t), & -\frac{T}{2} \le t \le \frac{T}{2} \\ 0, & \text{otherwise} \end{cases}$$

$$f_0 = \frac{1}{T}$$

and $X(nf_0)$ is the Fourier transform of $g(t)$, evaluated at the frequency $f = nf_0$. Using Eq. (1) to evaluate the integral

$$\frac{1}{T}\int_{-T/2}^{T/2}|x(t)|^2 dt$$

we write

$$I = f_0\int_{-T/2}^{T/2}\left(\sum_{n=-\infty}^{\infty} X(nf_0)\exp(j2\pi n f_0 t)\right)\left(\sum_{m=-\infty}^{\infty} X^*(mf_0)\exp(-j2\pi m f_0 t)\right)dt$$

$$= f_0 \sum_{n=-\infty}^{\infty} X(nf_0)X^*(mf_0)\int_{-T/2}^{T/2}\exp(j2\pi(n-m)f_0 t)dt \quad (2)$$

To evaluate the integral on the right-hand side of Eq. (2), we write

$$\int_{-T/2}^{T/2}\exp(j2\pi(n-m)f_0 t)dt = \frac{1}{j2\pi(n-m)f_0}\exp(j2\pi(n-m)f_0 t)\Big|_{t=-T/2}^{T/2}$$

Continued on next slide

Problem 2-18 continued

$$= \frac{1}{j2\pi(n-m)f_0}[\exp((j\pi(n-m)) - \exp - j\pi(n-m))]$$

$$= \frac{1}{\pi(n-m)f_0}\sin(\pi(n-m)) \quad (3)$$

Whenever the indices n and m are assigned different integer values, Eq. (3) assumes the value zero. On the other hand, whenever the indices are assigned the same integer value, the integral in Eq. (3) assumes the limiting value

$$\frac{1}{f_0} \lim_{n=m} \frac{\sin(\pi(n-m))}{\pi(n-m)} = \frac{1}{f_0}$$

Accordingly, we may simplify Eq. (3) as

$$\int_{-T/2}^{T/2} \exp(j2\pi(n-m)f_0 t)dt = \begin{cases} \frac{1}{f_0}, & n = m \\ 0, & \text{otherwise} \end{cases} \quad (4)$$

Hence, substituting Eq. (4) into (2), we get

$$I = \sum_{n=-\infty}^{\infty} |X(nf_0)|^2 \quad (5)$$

We finally write

$$\frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 dt = \sum_{n=-\infty}^{\infty} |X(nf_0)|^2$$

which is the desired result.

Problem 2.19

(a) The half-cosine pulse $g(t)$ of Fig. 2.40(a) may be considered as the product of the rectangular function $\text{rect}(t/T)$ and the sinusoidal wave $A\cos(\pi t/T)$. Since

$$\text{rect}\left(\frac{t}{T}\right) \rightleftharpoons T\,\text{sinc}(fT)$$

$$A\cos\left(\frac{\pi t}{T}\right) \rightleftharpoons \frac{A}{2}\left[\delta\left(f-\frac{1}{2T}\right)+\delta\left(f+\frac{1}{2T}\right)\right]$$

and multiplication in the time domain is transformed into convolution in the frequency domain, it follows that

$$G(f) = [T\,\text{sinc}(fT)]\bigstar\left\{\frac{A}{2}\left[\delta\left(f-\frac{1}{2T}+f+\frac{1}{2T}\right)\right]\right\}$$

where ★ denotes convolution. Therefore, noting that

$$\text{sinc}(fT)\bigstar\delta\left(f-\frac{1}{2T}\right) = \text{sinc}\left[T\left(f-\frac{1}{2T}\right)\right]$$

$$\text{sinc}(fT)\bigstar\delta\left(f+\frac{1}{2T}\right) = \text{sinc}\left[T\left(f+\frac{1}{2T}\right)\right]$$

we obtain the desired result

$$G(f) = \frac{AT}{2}\left[\text{sinc}\left(fT-\frac{1}{2}\right)+\text{sinc}\left(fT+\frac{1}{2}\right)\right]$$

(b) The half-sine pulse of Fig. 2.40(b) may be obtained by shifting the half-cosine pulse to the right by $T/2$ seconds. Since a time shift of $T/2$ seconds is equivalent to multiplication by $\exp(-j\pi fT)$ in the frequency domain, it follows that the Fourier transform of the half-sine pulse is

$$G(f) = \frac{AT}{2}\left[\text{sinc}\left(fT-\frac{1}{2}\right)+\text{sinc}\left(fT+\frac{1}{2}\right)\right]\exp(-j\pi fT)$$

(c) The Fourier transform of a half-sine pulse of duration aT is equal to

$$\frac{|a|AT}{2}\left[\text{sinc}\left(afT-\frac{1}{2}\right)+\text{sinc}\left(afT+\frac{1}{2}\right)\right]\exp(-j\pi faT)$$

(d) The Fourier transform of the negative half-sine pulse shown in Fig. 2.40(c) is obtained from the result by putting $a = -1$, and multiplying the result by -1, and so we find that its Fourier transform is equal to

$$-\frac{AT}{2}\left[\text{sinc}\left(fT+\frac{1}{2}\right)+\text{sinc}\left(fT-\frac{1}{2}\right)\right]\exp(j\pi fT)$$

Continued on next slide

Problem 2-19 continued

(e) The full-sine pulse of Fig. 2.40(d) may be considered as the superposition of the half-sine pulses shown in parts (b) and (c) of the figure. The Fourier transform of this pulse is therefore

$$G(f) = \frac{AT}{2}\left[\operatorname{sinc}\left(fT - \frac{1}{2}\right) + \operatorname{sinc}\left(fT + \frac{1}{2}\right)\right][\exp(-j\pi fT) - \exp(j\pi fT)]$$

$$= -jAT\left[\operatorname{sinc}\left(fT - \frac{1}{2}\right) + \operatorname{sinc}\left(fT + \frac{1}{2}\right)\right]\sin(\pi fT)$$

$$= -jAT\left[\frac{\sin\left(\pi fT - \frac{\pi}{2}\right)}{\pi fT - \frac{\pi}{2}} + \frac{\sin\left(\pi fT + \frac{\pi}{2}\right)}{\pi fT + \frac{\pi}{2}}\right]\sin(\pi fT)$$

$$= -jAT\left[-\frac{\cos(\pi fT)}{\pi fT - \frac{\pi}{2}} + \frac{\cos(\pi fT)}{\pi fT + \frac{\pi}{2}}\right]\sin(\pi fT)$$

$$= jAT\left[\frac{\sin(2\pi fT)}{2\pi fT - \pi} - \frac{\sin(2\pi fT)}{2\pi fT + \pi}\right]$$

$$= jAT\left[-\frac{\sin(2\pi fT - \pi)}{2\pi fT - \pi} + \frac{\sin(2\pi fT + \pi)}{2\pi fT + \pi}\right]$$

$$= jAT[\operatorname{sinc}(2fT + 1) - \operatorname{sinc}(2fT - 1)]$$

Problem 2.20

(a) The even part $g_e(t)$ of a pulse $g(t)$ is given by

$$g_e(t) = \frac{1}{2}[g(t) + g(-t)]$$

Therefore, for $g(t) = A\text{rect}\left(\frac{t}{T} - \frac{1}{2}\right)$ we obtain

$$g_e(t) = \frac{A}{2}\left[\text{rect}\left(\frac{t}{T} - \frac{1}{2}\right) + \text{rect}\left(-\frac{t}{T} - \frac{1}{2}\right)\right]$$

$$= \frac{A}{2}\left[\text{rect}\left(\frac{t}{2T}\right)\right]$$

which is shown illustrated in Fig. 1:

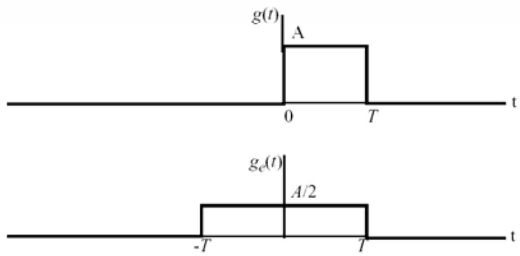


Figure 1

The odd part of $g(t)$ is defined by

$$g_o(t) = \frac{1}{2}[g(t) - g(-t)]$$

$$= \frac{A}{2}\left[\text{rect}\left(\frac{t}{T} - \frac{1}{2}\right) - \text{rect}\left(-\frac{t}{T} - \frac{1}{2}\right)\right]$$

which is illustrated in Fig. 2:

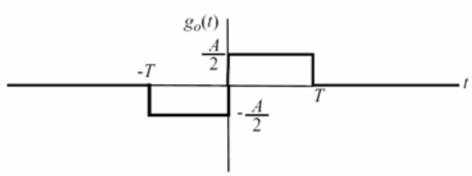


Figure 2

(b) The Fourier transform of the even part is

$$G_e(f) = AT\,\text{sinc}(2fT)$$

The Fourier transform of the odd part is

$$G_o(f) = \frac{AT}{2}\text{sinc}(fT)\exp(-j\pi fT) - \frac{AT}{2}\text{sinc}(fT)\exp(j\pi fT)$$

$$= \frac{AT}{j}\text{sinc}(fT)\sin(\pi fT)$$

Problem 2.21

Express $g(t)$ as

$$g(t) = g_1(t) + g_2(t)$$

where

$$g_1(t) = \frac{1}{\tau}\int_{t-T}^{0} \exp\left(-\frac{\pi u^2}{\tau^2}\right)du$$

$$g_2(t) = \frac{1}{\tau}\int_{0}^{t+T} \exp\left(-\frac{\pi u^2}{\tau^2}\right)du$$

Therefore,

$$g_1(t+T) = \frac{1}{\tau}\int_{t}^{0} \exp\left(-\frac{\pi u^2}{\tau^2}\right)du$$

$$= -\frac{1}{\tau}\int_{0}^{t} \exp\left(-\frac{\pi u^2}{\tau^2}\right)du$$

$$= -\frac{1}{\tau}\int_{-\infty}^{t} \exp\left(-\frac{\pi u^2}{\tau^2}\right)du + \frac{1}{2}$$

where we have made use of the fact that

$$\frac{1}{\tau}\int_{-\infty}^{t} \exp\left(-\frac{\pi u^2}{\tau^2}\right)du = \frac{1}{2}$$

Similarly

$$g_2(t-T) = \int_{0}^{t} \exp\left(-\frac{\pi u^2}{\tau^2}\right)du$$

$$= \frac{1}{\tau}\int_{-\infty}^{t} \exp\left(-\frac{\pi u^2}{\tau^2}\right)du = \frac{1}{2}$$

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Problem 2-21 continued

Next, noting the following four relationships

$$\mathbf{F}[g_1(t+T)] = G_1(f)\exp(j2\pi fT)$$

$$\mathbf{F}[g_2(t-T)] = G_2(f)\exp(-j2\pi fT)$$

$$\exp\left(-\frac{\pi t^2}{\tau^2}\right) \rightleftharpoons \tau\exp(-\pi\tau^2 f^2)$$

$$\int_{-\infty}^{t} g(u)du = \frac{1}{j2\pi f}G(f) + \frac{1}{2}G(0)\delta(f)$$

we find that taking the Fourier transforms of $g_1(t+T)$ and $g_2(t-T)$ respectively yields

$$G_1(f)\exp(j2\pi fT) = -\frac{1}{j2\pi f}\exp(-\pi\tau^2 f^2)$$

$$G_2(f)\exp(-j2\pi fT) = \frac{1}{j2\pi f}\exp(-\pi\tau^2 f^2)$$

Therefore,

$$G_1(f) = \frac{1}{j2\pi f}\exp(-\pi\tau^2 f^2)\exp(-j2\pi fT)$$

$$G_2(f) = \frac{1}{j2\pi f}\exp(-\pi\tau^2 f^2)\exp(j2\pi fT)$$

Thus the Fourier transform of $g(t)$ is

$$\begin{aligned} G(f) &= G_1(f) + G_2(f) \\ &= \frac{1}{j2\pi f}\exp(-\pi\tau^2 f^2)[\exp(-j2\pi fT) + \exp(j2\pi fT)] \\ &= \frac{1}{\pi f}\exp(-\pi\tau^2 f^2)\sin(2\pi fT) \\ &= 2T\exp(-\pi\tau^2 f^2)\operatorname{sinc}(2fT) \end{aligned}$$

When τ approaches zero, $G(f)$ approaches the limiting value $2T\operatorname{sinc}(2fT)$, which corresponds to the Fourier transform of a rectangular pulse of unit amplitude and duration $2T$, which is correct.

Problem 2.22

(a) $G(f) = \int_{-\infty}^{\infty} g(t)\exp(-j2\pi ft)dt$

$$= \int_{-\infty}^{0} g(t)\exp(-j2\pi ft)dt + \int_{0}^{\infty} g(t)\exp(-j2\pi ft)dt$$

$$= \int_{-\infty}^{0} g(t)\cos(2\pi ft)dt - \int_{-\infty}^{0} jg(t)\sin(2\pi ft)dt$$

$$+ \int_{0}^{\infty} g(t)\cos(2\pi ft)dt - \int_{0}^{\infty} jg(t)\sin(2\pi ft)dt$$

If $g(t)$ is even, then $g(t) = g(-t)$. Hence,

$$\int_{-\infty}^{0} g(t)\cos(2\pi ft)dt = \int_{0}^{\infty} g(t)\cos(2\pi ft)dt$$

$$\int_{-\infty}^{0} g(t)\sin(2\pi ft)dt = -\int_{0}^{\infty} g(t)\sin(2\pi ft)dt$$

and so

$G(f) = 2\int_{0}^{\infty} g(t)\cos(2\pi ft)dt$. which is purely real.

If, on the other hand, $g(t)$ is odd, $g(t) = -g(-t)$. Hence,

$$\int_{-\infty}^{0} g(t)\sin(2\pi ft)dt = \int_{0}^{\infty} g(t)\sin(2\pi ft)dt$$

$$\int_{-\infty}^{0} g(t)\cos(2\pi ft)dt = -\int_{0}^{\infty} g(t)\cos(2\pi ft)dt$$

and thus

$G(f) = -2j\int_{0}^{\infty} g(t)\sin(2\pi ft)dt$ which is purely imaginary.

(b) The Fourier transform of $g(t)$ is defined by

$$G(f) = \int_{-\infty}^{\infty} g(t)\exp(-j2\pi ft)dt$$

Differentiating both sides of this relation n times with respect to f:

$$\frac{d^n G(f)}{df^n} = (-j2\pi)^n \int_{-\infty}^{\infty} t^n \exp(-j2\pi ft)dt \qquad (1)$$

That is,

$$t^n g(t) \rightleftharpoons \left(\frac{j}{2\pi}\right)^n \frac{d^n G(f)}{df^n}$$

(c) Putting $f = 0$ in Eq. (1), we get

$$\int_{-\infty}^{\infty} t^n g(t)dt = \left(\frac{j}{2\pi}\right)^n G^{(n)}(0)$$

Continued on next slide

Problem 2.22 continued

where $G^{(n)}(f) = \dfrac{d^n G(f)}{d fn}$

(d) Since

$$g_2^*(t) \rightleftharpoons G_2^*(-f)$$

it follows that

$$g_1(t)g_2^*(t) \rightleftharpoons \int_{-\infty}^{\infty} G_1(\lambda)G_2^*(\lambda - f)d\lambda$$

From this result we deduce the Fourier transform

$$\mathbf{F}[g_1(t)g_2^*(t)] = \int_{-\infty}^{\infty} g_1(t)g_2^*(t)\exp(-j2\pi ft)dt$$

$$= \int_{-\infty}^{\infty} G_1(\lambda)G_2^*(\lambda - f)d\lambda \qquad (2)$$

Setting $f = 0$ in Eq. (2), we get the desired relation

$$\int_{-\infty}^{\infty} g_1(t)g_2^*(t) = \int_{-\infty}^{\infty} G_1(\lambda)G_2^*(\lambda)d\lambda$$

Problem 2.23

We are given the following inequalities:

$$|G(f)| \le \int_{-\infty}^{\infty} |g_1(t)| \quad dt$$

$$|j2\pi f G(f)| \le \int_{-\infty}^{\infty} \left|\frac{dg(t)}{dt}\right| dt$$

$$\left|j2\pi f^2 G(f)\right| \le \int_{-\infty}^{\infty} \left|\frac{d^2 g(t)}{dt}\right| dt$$

Considering the triangular pulse $g(t)$ of Fig. 2.41 in the text, its first and second derivatives with respect to time t are illustrated in Fig. 1:

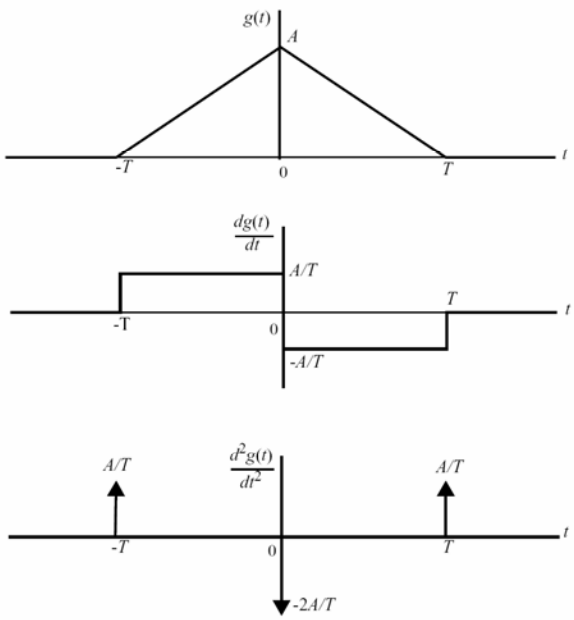


Figure 1

We thus have

$$\int_{-\infty}^{\infty} |g(t)| dt \;=\; AT$$

$$\int_{-\infty}^{\infty} \frac{dg(t)}{dt} dt \;=\; 2A$$

$$\int_{-\infty}^{\infty} \frac{d^2 g(t)}{dt} dt \;=\; \int_{-\infty}^{\infty} \frac{A}{T}|\delta(t+T) - 2\delta(t) + \delta(t-T)| dt$$

$$= \frac{4A}{T}$$

The bounds on the amplitude spectrum $|G(f)|$ are therefore as follows:

$$|G(f)| \le AT$$

Continued on next slide

Problem 2.23 continued

$$|G(f)| \le \frac{A}{\pi|f|}$$

$$|G(f)| \le \frac{A}{\pi^2 f^2 T}$$

which are shown plotted in Fig. 2.

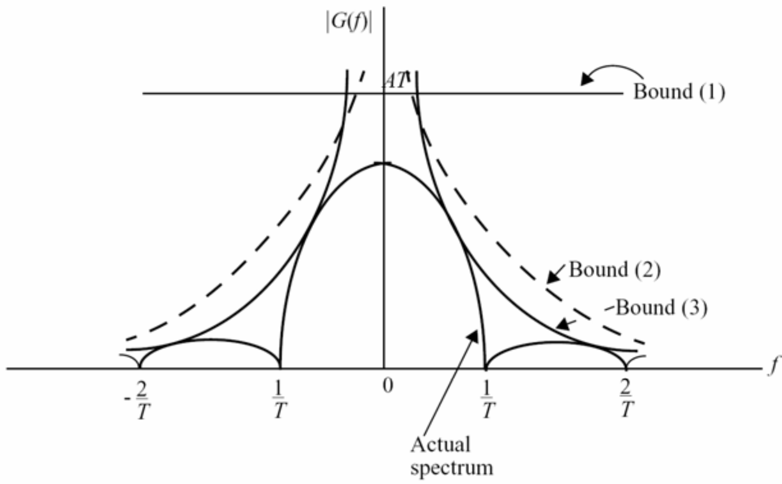


Figure 2

The actual amplitude spectrum of the triangular pulse is given by

$$|G(f)| = AT\,\text{sinc}^2(fT)$$

which is also plotted in Fig. 1. From this figure we see that bounds (1) and (3) define boundaries on the actual spectrum $|G(f)|$.

Problem 2.24

(a) The convolution of $g_1(t)$ and $g_2(t)$ is defined by

$$g_1(t)\bigstar g_2(t) = \int_{-\infty}^{\infty} g_1(\tau)g_2(t-\tau)d\tau \qquad (1)$$

Differentiating both sides of Eq. (1) with respect to time t:

$$\begin{aligned}\frac{d}{dt}[g_1(t)\bigstar g_2(t)] &= \int_{-\infty}^{\infty} g(\tau)\frac{d}{dt}g_2(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} g_1(\tau)\frac{d}{d(t-\tau)}g_2(t-\tau)d\tau \\ &= g_1(t)\bigstar\left[\frac{d}{dt}g_2(t)\right]\end{aligned}$$

Since convolution is commutative, we may also write

$$\frac{d}{dt}[g_1(t)\bigstar g_2(t)] = \left[\frac{d}{dt}g_1(t)\right]\bigstar g_2(t)$$

In other words, the derivative of a convolution produce of two signals is equivalent to the convolution of one of the signals and the derivative of the other.

(b) Changing variables in Eq. (1), we may write

$$g_1(t)\bigstar g_2(t) = \int_{-\infty}^{\infty} g_1(\lambda)g_2(t-\lambda)d\lambda \qquad (2)$$

Integrating both sides of Eq. (2) with respect to t:

$$\int_{-\infty}^{t}[g_1(\tau)\bigstar g_2(\tau)]d\tau = \int_{-\infty}^{t}\int_{-\infty}^{\infty} g_1(\lambda)g_2(\tau-\lambda)\lambda d\tau$$

Interchanging the order of integration and rearranging terms:

$$\int_{-\infty}^{t}[g_1(\tau)\bigstar g_2(\tau)]d\tau = \int_{-\infty}^{\infty} g_1(\lambda)\int_{-\infty}^{\infty} g_2(\tau-\lambda)\tau d\lambda d \qquad (3)$$

Recognizing that

$$\int_{-\infty}^{t} g_2(\tau-\lambda)d\lambda = \int_{-\infty}^{t\text{-}\lambda} g_2(\tau)d\tau$$

we may rewrite Eq. (3) as

$$\begin{aligned}\int_{-\infty}^{t}[g_1(\tau)\bigstar g_2(\tau)]d\tau &= \int_{-\infty}^{\infty} g_1(\lambda)\int_{-\infty}^{t\text{-}\lambda} g_2(\tau)d\tau d\lambda \\ &= g_1(t)\bigstar\left[\int_{-\infty}^{t} g_2(\tau)d\tau\right]\end{aligned}$$

In other words, the integral of a convolution product of two signals is equivalent to the convolution of one of the signals and the integral of the other.

Problem 2.25

Express $y(t)$ as

$$y(t) = x^2(t)$$
$$= x(t)x(t)$$

Since multiplication in the time domain corresponds to convolution in the frequency domain, we may express the Fourier transform of $y(t)$ as

$$Y(f) = \int_{-\infty}^{\infty} X(\lambda)X(f-\lambda)d\lambda$$

where $X(f)$ is the Fourier transform of $x(t)$. However, $X(f)$ is zero for $|f| > W$. Therefore,

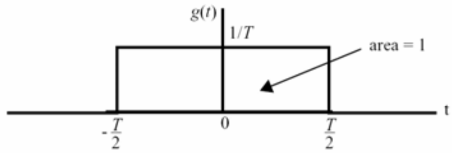
$$Y(f) = \int_{-W}^{W} X(\lambda)X(f-\lambda)d\lambda$$

In this integral we note that $X(f - \lambda)$ is limited to $-W \leq f - \lambda \leq W$. When $\lambda = -W$, we find that $-2W \leq f \leq 0$. When $\lambda = W$, we find that $0 \leq f \leq 2W$. Accordingly, the Fourier transform $Y(f)$ is limited to the frequency interval $-2W \leq f \leq 2W$.

Problem 2.26

(a) Consider a rectangular pulse $g(t)$ of duration T and amplitude $1/T$, centered at $t = 0$, as shown in Fig. 1:

Figure 1



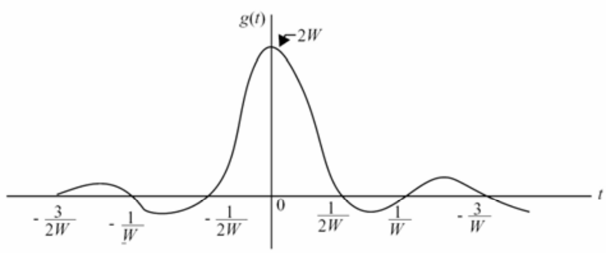
The Fourier transform of $g(t)$ is

$$G(f) = \frac{\sin(\pi f T)}{\pi f T}$$

As the duration T approaches zero, $g(t)$ approaches a delta function, and so we find that in the limit:

$$\lim_{T \to 0} G(f) = \lim_{T \to 0} \frac{\sin(\pi f T)}{\pi f T} = 1$$

(b) Consider next the sinc pulse $2W\,\text{sinc}(2Wt)$ of unit area, as shown in Fig. 2:



The Fourier transform of $g(t)$ is

$$G(f) = \text{rect}\left(\frac{f}{2W}\right)$$

which has unit amplitude and width $2W$, centered at $f = 0$. As W approaches infinity, $g(t)$ approaches a delta funct6ion, and the corresponding Fourier transform becomes equal to unity for all f.

Problem 2.27

The $G(f)$ is in the form of a unit step function defined in the frequency domain, as shown in Fig. 1

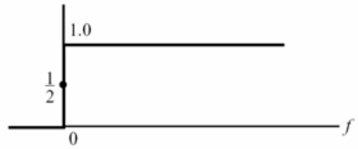


Figure 1

Now, for a unit step function defined in the time domain, we have

$$u(t) \rightleftharpoons \frac{1}{j2\pi f} + \frac{1}{2}\delta(f)$$

Applying the duality property of the Fourier transform to this relation, we get

$$-\frac{1}{j2\pi t} + \frac{1}{2}\delta(f) \rightleftharpoons u(f)$$

where we have used the fact that $\delta(-t) = \delta(t)$. Therefore, the time function $g(t)$ whose Fourier transform is depicted in Fig. 1, is given by

$$g(t) = -\frac{1}{j2\pi t} + \frac{1}{2}\delta(t)$$

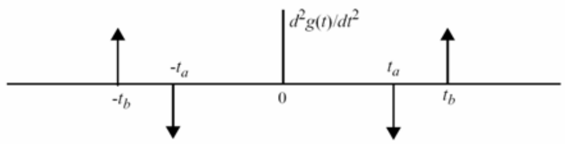
Problem 2.28

(a) Taking the Fourier transform of both sides of $\frac{d^2 g(t)}{dt^2} = \sum_i k_i \delta(t - t_i)$, we get

$$(j2\pi f)^2 G(f) = \sum_i k_i \exp(-j2\pi f t_i)$$

Therefore, $G(f) = \frac{-1}{4\pi^2 f^2} \sum_i k_i \exp(-j2\pi f t_i)$

(b) Differentiating the trapezoidal pulse of Fig. 2.42 twice, we get:



Hence,

$$G(f) = \frac{-A}{4\pi^2 f^2 (t_b - t_a)} [\exp(j2\pi f t_b) - \exp(j2\pi f t_a) - \exp(j2\pi f t_a) - \quad + \exp(j2\pi f t_b)]$$

$$= \frac{-A}{2\pi^2 f^2 (t_b - t_a)} [\cos(j2\pi f t_b) - \cos(j2\pi f t_a)]$$

$$= \frac{-A}{2\pi^2 f^2 (t_b - t_a)} \sin[\pi f (t_b - t_a) \sin \pi f (t_b + t_a)]$$

Problem 2.29

(a) From part (b) of Problem 2.28, we have

$$G(f) = \frac{A}{\pi^2 f^2 (t_b - t_a)} \sin[\pi f (t_b - t_a)] \sin[\pi f (t_b + t_a)] \tag{1}$$

As t_b approaches t_a, we get the following result:

$$\lim_{t_b \to t_a} \frac{1}{\pi f^2 (t_b - t_a)} \sin[\pi f (t_b - t_a)] = 1$$

and

$$\lim_{t_b \to t_a} \sin[\pi f (t_b + t_a)] = \sin(\pi f t_a)$$

Accordingly, the Fourier transform of Eq. (1) approaches the limiting value

$$\begin{aligned} \lim_{t_b \to t_a} G(f) &= \frac{A}{\pi f} \sin(2\pi f t_a) \\ &= 2t_a A \frac{\sin(2\pi f t_a)}{2\pi f t_a} \\ &= 2t_a A \,\mathrm{sinc}(2\pi f t_a) \end{aligned} \tag{2}$$

which is the desired result.

(b) The limiting Fourier transform of Eq. (2) is recognized as the Fourier transform of a rectangular pulse of amplitude A and duration $T = 2t_a$.

Problem 2.30

The transfer function $H(f)$ and impulse response $h(t)$ of a linear time-invariant filter are related by

$$H(f) = \int_{-\infty}^{\infty} h(t)\exp(-j2\pi ft)dt$$

Applying a special form of Schwarz's inequality (see Appendix 5), we may write

$$|H(f)| \le \int_{-\infty}^{\infty} |h(t)\exp(-j2\pi ft)|dt$$

Since $|\exp(-j2\pi ft)| = 1$, we may simplify this relation as

$$|H(f)| \le \int_{-\infty}^{\infty} |h(t)|dt$$

If the filter is stable, the impulse response is absolutely integrable:

$$\int_{-\infty}^{\infty} |h(t)|dt < \infty$$

Therefore, the amplitude response of a stable filter is bounded for every value of the frequency f, as shown by

$$|H(f)| = \infty$$

According to Rayleigh's energy theorem, the energy of the input signal $x(t)$ is given by

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

and the energy of the output signal $y(t)$ is

$$E_y = \int_{-\infty}^{\infty} |y(t)|^2 dt = \int_{-\infty}^{\infty} |Y(f)|^2 df$$

The Fourier transforms $Y(f)$ and $X(f)$ are related by

$$Y(f) = H(f)X(f)$$

Therefore,

$$E_y = \int_{-\infty}^{\infty} |H(f)|^2 |X(f)|^2 df \qquad (1)$$

For a stable filter, we may express $|H(f)|$ in the form $K|H_n(f)|$ where K is a scaling factor equal to the maximum value of $|H(f)|$ and $|H_n(f)| \le 1$ for all f. Thus, we may rewrite Eq. (1) in the form:

$$E_y = K^2 \int_{-\infty}^{\infty} |H_n(f)|^2 |X(f)|^2 df$$

Since $|H_n(f)| \le 1$ for all f, it follows that

$$\int_{-\infty}^{\infty} |H(f)|^2 |X(f)|^2 df \le \int_{-\infty}^{\infty} |X(f)|^2 df$$

or equivalently

$$E_y \le K^2 \int_{-\infty}^{\infty} |X(f)|^2 df$$

If the input signal has finite energy, we then have

$$\int_{-\infty}^{\infty} |X(f)|^2 df = \infty$$

Accordingly, we find that $E_y < \infty$, which means that the output signal $y(t)$ also has finite energy.

Problem 2.31

(a) The transfer function of the *i*th stage of the system of Fig. 2.43 is

$$H_i(f) = \frac{1}{1 + j2\pi fRC}$$

$$= \frac{1}{1 + j2\pi f\tau_0}, \qquad T_0 = RC$$

where it is assumed that the buffer amplifier has a constant gain of unity. The overall transfer function of the system is therefore

$$H(f) = \prod_{i=1}^{N} H_i(f)$$

$$= \frac{1}{(1 + j2\pi f\tau_0)^N}$$

The corresponding amplitude response is

$$|H(f)| = \frac{1}{[1 + (2\pi f\tau_0)^2]^{N/2}} \qquad (1)$$

(b) Let

$$\tau_0^2 = \frac{T^2}{4\pi^2 N}$$

Then, we may rewrite Eq. (1) for the amplitude response as

$$|H(f)| = \left[1 + \frac{1}{N}(fT)^2\right]^{-N/2}$$

In the limit, as N approaches infinity we have

$$|H(f)| = \lim_{N \to \infty}\left[1 + \frac{1}{N}(fT)^2\right]^{-N/2}$$

$$= \exp\left[\frac{N}{2} \cdot \frac{1}{N}(fT)^2\right]$$

$$= \exp\left(-\frac{f^2T^2}{2}\right)$$

Problem 2.32

(a) The integrator output is

$$y(t) = \int_{t-T}^{t} x(\tau)d\tau$$

Let $x(t) \rightleftharpoons X(f)$; then

$$x(t) = \int_{-\infty}^{\infty} X(f)\exp(j2\pi ft)df$$

Therefore,

$$y(t) = \int_{t-T}^{t}\left[\int_{-\infty}^{\infty} X(f)\exp(j2\pi f\tau)df\right]d\tau$$

Interchanging the order of integration:

$$y(t) = \int_{-\infty}^{\infty} X(f)\left[\int_{t-T}^{t}\exp(j2\pi f\tau)d\tau\right]df$$

$$= \int_{-\infty}^{\infty} [TX(f)\,\mathrm{sinc}(fT)\exp(-j\pi fT)]\exp(j2\pi ft)df$$

The Fourier transform of the integrator output is therefore

$$Y(f) = TX(f)\,\mathrm{sinc}(fT)\exp(-j\pi fT) \qquad (1)$$

Equation (1) shows that $y(t)$ can be obtained by passing the input signal $x(t)$ through a linear filter whose transfer function is equal to $T\mathrm{sinc}(fT)\exp(-j\pi fT)$.

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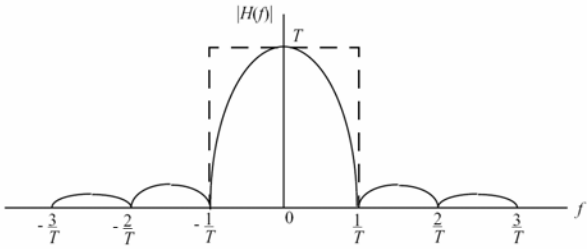
Problem 2-32 continued

(b) The amplitude response of this filter is shown in Fig. 1:

Figure 1



The approximation with an ideal low-pass filter of bandwidth $1/T$, gain T, and delay $T/2$, is shown dashed in Fig. 1. The response of this ideal filter to a unit step function applied at $t = 0$ is given by

$$y_{\text{ideal}}(t) = \frac{T}{\pi}\int_{-\infty}^{\frac{2\pi}{T}\left(t-\frac{T}{2}\right)} \frac{\sin\lambda}{\lambda}d\lambda$$

At time $t = T$. we therefore have

$$\begin{aligned} y_{\text{ideal}}(t) &= \frac{T}{\pi}\int_{-\infty}^{\pi} \frac{\sin\lambda}{\lambda}d\lambda \\ &= \frac{T}{\pi}\left[\int_{-\infty}^{0} \frac{\sin\lambda}{\lambda}d\lambda + \int_{0}^{\pi} \frac{\sin\lambda}{\lambda}d\lambda\right] \\ &= \frac{T}{\pi}[\text{Si}(\infty) + \text{Si}(\pi)] \\ &= \frac{T}{\pi}\left(\frac{\pi}{2} + 1.85\right) \\ &= 1.09T \qquad (2) \end{aligned}$$

On the other hand, the output of the ideal integrator to a unit step function, evaluated at time $t = T$, is given by

$$\begin{aligned} y(T) &= \int_0^T u(\tau)d\tau \\ &= T \qquad (3) \end{aligned}$$

Thus, comparing Eqs. (2) and (3) we see that the ideal low-pass filter output exceeds the ideal integrator output by only nine percent for $T = 1$.

Problem 2.33

The half cosine pulse in Fig. 2.33(a) is

$$g(t) = A\,\text{rect}\left(\frac{t}{T}\right)\cos\left(\frac{\pi t}{T}\right)$$

Fourier transforming both sides gives

$$G(f) = AT\frac{\sin[\pi fT]}{\pi fT} \star \left\{\frac{1}{2}\left[\delta\left(f-\frac{1}{2T}\right)+\delta\left(f+\frac{1}{2T}\right)\right]\right\}$$

$$= \frac{AT\sin\left[\pi fT-\frac{\pi}{2}\right]}{2\left(\pi fT-\frac{\pi}{2}\right)} + \frac{AT\sin\left[\pi fT+\frac{\pi}{2}\right]}{2\left(\pi fT+\frac{\pi}{2}\right)}$$

$$= \frac{2AT\cos(\pi fT)}{\pi(1-2fT)(1+2fT)}$$

Therefore, the energy density of $g(t)$ is

$$\Psi(f) = |G(f)|^2 = \frac{4A^2T^2\cos^2(\pi fT)}{\pi^2(1-4f^2T^2)^2} = \frac{4A^2T^2\cos^2(\pi fT)}{\pi^2(4T^2f^2-1)^2} \quad (1)$$

Consider next the half-sine pulse in Fig. 2.33(b), which is the same as that of Fig. 2.33(b) shifted to the right by $T/2$. This time-shift corresponds to multiplication by $\exp(-j2\pi fT)$, which has unit amplitude for all f. Therefore, both pulses have exactly the same energy density defined in Eq. (1).

Problem 2.34

The autocorrelation function of a deterministic signal $g(t)$ is defined by

$$R_g(\tau) = \int_{-\infty}^{\infty} g(t)g(t-\tau)dt \qquad (1)$$

This formula applies to a real-valued signal, which is satisfied by all three signals specified under parts (a) through (c).

(a) $g(t) = \exp(-at)u(t)$, $u(t)$: unit step function

Applying Eq. (1) yields

$$R_g(\tau) = \int_{\tau}^{\infty} \exp(-at)\exp(-a(t-\tau))dt$$

$$= \exp(a\tau)\int_{\tau}^{\infty} \exp(-2at)dt$$

$$= \exp(a\tau)\left[-\frac{1}{2a}\exp(-2at)\right]_{t=\tau}^{\infty}$$

$$= \frac{1}{2a}\exp(-a\tau)$$

which is depicted in Fig. 1

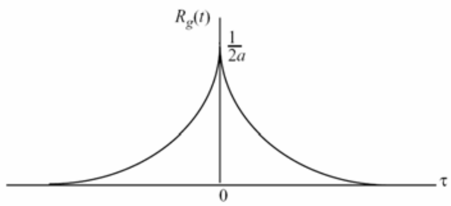


Figure 1

Continued on next slide

Problem 2-34 continued

(b) $g(t) = \exp(-a|t|)$

which is sketched in Fig. 2(a). Part (b) of the figure sketches $g(t - \tau) = \exp(-a|t - \tau|)$

(a)

(b)

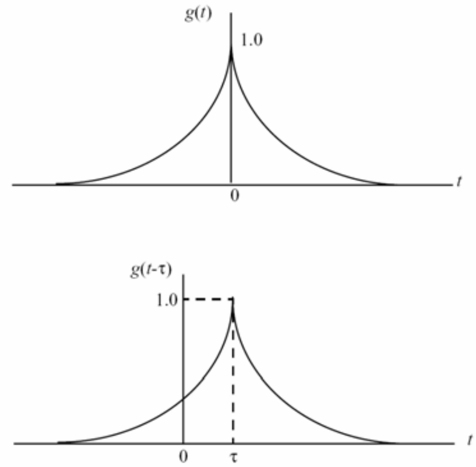


Figure 2

In light of Fig. 1, applying Eq. (1):

$$R_g(\tau) = \int_\tau^\infty \exp(-at)\exp(-a(t-\tau))dt \quad + \int_0^\tau \exp(-at)\exp(-a(t-\tau))dt$$

$$+ \int_0^\tau \exp(at)\exp(a(t-\tau))dt$$

$$= \frac{1}{2a}\exp(-at) + \tau\exp(-at) + \frac{1}{2a}\exp(-a\tau)$$

$$= \left(\frac{1}{a} + \tau\right)\exp(-a\tau)$$

which is sketched in Fig. 3.

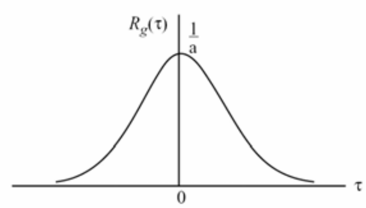


Figure 3

(c) $g(t) = \exp(-at)u(t) - \exp(at)u(-t)$

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Problem 2-34 continued

which is sketched in Fig. 4(a). Part (b) of the figure sketches $g(t - \tau)$

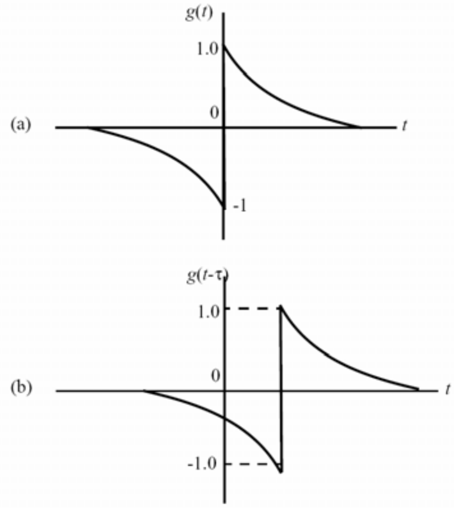

Figure 4

In light of Fig. 4, applying Eq. (1) for $\tau \geq 0$:

$$R_g(\tau) = \int_{\tau}^{\infty} \exp(-at)\exp(-a(t-\tau))dt$$

$$+ \int_{0}^{\tau} \exp(-at)[-\exp(a(t-\tau))]dt$$

$$+ \int_{-\infty}^{0} [-\exp(-at)][-\exp(a(t-\tau))]dt$$

$$= \frac{1}{2a}\exp(-a\tau) - \tau\exp(-a\tau) + \frac{1}{2a}\exp(-a\tau)$$

$$= \left(\frac{1}{a} - \tau\right)\exp(-a\tau), \qquad \tau \geq 0$$

Similarly, for $\tau \leq 0$ we have

$$R_g(\tau) = \left(\frac{1}{a} + \tau\right)\exp(a\tau)$$

Summing up these two results:

$$R_g(\tau) = \begin{cases} \left(\frac{1}{a} - \tau\right)\exp(-a\tau), & \tau \geq 0 \\ \left(\frac{1}{a} + \tau\right)\exp(a\tau), & \tau \leq 0 \end{cases}$$

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Problem 2-34 continued

which is sketched in Fig. 5.

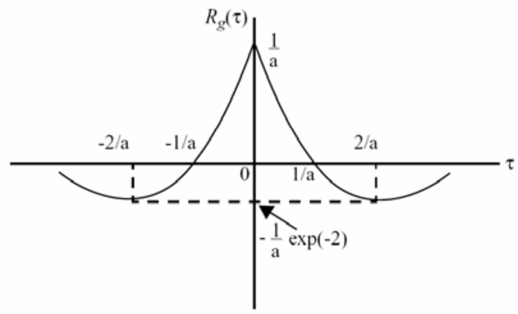


Figure 5

Problem 2.35

Applying the formula for the autocorrelation function

$$R_g(\tau) = \int_{-\infty}^{\infty} g(t)g^*(t-\tau)dt$$

to the specified signal

$$g(t) = \frac{1}{t_0}\exp\left(-\frac{\pi t^2}{t_0^2}\right), \qquad -\infty < t < \infty$$

we get

$$R_g(\tau) = \int_{-\infty}^{\infty} \frac{1}{t_0^2}\exp\left[\frac{\pi}{t_0^2}(t^2 + (t-\tau)^2)\right]dt$$

$$= \frac{1}{t_0^2}\int_{-\infty}^{\infty} \exp\left[\left(-\frac{\pi}{t_0^2}\right)(2t^2 - 2t\tau + \tau^2)\right]dt$$

$$= \frac{1}{t_0^2}\int_{-\infty}^{\infty} \exp\left[\left(-\frac{\pi}{t_0^2}\right)\left(\sqrt{2}t - \frac{\tau}{\sqrt{2}}\right) - \frac{\pi}{t_0^2}\ \frac{\tau^2}{2}\right]dt$$

$$= \frac{1}{t_0^2}\exp\left(-\frac{\pi\tau^2}{2t_0^2}\right)\int_{-\infty}^{\infty} \exp\left[\left(-\frac{\pi}{t_0^2}\right)\left(\sqrt{2}t - \frac{\tau}{\sqrt{2}}\right)^2\right]dt \tag{1}$$

Let $x = \frac{1}{t_0}\left(\sqrt{2}t - \frac{\tau}{\sqrt{2}}\right)$, and therefore (for fixed τ)

$$dt = \frac{t_0}{\sqrt{2}}dx$$

We may then rewrite Eq. (1) as

$$R_g(\tau) = \frac{1}{\sqrt{2}t_0}\exp\left(-\frac{\pi\tau^2}{2t_0^2}\right)\int_{-\infty}^{\infty} \exp(-\pi x^2)dx \tag{2}$$

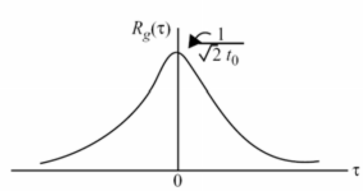
Recognizing that

$$\int_{-\infty}^{\infty} \exp(-\pi x^2)dx = 1$$

we find that Eq. (2) simplifies to

$$R_g(\tau) = \frac{1}{\sqrt{2}t_0}\exp\left(-\frac{\pi\tau^2}{2t_0^2}\right)$$

which has the same form as the bell-shaped Gaussian curve:



Problem 2.36

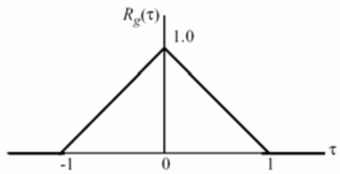
We are given the Fourier transform

$G(f) = \text{sinc}(f)$

Using the transform pair

$R_g(\tau) \rightleftharpoons |G(f)|^2$

we may therefore express the autocorrelation function $R_g(\tau)$ as the inverse Fourier transform of $\text{sinc}^2(f)$. From Eq. (2.43) in the text, we readily deduce that $R_g(\tau)$ has the triangular form



Problem 2.37

Recognizing two facts:

1. $R_g(\tau) = |G(f)|^2$, and
2. the spectrum $G(f)$ is invariant to a time shift,

we infer that a signal $g(t)$ and the time-shifted version $g(t - t_0)$ for any t_0 will have exactly the same autocorrelation function.

Problem 2.38

(a) We are given the power signal

$$g(t) = A_0 + A_1 \cos(2\pi f_1 t + \theta_1) + A_2 \cos(2\pi f_2 t)$$

The three components of $g(t)$ are uncorrelated with each other. Therefore, the power spectral density of $g(t)$ is the sum of the power spectral densities of the three constituent components, as shown by

$$S_g(f) = \frac{A^2}{2}\delta(f) + \frac{A_1^2}{4}[\delta(f - f_1) + \delta(f + f_1)] + \frac{A_2^2}{4}[\delta(f - f_2) + \delta(f + f_2)]$$

Correspondingly, the autocorrelation function $R_g(\tau)$ is given by

$$R_g(\tau) = \frac{A^2}{2} + \frac{A_1^2}{2}\cos(2\pi f_1 \tau) + \frac{A_2^2}{2}\cos(2\pi f_2 \tau)$$

(Here we are postulating a fundamental result that, as with energy signals, the autocorrelation function and power spectral density of a power signal constitute a Fourier-transform pair).

(b) $R_g(0) = \frac{A^2}{2}$.

(c) In calculating the autocorrelation function, information about the phase shifts θ_1 and θ_2 is completely lost.

Problem 2.39

We will determine the autocorrelation function of the signal $g(t)$ depicted in Fig. 2.45 by proceeding on a segment-by-segment basis:

1. The maximum value of $R_g(\tau)$ occurs at $\tau = 0$, for then $g(t)$ and $g(t\text{-}\tau)$ overlap exactly, yielding

$$R_g(0) = 3(A^2)(T) = 3A^2T$$

2. For $0 < \tau < (T/2)$, we have the picture depicted in Fig. 1. From this figure, we obtain

$$\begin{aligned} R_g(\tau) &= \int_{-(3T/2)+\tau}^{-T/2}(+A)(+A)dt + \int_{-T/2}^{-(T/2)+\tau}(+A)(\text{-}A)dt \\ &\quad + \int_{-(T/2)+\tau}^{T/2}(+A)(+A)dt + \int_{T/2}^{(T/2)+\tau}(\text{-}A)(+A)dt \\ &\quad + \int_{(T/2)+\tau}^{3T/2}(\text{-}A)(\text{-}A)dt \\ &= A^2(T-\tau) - A^2\tau + A^2(T-\tau) - A^2\tau + A^2(T-\tau) \\ &= A^2(3T-5\tau), \qquad 0 < |\tau| < (T/2) \end{aligned}$$

where the use of $|\tau|$ is invoked in light of the symmetric property of the autocorrelation function.

3. Next, for $(T/2) < \tau < T$, we have the picture depicted in Fig. 2, from which we obtain

$$\begin{aligned} R_g(\tau) &= \int_{-(3T/2)+\tau}^{-T/2}(\text{-}A)(\text{-}A)dt + \int_{-T/2}^{-(T/2)+\tau}(+A)(\text{-}A)dt \\ &\quad + \int_{-(T/2)+\tau}^{T/2}(+A)(+A)dt + \int_{T/2}^{(T/2)+\tau}(\text{-}A)(+A)dt \\ &\quad + \int_{(T/2+\tau)}^{3T/2}(\text{-}A)(\text{-}A)dt \\ &= A^2(T-\tau) - A^2\tau + A^2(T-\tau) - A^2\tau + A^2(T-\tau) \\ &= A^2(3T-5\tau), \qquad \frac{T}{2} < |\tau| < T \end{aligned}$$

4. Next, for $T < \tau < 3T/2$, we have the picture depicted in Fig. 3, from which we obtain

$$\begin{aligned} R_g(\tau) &= \int_{-(3T/2)+\tau}^{T/2}(+A)(\text{-}A)dt + \int_{T/2}^{-(T/2)+\tau}(\text{-}A)(\text{-}A)dt \\ &\quad + \int_{-(T/2)+\tau}^{3T/2}(\text{-}A)(+A)dt \\ &= -A^2(2T-\tau) + A^2(-T+\tau) - A^2(2T-\tau) \\ &= A^2(-5T+3\tau) \qquad \text{for } T < |\tau| < 3T/2 \end{aligned}$$

Continued on next slide

Problem 2-39 continued

5. For $(3T/2) < \tau < 2T$ we have the picture depicted in Fig. 4, from which we obtain

$$R_g(\tau) = \int_{-(3T/2)+\tau}^{T/2} (+A)(-A)dt + \int_{T/2}^{-(T/2)+\tau} (-A)(-A)dt + \int_{-(T/2)+\tau}^{3T/2} (-A)(+A)dt$$

$$= -A^2(2T-\tau) + A^2(-T+\tau) - A^2(2T-\tau)$$

$$= A^2(-5T+3\tau) \qquad \text{for } (3T/2) + |\tau| < 2T$$

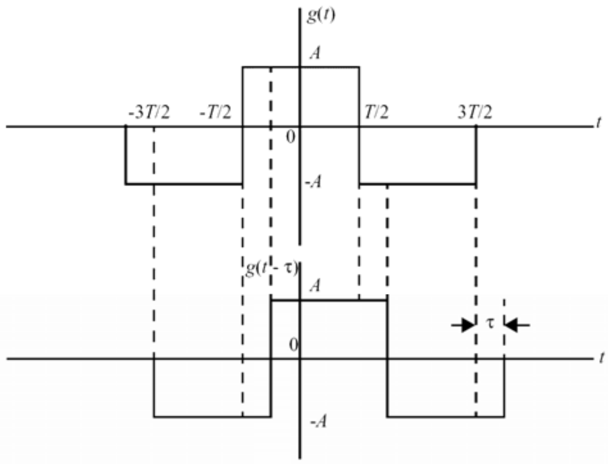


Figure 1: $-(T/2) < \tau < (T/2)$

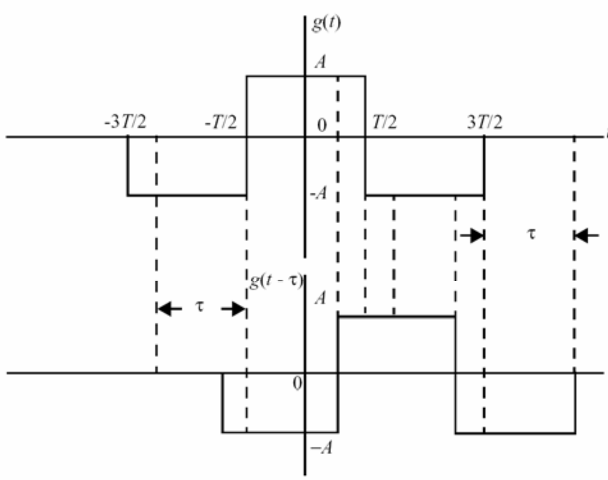


Figure 2: $(T/2) < \tau < (3T/2)$

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Problem 2-39 continued

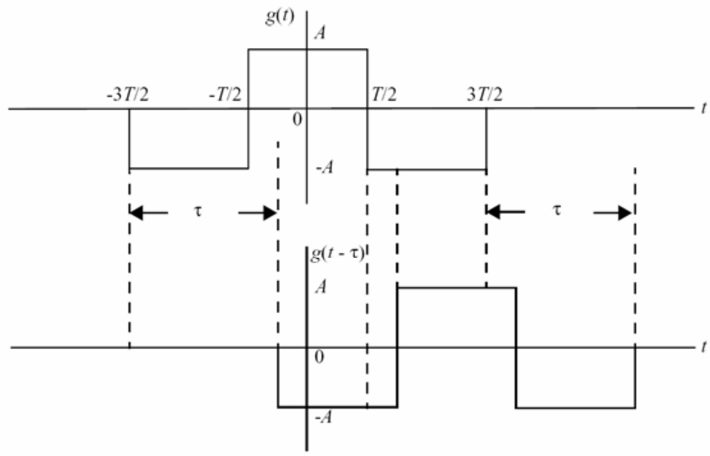

Figure 3: $T < \tau < (3T/2)$

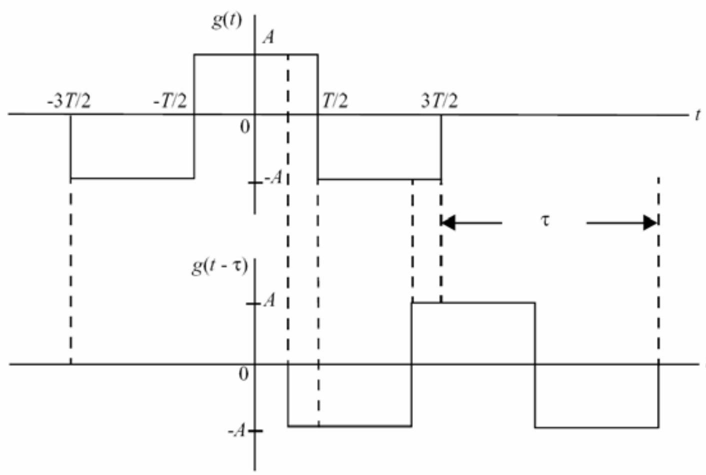

Figure 4: $(3T/2) < \tau < 2T$

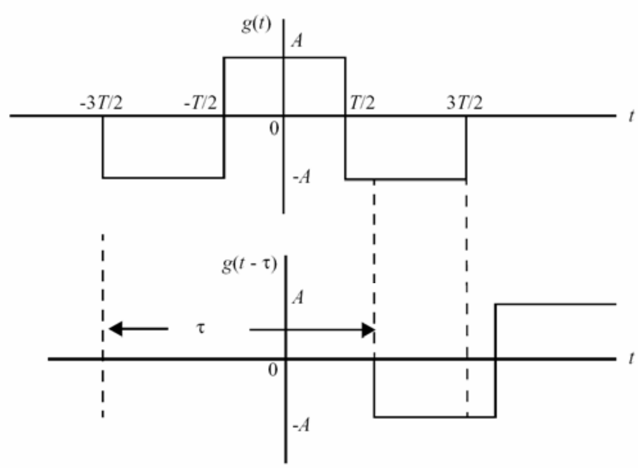

Figure 5: $2T < \tau < 3T$

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Problem 2-39 continued

7. Finally, for $|\tau| > 3T$, we find that $R_g(\tau) = 0$.

Putting all these pieces together, we get the autocorrelation function $R_g(\tau)$ plotted in Fig. 6, which is symmetric about the origin $\tau = 0$.

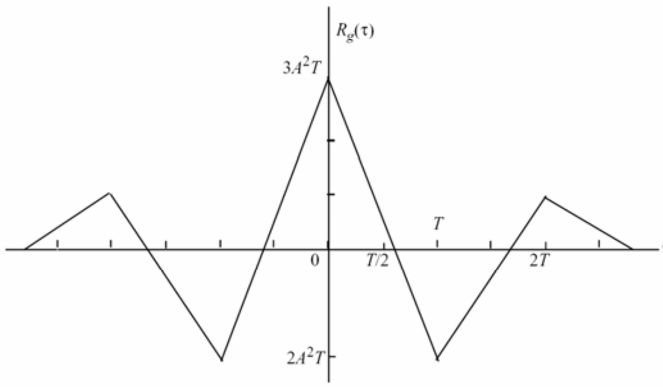

Figure 6: Plot of the autocorrelation function $R_g(t\tau)$

Problem 3.1.

For 100 percent modulation, is it possible for the envelope of AM to become zero for some time t? Justify your answer.

Solution

By definition, the envelope of AM signals is $A_c|t + k_a m(t)|$, where A_c is the carrier amplitude, k_a is the amplitude sensitivity of the modulator, and $m(t)$ is the message signal. The envelope will assure the zero value if and only if

$$k_a m(t) = 1$$

So long as this condition is satisfied, then the envelope of the AM signal will assume the value zero.

Problem 3.2.

For a particular case of AM using sinusoidal modulating wave, the percentage modulation is 20 percent. Calculate the average power in (a) the carrier and (b) each side frequency, expressing your results as percentages of total transmitted power.

Solution

For sinusoidal modulation, the AM wave is defined by

$$s(t) = A_c(1 + k_a m(t))\cos(2\pi f_c t)$$

For $m(t) = A_m\cos(2\pi f_m t)$, we have (see Example 3.1)

$$s(t) = A_c\cos(2\pi f_c t) + \frac{1}{2}\mu A_c\cos[2\pi(f_c + f_m)t] + \frac{1}{2}\mu A_c\cos[2\pi(f_c - f_m)t]$$

(a) The average power in the carrier, expressed as a percentage of the total transmitted power, is (with $\mu = 20\%$)

$$\frac{\frac{1}{2}A_c^2}{\frac{1}{2}A_c^2 + \frac{1}{4}\mu^2 A_c^2} = \frac{1}{1 + 0.5\mu^2} = \frac{1}{1 + 0.5 \times 0.2^2} = \frac{1}{1 + 0.02} \approx 0.98$$

Expressing this result as a percentage, the result reads as 98%.

(b) The average power in each side frequency is therefore approximately 1%.

Problem 3.3

In AM, *spectral overlap* is said to occur if the lower sideband for positive frequencies overlaps with its *image* for negative frequencies. What condition must the modulated wave satisfy if we are to avoid spectral overlap? Assume that the message signal $m(t)$ is of a low-pass kind with bandwidth W.

Solution

The lowest frequency of the lower sideband is f_c - W, where f_c is the carrier frequency and W is the message bandwidth. To avoid spectral overlap, we must therefore satisfy the condition:

$f_c - W > 0$

Hence, f_c must always be greater than the message bandwidth W.

Problem 3.4

A *square-law modulator* for generating an AM wave relies on the use of a nonlinear device (e.g., diode); Fig. 3.8 depicts the simplest form of such a modulator. Ignoring higher order terms, the input-output characteristic of the diode-load resistor combination in this figure is represented by the *square law*:

$$v_2(t) = a_1 v_1(t) + a_2 v_1^2(t)$$

where

$$v_1(t) = A_c \cos(2\pi f_c t) + m(t)$$

is the input signal, $v_2(t)$ is the output signal developed across the load resistor, and a_1 and a_2 are constants.

(a) Determine the spectral content of the output signal $v_2(t)$.

(b) To extract the desired AM wave from $v_2(t)$, we need a band-pass filter (not shown in Fig. 3.8). Determine the cutoff frequencies of the required filter, assuming that the message signal is limited to the band-$W \le f \le W$.

(c) To avoid *spectral distortion* by the presence of undesired modulation products in $v_2(t)$, the condition $f_c > 2W$ must be satisfied; validate this condition.

Solution

The output signal is

$$\begin{aligned} v_2(t) &= a_1 v_1(t) + a_2 v_1^2(t) \\ &= a_1(A_c \cos(2\pi f_c t) + m(t)) + a_2(A_c \cos(2\pi f_c t) + m(t))^2 \\ &= [a_1 + 2a_2 m(t)] A_c \cos(2\pi f_c t) \\ &\quad + [a_1 m(t) + a_2 A_c^2 \cos^2(2\pi f_c t) + a_2 m^2(t)] \end{aligned} \qquad (1)$$

(a) The expression inside the first set of square brackets defines the desired AM wave:

$$\begin{aligned} s(t) &= A_c[a_1 + 2a_2 m(t)] \cos(2\pi f_c t) \\ &= a_1 A_c \left[1 + \frac{2a_2}{a_1} m(t)\right] \cos(2\pi f_c t) \end{aligned}$$

which represents an AM wave with

$$k_a = \frac{2a_2}{a_1}$$

defining the amplitude sensitivity of the modulator.

Continued on next slide

Problem 3.4 continued

(b) The required band-pass filter must have a passband centered on f_c and a bandwidth equal to $2W$.

(c) The expression inside the second set of square brackets of Eq. (1) defines the undesired modulated products. The terms that matter are:

- The term $a_2 m^2(t)$, whose highest frequency component is 2W.
- The term $a_2 A_c^2 \cos^2(2\pi f_c t)$, whose frequency is $2f_c$.

To extract the desired AM wave we therefore require:
Condition 1:
$(f_c + W) < 2f_c$
or $f_c > W$
Condition 2:
$(f_c - W) > 2W$
or $f_c > 3W$
If therefore we satisfy condition 2, then condition 1 is automatically satisfied.

Problem 3.5

For the sinusoidally DSB-SC modulation considered in Problem 3.5, what is the average power in the lower or upper side-frequency, expressed as a percentage of the average power in the DSB-SC modulated wave?

Solution

With the carrier suppressed at the modulator output, the average power in either side frequency is 50% of the average power of the modulated wave.

Problem 3.6.

The sinusoidally modulated DSB-SC wave of Example 3.2 is applied to a product modulator using a locally generated sinusoid of unit amplitude, and which is synchronous with the carrier used in the modulation.

(a) Determine the output of the product modulator, denoted by $v(t)$.

(b) Identify the two sinusoidal terms in $v(t)$ that are produced by the upper side frequency of the DSB-SC modulated wave, and the remaining two sinusoidal terms produced by the lower side frequency.

Solution

(a) From Example 3.2, the DSB-SC modulated is defined by

$$s(t) = \frac{1}{2}A_cA_m\cos(2\pi(f_c+f_m)t) + \frac{1}{2}A_cA_m\cos(2\pi(f_c-f_m)t)$$

Applying $s(t)$ and $\cos(2\pi f_c t)$ to a product modulator yields

$$v(t) = s(t)\cos(2\pi f_c t)$$

$$= \frac{1}{2}A_cA_m\cos(2\pi(f_c+f_m)t)\cos(2\pi f_c t) + \frac{1}{2}A_cA_m\cos(2\pi(f_c-f_m)t)\cos(2\pi f_c t)$$

$$= \frac{1}{4}A_cA_m[\cos(2\pi(2f_c+f_m)t) + \cos(2f_mt)]$$

$$+ \frac{1}{4}A_cA_m[\cos(2\pi(2f_c-f_m)t) + \cos(2f_mt)] \qquad (1)$$

(b) The two sinusoidal terms inside the first set of square brackets in Eq. (1) are produced by the upper side frequency at $f_c + f_m$. The other two sinusoidal terms inside the second set of brackets are produced by the lower side frequency f_c - f_m.

Note that with $f_c > f_m$, the first and third terms in $v(t)$, both of which relate to carrier frequency $2f_c$ are removed by a low-pass filter. This would then leave the second and fourth sinusoidal terms, both of frequency f_m, as the only output of the filter. The coherent detector thus reproduces the original modulating wave of frequency f_m, with the output consisting of two contributions, one due to the upper side frequency and the other due to the lower side frequency.

Problem 3.7

The coherent detector for the demodulation of DSB-SC fails to operate satisfactorily if the modulator experiences spectral overlap. Explain the reason for this failure.

Solution

The DSB-SC modulated wave is defined by

$$s(t) = A_c m(t)\cos(2\pi f_c t)$$

Spectral overlap occurs if the condition $f_c > W$ is violated, in which case the lower sideband overlaps with its image.

However, when $s(t)$ is applied to a coherent detector, the resulting output is

$$v(t) = s(t)\cos(2\pi f_c t)$$

$$= A_c m(t)\cos^2(2\pi f_c t)$$

$$= \frac{1}{2}A_c m(t)[1 + \cos(4\pi f_c t)]$$

The spectral description of $v(t)$ is shown in Fig. 1, assuming that $f_c < W$:

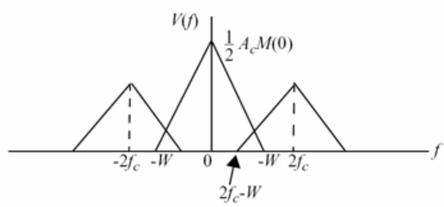

Figure 1

Recovery of the original signal is possible only if

$$2f_c - W > W$$

or

$$f_c > W$$

But this condition is being violated because of the spectral overlap.

Hence, once spectral overlap is permitted, no coherent detector can recover the original modulating signal.

Problem 3.8

As just mentioned, the phase discriminator in the Costas receiver of Fig. 3.16 consists of a multiplier followed by a time-averaging unit. Referring to this figure, do the following:

(a) Assuming that the phase error ϕ is small compared to one radian, show that the output $g(t)$ of the multiplier component is approximately $\frac{1}{4}\phi m^2(t)$.

(b) Furthermore, passing $g(t)$ through the time-averaging unit defined by

$$\frac{1}{2T}\int_{-T}^{T} g(t)dt$$

where the averaging interval $2T$ is long enough compared to the reciprocal of the bandwidth of $g(t)$, show that the output of the phase discriminator is proportional to the phase-error ϕ multiplied by the dc (direct current) component of $m^2(t)$. The amplitude of this signal (acting as the control signal applied to the voltage-controlled oscillator in Fig. 3.16) will therefore always have the same algebraic sign as that of the phase error ϕ, which is how it should be.

Solution

(i) Referring to the Costas receiver in Fig. 3.16 in the text, we see that the output of the in-phase channel is $\frac{1}{2}A_c\cos\phi m(t)$ and the output of the quadrature channel is $\frac{1}{2}A_c\sin\phi m(t)$. The output of the multiplier in the phase discriminator is therefore

$$\begin{aligned} g(t) &= \left(\frac{1}{2}A_c\cos\phi m(t)\right)\left(\frac{1}{2}A_c\sin\phi m(t)\right) \\ &= \frac{1}{4}\sin\phi\cos\phi m^2(t) \end{aligned} \tag{1}$$

If the phase error ϕ is small compared to one radian, we may use the approximations:

$\sin\phi \approx \phi$

$\cos\phi \approx 1$

in which case the multiplier output $g(t)$ simplifies approximately to $\frac{1}{4}\phi m^2(t)$.

(ii) Passing $g(t)$ through the time-averaging unit yields the phase discriminator output

$$\begin{aligned} v(t) &= \frac{1}{2T}\int_{-T}^{T} g(t)dt \\ &\approx \frac{1}{2T}\int_{-T}^{T} \frac{1}{4}\phi m^2(t)dt \\ &= \frac{\phi}{8T}\int_{-T}^{T} m^2(t)dt \\ &= \frac{1}{4}\phi P_0 \end{aligned}$$

where

$$P_0 = \frac{1}{2T}\int_{-T}^{T} m^2(t)dt$$

is the dc component of $m^2(t)$ or, equivalently, the average power of $m(t)$.

Problem 3.9

Verify that the outputs of the receiver in Fig. 3.17(b) are as indicated in the figure, assuming perfect synchronism between the receiver and transmitter.

Solution

The transmitted signal is

$$s(t) = A_c m_1(t)\cos(2\pi f_c t) + A_c m_2(t)\sin(2\pi f_c t)$$

Hence, the product modulator output of upper channel in Fig. 3.17(b) is

$$\begin{aligned} v_1(t) &= A'_c\cos(2\pi f_c t)s(t) \\ &= A_c A'_c m_1(t)\cos(2\pi f_c t) + A_c A'_c m_2(t)\cos(2\pi f_c t)\sin(2\pi f_c t) \\ &= \frac{1}{2}A_c A'_c m(t)[1 + \cos(4\pi f_c t)] + \frac{1}{2}A_c A'_c m(t)\sin(4\pi f_c t) \end{aligned}$$

Passing $v_1(t)$ through the low-pass filter yields $\frac{1}{2}A_c A'_c m(t)$, so long as there is no spectral overlap, that is, $f_c > W$.

Consider next the lower channel of the figure. The product-modulator output is

$$\begin{aligned} v_2(t) &= A'_c\cos(2\pi f_c t)s(t) \\ &= A_c A'_c m_1(t)\sin(2\pi f_c t)\cos(2\pi f_c t) + A_c A'_c m_2(t)\sin^2(2\pi f_c t) \\ &= \frac{1}{2}A_c A'_c m(t)\sin(4\pi f_c t) + \frac{1}{2}A_c A'_c m_2(t)[1 - \cos(4\pi f_c t)] \end{aligned}$$

Passing $v_2(t)$ through the low-pass filter yields $\frac{1}{2}A_c A'_c m(t)$, as indicated in the figure.

Problem 3.10

Using Eqs. (3.22) and (3.23), show that for positive frequencies the spectra of the two kinds of SSB modulated waves are defined as follows:

(a) For the upper SSB,

$$S(f) = \begin{cases} \frac{A_c}{2}M(f-f_c) & \text{for } f \geq f_c \\ 0 & \text{for } 0 < f \leq f_c \end{cases}$$

(b) For the lower SSB,

$$S(f) = \begin{cases} 0 & \text{for } f > f_c \\ \frac{A_c}{2}M(f-f_c) & \text{for } 0 < f \leq f_c \end{cases}$$

(c) Write down the formulas for these two kinds of SSB modulation that pertain to negative frequencies.

Solution

According to Eq. (3.24):

$$s(t) = \frac{A_c}{2}m(t)\cos(2\pi f_c t) \mp \frac{A_c}{2}\hat{m}(t)\sin(2\pi f_c t)$$

where $\hat{m}(t)$ is the Hilbert transform of $m(t)$. Taking the Fourier transform of $s(t)$:

$$S(f) = \frac{A_c}{4}(M(f-f_c) + M(f+f_c)) \pm \frac{A_c}{4j}(\hat{M}(f-f_c) - \hat{M}(f+f_c))$$

From Eq. (3.22):

$$\hat{M}(f) = -jM(f)\operatorname{sgn}(f)$$

Hence,

$$S(f) = \frac{A_c}{4}(M(f\text{-}f_c) + M(f\text{+}f_c)) \pm (\frac{A_c}{4j}(-jM(f\text{-}f_c)\operatorname{sgn}(f\text{-}f_c) + jM(f\text{+}f_c)\operatorname{sgn}(f\text{+}f_c))$$

$$= \frac{A_c}{4}(M(f-f_c) \mp M(f-f_c)\operatorname{sgn}(f-f_c)) + \frac{A_c}{4}(M(f+f_c) \pm M(f+f_c)\operatorname{sgn}(f+f_c))$$

$$= \frac{A_c}{4}(1 \mp \operatorname{sgn}(f-f_c))M(f-c_c) + \frac{A_c}{4}(1 \pm \operatorname{sgn}(f+f_c))M(f+f_c) \qquad (1)$$

By definition:

$$\operatorname{sgn}(f-f_c) = \begin{cases} 1 & \text{for } f > f_c \\ -1 & \text{for } f < f_c \end{cases}$$

and

$$\operatorname{sgn}(f+f_c) = \begin{cases} 1 & \text{for } f > -f_c \\ -1 & \text{for } f < -f_c \end{cases}$$

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Problem 3.10 continued

(a) From Eq. (3.24), recall that the minus sign in this formula corresponds to upper SSB. Hence, for the upper SSB, we have

$$S(f) = \frac{A_c}{4}(1+\operatorname{sgn}(f-f_c))M(f-f_c) + \frac{A_c}{4}(1-\operatorname{sgn}(f+f_c))M(f+f_c) \qquad (2)$$

where the term containing $M(f - f_c)$ pertains to positive frequencies and the term containing $M(f + f_c)$ pertains to negative frequencies. Therefore for positive frequencies and $f \geq f_c$, Eq. (2) simplifies to

$$\begin{aligned} S(f) &= \frac{A_c}{4}(1+1)M(f-f_c) \\ &= \frac{A_c}{2}M(f-f_c) \end{aligned} \qquad (3)$$

For $0 \leq f \leq f_c$, $S(f) = 0$.

(b) From Eq. (3.24), also recall that the plus sign in this formula corresponds to the lower SSB, for which we find that for $f \leq f_c$:

$$S(f) = \frac{A_c}{4}(1-\operatorname{sgn}(f-f_c))M(f-f_c) + \frac{A_c}{4}(1+\operatorname{sgn}(f+f_c))M(f+f_c)$$

Therefore for positive frequencies and $f \leq f_c$, we have

$$\begin{aligned} S(f) &= \frac{A_c}{4}(1+1)M(f-f_c) \\ &= \frac{A_c}{2}M(f-f_c) \end{aligned} \qquad (4)$$

On the other hand, for $f > f_c$ we have $S(f) = 0$.

(c) For negative frequencies, we focus on terms containing $M(f + f_c)$, in light of which we get the following results:

(i) For upper SSB:

$$S(f) = \begin{cases} \frac{A_c}{4}M(f+f_c) & \text{for } f \leq -f_c \\ 0 & \text{for } -f_c < f < 0 \end{cases} \qquad (5)$$

(ii) For lower SSB:

$$S(f) = \begin{cases} 0 & \text{for } f < -f_c \\ \frac{A_c}{4}M(f+f_c) & \text{for } -f_c < f < 0 \end{cases} \qquad (6)$$

Problem 3.11

Show that if the message signal $m(t)$ is low-pass, then the Hilbert transform $\hat{m}(t)$ is also low-pass with the same bandwidth as $m(t)$.

Solution

The Fourier transform of the Hilbert transform $\hat{m}(t)$ is defined by

$$\hat{M}(f) = -j\,\mathrm{sgn}(f)M(f)$$

where $M(f) = \mathbf{F}[m(t)]$. To illustrate, let the spectrum $M(f)$ be as shown in Fig. 1(a). Then, the corresponding spectrum $\hat{M}(f)$ is as shown in part (b) of the figure. The spectrum $\hat{M}(f)$ is therefore also low-pass, occupying the frequency band $-W \le f \le W$ just like $M(f)$.

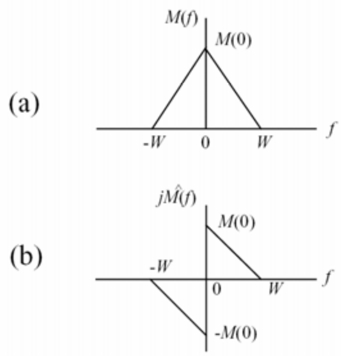

Figure 1

Problem 3.12

Starting with Eq. (3.23) for a SSB modulated wave, show that the product-modulator output in the coherent detector of Fig. 3.12 (assuming perfect synchronism with the transmitter) in response to this modulated wave contains a new SSB modulated wave with carrier frequency $2f_c$.

Solution

Suppose we focus on the upper SSB in Eq. (3.23) (i.e., use the minus sign in this formula). Then

$$s(t) = \frac{A_c}{2}m(t)\cos(2\pi f_c t) - \frac{A_c}{2}\hat{m}(t)\sin(2\pi f_c t)$$

Applying this modulated wave to the coherent detector of Fig. 3.12 with the phase error $\phi = 0$, we first get the product-modulated output

$$\begin{aligned} v(t) &= A'_c s(t)\cos(2\pi f_c t) \\ &= \frac{A_c A'_c}{2}m(t)\cos^2(2\pi f_c t) - \frac{A_c A'_c}{2}\hat{m}(t)\sin(2\pi f_c t)\cos(2\pi f_c t) \\ &= \frac{A_c A'_c}{4}m(t)[1 + \cos(4\pi f_c t)] - \frac{A_c A'_c}{4}\hat{m}(t)\sin(4\pi f_c t) \\ &= \frac{A_c A'_c}{4}m(t) + \frac{A_c A'_c}{4}[m(t)\cos(4\pi f_c t) - \hat{m}(t)\sin(4\pi f_c t)] \end{aligned}$$

Comparing this formula with that for $s(t)$, we see that $v(t)$ contains a new upper SSB modulated wave with carrier frequency $2f_c$. This same statement also applies to the lower SSB modulated wave $s(t)$.

Problem 3.13

For the low-pass filter in Fig. 3.12 (assuming perfect synchronism) to suppress the undesired SSB wave, the following condition must hold

$f_c > W, f_c$ = carrier frequency, and W = message bandwidth

Justify this condition

Solution

Continuing with the solution to Problem 3.12, we see that the product-modulator output $v(t)$ also contains a scaled version of the original message signal $m(t)$. For positive frequencies,the highest frequency component of $m(t)$ is W, and the lowest frequency of the new upper SSB modulated wave is $2f_c$ - W. For the low-pass filter to reject this SSB modulated wave, we require that

$2f_c - W > W$, or simply $f_c > W$. Under this condition, the detector output is

$$v_o(t) = \frac{A_c A'_c}{4} m(t)$$

Problem 3.14

Validate the statement that the high-frequency components in Eq. (3.36) represent a VSB wave modulated onto a carrier of frequency $2f_c$.

Solution

The high-frequency components in Eq. (3.36) are defined by the formula

$$G(f) = \frac{1}{4}A_cA'_c[M(f-2f_c)H(f-f_c)+M(f+2f_c)H(f+f_c)] \quad (1)$$

Referring to Eq. (3.33), the spectrum of the incoming VSB modulated wave is

$$\begin{aligned} S(f) &= \frac{1}{2}A_c[M(f-f_c)+M(f+f_c)]H(f) \\ &= \frac{1}{2}A_cM(f-f_c)H(f)+\frac{1}{2}A_cM(f+f_c)H(f) \quad (2) \end{aligned}$$

Examining Eqs. (1) and (2), as labelled here, we see that (ignoring the scaling factors)

1. The first term in Eq. (1), namely, $M(f - 2f_c)H(f - f_c)$ is equal to the first term in Eq. (2), namely $M(f - f_c)H(f)$ shifted to the right by f_c.
2. By the same token, the second term in Eq. (1), namely, $M(f + 2f_c)H(f + f_c)$ is equal to the second term in Eq. (2), namely, $M(f + f_c)H(f)$ shifted to the left by f_c.

Since Eq. (2) represents a VSB wave modulated onto carrier frequency f_c, it follows that Eq. (1) represents a VSB wave modulated onto the new carrier frequency $2f_c$.

Problem 3.15

Derivation of the synthesizer depicted in Fig. 3.25(b) follows directly from Eq. (3.39). However, derivation of the analyzer depicted in Fig. 3.25(a) requires more detailed consideration. Given that $f_c > W$ and

$$\cos^2(2\pi f_c t) = \frac{1}{2}[1 + \cos(4\pi f_c t)]$$

and

$$\sin(2\pi f_c t)\cos(2\pi f_c t) = \frac{1}{2}\sin(4\pi f_c t),$$

show that the analyzer of Fig. 3.25(a) yields $s_I(t)$ and $s_Q(t)$ as its two outputs.

Solution

Consider first the upper channel in Fig. 3.25(a). Multiplying (see Eq. (3.39))

$$s(t) = s_I(t)\cos(2\pi f_c t) - s_Q\sin(2\pi f_c t)$$

by the carrier $2\cos(2\pi f_c t)$, we get

$$\begin{aligned} v_1(t) &= 2s(t)\cos(2\pi f_c t) \\ &= 2s_I(t)\cos^2(2\pi f_c t) - 2s_Q(t)\sin(2\pi f_c t)\cos(2\pi f_c t) \\ &= s_I(t)[1 + \cos(4\pi f_c t)] - s_Q(t)\sin(4\pi f_c t) \\ &= s_I(t) + s'(t) \end{aligned}$$

where

$$s'(t) = s_I(t)\cos(4\pi f_c t) - s_Q(t)\sin(4\pi f_c t)$$

represents a new linearly modulated signal with carrier frequency $2f_c$. Provided that both $s_I(t)$ and $s_Q(t)$ are limited to the band $-W \le f \le W$ and we pass $v_1(t)$ through a low-pass filter of cutoff frequency W as in Fig. 3.25(a), then $s'(t)$ is rejected provided that $f_c > W$.

Consider next the lower channel in Fig. 3.25(a). Multiplying $s(t)$ by $-2\sin(2\pi f_c t)$, we get

$$\begin{aligned} v_2(t) &= -2s(t)\sin(2\pi f_c t) \\ &= -2s_I(t)\sin(2\pi f_c t)\cos(2\pi f_c t) + 2s_Q(t)\sin^2(2\pi f_c t) \\ &= -s_I(t)\sin(4\pi f_c t) + [1 - \cos(4\pi f_c t)]s_Q(t) \\ &= s_Q(t) - s''(t) \end{aligned}$$

where

$$s''(t) = s_I(t)\sin(4\pi f_c t) + s_Q(t)\cos(4\pi f_c t)$$

is a new linearly modulated signal with carrier frequency $2f_c$. Hence, passing $v_2(t)$ through a low-pass filter as in Fig. 3.25(a), $s''(t)$ is rejected again provided that the cutoff frequency W of the low-pass filter satisfies the condition $f_c > W$.

Problem 3.16
Starting with the complex low-pass system depicted in Fig. 3.26(c), show that the $y(t)$ derived in Eq. (3.45) is identical to the actual output $y(t)$ in Fig. 3.26(a).

Solution
According to Fig. 3.25(a), we have

$$Y(f) = H(f)S(f) \tag{1}$$

and according to Fig. 3.25(b),

$$2\tilde{Y}(f) = \tilde{H}(f)\tilde{S}(f) \tag{2}$$

From Eq. (3.44) we note that

$$\tilde{H}(f - f_c) = 2H(f) \qquad \text{for } f > 0 \tag{3}$$

Therefore, substituting Eq. (3) into (2) and cancelling the common factor 2, we get

$$\tilde{Y}(f - f_c) = H(f)\tilde{S}(f - f_c), \qquad f > 0 \tag{4}$$

Finally, noting that for $f > 0$

$$Y(f) = \tilde{Y}(f - f_c)$$

and

$$S(f) = \tilde{S}(f - f_c),$$

we readily see that Eq. (3) is a rewrite of Eq. (1), which validates the outputs displayed in Fig. 3.26.

Problem 3.17

(a) We are given

$c(t) = A_c \sin(2\pi f_c t)$

and

$m(t) = A_m \sin(2\pi f_m t)$

Invoking the definition of AM wave

$s(t) = [1 + k_a m(t)]c(t)$

we now write

$$s(t) = A_c[1 + k_a A_m \sin(2\pi f_m t)]\sin(2\pi f_c t)$$
$$= A_c \sin(2\pi f_c t) + \mu A_c \sin(2\pi f_m t)\sin(2\pi f_c t) \quad (1)$$

where

$\mu = k_a A_m$

is the modulation factor. Next, we use the trigonometric identity

$$\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$$

Hence, we may rewrite Eq. (1) as

$$s(t) = A_c \sin(2\pi f_c t) + \frac{1}{2}\mu A_c[\cos(2\pi(f_c - f_m)t) - \cos(2\pi(f_c + f_m)t)] \quad (2)$$

The spectrum of the AM wave s(t) consists of three components:

(i) Carrier: $A_c \sin(2\pi f_c t)$

(ii) Lower side-frequency: $\frac{1}{2}\mu A_c \cos(2\pi(f_c - f_m)t)$

(iii) Upper side-frequency: $-\frac{1}{2}\mu A_c \cos(2\pi(f_c + f_m)t)$

This spectrum is depicted in Fig. 1.

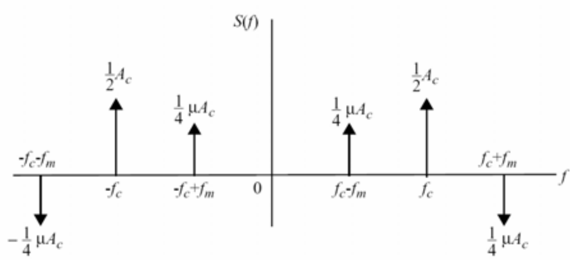

Figure 1

(b) Comparing the AM spectrum of Fig. 1 with the corresponding AM spectrum of Fig. 3.3(c) on page 105 of the text, we may make two observations:

- The frequency locations of the spectral components of these two AM waves are identical.
- The only difference between them is that the upper side-frequency $f_c + f_m$ in Fig. 1 is the negative of the upper side-frequency $f_c + f_m$ in Fig. 3.3(c).

Note: The following correction in the first printing of the book should be made. The modulating wave should read as follows:

$m(t) = A_m \sin(2\pi f_m t)$

Problem 3.18

(a) We are given

$c(t) = 50\cos(100\pi t)$ volts, $\quad f_c = 50\text{Hz}$

and

$m(t) = 20\cos(2\pi t)$ volts, $\quad f_m = 1\text{Hz}$

The resulting AM wave is

$$s(t) = [1 + k_a m(t)]c(t)$$
$$= 50[1 + 20k\cos(2\pi t)]\cos(100\pi t) \quad (1)$$

A percentage modulation of 75% corresponds

$20k = 0.75$

or

$k = 0.0375$

Accordingly, we may rewrite Eq. (1) as

$$s(t) = 50[1 + 0.75\cos(2\pi t)]\cos(100\pi t) \quad (2)$$

Equation (2) is plotted in Fig. 1.

(b) Expanding the AM wave s(t) of Eq. (2) into its spectral components, we write

$$s(t) = 50\cos(100\pi t) + 37.5\cos(2\pi t)\cos(100\pi t)$$
$$= 50\cos(100\pi t) + 18.75[\cos(102\pi t) + \cos(98\pi t)]\text{volts}$$

The power developed across a 100-ohm load by this AM wave is therefore

$$P = \frac{1}{2}\ \frac{(50)^2}{100} + \frac{1}{2}\ \frac{(18.75)^2}{100} + \frac{1}{2}\ \frac{(18.75)^2}{100}$$
$$= 12.5 + 3.426$$
$$= 15.926 \text{ watts}$$

This result shows that the carrier contributes about 80% of the power delivered to the load.

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Problem 3.18 continued

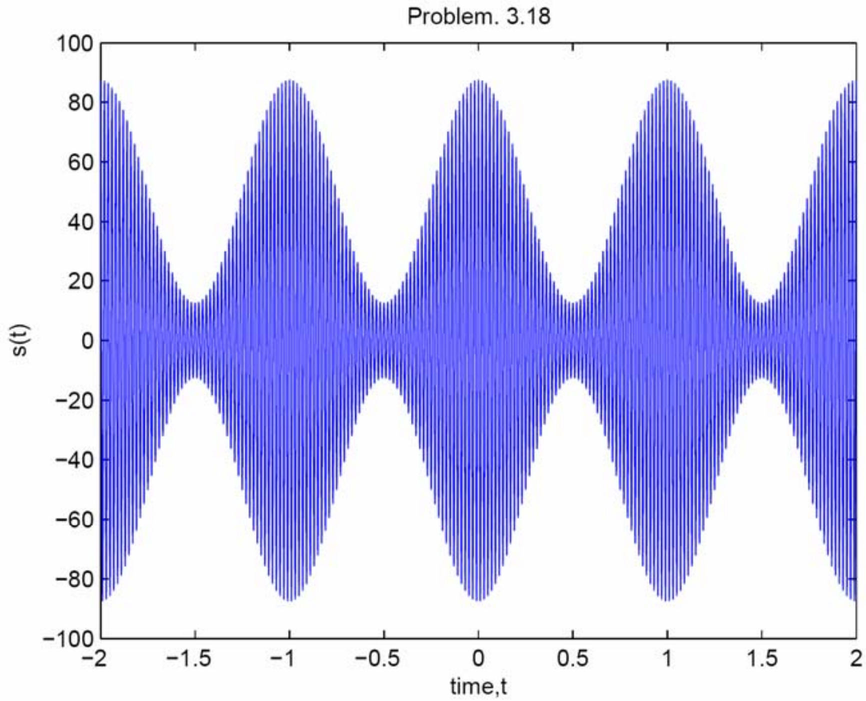

Figure 1: Problem 3.18

Problem 3.19

We are given

$$m(t) = \frac{t}{1+t^2} \tag{1}$$

The AM wave is therefore defined by

$$s(t) = A_c[1 + k_a m(t)]\cos(2\pi f_c t)$$
$$= A_c\left(1 + \frac{k_a t}{1+t^2}\right)\cos(2\pi f_c t) \tag{2}$$

The message signal $m(t)$ is plotted in Fig. 1(a) with its maximum value of 1/2 and minimum value of -1/2 at $t = 1$ and $t = -1$, respectively.

(a) Percentage modulation = 50%

$$k_a|m(t)|_{\max} = 0.5$$

with $|m(t)|_{\max} = \frac{1}{2}$, it follows that $k_a = 1$ for 50% modulation. For this example, Eq. (1) takes the form

$$s(t) = A_c\left(1 + \frac{t}{1+t^2}\right)\cos(2\pi f_c t) \tag{3}$$

Let t be measured in seconds. Then, for the envelope of the AM wave to be clearly visible, the period of the carrier, $1/f_c$, must be small compared to the time taken for the message signal $m(t)$ to reach its peak value. To satisfy this requirement, we let

$$\frac{1}{f_c} = 10\text{Hz}$$

which corresponds to

$$f_c = 10\text{Hz}$$

Setting the carrier amplitude $A_c = 1$ volt, and $f_c = 10$Hz, Eq. (3) is plotted in part (b) of Fig. 1.

(b) Percentage modulation = 100%

In this case, we have $k_a = 2$. Correspondingly, Eq. (1) assumes the form

$$s(t) = A_c\left(1 + \frac{2t}{1+t^2}\right)\cos(2\pi f_c t) \tag{4}$$

Keeping $A_c = 1$ volt, and $f_c = 10$Hz as in case (a), Eq. (4) is plotted in Fig. 1(c).

(c) Percentage modulation = 125%

In this third and final example, we have $k_a = 2.5$. Hence, Eq. (1) now assumes the form

$$s(t) = A_c\left(1 + \frac{2.5t}{1+t^2}\right)\cos(2\pi f_c t) \tag{5}$$

Keeping $_c = 1$ volt, and $f_c = 10$Hz as before, Eq. (5) is plotted in Fig. 1(d).

Comparing the AM waveforms plotted in parts (b), (c) and (d) of Fig. 1, we may make the following observations:

- The AM wave of Fig. 1(b) is undermodulated
- The AM wave of Fig. 1(c) is on the verge of overmodulation
- The AM wave of Fig. 1(d) is overmodulated

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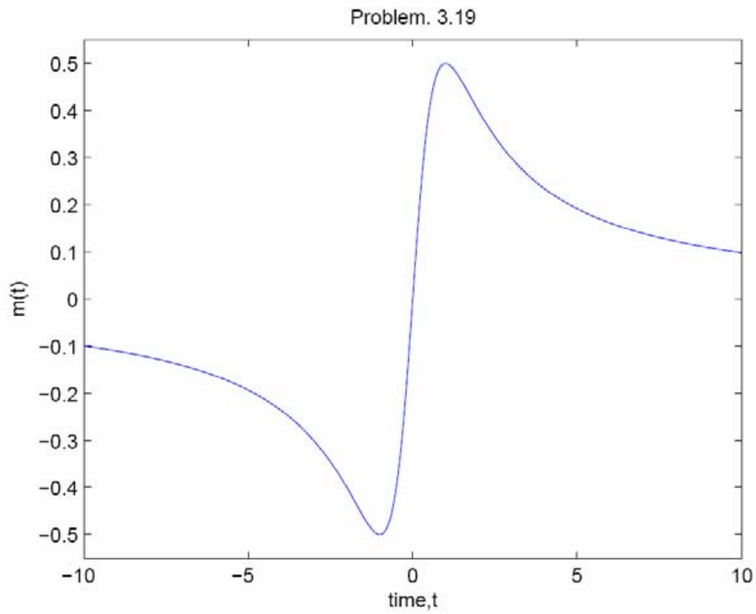
Problem 3-19 continued

(a)



(b)

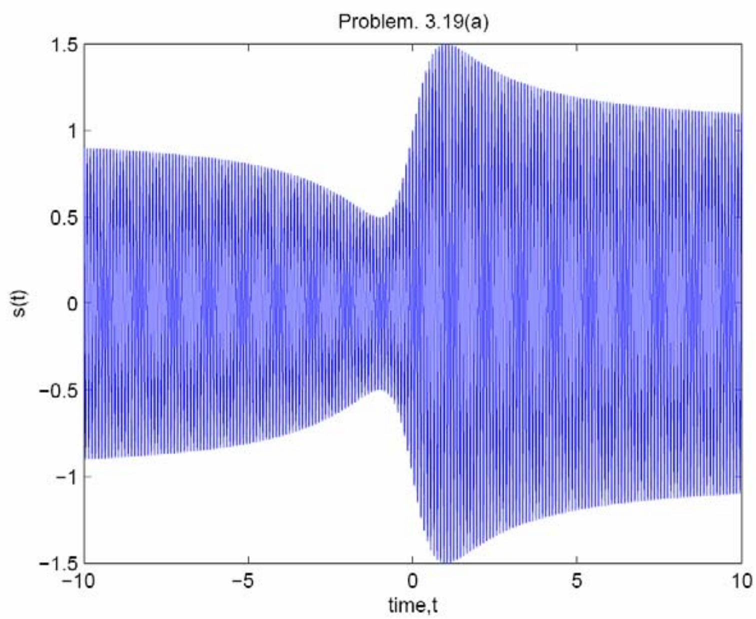


Figure 1: Problem 3.19

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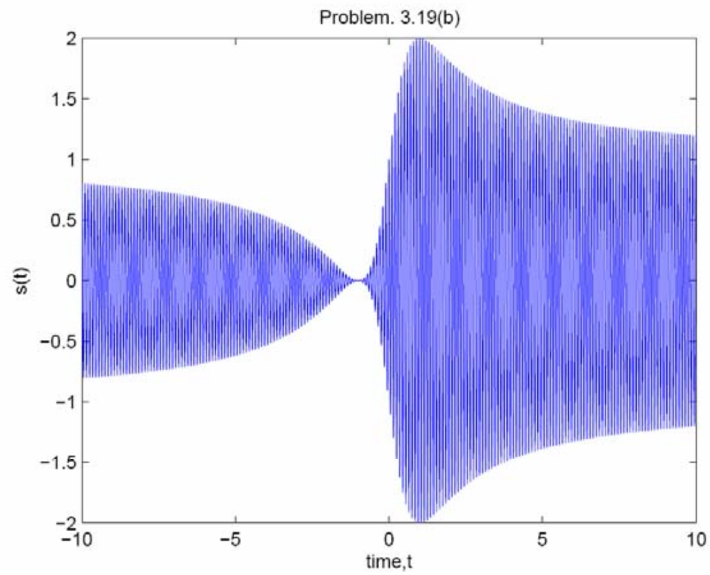
Problem 3-19 continued

(c)



(d)

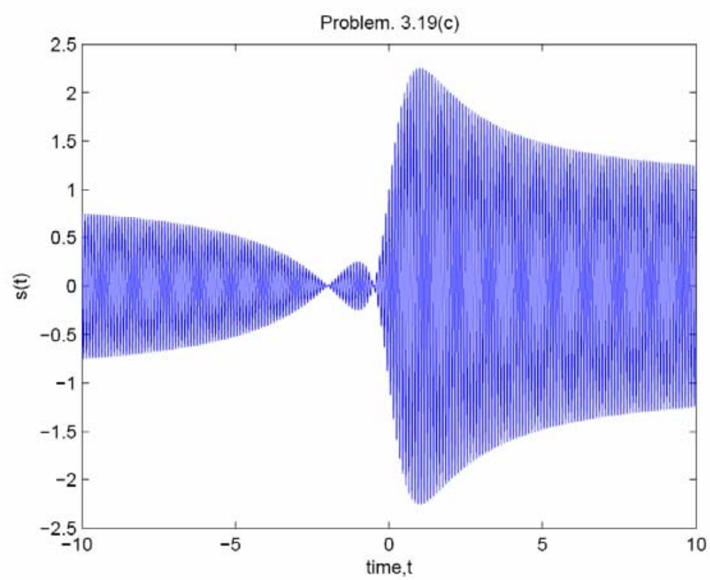


Figure 1 (continued): Problem 3.19

Problem 3.20

(a) Let the input voltage v_i consist of a sinusoidal wave of frequency $\frac{1}{2}f_c$ (i.e., half the desired carrier frequency) and the message signal $m(t)$, as shown by

$$v_i = A_c\cos(\pi f_c t) + m(t) \qquad (1)$$

Then, the output current i_o is

$$\begin{aligned} i_o &= a_1 v_i + a_3 v_i^3 \\ &= a_1[A_c\cos(\pi f_c t) + m(t)] + a_3[A_c\cos(\pi f_c t) + m(t)]^3 \\ &= a_1[A_c\cos(\pi f_c t) + m(t)] + \frac{1}{4}a_3 A_c^3[\cos 3(\pi f_c t) + 3\cos(\pi f_c t)] \\ &\quad + \frac{3}{2}a_3 A_c^2 m(t)[1 + \cos(2\pi f_c t)] + 3a_3 A_c\cos(\pi f_c t)m^2(t) + a_3 m^3(t) \end{aligned}$$

Assume that $m(t)$ occupies the frequency interval $-W \le f \le W$. Then, the amplitude spectrum of the output current i_o is as shown Fig. 1:

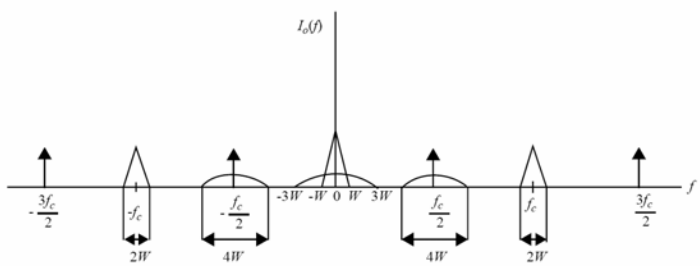


Figure 1

From this spectrum we see that in order to extract a DSB-SC wave with carrier frequency f_c from i_o, we need a bandpass filter with mid-band frequency f_c and bandwidth $2W$, the two of which satisfy the requirement:

$$f_c - W > \frac{f_c}{2} + 2W$$

that is, $f_c > 6W$

Therefore, to use the given nonlinear device as a product modulator, we may use the configuration: shown in Fig. 2.

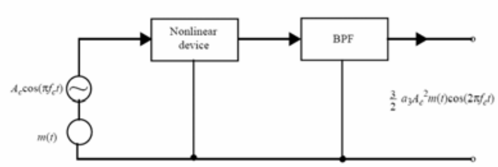


Figure 2

Continued on next slide

Problem 3-20 continued

(b) To generate an AM wave with carrier frequency f_c, we require a sinusoidal component of frequency f_c to be added to the DSB-SC generated in the manner described under (a). To achieve this requirement, we may use a configuration involving a pair of the nonlinear devices and a pair of identical bandpass filters, as depicted in Fig. 3.

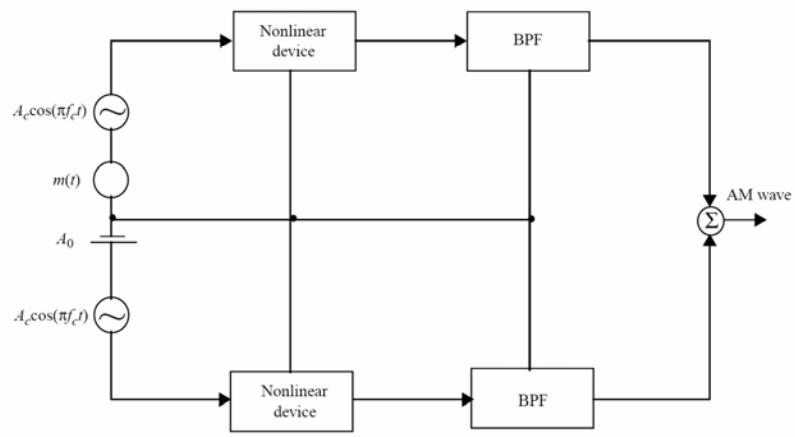


Figure 3

The resulting AM wave is therefore $\frac{3}{2}a_3A_c^2[A_0+m(t)]\cos(2\pi f_c t)$. Thus, the choice of the dc level A_0 at the input of the lower branch controls the percentage modulation of the AM wave.

The nonlinear device defined in Eq. (1) cannot be used for demodulation. The reason for saying so is that Eq. (1) lacks a square-law term, which is essential for demonstration (i.e., recovery of the message signal from an incoming AM wave).

Problem 3.21

We are given

$m(t) = A_c \cos(2\pi f_m t)$

and

$c(t) = A_c \cos(2\pi f_c t + \phi)$

The AM wave is therefore

$$\begin{aligned} s(t) &= [1 + k_a m(t)]c(t) \\ &= A_c[1 + k_a A_m \cos(2\pi f_m t)]\cos(2\pi f_c t + \phi) \\ &= A_c[1 + \mu \cos(2\pi f_m t)]\cos(2\pi f_c t + \phi) \qquad (1) \end{aligned}$$

The focus in this problem is to see how varying the phase ϕ affects the waveform of the AM signal $s(t)$. We may thus set the following parameters:

$\mu = k_a A_m = 0.5$

$A_c = 1$ volt

$f_m = 1$Hz

and

$f_c = 5$Hz

Then, Eq. (1) assumes the form

$$s(t) = [1 + 0.5\cos(2\pi t)]\cos(10\pi t + \phi) \qquad (2)$$

Equation (2) is plotted in Fig. 1 for the prescribed values $\phi = 0^o$, 45^o, 90^o, and 135^o. Examining these four waveforms, we may make the following observation:

- Insofar as the envelope of the AM wave is concerned, varying the carrier phase ϕ has no effect whatsoever on the waveform of the envelope, which is intuitively satisfying.
- The only visible effect of varying the carrier phase ϕ is a shift in the uniformly spaced zero-crossings of the AM wave.

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Problem 3-21 continued

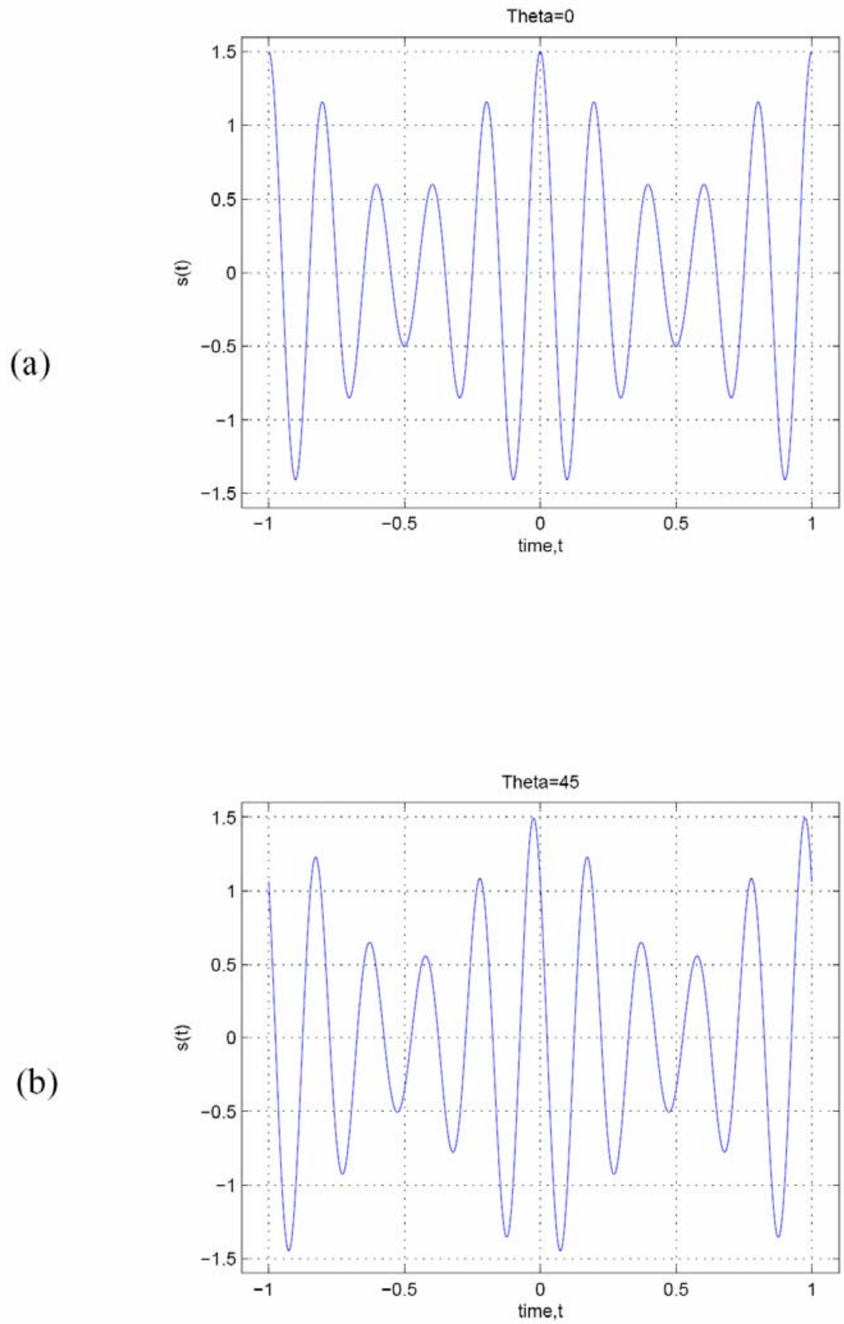

Figure 1: Problem 3.21

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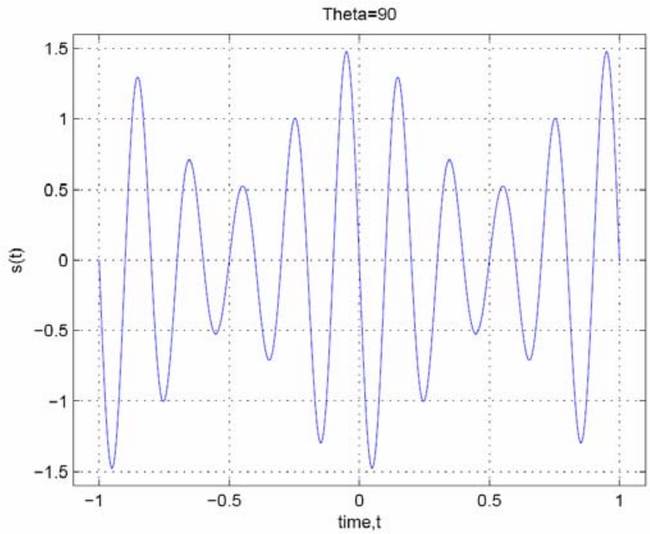
Problem 3-21 continued

(c)



(d)

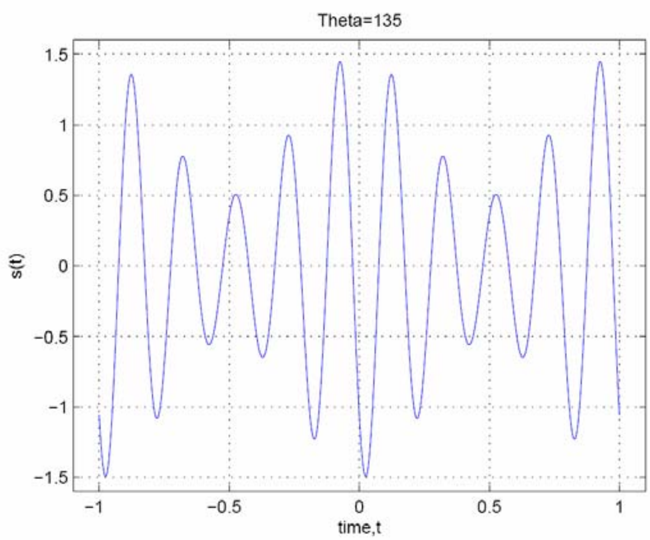


Figure 1 (continued): Problem 3.21

Problem 3.22

For the solution to this problem, see Fig. 2 in the solution to Problem 3.20.

Problem 3.23

(a) For f_c = 1.25 kHz, the spectra of the message signal $m(t)$, the product modulator output $s(t)$, and the coherent detector output $v(t)$ are as shown in Fig. 1, respectively:

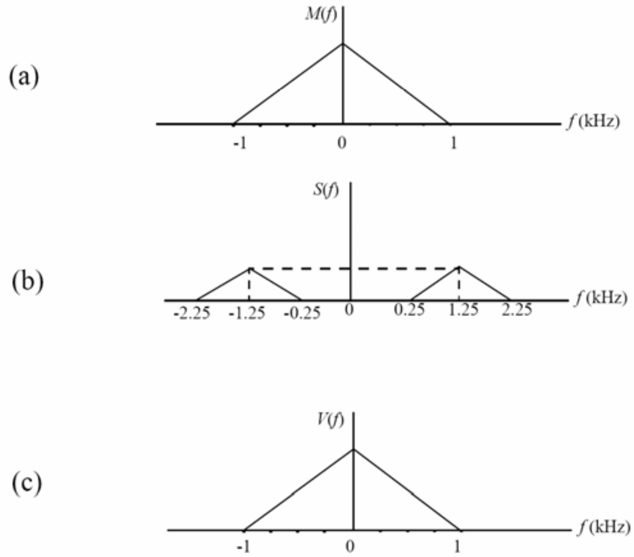


Figure 1

(b) For the case when f_c = 0.75, the respective spectra are as shown in Fig. 2:

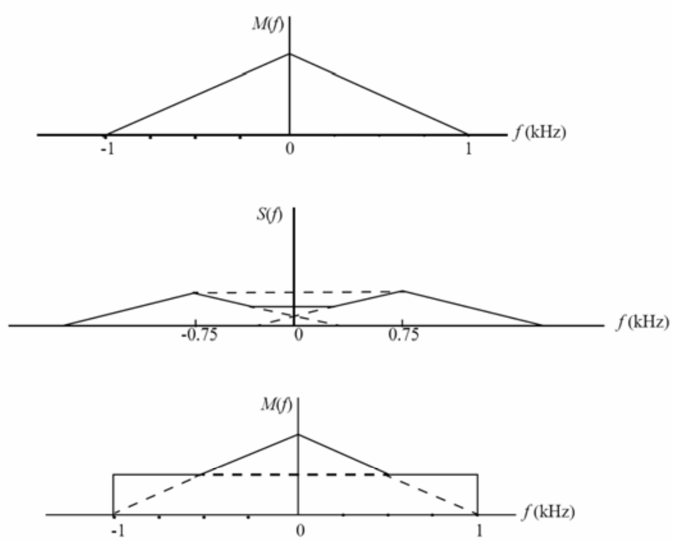


Figure 2

To avoid sideband-overlap, the carrier frequency f_c must be equal to or greater than 1 kHz. The lowest carrier frequency is therefore 1 kHz for each sideband of the modulated wave $s(t)$ to be uniquely determined by $m(t)$.

Problem 3.24

The noncoherent carrier is

$c(t) = A_c \cos(2\pi f_c t + \phi)$

and the DSB-SC modulated wave is $m(t)\cos(2\pi f_c t)$. The composite signal is therefore

$$s(t) = A_c \cos(2\pi f_c t + \phi) + m(t)\cos(2\pi f_c t)$$
$$= [A_c \cos\phi + m(t)]\cos(2\pi f_c t) - A_c \sin\phi \sin(2\pi f_c t)$$
$$= s_I(t)\cos(2\pi f_c t) - s_Q(t)\sin(2\pi f_c t)$$

where

$$s_I(t) = A_c \cos\phi + m(t)$$
$$s_Q(t) = A_c \sin\phi$$

Applying the composite signal $s(t)$ to an ideal envelope detector produces the output

$$a(t) = [s_I^2(t) + s_Q^2(t)]^{1/2}$$
$$= [(A_c \cos\phi + m(t))^2] + (A_c \sin\phi)^{2\,1/2}$$
$$= [A_c^2 \cos^2\phi + 2A_c \cos\phi m(t) + m^2(t) + A_c^2 \sin^2\phi]^{1/2}$$
$$= [A_c^2 + 2A_c \cos\phi m(t) + m^2(t)]^{1/2} \qquad (1)$$

(a) For $\phi = 0$, Eq. (1) reduces to

$$a(t) = [A_c^2 + 2A_c m(t) + m^2(t)]^{1/2}$$
$$= A_c + m(t) \qquad (2)$$

which consists of the message signal $m(t)$ plus a dc bias equal to the carrier amplitude.

(b) For $\phi \neq 0$ and $|m(t)| << A_c/2$, we may approximate Eq. (1) as follows:

$$a(t) \approx [A_c^2 + 2A_c \cos\phi m(t)]^{1/2}$$
$$= A_c\left[1 + \frac{2}{A_c}\cos\phi m(t)\right]^{1/2} \qquad (3)$$

With $|\cos\phi| \leq 1$, and $|m(t)| << A_c/2$, we may approximate Eq. (3) further as

$$a(t) \approx A_c\left[1 + \frac{1}{A_c}\cos\phi m(t)\right]$$
$$= A_c + \cos\phi m(t) \qquad (4)$$

When ϕ is close to zero, the detector output in Eq. (4) is very close to the value defined in Eq. (2). However, when ϕ approaches 90°, $\cos\phi$ approach zero, then the envelope detector output in Eq. (4) reduces to a dc component equal to A_c with no significant trace of the message signal $m(t)$ being visible. If therefore the phase error ϕ is variable, then the envelope detector output a(t) varies in a corresponding way, which could be undesirable.

Problem 3.25

(a) The effect of a frequency error Δf in the local oscillator used in the coherent detector shows itself as follows:

$$c'(t) = \cos(2\pi(f_c + \Delta f)t)$$

Applying the DSB-SC modulated wave $s(t)$

$$s(t) = A_c \cos(2\pi f_c t) m(t)$$

to a coherent detector employing $c'(t)$ yields the product modulator output (see Fig. 1)

$$\begin{aligned} v(t) &= s(t)c'(t) \\ &= A_c \cos(2\pi f_c t)\cos(2\pi f_c t + 2\pi\Delta f t) m(t) \end{aligned} \quad (1)$$

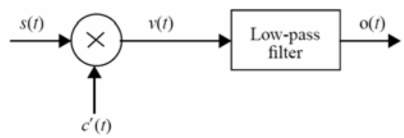


Figure 1

Using the trigonometric identity

$$\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$$

we may rewrite Eq. (1) as

$$v(t) = \frac{1}{2}A_c[\cos(4\pi f_c t + 2\pi\Delta f t) + \cos(2\pi\Delta f t)]m(t) \quad (2)$$

Next, passing $v(t)$ through the low-pass filter in Fig. 1 removes the high-frequency component, producing the output

$$o(t) = \frac{1}{2}A_c \cos(2\pi\Delta f t) m(t) \quad (3)$$

which exhibits beats at the error frequency Δf.

Problem 3.26

The message signal is defined by the rectangular pulse

$$m(t) = \begin{cases} A, & -T/2 \leq t \leq T/2 \\ 0, & \text{otherwise} \end{cases} \quad (1)$$

The SSB modulated wave is defined by

$$s(t) = \frac{A_c}{2} m(t) \cos(2\pi f_c t) \mp \frac{A_c}{2} \hat{m}(t) \sin(2\pi f_c t)$$

where $\hat{m}(t)$ is the Hilbert transform of $m(t)$. The in-phase and quadrature components of $s(t)$ are respectively defined by

$$s_I(t) = \frac{A_c}{2} m(t)$$

$$s_Q(t) = \pm \frac{A_c}{2} \hat{m}(t)$$

The envelope of $s(t)$ is therefore

$$\begin{aligned} a(t) &= [s_I^2(t) + S_Q^2(t)]^{1/2} \\ &= \frac{A_c}{2}[m^2(t) + \hat{m}^2(t)]^{1/2} \end{aligned} \quad (2)$$

The Hilbert transform of the rectangular pulse of Eq. (1) was determined in Problem 2.52 of Chapter 2; it is reproduced here for a pulse of unit amplitude and duration T:

$$\hat{m}(t) = -\frac{1}{\pi} \ln \left| \frac{t - (T/2)}{t + (T/2)} \right| \quad (3)$$

where ln denotes the natural logarithm. From Eq. (3) we see that $\hat{m}^2(t)$ assumes an infinitely large value at $t = T/2$ and $t = -T/2$. Correspondingly, the envelope of the SSB modulator exhibits peaks at the beginning and end of the input pulse.

Problem 3.27

(a) The frequency error $\Delta f = 20$ Hz. Since this frequency error is positive and the incoming SSB wave contains the upper sideband, the frequency components of the demodulated signal are shifted downward by Δf, compared with the message signal. The demodulated signal therefore consists of three frequency components: 80, 180, and 380 Hz.

(b) When the lower sideband is transmitted, the frequency components of the demodulated signal are shifted upward by Δf, compared with the message signal. The demodulated signal therefore consists of three frequency components: 120, 220, and 420 Hz.

Problem 3.28

The energy of the carrier over a bit duration is defined by

$$E = \int_0^{T_b} c^2(t)dt$$

$$= A_c^2 \int_0^{T_b} \cos^2(2\pi f t)dt \qquad (1)$$

Using the identity

$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$$

we rewrite Eq. (1) as

$$E = \frac{1}{2}A_c^2 \int_0^{T_b} [1 + \cos(4\pi f_c t)]dt$$

$$= \frac{1}{2}A_c^2 \int_0^{T_b} dt + \frac{1}{2}A_c^2 \int_0^{T_b} \cos(4\pi f_c t)dt \qquad (2)$$

Typically, the carrier frequency f_c is high compared to the bit rate $1/T_b$; we may therefore set the integral term in Eq. (2) approximately equal to zero, in which case we write

$$E \approx \frac{1}{2}A_c^2 \int_0^{T_b} dt$$

$$= \frac{1}{2}A_c^2 T_b \qquad (3)$$

For the energy E to equal unity, we may solve Eq. (3) for the carrier amplitude A_c, obtaining

$$A_c = \sqrt{\frac{2}{T_b}}$$

which is the desired result. On this basis, we express the carrier as

$$c(t) = \sqrt{\frac{2}{T_b}}\cos(2\pi f_c t)$$

Problem 3.29

(a) Using the terminated series expansion $\exp(-x) \approx 1 - x + \frac{x^2}{2} - \frac{x^3}{6}$ we may express the diode current i, normalized with respect to I_0, as

$$\frac{i}{I_0} = \exp\left(-\frac{v}{V_T}\right) - 1$$

$$= -\frac{v}{V_T} + \frac{1}{2}\left(\frac{v}{V_T}\right)^2 - \frac{1}{6}\left(\frac{v}{V_T}\right)^3 \quad (1)$$

(b) Given

$$\frac{v}{V_T} = \frac{0.01}{0.026}[\cos(2\pi f_m t) + \cos(2\pi f_c t)]$$

$$\approx 0.385[\cos(2\pi f_m t) + \cos(2\pi f_c t)] \quad (2)$$

we find that substitution of Eq. (2) into (1) yields

$$\frac{i}{I_0} \approx -0.385[\cos(2\pi f_m t) + \cos(2\pi f_c t)]$$

$$+ 0.074[\cos(2\pi f_m t) + \cos(2\pi f_c t)]^2$$

$$- 0.0095[\cos(2\pi f_m t) + \cos(2\pi f_c t)]^3 \quad (3)$$

Next, using the identities

$$\cos^2\theta = \frac{1}{2}[1 + \cos(2\theta)]$$

$$\cos^3\theta = \frac{3}{4}\cos\theta + \frac{1}{4}\cos(3\theta)$$

$$\cos\theta\cos\phi = \frac{1}{2}[\cos(\theta+\phi) + \cos(\theta-\phi)]$$

we may rewrite Eq. (3) in the form:

$$\frac{i}{I_0} = 0.074 - 0.406[\cos(2\pi f_m t) + \cos(2\pi f_c t)]$$

$$+ 0.037\{\cos(4\pi f_m t) + \cos(4\pi f_c t) + \cos[2\pi(f_c + f_m)t] + \cos[2\pi(f_c - f_m)t]\}$$

$$- 0.0016[\cos(6\pi f_m t) + \cos(6\pi f_c t)]$$

$$- 0.0071\{\cos[2\pi(f_c + 2f_m)t] + \cos[2\pi(f_c - 2f_m)t]$$

$$+ \cos[2\pi(2f_c + f_m)t] + \cos[2\pi(2f_c - f_m)t]\}$$

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Problem 3-29 continued

For f_m - 1 kHz and f_c = 100 kHz, we thus find that the discrete amplitude spectrum of the diode current i (for $f \geq 0$) is as shown in Fig. 1.

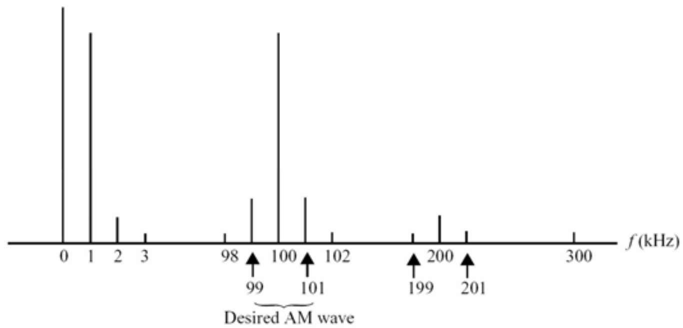

Figure 1

(c) From the amplitude spectrum of Fig. 1 we see that in order to extract an AM wave with carrier frequency f_c from the diode current i, we need a band-pass filter that passes only the frequency components: 99, 100 and 101 kHz, corresponding to f_c - f_m, f_c, and $f_c + f_m$, respectively. We therefore require a band-pass filter with center frequency 100 kHz and bandwidth 2 kHz.

(d) The resulting band-pass filter output is

$$\frac{i}{I_0} = -0.406\cos(2\pi f_c t) + 0.148\cos(2\pi f_c t)\cos(2\pi f_m t)$$

$$= -0.406[1 - 0.362\cos(2\pi f_m t)]\cos(2\pi f_c t)$$

The percentage modulation is therefore 36.2 percent.

Problem 3.30

The multiplexed signal is defined by

$$s(t) = A_c m_1(t)\cos(2\pi f_c t) + A_c m_2(t)\sin(2\pi f_c t)$$

Therefore, the spectrum of $s(t)$ is

$$S(f) = \frac{A_c}{2}[M_1(f-f_c) + M_1(f+f_c)] + \frac{A_c}{2j}[M_2(f-f_c) - M_2(f+f_c)]$$

where $M_1(f) = \mathbf{F}(m_1(t)]$ and $M_2(f) = \mathbf{F}(m_2(t)]$. The spectrum of the received signal is therefore

$$R(f) = H(f)S(f)$$

$$= \frac{A_c}{2}H(f)\left[M_1(f-f_c) + M_1(f+f_c) + \frac{1}{j}M_2(f-f_c) - \frac{1}{j}M_2(f+f_c)\right]$$

To recover $m_1(t)$, we multiply $r(t)$ [i.e., the inverse Fourier transform of $R(f)$] by $\cos(2\pi f_c t)$ and then pass the resulting output through a low-pass filter, which is designed to have a cutoff frequency equal to the message bandwidth W. The signal produced at the filter output has the following spectrum

$$\mathbf{F}[r(t)\cos(2\pi f_c t)] = \frac{1}{2}[R(f-f_c) + R(f+f_c)]$$

$$= \frac{A_c}{4}H(f-f_c)[M_1(f-2f_c) + M_1(f) + \frac{1}{j}M_2(f-2f_c) - \frac{1}{j}M_2(f)]$$

$$+ \frac{A_c}{4}H(f+f_c)\left[M_1(f) + M_1(f+2f_c) + \frac{1}{j}M_2(f) - \frac{1}{j}M_2(f+2f_c)\right] \quad (1)$$

The condition $H(f_c+f) = H^*(f_c-f)$ is equivalent to $H(f+f_c) = H(f-f_c)$; this follows from the fact that for a real-valued impulse response $h(t)$, we have $H(-f) = H^*(f)$. Hence, substituting this condition in Eq. (1), we get

$$\mathbf{F}[r(t)\cos(2\pi f_c t)] = \frac{A_c}{2}H(f-f_c)M_1(f)$$

$$+ \frac{A_c}{4}H(f-f_c)\left[M_1(f-2f_c) + \frac{1}{j}M_2(f-2f_c) + M_1(f+2f_c) - \frac{1}{j}M_2(f+2f_c)\right]$$

The low-pass filter output therefore has a spectrum equal to $(A_c/2)H(f-f_c)M_1(f)$.

Similarly, to recover $m_2(t)$, we multiply $r(t)$ by $\sin(2\pi f_c t)$, and then pass the resulting signal through a low-pass filter. In this case, we get an output with a spectrum equal to $(A_c/2)H(f-f_c)M_2(f)$.

Problem 3.31

(a) The SSB wave $s_u(t)$ is defined by

$$s_u(t) = \frac{A_c}{2}[m(t)\cos(2\pi f_c t) - \hat{m}(t)\sin(2\pi f_c t)] \qquad (1)$$

and its Hilbert transform is defined by

$$s_u(t) = \frac{A_c}{2}[m(t)\sin(2\pi f_c t) - \hat{m}(t)\cos(2\pi f_c t)] \qquad (2)$$

In Eq. (2), we have used the following properties of the Hilbert transform:

(a) The Hilbert transform of $m(t)\cos(2\pi f_c t)$ is $m(t)\sin(2\pi f_c t)$

(b) The Hilbert transform of $\hat{m}(t)\sin(2\pi f_c t)$ is $-\hat{m}(t)\cos(2\pi f_c t)$

We may therefore use Eqs. (1) and (2) to write

$$s_u(t) = \cos(2\pi f_c t) = \frac{A_c}{2}[m(t)\cos^2(2\pi f_c t) - \hat{m}(t)\sin(2\pi f_c t)\cos(2\pi f_c t)] \qquad (3)$$

$$s_u(t)\sin(2\pi f_c t) = \frac{A_c}{2}[m(t)\sin^2(2\pi f_c t) + \hat{m}(t)\sin(2\pi f_c t)\cos(2\pi f_c t)] \qquad (4)$$

Adding Eqs. (3) and (4) and solving for $m(t)$, we get

$$m(t) = \frac{A_c}{2}[s_u(t)\cos(2\pi f_c t) + \hat{s}_u(t)\sin(2\pi f_c t)] \qquad (5)$$

Next, we use Eqs. (1) and (2) to write

$$s_u(t)\sin(2\pi f_c t) = \frac{A_c}{2}[m(t)\cos(2\pi f_c t)\sin(2\pi f_c t) - \hat{m}(t)\sin^2(2\pi f_c t)] \qquad (6)$$

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Problem 3-31 continued

$$s_u(t)\cos(2\pi f_c t) = \frac{A_c}{2}[m(t)\sin(2\pi f_c t)\cos(2\pi f_c t) + \hat{m}(t)\cos^2(2\pi f_c t)] \quad (7)$$

Subtracting Eq. (6) from Eq. (7) and then solving for $\hat{m}(t)$, we get

$$\hat{m}(t) = \frac{2}{A_c}[\hat{s}_u(t)\cos(2\pi f_c t) - s_u(t)\sin(2\pi f_c t)] \quad (8)$$

Equations (5) and (8) are the desired results for part (a) of the problem.

(b) The SSB wave $s_l(t)$ is defined by

$$s_l(t) = \frac{A_c}{2}[m(t)\cos(2\pi f_c t) + \hat{m}(t)\sin(2\pi f_c t)] \quad (9)$$

and its Hilbert transform is defined by

$$\hat{s}_l(t) = \frac{A_c}{2}[m(t)\sin(2\pi f_c t) - \hat{m}(t)\cos(2\pi f_c t)] \quad (10)$$

where again we have made use of the above-mentioned properties of the Hilbert transform. Therefore, using Eqs. (9) and (10) we write

$$s_l(t)\cos(2\pi f_c t) = \frac{A_c}{2}[m(t)\cos^2(2\pi f_c t) + \hat{m}(t)\sin(2\pi f_c t)\cos(2\pi f_c t)] \quad (11)$$

$$\hat{s}_l(t)\sin(2\pi f_c t) = \frac{A_c}{2}[m(t)\sin^2(2\pi f_c t) - \hat{m}(t)\cos(2\pi f_c t)\sin(2\pi f_c t)] \quad (12)$$

Adding Eqs. (11) and (12) and then solving for $m(t)$, we get

$$m(t) = \frac{2}{A_c}[s_l(t)\cos(2\pi f_c t) + \hat{s}_l(t)\sin(2\pi f_c t)] \quad (13)$$

Next, we use Eqs. (11) and (12) to write

$$s_l(t)\sin(2\pi f_c t) = \frac{A_c}{2}[m(t)\cos(2\pi f_c t)\sin(2\pi f_c t) + \hat{m}(t)\sin^2(2\pi f_c t)] \quad (14)$$

$$\hat{s}_l(t)\cos(2\pi f_c t) = \frac{A_c}{2}[m(t)\sin(2\pi f_c t)\cos(2\pi f_c t) - \hat{m}(t)\cos^2(2\pi f_c t)] \quad (15)$$

Subtracting Eq. (15) from Eq. (14) and then solving for $\hat{m}(t)$, we get

$$\hat{m}(t) = \frac{2}{A_c}[s_l(t)\sin(2\pi f_c t) - \hat{s}_l(t)\cos(2\pi f_c t)] \quad (16)$$

Equations (13) and (16) are the desired results for part (b) of the problem.

(c) From Eqs. (15) and (16), we see that the message signal $m(t)$ may be recovered from $s_u(t)$ or $s_l(t)$ by using the scheme shown in Fig. 1.

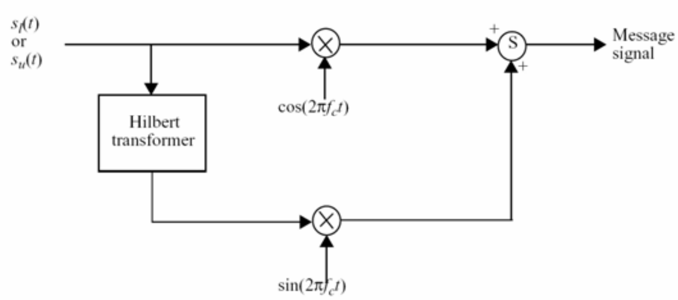

Figure 1

Problem 3.32

We will approach the solution to this problem by showing that, as postulated in the problem, if the in-phase component $H_I(f)$ of the complex low-pass filter's transfer function and its quadrature component $H_Q(f)$ satisfy the following relations

$$H_I(f) = 1 \qquad \text{for } -W \le f \le W \tag{1}$$

and

$$H_Q(-f) = -H_Q(f) \qquad \text{for } -W \le f \le W \tag{2}$$

then, starting with the frequency-discrimination basis for generating a VSB modulated wave $s(t)$, we may express $s(t)$ containing a vestige of the lower sideband as follows:

$$s(t) = \frac{A_c}{2}m(t)\cos(2\pi f_c t) - \frac{A_c}{2}m'(t)\sin(2\pi f_c t) \tag{3}$$

where $m'(t)$ is obtained by passing the message signal $m(t)$ through the quadrature filter defined by $H_Q(f)$.

To proceed, from Eq. (3.44) in the text, recall the relation

$$\frac{1}{2}\tilde{H}(f - f_c) = H(f), \qquad f > 0 \tag{4}$$

The corresponding relation for negative frequencies is described by

$$\frac{1}{2}\tilde{H}^*(f + f_c) = H(f), \qquad f < 0 \tag{5}$$

Using frequency discrimination as the basis for generating the VSB modulated wave $s(t)$, we express the spectrum of $s(t)$ as

$$S(f) = \frac{A_c}{2}[M(f - f_c) + M(f + f_c)]H(f) \tag{6}$$

where $M(f) = \mathbf{F}[s(t)]$. Next, using Eqs., (4) and (5) in (6), we write

$$\begin{aligned} S(f) &= \frac{A_c}{4}[M(f - f_c) + M(f + f_c)][\tilde{H}(f - f_c)\tilde{H}^*(f + f_c)] \\ &= \frac{A_c}{4}M(f - f_c)\tilde{H}(f - f_c) + \frac{A_c}{4}M(f + f_c)\tilde{H}^*(f + f_c) \end{aligned} \tag{7}$$

Continued on next slide

where it is recognized that the cross-product terms

$M(f-f_c)\tilde{H}^*(f+f_c)$ and $M(f+f_c)\tilde{H}^*(f-f_c)$ are both zero, because the individual factors in each product term occupy completely disjoint frequency bands. Setting

$$\tilde{H}(f) = H_I(f) + jH_Q(f)$$

and

$$\tilde{H}^*(f) = H_I(f) - jH_Q(f)$$

we expand Eq. (7) as

$$S(f) = \frac{A_c}{4}[M(f-f_c)H_I(f-f_c) + M(f+f_c)H_I(f+f_c)]$$
$$+ j\frac{A_c}{4}[M(f-f_c)H_Q(f-f_c) - M(f+f_c)H_Q(f+f_c)] \quad (8)$$

Using the all-pass property of $H_I(f)$ defined in Eq. (1) and the odd-function property of $H_Q(f)$ defined in Eq. (2), we may simplify Eq. (8) as

$$S(f) = \frac{A_c}{4}[M(f-f_c) + M(f+f_c)]$$
$$+ j\frac{A_c}{4}[M(f-f_c) - M(f+f_c)]H_Q(f) \quad (9)$$

Transforming Eq. (9) into the time domain, we obtain the formula of Eq. (3) for the VSB modulated wave $s(t)$.

As noted earlier, $m'(t)$ is obtained by passing the message signal $m(t)$ through the quadrature filter. In accordance with the description of $H_Q(f)$ depicted in the problem, we may depict the frequency response of the quadrature filter as in Fig. 1, where f_v denotes the vestigial bandwidth.

The important point to note from the solution to this problem is that Eq. (3) includes SSB modulation as a special case. Specifically, if $f_v = 0$, then the frequency response depicted in Fig. 1 simplifies to a signum function. Correspondingly, Eq. (3) reduces to a SSB modulated wave containing the upper sideband.

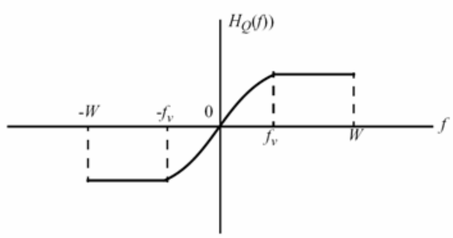

Figure 1

Problem 4.1.
Using Eq. (4.7), show that FM waves also violate the principle of superposition.

Solution
From Eq. (4.7), the FM wave is defined by

$$s(t) = A_c \cos\left[2\pi f_c t + 2\pi k_f \int_0^t m(\tau)d\tau\right]$$

Suppose $m(t) = m_1(t) + m_2(t)$. Then,

$$s(t) = A_c \cos\left[2\pi f_c t + 2\pi k_f \int_0^t m_1(\tau)d\tau + 2\pi k_f \int_0^t m_2(\tau)d\tau\right] \quad (1)$$

Suppose next the two message signals $m_1(t)$ and $m_2(t)$ are applied individually to the frequency modulator. Then in response to $m_1(t)$, we have

$$s_1(t) = A_c \cos\left[2\pi f_c t + 2\pi k_f \int_0^t m_1(\tau)d\tau\right] \quad (2)$$

Likewise, for $m_2(t)$ we have

$$s_2(t) = A_c \cos\left[2\pi f_c t + 2\pi k_f \int_0^t m_2(\tau)d\tau\right] \quad (3)$$

From Eqs. (1) through (3), we readily see that

$$s(t) \neq s_1(t) + s_2(t)$$

In other words, the principle of superposition (basic to linear systems) is violated. Hence, frequency modulation is a nonlinear process.

Problem 4.2

Suppose that the linear modulating wave

$$m(t) = \begin{cases} at & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

is applied to the scheme shown in Fig. 4.3(a). The phase modulator is defined by Eq. (4.4). Show that if the resulting FM wave is to have exactly the form as that defined in Eq. (4.7), then the phase-sensitivity factor k_p of the phase modulator is related to the frequency sensitivity factor k_f in Eq. (4.7) by the formula

$$k_p = 2\pi k_f T$$

where T is the interval over which the integration in Fig. 4.3(a) is performed. Justify the dimensionality of this expression.

Solution

According to Fig. 4.3(a), the FM wave is defined by

$$s(t) = A_c \cos\left[2\pi f_c t + \frac{1}{T} k_p \int_0^t m(\tau)d\tau\right] \qquad (1)$$

where T is an integration constant.

According to Eq. (4.7), the FM wave is defined by

$$s(t) = A_c \cos\left[2\pi f_c t + 2\pi k_f \int_0^t m(\tau)d\tau\right] \qquad (2)$$

If Eqs. (1) and (2) are to be identical, then we require that

$$k_p = 2\pi k_f T$$

Dimensionality of this expression is justified as follows:

1. k_f is measured in hertz per volt. Therefore, $2\pi k_f T$ has the dimensions of cycles, per volt and therefore radians per volt.
2. k_p is itself measured in radians per volt.

Problem 4.3

The *Cartesian baseband representation* of band-pass signals discussed in Section 3.8.1 is well-suited for linear modulation schemes exemplified by the amplitude modulation family. On the other hand, the *polar baseband representation*

$$s(t) = a(t)\cos[2\pi f_c t + \phi(t)]$$

is well-suited for nonlinear modulation schemes exemplified by the angle modulation family. The $a(t)$ in this new representation is the envelope of $s(t)$ and $\phi(t)$ is its phase.

Starting with the baseband representation [see Eq. (3.39)]

$$s(t) = s_I(t)\cos 2\pi f_c t - s_Q(t)\sin(2\pi f_c t)$$

where $s_I(t)$ is the in-phase component and $s_Q(t)$ is the quadrature component, we may write

$$a(t) = [s_I^2(t) + s_Q^2(t)]^{1/2}$$

and

$$\phi(t) = \tan^{-1}\left[\frac{s_Q(t)}{s_I(t)}\right]$$

Show that the polar representation of $s(t)$ in terms of $a(t)$ and $\phi(t)$ is exactly equivalent to its Cartesian representation in terms of $s_I(t)$ and $s_Q(t)$.

Solution

We are given

$$a(t) = [s_I^2(t) + s_Q^2(t)]^{1/2}$$

and

$$\phi(t) = \tan^{-1}\left[\frac{s_Q(t)}{s_I(t)}\right]$$

Hence, expanding the polar representation of $s(t)$, we write

$$\begin{aligned} s(t) &= a(t)\cos[\theta t] \\ &= a(t)\cos[2\pi f_c t + \phi(t)] \end{aligned}$$

Continued on next slide

Problem 4-3 continued

$$= a(t)\cos(\phi(t))\cos(2\pi f_c t) - a(t)\sin(\phi(t))\sin(2\pi f_c t) \quad (1)$$

Since $\tan[\phi(t)] = \left[\frac{s_Q(t)}{s_I(t)}\right]$, it follows that

$$\cos\phi(t) = \frac{s_I(t)}{[s_I^2(t) + s_Q^2(t)]^{1/2}} = \frac{s_I(t)}{a(t)}$$

and

$$\sin\phi(t) = \frac{s_Q(t)}{[s_I^2(t) + s_Q^2(t)]^{1/2}} = \frac{s_Q(t)}{a(t)}$$

Hence,

$$a(t)\cos\phi(t) = s_I(t) \quad (2)$$

and

$$a(t)\sin\phi(t) = s_Q(t) \quad (3)$$

Substituting Eqs. (2) and (3) into (1), we get

$$s(t) = s_I(t)\cos(2\pi f_c t) - s_Q(t)\sin(2\pi f_c t)$$

which is the Cartesian representation of $s(t)$.

Problem 4.4

Consider the narrow-band FM wave approximately defined by Eq. (4.17). Building on Problem 4.3, do the following:

(a) Determine the envelope of this modulated wave. What is the ratio of the maximum to the minimum value of this envelope?

(b) Determine the average power of the narrow-band FM wave, expressed as a percentage of the average power of the unmodulated carrier wave.

(c) By expanding the angular argument $\theta(t) = 2\pi f_c t + \phi(t)$ of the narrow-band FM wave $s(t)$ in the form of a power series and restricting the modulation index β to a maximum value of 0.3 radian, show that

$$\theta(t) \approx 2\pi f_c t + \beta \sin(2\pi f_m t) - \frac{\beta^3}{3}\sin^3(2\pi f_m t)$$

What is the value of the harmonic distortion for $\beta = 0.3$ radian?

Hint: For small x, the following power series approximation

$$\tan^{-1}(x) \approx x - \frac{1}{3}x^3$$

holds. In this approximation, terms involving x^5 and higher order ones are ignored, which is justified when x is small compared to unity.

Solution

(a) From Eq. (4.17), the narrow-band FM wave is approximately defined by

$$s(t) \approx A_c \cos((2\pi f_c t) - \beta A_c \sin(2\pi f_c t)\sin(2\pi f_m t)) \qquad (1)$$

The envelope of $s(t)$ is therefore

$$a(t) = A_c(1 + \beta^2 \sin^2(2\pi f_m t))^{1/2}$$

$$\approx A_c\left(1 + \frac{1}{2}\beta^2 \sin^2(2\pi f_m t)\right)^{1/2} \qquad \text{for small } \beta$$

The maximum value of $a(t)$ occurs when $\sin^2(2\pi f_m t) = 1$, yielding

$$A_{\max} \approx A_c\left(1 + \frac{1}{2}\beta^2\right)$$

The minimum value of $a(t)$ occurs when $\sin^2(2\pi f_m t) = 0$, yielding

$$A_{\min} = A_c$$

The ratio of the maximum to the minimum value is therefore

$$\frac{A_{\max}}{A_{\min}} \approx \left(1 + \frac{1}{2}\beta^2\right)$$

Continued on next slide

Problem 4-4 continued

(b) Expanding Eq. (1) into its individual frequency components, we may write

$$s(t) \approx A_c\cos(2\pi f_c t) + \frac{1}{2}\beta A_c\cos(2\pi(f_c + f_m)t) - \frac{1}{2}\beta A_c\cos(2\pi(f_c - f_m)t)$$

The average power of $s(t)$ is therefore

$$P_{av} = \frac{1}{2}A_c^2 + \left(\frac{1}{2}\beta A_c\right)^2 + \left(\frac{1}{2}\beta A_c\right)^2$$

$$= \frac{1}{2}A_c^2(1 + \beta^2)$$

The average power of the unmodulated carrier is

$$P_c = \frac{1}{2}A_c^2$$

Hence,

$$\frac{P_{av}}{P_c} = 1 + \beta^2$$

(c) The angle $\theta(t)$ is defined by

$$\theta(t) = 2\pi f_c t + \phi(t)$$

$$= 2\pi f_c t + \tan^{-1}(\beta\sin(2\pi f_m t))$$

Setting $\beta = \sin(2\pi f_m t)$

and using the approximation (based on the Hint), we may approximate $\theta(t)$ as

$$\theta(t) \approx 2\pi f_c t + \beta\sin(2\pi f_m t) - \frac{1}{3}\beta^3\sin(2\pi f_m t)$$

Ideally, we should have (see Eq. (4.15))

$$\theta(t) = 2\pi f_c t + \beta\sin(2\pi f_m t)$$

The harmonic distortion produced by using the narrow-band approximation is therefore

$$D(t) = \frac{\beta^3}{3}\sin^3(2\pi f_m t)$$

The maximum absolute value of $D(t)$ for $\beta = 0.3$ is therefore

$$D_{max} = \frac{\beta^3}{3}$$

$$= \frac{0.3^3}{3} = 0.009 \approx 1\%$$

which is small enough for it to be ignored in practice.

Problem 4.5

Strictly speaking, the FM wave of Eq. (4.15) produced by a sinusoidal modulating wave is a nonperiodic function of time t. Demonstrate this property of frequency modulation.

Solution

Starting with Eq. (4.15) we write

$$s(t) = A_c[\cos(2\pi f_c t) + \beta\sin(2\pi f_m t)] \tag{1}$$

For the FM wave $s(t)$ to be a periodic function of time, we require that the condition

$$s\left(t + \frac{1}{f_m}\right) = s(t) \tag{2}$$

be satisfied for a period equal to $1/f_m$. Replacing t with $t + (1/f_m)$ in Eq. (1), we write

$$s\left(t + \frac{1}{f_m}\right) = A_c\cos\left[2\pi f_c\left(t + \frac{1}{f_m}\right) + \beta\sin\left(2\pi f_m t + \frac{1}{f_m}\right)\right]$$

$$= A_c\cos[2\pi f_c + (2\pi f_c / f_m) + \beta\sin(2\pi f_m t + 2\pi)]$$

$$= A_c\cos[2\pi f_c + (2\pi f_c / f_m) + \beta\sin(2\pi f_m t)] \tag{3}$$

In general, the carrier frequency f_c is a noninteger multiple of the modulation frequency f_m. Accordingly, $s(t + (1/f_m)) \neq s(t)$ and therefore the condition of Eq. (2) for periodicity is violated.

Problem 4.6
Using a well-known trigonometric identity involving the product of the sine of an angle and the cosine of another angle, demonstrate the two results just described under points 1 and 2.

Solution
The incoming FM wave is defined by (see Eq. (4.57))

$$s(t) = A_v \cos[2\pi f_c t + \phi_1(t)] \quad (1)$$

The internally generated output of the VCO is defined by (see Eq. (4.59))

$$r(t) = A_v \cos[2\pi f_c t + \phi_2(t)] \quad (2)$$

Multiplying $s(t)$ by $r(t)$ yields

$$s(t)r(t) = A_c A_v \sin[2\pi f_c t + \phi_1(t)] \cos[2\pi f_c t + \phi_2(t)] \quad (3)$$

Using the trigonometric identity

$$\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$$

we may rewrite Eq. (3) as

$$s(t)r(t) = \frac{1}{2}A_c A_v \sin[4\pi f_c t + \phi_1(t) + \phi_2(t)] + \frac{1}{2}A_c A_v \sin[\phi_1(t) - \phi_2(t)] \quad (4)$$

Except for a scaling factor, the first term of Eq. (4) defines the double-frequency term (identified under point 1 on page 179) and the second term of the equation defines the difference-frequency term (identified under point 2 of the same page).

Problem 4.7

Using the linearized model of Fig. 4.15(a), show that the model is approximately governed by the integro-differential equation

$$\frac{d\phi_e(t)}{dt} + 2\pi K_0\int_{-\infty}^{\infty}\phi_e(\tau)h(t-\tau)d\tau \approx \frac{d\phi_1(t)}{dt}$$

Hence, derive the following two approximate results in the frequency domain:

(a) $\Phi_e(f) = \frac{1}{1+L(f)}\Phi_1(f)$

(b) $V(f) = \frac{jf}{k_v}\,\frac{L(f)}{1+L(f)}\Phi_1(f)$

where

$$L(f) = K_0\frac{H(f)}{jf}$$

is the open-loop transfer function. Finally, show that when *L*(*f*) is large compared with unity for all frequencies inside the message band, the time-domain version of the formula in part (b) reduces to the approximate form in Eq. (4.68).

Solution

According to condition 1 stated on p.178 of the text, the frequency of the VCO is set equal to the carrier frequency f_c. According to condition 2 on the same page, the VCO output has a 90° phase shift with respect to the unmodulated carrier. In light of these two conditions, we note starting with the equation

$$\frac{d\phi_e(t)}{dt} + 2\pi K_0\int_{-\infty}^{\infty}\phi_e(\tau)h(t-\tau)d\tau \approx \frac{d\phi_1(t)}{dt}$$

the integral in the left-hand side of the equation is the convolution of $\phi_e(t)$ and $h(t)$. Therefore, applying the Fourier transform to this equation and using two properties of the fourier transform pertaining to differentiation and convolution, we get

$$j2\pi f\,\Phi_e(f) + 2\pi K_0\Phi_e(f)H(f) \approx j2\pi f\,\Phi_1(f) \qquad (1)$$

where

$\Phi_e(f) = \mathbf{F}[\phi_e(t)]$ and $\Phi_1(f) = \mathbf{F}[\phi_1(t)]$

(a) Solving Eq. (1) for $\Phi_e(f)$, we get

$$\Phi_e(f) \approx \frac{j2\pi f}{j2\pi f + 2\pi K_0 H(f)}\Phi_1(f)$$

$$= \frac{1}{1+K_0\frac{H(f)}{jf}}\Phi_1(f)$$

$$= \frac{1}{1+L(f)}\Phi_1(f) \qquad (2)$$

where $L(f) = \frac{H(f)}{jf}$

Continued on next slide

Problem 4-7 continued

(b) Next, from Eq. (4.63) we have

$$e(t) = \frac{K_0}{k_v}\phi_e(t)$$

Therefore

$$E(f) = \frac{K_0}{k_v}\Phi_e(f)$$

And, from Eq. (4.65) we have

$$v(t) = \int_{-\infty}^{\infty} e(\tau)h(t-\tau)d\tau$$

Therefore

$$V(f) = E(f)H(f)$$

Eliminating $E(f)$ between these two transform-related equations, we get

$$V(f) = \frac{K_0}{k_v}H(f)\Phi_e(f) \qquad (3)$$

Eliminating $\Phi_e(f)$ between Eqs. (1) and (3), we get

$$V(f) = \frac{K_0}{k_v}H(f)\cdot\frac{1}{1+L(f)}\Phi_1(f)$$

Since

$$L(f) = K_0\frac{H(f)}{jf}$$

then

$$\frac{K_0}{k_v}H(f) = \frac{jf}{k_v}L(f)$$

and so we get the desired result

$$V(f) \approx \frac{jf}{k_v}\ \frac{L(f)}{1+L(f)}\Phi_1(f) \qquad (4)$$

Finally, when $L(f) >> 1$ for all f, Eq. (4) simplifies further as

$$(f) \approx \frac{jf}{k_v}\Phi_1(f) \approx \frac{j2\pi f}{2\pi k_v}\Phi_1(f)$$

The time-domain version of this formula reads as follows

$$v(t) \approx \frac{1}{2\pi k_v}\ \frac{d\phi_1(t)}{dt}$$

which is a repeat of Eq. (4.67).

Problem 4.8

For the PM case, we have by definition
$s(t) = A_c \cos[2\pi f_c t + k_p m(t)]$.
whose angle is
$\theta_i(t) = 2\pi f_c t + k_p m(t)$.
The instantaneous frequency is therefore

$$
\begin{aligned}
f_i(t) &= \frac{1}{2\pi} \frac{d\theta_i(t)}{dt}, \\
&= f_c + \frac{k_p}{2\pi} \frac{dm(t)}{dt} \\
&= f_c + \frac{Ak_p}{2\pi T_0} - \frac{Ak_p}{2\pi} \sum_n \delta(t - nT_0) \qquad (1)
\end{aligned}
$$

which is equal to $f_c + Ak_p/2\pi T_0$ except for the instants that the message signal has discontinuities. At these instants, the phase shifts by $-k_p A/T_0$ radians.Accordingly, the PM wave has the waveform depicted in Fig. 1

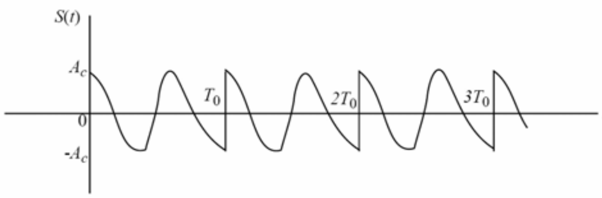

Figure 1

For the FM case, we have
$f_i(t) = f_c + k_f m(t)$
and the modulated wave is defined by

$$s(t) = A_c \cos\left[2\pi f_c t + 2\pi k_f \int_0^t m(\tau) dt\right]$$

The modulated wave is therefore depicted in Fig. 2.

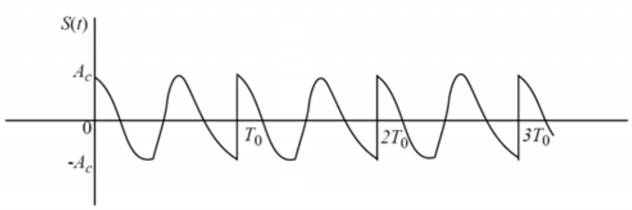

Figure 2

Problem 4.9

The instantaneous frequency of the mixer output is as shown in Fig. 1:

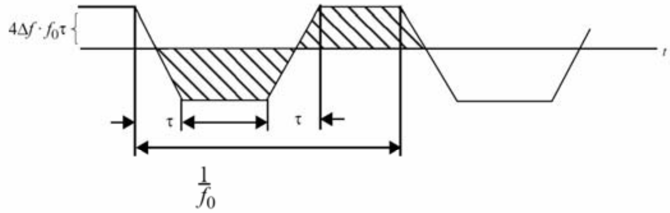

Figure 1

The presence of negative frequency merely indicates that the phasor representing the difference frequency at the mixer output has reversed its direction of rotation.

Let N denote the number of beat cycles in one period. Then, noting that N is equal to the number of shaded areas shown in Fig. 1, we deduce that

$$N = 2\left[4\Delta f \cdot f_0\tau\left(\frac{1}{2f_0} - \tau\right) + 2\Delta f \cdot f_0\tau^2\right]$$

$$= 4\Delta f \cdot \tau(1 - f_0\tau)$$

Since $f_0\tau << 1$, we have the approximate result

$$N \approx 4\Delta f \cdot \tau$$

Therefore, the number of beat cycles counted over one second is equal to

$$\frac{N}{1/f_0} = 4\Delta f \cdot f_0\tau.$$

Problem 4.10

By definition, the instantaneous frequency f_i is related to the phase $\theta(t)$ as

$$f_i = \frac{1}{2\pi}\frac{d\theta}{dt}$$

which may be rewritten as

$$f_i \approx \frac{1}{2\pi}\frac{\Delta\theta}{\Delta t} \qquad (1)$$

where $\Delta\theta$ and Δt are small changes in the phase $\theta(t)$ and time t. We are given

$$\theta(t+\Delta t) - \theta(t) = \pi$$

from which we infer that

$$\Delta\theta = \pi \qquad (2)$$

Substituting Eq. (2) into (1) yields

$$f_i \approx \frac{1}{2\pi}\cdot\frac{\Delta\theta}{\Delta t} = \frac{1}{2\Delta t}$$

which is the desired result.

Problem 4.11

The phase-modulated wave is defined by

$$
\begin{aligned}
s(t) &= A_c\cos[2\pi f_c t + k_p A_m \cos(2\pi f_m t)] \\
&= A_c\cos[2\pi f_c t + \beta_p \cos(2\pi f_m t)], \qquad \beta_p = k_p A_m \\
&= A_c\cos(2\pi f_c t)\cos[\beta_p\cos(2\pi f_m t)] - A_c\sin(2\pi f_c t)\sin[\beta_p\cos(2\pi f_m t)] \qquad (1)
\end{aligned}
$$

If $\beta_p \le 0.3$, then for all time t we approximately have

$$\cos[\beta_p\cos(2\pi f_m t)] \approx 1$$

$$\sin[\beta_p\cos(2\pi f_m t)] \approx \beta_p\cos(2\pi f_m t)$$

Correspondingly, we may approximate Eq. (1) as follows:

$$
\begin{aligned}
s(t) &\approx A_c\cos(2\pi f_c t) - \beta_p A_c \sin(2\pi f_c t)\cos(2\pi f_m t) \\
&= A_c\cos(2\pi f_c t) - \frac{1}{2}\beta_p A_c \sin[2\pi(f_c + f_m)t] - \frac{1}{2}\beta_p A_c\sin[2\pi(f_c - f_m)t] \qquad (2)
\end{aligned}
$$

The spectrum of $s(t)$ is therefore

$$
\begin{aligned}
S(f) \approx & \frac{1}{2}A_c[\delta(f - f_c) + \delta(f + f_c)] \\
& -\frac{1}{4j}\beta_p A_c[\delta(f - f_c - f_m) - \delta(f + f_c + f_m)] \\
& -\frac{1}{4j}\beta_p A_c[\delta(f - f_c + f_m) - \delta(f + f_c - f_m)]
\end{aligned}
$$

Problem 4.12

(a) From Table A3.1 in Appendix 3, we find (by interpolation) that $J_0(\beta)$ is zero for the following values of modulation index:
$\beta = 2.44$,
$\beta = 5.52$,
$\beta = 8.65$,
$\beta = 11.8$,
and so on.

(b) The modulation index is defined by

$$\beta = \frac{\Delta f}{f_m} = \frac{k_f A_m}{f_m}$$

Therefore, the frequency sensitivity factor is

$$k_f = \frac{\beta f_m}{A_m} \tag{1}$$

We are given $f_m = 1$ kHz and $A_m = 2$ volts. Hence, with $J_0(\beta) = 0$ for the first time when $\beta = 2.44$, the use of Eq. (1) yields

$$k_f = \frac{2.44 \times 10^3}{2}$$

$$= 1.22 \times 10^3 \text{ hertz/volt}$$

Next, we note that $J_0(\beta) = 0$ for the second time when $\beta = 5.52$. Hence, the corresponding value of A_m for which the carrier component is reduced to zero is

$$A_m = \frac{\beta f_m}{k_f}$$

$$= \frac{5.52 \times 10^3}{1.22 \times 10^3}$$

$$= 4.52 \text{ volts}$$

Problem 4.13

(a) The frequency deviation is

$$\Delta f = k_f A_m = 25 \times 10^3 \times 20 = 5 \times 10^5 \text{Hz}$$

The corresponding value of the modulation index is

$$\beta = \frac{\Delta f}{f_m} = \frac{5 \times 10^5}{10^5} = 5$$

Using Carson's rule, the transmission bandwidth of the FM wave is therefore

$$B_T = 2f_m(1 + \beta) = 2 \times 100(1 + 5) = 1200\text{kHz} = 1.2\text{MHz}$$

(b) Using the universal curve of Fig. 4.9, we find that for $\beta = 5$:

$$\frac{B_T}{\Delta f} = 3$$

Therefore, the transmission bandwidth is

$$B_T = 3 \times 500 = 1500\text{kHz} = 1.5\text{MHz}$$

which is greater than the value calculated by Carson's rule.

(c) If the amplitude of the modulating wave is doubled, we find that

$$\Delta f = 1\text{MHz} \text{ and } \beta = 10$$

Thus, using Carson's rule we now obtain the transmission bandwidth

$$B_T = 2 \times 100(1 + 10) = 2200\text{kHz} = 2.2\text{MHz}$$

On the other hand, using the universal curve of Fig. 4.9, we get

$$\frac{B_T}{\Delta f} = 2.75$$

and $B_T = 2.75$ MHz.

(d) If f_m is doubled, $\beta = 2.5$. Then, using Carson's rule, $B_T = 1.4$ MHz. Using the universal curve, $(B_T/\Delta f) = 4$, and

$$B_T = 4\Delta f = 2\text{MHz}$$

Problem 4.14

(a) The angle of the PM wave is defined by

$$\begin{aligned}\theta_i(t) &= 2\pi f_c t + k_p m(t) \\ &= 2\pi f_c t + k_p A_m \cos(2\pi f_m t) \\ &= 2\pi f_c t + \beta_p \cos(2\pi f_m t)\end{aligned}$$

where $\beta_p = k_p A_m$. The instantaneous frequency of the PM wave is therefore

$$\begin{aligned}f_i(t) &= \frac{1}{2\pi} \frac{d\theta_i(t)}{dt} \\ &= f_c - \beta_p f_m \sin(2\pi f_m t) \qquad (1)\end{aligned}$$

Based on Eq. (1), we see that the maximum frequency deviation in a PM wave varies linearly with the modulation frequency f_m.

Using Carson's rule, we find that the transmission bandwidth of the PM wave is approximately (for the case when β_p is small compared to unity)

$$B_T \approx 2(f_m + \beta_p f_m) = 2f_m(1 + \beta_p) \approx 2f_m \beta_p. \qquad (2)$$

Equation (2) shows that B_T varies linearly with the modulation frequency f_m.

(b) In an FM wave, the transmission bandwidth B_T is approximately equal to $2\Delta f$, assuming that the modulation index β is small compared to unity. Therefore, for an FM wave, B_T is effectively independent of the modulation frequency f_m.

Problem 4.15

Consider first the action of the mixer with the two inputs: voltage-controlled oscillator (VCO) output and crystal oscillator output. The mixer produces an output of its own whose frequency is the difference between the instantaneous frequency of the VCO and the crystal oscillator frequency.

The mixer output is applied to the frequency discriminator followed by a low-pass filter. By design, the output produced by the frequency discriminator has an instantaneous amplitude that is proportional to the instantaneous frequency of the FM signal applied to its input. Accordingly, the amplitude of the signal produced by the frequency discriminator is proportional to the difference between the VCO frequency and the crystal oscillator frequency.

In light of these considerations, we may now make the following statements:

- When the FM signal $s(t)$ produced at the VCO output has exactly the correct frequency, the low-pass filter output is zero.
- Deviations in the carrier frequency of the FM signal $s(t)$ from its assigned value will cause the frequency discriminator-filter output to produce a dc output with a polarity determined by the sense of the carrier-frequency drift in the FM signal $s(t)$. This dc signal, after suitable amplification is, in turn, applied to the VCO in such a way as to modify the instantaneous frequency of the VCO in a direction that tends to restore the carrier frequency of the FM signal $s(t)$ to its correct value.

In summary, the application of feedback applied to the VCO in the manner described in Fig. 4.19 has the beneficial effect of stabilizing the carrier frequency of the FM signal produced at the VCO output.

Problem 4.16

From Fig. 4.20, we see that the envelope detector input is

$$v(t) = s(t) - s(t-T)$$
$$= A_c\cos[2\pi f_c t + \phi(t)] - A_c\cos[2\pi f_c(t-T) + \phi(t-T)]$$

Using a well-known trigonometric identity, we write

$$v(t) = -2A_c\sin\left[\frac{2\pi f_c(2t-T) + \phi(t) + \phi(t-T)}{2}\right]\sin\left[\frac{2\pi f_c T + \phi(t) - \phi(t-T)}{2}\right] \qquad (1)$$

For $\phi(t)$, we have

$$\phi(t) = \beta\sin(2\pi f_m t)$$

Correspondingly, the phase difference $\phi(t)$ - $\phi(t - T)$ is given by

$$\phi(t) - \phi(t-T) = \beta\sin(2\pi f_m t) - \beta\sin[2\pi f_m(t-T)]$$
$$= \beta[\sin(2\pi f_m t) - \sin(2\pi f_m t)\cos(2\pi f_m T) + \cos(2\pi f_c t)\sin(2\pi f_m T)] \qquad (2)$$

Using the approximations:

$$\cos(2\pi f_m T) \approx 1$$
$$\sin(2\pi f_m T) \approx 2\pi f_m T$$

we may approximate Eq. (2) as

$$\phi(t) - \phi(t-T) \approx \beta[\sin(2\pi f_m t) - \sin(2\pi f_m t) + 2\pi f_m T\cos(2\pi f_m t)]$$
$$= 2\pi\Delta f T\cos(2\pi f_m t) \qquad (3)$$

where

$$\Delta f = \beta f_m.$$

Therefore, recognizing that $2\pi f_c T = \pi/2$, we may write

$$\sin\left(\frac{2\pi f_c T + \phi(t) - \phi(t-T)}{2}\right) \approx \sin(\pi f_c T + \pi\Delta f T\cos(2\pi f_m t))$$
$$= \sin\left(\frac{\pi}{4} + \pi\Delta f T\cos(2\pi f_m t)\right)$$
$$= \sqrt{2}\cos(\pi\Delta f T\cos(2\pi f_m t)) + \sqrt{2}\sin(\pi\Delta f T\cos(2\pi f_m t))$$
$$= \sqrt{2} + \sqrt{2}\pi\Delta f T\cos(2\pi f_m t)$$

where we have made use of the fact that $\pi\Delta f T << 1$. We may therefore rewrite Eq. (1) as

$$v(t) \approx -2\sqrt{2}A_c(1 + \pi\Delta f T\cos(2\pi f_m t))\sin\left(\pi f_c(2t-T) + \frac{\phi(t) + \phi(t-T)}{2}\right) \qquad (4)$$

Accordingly, the envelope detector output is the envelope of $v(t)$, namely,

$$a(t) \approx 2\sqrt{2}A_c(1 + \pi\Delta f T\cos(2\pi f_m t))$$

which, except for a bias term, is proportional to the modulating wave.

Problem 4.17

Consider first the message signal

$$m_1(t) = \begin{cases} a_1 t + a_0, & t \geq 0 \\ 0, & t = 0 \end{cases}$$

applied to a frequency modulator. The signal produced by this modulator is defined by

$$\begin{aligned} s_1(t) &= A_c \cos\left[2\pi f_c t + 2\pi k_f \int_0^t m_1(\tau) d\tau\right] \\ &= A_c \cos\left[2\pi f_c t + 2\pi k_f \int_0^t (a_1 t + a_0) d\tau\right] \\ &= A_c \cos\left[2\pi f_c t + 2\pi k_f\left(\frac{1}{2} a_1 t^2 + a_0 t + C\right)\right], \qquad t \geq 0 \qquad (1) \end{aligned}$$

where C is the constant of integration.

Consider next the message signal

$$m_2(t) = \begin{cases} b_2 t^2 + b_1 t + b_0 & t \geq 0 \\ 0, & t = 0 \end{cases}$$

applied to a phase modulator. The signal produced by this second modulator is defined by

$$\begin{aligned} s_2(t) &= A_c \cos[2\pi f_c t + k_p m_2(t)] \\ &= A_c \cos[2\pi f_c t + k_p(b_2 t^2 + b_1 t + b_0)], \qquad t \geq 0 \qquad (2) \end{aligned}$$

For the FM signal $s_1(t)$ of Eq. (1) and the PM signal of Eq. (2) to be exactly equal for $t \geq 0$, we require that the following conditions be satisfied:

(i) $\pi k_f a_1 = k_p b_2$

(ii) $2\pi k_f a_0 = k_p b_1$

(iii) $2\pi k_f C = k_p b_0$

Problem 4.18

We are given that the IF filter has a bandwidth of 200 kHz centered on the frequency f_{IF} = 10.7 MHz. This filter will therefore pass frequencies inside the range defined by the two extremes:

low-end: 10.7 - 0.2 = 10.5 MHz

high-end: 10.7 + 0.2 = 10.9 MHz

The image lies inside the band 109.4 to 129.4 MHz, which is positioned well outside the passband of the IF filter. Therefore, the IF filter will suppress the translated band centered on the image frequency f_{image}.

Problem 4.19

The instantaneous frequency of the modulated wave $s(t)$ is shown in Fig. 1

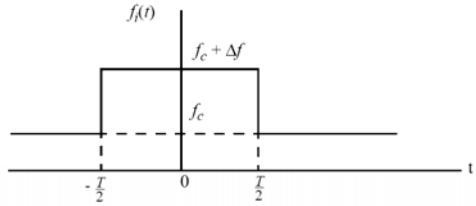

Figure 1

We may thus express $s(t)$ as follows

$$s(t) = \begin{cases} \cos(2\pi f_c t), & t < -\frac{T}{2} \\ \cos[2\pi(f_c + \Delta f)t], & -\frac{T}{2} \le t \le \frac{T}{2} \\ \cos(2\pi f_c t), & \frac{T}{2} < t \end{cases} \tag{1}$$

The Fourier transform of $s(t)$ is therefore

$$\begin{aligned} S(f) &= \int_{-\infty}^{-T/2} \cos(2\pi f_c t)\exp(-j2\pi ft)dt \\ &\quad + \int_{-T/2}^{T/2} \cos[2\pi(f_c + \Delta f)t]\exp(-j2\pi ft)dt \\ &\quad + \int_{T/2}^{\infty} \cos(2\pi f_c t)\exp(-j2\pi ft)dt \\ &= \int_{-\infty}^{\infty} \cos(2\pi f_c t)\exp(-j2\pi ft)dt \\ &\quad + \int_{-T/2}^{T/2} \{\cos[2\pi(f_c + \Delta f)t - \cos(2\pi f_c t)]\}\exp(-j2\pi ft)dt \end{aligned} \tag{2}$$

The second term of Eq. (2) is recognized as the difference between the Fourier transforms of two RF pulses of unit amplitude, one having a frequency equal to $f_c + \Delta f$ and the other having a frequency equal to f_c. Hence, assuming that $f_c T >> 1$, we may express the Fourier transform $S(f)$ of Eq. (2) as follows:

$$S(f) \approx \begin{cases} \frac{1}{2}\delta(f - f_c) + \frac{T}{2}\text{sinc}[T(f - f_c - \Delta f)] - \frac{T}{2}\text{sinc}[T(f - f_c)], & f > 0 \\ \frac{1}{2}\delta(f + f_c) + \frac{T}{2}\text{sinc}[T(f + f_c + \Delta f)] - \frac{T}{2}\text{sinc}[T(f + f_c)], & f < 0 \end{cases} \tag{3}$$

Problem 4.20

The filter input is

$$v_1(t) = g(t)s(t)$$
$$= g(t)\cos(2\pi f_c t - \pi k t^2)$$

The complex envelope of $v_1(t)$ is

$$\tilde{v}_1(t) = g(t)\exp(-j\pi k t^2)$$

The impulse response $h(t)$ of the filter is defined in terms of the complex impulse response $\tilde{h}(t)$ as follows

$$h(t) = \mathbf{Re}[\tilde{h}(t)\exp(j2\pi f_c t)]$$

With $h(t)$ defined by

$$h(t) = \cos(2\pi f_c t + \pi k t^2),$$

we have

$$\tilde{h}(t) = \exp(j\pi k t^2)$$

The complex envelope of the filter output is therefore (except for a scaling factor)[1]

$$\tilde{v}_o(t) = \tilde{h}(t) \star \tilde{v}_i(t)$$
$$= \int_{-\infty}^{\infty} g(\tau)\exp(-j\pi k\tau^2)\exp[j\pi k(t-\tau)]^2 d\tau$$
$$= \exp(j\pi k t^2)\int_{-\infty}^{\infty} g(\tau)\exp(-2j\pi k t\tau)d\tau$$
$$= \exp(j\pi k t^2)G(kt) \qquad (1)$$

where in the last line we have used the definition of the Fourier transform to write

$$G(kt) = \int_{-\infty}^{\infty} g(\tau)\exp(-j2\pi k t\tau)d\tau$$

Hence, from Eq. (1), we obtain the

1. It turns out that the scaling factor equals 1/2; to be exact, we should write
$\tilde{v}_o(t) = \frac{1}{2}\tilde{h}(t) \star \tilde{v}_i(t)$
For details, see the 4th edition of the book:
S. Haykin, Communication Systems, pp. 725-734, 4th edition, Wiley.

$$\tilde{v}_0(t) = |G(kt)| \qquad (2)$$

Equation (2) shows that the envelope of the filter output is, except for a scaling factor, equal to the magnitude of the Fourier transform of the input signal $g(t)$ with kt playing the role of frequency f.

Problem 4.21

For convenience of the discussion, we assume time-domain symmetry around the origin $t = 0$. Accordingly, in theory, the signal produced by the amplitude limiter component of the band-pass limiter due to $s_1(t)$ consists of an infinite sequence of harmonically related angle-modulated components with two properties:

- The components are centered on odd multiples of the carrier frequency f_c.
- The components have progressively decreasing amplitudes.

Typically, the carrier frequency f_c of an FM signal is large compared to the transmission bandwidth B_T of the FM signal. It follows therefore that provided this condition is satisfied, that is, f_c is large enough compared to B_T, then the filter component of the band-pass limiter will effectively suppress all the spectral components coming out of the limiter except for the one component centered on f_c.

In light of these observations that are intuitively satisfying, we may now state that if f_c is large enough compared to B_T, then the output $s_2(t)$ produced by the band-pass limiter in response to the input $s_1(t)$ is defined by the FM signal

$$s_2(t) = A\cos\left[2\pi f_c t + 2\pi k_f \int_0^t m(\tau)d\tau\right]$$

where the amplitude A is a constant.

Problem 4.22

(a) Starting with Eq. (4.15) for sinusoidal FM, we write

$$s(t) = A_c \cos[2\pi f_c t + \beta \sin(2\pi f_m t)] \quad (1)$$

where f_m is the modulation frequency and β is the modulation index. Correspondingly, the Fourier transform of $s(t)$ is defined for an arbitrary value of β (see Eq. (4.31))

$$S(f) = \frac{A_c}{2} \sum_{n=-\infty}^{\infty} J_n(\beta)[\delta(f - f_c - nf_m) + \delta(f + f_c + nf_m)] \quad (2)$$

where $J_n(\beta)$ is the nth order Bessel function of the first kind. Passing $s(t)$ through a linear channel of transfer function $H(f)$ produces an output signal $y(t)$ whose Fourier transform is defined by

$$Y(f) = H(f)S(f)$$

$$= \frac{A_c}{2} \sum_{n=-\infty}^{\infty} J_n(\beta)[H(f_c + nf_m)\delta(f - f_c - nf_m) + H(-f_c - nf_m)\delta(f + f_c + nf_m)] \quad (3)$$

Applying the inverse Fourier transform to $Y(f)$ yields the output signal

$$y(t) = \frac{A_c}{2} \sum_{n=-\infty}^{\infty} J_n(\beta)[H(f_c + nf_m)\exp(j2\pi(f_c + nf_m)t)]$$

$$+ H(-f_c - nf_m)\exp(-j2\pi(f_c + nf_m)t) \quad (4)$$

For a channel with real-valued impulse response, we have $H(f) = H^*(-f)$ where the asterisk denotes complex conjugation. We may therefore rewrite Eq. (4) as

$$y(t) \approx \frac{1}{2} A_c \sum_{n=-\infty}^{\infty} J_n(\beta)[H(f_c + nf_m)\exp(j2\pi(f_c + nf_m)t)]$$

$$+ H^*(f_c + nf_m)\exp(-j2\pi(f_c + nf_m)t)$$

$$= A_c \sum_{n=-\infty}^{\infty} J_n(\beta)\mathbf{Re}[H(f_c + nf_m)\exp(j2\pi(f_c + nf_m)t)] \quad (5)$$

where **Re** denotes the real-time operator.

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Problem 4-22 continued

(b) Following the development of the universal curve plotted in Fig. 4.9, let n_{max} denote the largest value of n in Eq. (5) for which the condition

$|J_n(\beta)| > 0.1$

is satisfied so as to preserve the effective frequency content of the FM signal $s(t)$. We may then approximate Eq. (5) as

$$y(t) \approx A_c \sum_{n=-n_{max}}^{n_{max}} J_n(\beta)\mathbf{Re}[H(f_c + nf_m)\exp(j2\pi(f_c + nf_m)t)] \quad (6)$$

Expressing the transfer function $H(f)$ in the polar form

$$H(f) = |H(f)|\exp(j\phi(f)) \quad (7)$$

we may rewrite Eq. (6) as

$$y(t) \approx A_c \sum_{n=-n_{max}}^{n_{max}} J_n(\beta)|H(f_c + nf_m)|\cos(2\pi(f_c + nf_m)t + \phi(f_c + nf_m)) \quad (8)$$

From the discussion presented in Section 2.7, recall that the transmission of a signal through a linear channel (filter) is distortionless provided that two conditions are satisfied:

(i) The amplitude response $|H(f)|$ is constant over the band $-B \le f \le B$, where B is the channel bandwidth.

(ii) The phase response $\phi(f)$ is a linear function of the frequency f inside the band $-B \le f \le B$.

Accordingly, in the context of our present discussion, the FM transmission through the channel of transfer function $H(f)$ introduces two forms of linear distortion:

(i) *Amplitude distortion*, which arises when the condition

$|H(f_c + nf_m)|$ is constant for $0 \le n \le n_{max}$

is violated.

(ii) *Phase distortion*, when the condition

$\theta(f_c + nf_m)$ is a linear function of n for $0 \le n \le n_{max}$

is violated.

Problem 4.23

(a) Consider the FM version of angle modulation. Let the instantaneous frequency of the modulator be a linear function of the first derivative of the message signal $m(t)$, as shown by

$$f_i(t) = f_c + k_1 \frac{d}{dt} m(t)$$

Then, correspondingly, the instantaneous phase is defined by

$$\theta_i(t) = 2\pi \int_0^t f_i(t) dt$$

$$= 2\pi f_c t + 2\pi k_1 m(t)$$

where it is assumed that $m(0) = 0$. In this scenario, the modulated signal is defined by

$$s(t) = A_c \cos(\theta_i(t))$$

$$= A_c \cos[2\pi f_c t + \theta_i(t) m(t)]$$

which is recognized as phase modulation.

Suppose next that $f_i(t)$ is a linear function of the second derivative of $m(t)$, as shown by

$$f_i(t) = f_c + k_2 \frac{d^2 m(t)}{dt^2}$$

Correspondingly, we have

$$\theta_i(t) = 2\pi f_c t + 2\pi k_2 \frac{dm(t)}{dt}$$

where it is assumed that $dm(t)/dt$ is zero at $t = 0$. The modulated wave assumes the new form

$$s(t) = A_c \cos\left(2\pi f_c t + 2\pi k_2 \frac{dm(t)}{dt}\right)$$

We may generalize these results by stating that if the input to a frequency modulator is the nth derivative of the message signal $m(t)$, then the corresponding modulated wave is defined by

$$s(t) = A_c \cos\left(2\pi f_c t + 2\pi k_n \frac{d^{n-1} m(t)}{dt^{n-1}}\right)$$

where it is assumed that $d^{n-1}m(t)/dt^{n-1}$ is zero at time $t = 0$.

Consider next the scenario where the input to the frequency modulator involves integrals of the message signal $m(t)$. Starting with $\int_0^t m(\tau) d\tau$ as the input to the modulator, we write

$$f_i(t) = f_c + c_1 \int_0^t m(\tau) d\tau$$

and, correspondingly,

$$\theta_i(t) = 2\pi f_c t + 2\pi c_1 \left(\int_0^\lambda m(\tau) d\tau\right) d\lambda$$

The resulting modulated signal is defined by

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Problem 4-23 continued

$$s(t) = A_c \cos\left[2\pi f_c t + 2\pi c_1 \int_0^t \left(\int_0^\lambda m(\tau) d\tau\right) d\lambda\right]$$

Unlike the modulation scenario involving derivatives of $m(t)$, we can see that when considering the scenario involving integrals of $m(t)$, mathematical formulation of the modulated signal $s(t)$ becomes increasingly more complicated.

(b) There could be a practical benefit from using a frequency-modulation strategy involving integrals of the message signal $m(t)$ if $m(t)$ happens to be corrupted by additive noise. In such a scenario, the integration process tends to reduce the corruptive influence of the additive noise by smoothing it out. However, the drawback of such a modulation strategy is two-fold:

(i) Mathematical analysis of ordinary FM is complicated enough. Using integrals of the message signal as the input to the frequency modulator makes the problem even more complicated.

(ii) Likewise, designs of the transmitter and receiver become even more complicated.

Statements similar to (i) and (ii) apply to the use of second and higher derivatives of the message signal $m(t)$ as the input to the frequency modulator. The only exception here is the first derivative of $m(t)$, in which case the frequency modulator produces a phase modulated version of the signal. One other point to note is that if the message signal $m(t)$ is corrupted by additive noise, the operation of differentiation will enhance the presence of the noise component, which is undesirable.

To conclude, the "simple" forms of angle modulation exemplified by the ordinary FM and ordinary PM discussed in the chapter are good enough from a theoretical as well as practical perspective.

Problem 4.24

(a) We are given a nonlinear channel's input-output relation:

$$v_o(t) = a_1 v_i(t) + a_2 v_i^2(t) + a_3 v_i^3(t) \tag{1}$$

where $v_i(t)$ is the input and $v_o(t)$ is the output; a_1, a_2, and a_3 are fixed parameters. The input signal is defined by

$$v_i(t) = A_c \cos(2\pi f_c t + \phi(t)) \tag{2}$$

where

$$\phi(t) = 2\pi k_f \int_0^t m(\tau) d\tau \tag{3}$$

where $m(t)$ is the message signal and k_f is the frequency sensitivity of the frequency modulator. Substituting Eq. (2) into (1) yields

$$v_o(t) = a_1 A_c \cos(2\pi f_c t + \phi(t)) + a_2 A_c^2 \cos^2(2\pi f_c t + \phi(t))$$
$$+ a_3 A_c^3 \cos^3(2\pi f_c t + \phi(t)) \tag{4}$$

Using the trigonometric identities:

$$\cos^2\theta = \frac{1}{2}(1 + \cos(2\theta))$$

$$\cos^3\theta = \frac{1}{4}(1 + \cos(3\theta))$$

we may rewrite Eq. (4) as

$$v_o(t) = \frac{1}{2}a_2 A_c^2 + \left(a_1 A_c + \frac{3}{4}a_3 A_c^3\right)\cos(2\pi f_c t + \phi(t))$$
$$+ \frac{1}{2}a_2 A_c^2 \cos(4\pi f_c t + 2\phi(t)) + \frac{1}{4}a_3 A_c^3 \cos(6\pi f_c t + 3\phi(t)) \tag{5}$$

Equation (5) shows that the channel output consists of the following components:

- A dc component, $\frac{1}{2}a_2 A_c^2$
- Frequency modulated component of frequency f_c, phase $\phi(t)$ and amplitude $\left(a_1 A_c + \frac{3}{4}a_3 A_c^3\right)$
- Frequency modulated component of frequency $2f_c$, phase $2\phi(t)$ and amplitude $\frac{1}{2}a_2 A_c^2$
- Frequency modulated component of frequency $3f_c$, phase $3\phi(t)$ and amplitude $\frac{1}{4}a_3 A_c^3$

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Problem 4-24 continued

(b) To remove the nonlinear distortion and thereby extract a replica of the original FM signal $v_i(t)$, it is necessary to separate the FM component with carrier frequency f_c in $v_o(t)$ from the higher order FM components. Let Δf denote the frequency deviation of the original FM signal and W denote the highest frequency component of the message signal $m(t)$. Then, using Carson's rule and noting that the frequency deviation above $2f_c$ is doubled (which is the component nearest to the original FM signal), we find that the necessary condition for separating the desired FM signal with carrier frequency f_c from that with carrier frequency $2f_c$ is

$$2f_c - (2\Delta f + W) > f_c + \Delta f - W$$

or

$$f_c > 3\Delta f + 2W \tag{6}$$

(c) To extract a replica of the original FM signal $v_i(t)$, we need to pass the channel output $v_o(t)$ through a band-pass filter of midband frequency f_c and bandwidth $2(\Delta f + W)$. The resulting filter output is

$$v'_o(t) = \left(a_1 A_c + \frac{3}{4} a_3 A_c^3\right)\cos(2\pi f_c t + \phi(t)) \tag{7}$$

where $\phi(t)$ is defined by Eq. (3).

Problem 4.25

(a) The loop filter in the second-order phase-locked loop (PLL) is defined by

$$H(f) = 1 + \frac{a}{jf} \tag{1}$$

where a is a filter parameter. The Fourier transform of phase error $\phi_e(t)$ (i.e., the phase difference between the phase of the FM signal applied to the PLL and the phase of the FM signal produced by the VCO) in the PLL is defined by (see part (a) of Problem 4.7)

$$\Phi_e(f) = \frac{1}{1 + L(f)}\Phi_1(f) \tag{2}$$

where the loop transfer function is itself defined by

$$L(f) = K_0\frac{H(f)}{jf} \tag{3}$$

The $\Phi_1(f)$ in Eq. (2) is the Fourier transform of the angle $\phi_1(t)$ in the FM signal applied to the PLL. Substituting Eq. (1) into (3) and expanding terms, we get

$$\Phi_e(f) = \left(\frac{(jf)^2/aK_0}{1 + [(jf)/a] + [(jf)^2/aK_0]}\right)\Phi_1(f) \tag{4}$$

Define the *natural frequency* of the loop

$$f_n = \sqrt{aK_0} \tag{5}$$

and the *dampling factor*

$$\zeta = \sqrt{\frac{K_0}{4a}} \tag{6}$$

We may then recast Eq. (4) in terms of the loop parameters f_n and ζ as follows:

$$\Phi_e(f) = \left(\frac{(jf/f_n)}{1 + 2\zeta(jf/f_n) + (jf/f_n)^2}\right) \tag{7}$$

(b) Suppose the FM signal applied to the PLL is a single-tone modulating signal, for which the phase input is defined by

$$\phi_1(t) = \beta\sin(2\pi f_m t) \tag{8}$$

Then, invoking the use of Eq. (7), we find that the corresponding phase error $\phi_e(t)$ is defined by

$$\phi_e(t) = \phi_{eo}\cos(2\pi f_m t + \psi) \tag{9}$$

where

$$\phi_{eo} = \frac{(\Delta f/f_m)(f_m/f_n)}{[(1-(f_m/f_n)^2)^2 + 4\zeta^2(f_m/f_n)^2]^{1/2}} \tag{10}$$

and

$$\psi = \frac{\pi}{2} - \tan^{-1}\left[\frac{2\zeta(f_m/f_n)}{1-(f_m/f_n)^2}\right] \tag{11}$$

Continued on next slide

Problem 4-25 continued

One other thing we need to do is to evaluate the Fourier transform of the PLL output $v(t)$. For this purpose, we first note that the Fourier transform of $v(t)$ is related to $\Phi_1(f)$ as follows (see part (b) of Problem 4.7)

$$V(f) = \frac{jf}{k_v} \frac{L(f)}{1+L(f)}\Phi_1(f) \tag{12}$$

where k_v is the frequency sensitivity of the VCO. Using Eqs. (1), (3), and (12), we may write

$$V(f) = \left(\frac{(jf/f_n)[1+2\zeta(jf/f_n)]}{1+2\zeta(jf/f_n)+(jf/f_n)^2}\right)\Phi_1(f) \tag{13}$$

In light of the PLL theory presented herein, we may make two important observations for an incoming FM signal of fixed frequency deviation produced by a sinusoidal modulating signal $m(t)$:

(i) The frequency response that defines the phase error $\phi_e(t)$ is representative of a band-pass filter, as shown by Eq. (10).

(ii) The frequency response of the PLL output $v(t)$ is representative of a low-pass filter, as shown by Eq. (13).

Therefore, by appropriately choosing the damping factor ζ and natural frequency f_n, which determine the frequency response of the PLL, it is possible to restrain the phase error $\phi_e(t)$ to always remain small and yet, at the same time, the modulating (message) signal is reproduced at the PLL output with minimum distortion.

Problem 5.1

(a) Using the material presented in Section 2.5, justify the mathematical relationships listed at the bottom of the left-hand side of Table 5.1, which pertain to ideal sampling in the frequency domain.

(b) Applying the duality property of the Fourier transform to part (a), justify the mathematical relationships listed at the bottom of the right-hand side of this table, which pertain to ideal sampling in the time-domain.

Solution

1. Entry 1 on the left-hand side of Table 5.1:
   - The relationship

     $$\sum_{m=\infty}^{\infty} g(t-mT_s) = f_s \sum_{n=-\infty}^{\infty} G(nf_s)e^{j2\pi nf_st}$$

     where $g(t) \rightleftharpoons G(f)$ and $f_s = 1/T_s$, is a rewrite of Eq. (2.87) with one trivial change, namely, the replacements of T_o and f_o by T_s and f_s, respectively.
   - The Fourier transform pair

     $$\sum_{m=\infty}^{\infty} g(t)-m(T_s) \rightleftharpoons f_s \sum_{n=-\infty}^{\infty} G(nf_s)\delta(f-f_s)$$

     is also a rewrite of Eq. (2.88) except for the replacement of T_o and f_o with T_s and f_s, respectively.
2. Entry 2 on the right-hand side of Table 5.2:
   - The relationship

     $$\sum_{n=\infty}^{\infty} g(nT_s)e^{j2\pi nf_st} = f_s \sum_{m=-\infty}^{\infty} G(nf_sf-mf_s)$$

     is an exact reproduction of the equality in Eq. (5.2).
   - The Fourier-transform pair

     $$\sum_{n=\infty}^{\infty} g(nT_s)\delta(t-nT_s) \rightleftharpoons f_s \sum_{m=-\infty}^{\infty} G(f-mf_s)$$

     is an exact reproduction of the Fourier-transform pair listed in Eq. (5.2).

Problem 5.2

Show that as the sampling period T_s approaches zero, the formula for the discrete-time Fourier transform $G_\delta(f)$ approaches the Fourier transform $G(f)$.

Solution

From Eq. (5.3), we have

$$G_\delta(f) = \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right)\exp\left(-\frac{j\pi nf}{W}\right)$$

The sampling period T_s = -1/(2W). We may therefore rewrite this equation as

$$G_\delta(f) = \sum_{n=\infty}^{\infty} g(nT_s)\exp(-j2\pi nT_s f)$$

In the limit, as T_s approaches zero, the discrete time nT_s, approaches continuous time t. Moreover, the summation over n approaches the integral

$$\int_{-\infty}^{\infty} g(t)\exp(-j2\pi tf)dt$$

Correspondingly, $G_\delta(f)$ approaches the continuous Fourier transform $G(f)$. We may therefore state that the formula for the discrete Fourier transform $G_\delta(f)$ given in Eq. (5.3) approaches the formula for the Fourier transform:

$$G(f) = \int_{-\infty}^{\infty} g(t)\exp(-j2\pi tf)dt$$

as the sampling period T_s approaches zero.

Problem 5.3

Show that

$$\frac{1}{2W}\int_{-W}^{W} \exp\left[j2\pi f\left(t-\frac{n}{2W}\right)\right]df = \frac{\sin(2\pi Wt - n\pi)}{(2\pi Wt - n\pi)}$$
$$= \text{sinc}(2Wt-n)$$

Solution

$$\frac{1}{2W}\int_{-W}^{W} \exp\left[j2\pi f\left(t-\frac{n}{2W}\right)\right]df = \frac{1}{2W}\cdot\frac{1}{j2\pi(t-n/2W)}\cdot\exp\left[j2\pi f\left(t-\frac{n}{2W}\right)\right]_{f=-W}^{W}$$
$$= \frac{1}{j4\pi W(t-n/2W)}\cdot[\exp j\pi(2Wt-n) - \exp(-j\pi(2Wt-n))]$$
$$= \frac{\sin(\pi(2Wt-n))}{\pi(2Wt-n)}$$
$$= \sin c(2Wt-n)$$

Problem 5.4

This problem is intended to identify a linear filter for satisfying the interpolation formula of Eq. (5.7), albeit in a non-physically realizable manner. Equation (5.7) is based on the premise that the signal $g(t)$ is strictly limited to the band $-W \leq f \leq W$. With this specification in mind, consider an ideal low-pass filter whose frequency response $H(f)$ is as depicted in Fig. 5.2(c). The impulse response of this filter is defined by (see Eq. (2.25))

$$h(t) = \operatorname{sinc}(2Wt), \qquad -\infty < t < \infty$$

Suppose that the correspondingly instantaneously sampled signal $g_\delta(t)$ defined in Eq. (5.1) is applied to this ideal low-pass filter. With this background, use the convolution integral to show that the resulting output of the filter is defined exactly by the interpolation formula of Eq. (5.7).

Solution

From Eq. (5.5), we have

$$G(f) = \frac{1}{2W}\sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right)\exp\left(-\frac{j\pi nf}{W}\right), \qquad -W < f < W$$

According to this equation, $G(f)$ is low-pass with its frequency content confined to the range $-W < f < W$. Since $G(f)$ is the Fourier transform of $g(t)$, we can also write

$$\sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right)\operatorname{sinc}(2Wt - n) \rightleftharpoons \frac{1}{2W}\sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right)\exp\left(-\frac{j\pi nf}{W}\right), \qquad -W < f < W$$

Hence, the reconstruction filter defined by the left-hand side of this Fourier-transform pair is low-pass with its passband confined to the range $W < f < W$.

Problem 5.5

Specify the Nyquist rate and the Nyquist interval for each of the following signals:

(a) $g(t) = \text{sinc}(200t)$

(b) $g(t) = \text{sinc}^2(200t)$

(c) $g(t) = \text{sinc}(200t) + \text{sinc}^2(200t)$

Solution

(a) The highest frequency component of

$$g(t) = \text{sinc}(200t) = \frac{\sin(200\pi t)}{200\pi t}$$

is 100 Hz. Hence, the Nyquist rate is 200 Hz and the Nyquist interval is 5 ms.

(b) The highest frequency component of

$$g(t) = \text{sinc}^2(200t)$$

is twice that of $g(t)$ in part (a); it is so because squaring a band-limited single has the effect of doubling its highest frequency component. Hence, the Nyquist rate of

$$g(t) = \text{sinc}^2(2Wt)$$

is 400 Hz and the Nyquist interval is 2.5 ms.

(c) The highest frequency component of the composite signal

$$g(t) = \text{sinc}(200t) + \text{sinc}^2(200t)$$

is determined by the component $\text{sinc}^2(200t)$. Hence, the Nyquist rate of this third signal is 400 Hz and the Nyquist interval is 2.5 ms.

Problem 5.6

Consider uniform sampling of the sinusoidal wave

$g(t) = \cos(\pi t)$

Determine the Fourier transform of the sampled waveform for the following sampling period:

(a) $T_s = 0.25$s

(b) $T_s = 1$s

(c) $T_s = 1.5$s

Solution

We are given

$g(t) = \cos(\pi t)$

the frequency of which is 0.5 Hz.

(a) For the sampling period $T_s = 0.25$, we have

$$g_\delta(t) = \sum_{n=-\infty}^{\infty} g(nT_s)\delta(t - nT_s)$$

$$= \sum_{n=-\infty}^{\infty} \cos\left(\frac{n\pi}{4}\right)\delta(t - nT_s)$$

(b) For $T_s = 1$s,

$$g_\delta(t) = \sum_{n=-\infty}^{\infty} \cos(n\pi)\delta(t - nT_s)$$

$$= \sum_{n=-\infty}^{\infty} (-1)^n\delta(t - nT_s)$$

(c) For $T_s = 1.5$,

$$g_\delta(t) = \sum_{n=-\infty}^{\infty} \cos(1.5n\pi)\delta(t - nT_s)$$

Problem 5.7
Consider a continuous-time signal defined by

$$g(t) = \frac{\sin(\pi t)}{\pi t}$$

The signal $g(t)$ is uniformly sampled to produce the infinite sequence $\{g(nT_s)\}_{n=-\infty}^{\infty}$. Determine the condition which the sampling period T_s must satisfy so that the signal $g(t)$ is uniquely recovered from the sequence $\{g(nT_s)\}$.

Solution
The signal

$$g(t) = \frac{\sin(\pi t)}{\pi t} = \text{sinc}(t)$$

is limited to the band $-0.5 < f < 0.5$ Hz. The Nyquist rate for this signal must therefore exceed $2 \times 0.5 = 1$ Hz. Correspondingly, the permissible sampling interval must satisfy the condition $T_s < 1$s.

Problem 5.8

Starting with Eq. (5.9), show that the Fourier transform of the rectangular pulse $h(t)$ is given by

$H(f) = T\,\text{sinc}(fT)\exp(-j\pi fT)$

What happens to $H(f)/T$ as the pulse duration T approaches zero?

Solution

Given

$$h(t) = \begin{cases} 1, & 0 < t < T \\ \frac{1}{2}, & t = 0, t = T \\ 0, & \text{otherwise} \end{cases}$$

the required Fourier transform is

$$\begin{aligned} H(f) &= \int_{-\infty}^{\infty} h(t)e^{-j2\pi ft}dt \\ &= \int_0^T 1 \cdot \exp(-j2\pi ft)dt \\ &= \left[\frac{\exp(-j2\pi ft)}{-j2\pi f}\right]_{t=0}^{T} \\ &= \frac{1}{j2\pi f} - \frac{\exp(-j2\pi fT)}{j2\pi f} \\ &= \frac{\exp(-j2\pi fT)}{j2\pi f}[\exp(j\pi fT) - \exp(-j\pi fT)] \end{aligned}$$

Since

$$\sin(\pi fT) = \frac{1}{2j}[\exp(j\pi fT) - \exp(-j\pi fT)]$$

it follows that

$$\begin{aligned} H(f) &= \frac{\sin(\pi fT)}{\pi f}\exp(-j\pi fT) \\ &= T \cdot \frac{\sin(\pi fT)}{\pi fT}\exp(-j\pi fT) \\ &= T\,\text{sinc}(fT)\exp(-j\pi fT) \end{aligned}$$

Problem 5.9

Using Eqs. (5.23) and (5.25), respectively, derive the slope characteristics of Eqs. (5.24) and (5.26).

Solution

(a) The logarithmic law is defined by (see Eq. (5.23)

$$|v| = \frac{\log(1+\mu|m|)}{\log(1+\mu)}$$

Therefore, differentiation with respect to $|m|$ yields

$$\frac{d|v|}{d|m|} = \frac{1}{\log(1+\mu)} \cdot \frac{\mu}{1+\mu|m|}$$

Equivalently, we may write

$$\frac{dm}{d|v|} = \log(1+\mu)\frac{1+\mu|m|}{\mu}$$

(b) The A-law is defined by (see Eq. (5.25):

$$|v| = \begin{cases} \dfrac{A|m|}{1+\log A}, & 0 \le |m| \le \dfrac{1}{A} \\ \dfrac{1+\log(A|m|)}{1+\log A}, & \dfrac{1}{A} \le |m| \le 1 \end{cases}$$

Hence, differentiation of $|v|$ with respect to $|m|$ yields

$$\frac{d|v|}{d|m|} = \begin{cases} \dfrac{A}{1+\log A}, & 0 \le |m| \le \dfrac{1}{A} \\ \dfrac{A}{|m|(1+\log A)}, & \dfrac{1}{A} \le |m| \le 1 \end{cases}$$

Equivalently, we may write

$$\frac{d|m|}{d|v|} = \begin{cases} \dfrac{1+\log A}{A} \quad , & 0 \le |m| \le \dfrac{1}{A} \\ \left(\dfrac{1+\log A}{A}\right)|m|, & \dfrac{1}{A} \le |m| \le 1 \end{cases}$$

Problem 5.10
The best that a linear delta modulator can do is to provide a compromise between slope-overload distortion and granular noise. Justify this statement.

Solution

(a) In linear delta modulation, if we make the step-size Δ too small, then the system suffers from slope overload distortion.

(b) On the other hand, if we make the step-size Δ too large relative to the local slope characteristic of the message signal $m(t)$, then the system suffers from granular distortion.

For a *fixed* sampling rate $1/T_s$ and with Δ *as the only variable*, the best that the linear delta modulator can do is to choose a step-size Δ that will provide a compromise between these two forms of quantization noise.

Problem 5.11

Justify the two statements just made on sources of noise in a DPCM system.

Solution

1. When the step-size is too small and the input signal is changing too rapidly, the DPCM is unable to track the input signal, resulting in slope-overload distortion similar to linear delta modulation.
2. DPCM uses a quantizer in the transmitter. Hence, like pulse-code modulation, DPCM suffers from quantization noise.

Problem 5.12

(a) The PAM wave is defined by

$$s(t) = \sum_{n=-\infty}^{\infty} [1 + \mu m'(nT_s)] g(t - nT_s), \tag{1}$$

where $g(t)$ is the pulse shape, $m'(t) = m(t)/A_m = \cos(2\pi f_m t)$ and μ is the modulation factor. The PAM wave is equivalent to the convolution of the instantaneously sampled signal $[1 + \mu m'(t)]$ and the pulse shape $g(t)$, as shown by

$$s(t) = \left\{ \sum_{n-\infty}^{\infty} [1 + \mu m'(nT_s)] \delta(t - nT_s) \right\} \star g(t)$$

$$= \left\{ 1 + \mu m'(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \right\} \star g(t) \tag{2}$$

Let $m'(t) \rightleftharpoons M'(f)$, $g(t) \rightleftharpoons G(f)$, and $s(t) \rightleftharpoons S(f)$.

The spectrum of the PAM wave is therefore,

$$S(f) = \left\{ [\delta(f) + \mu M'(f)] \star \frac{1}{T_s} \sum_{m=-\infty}^{\infty} \delta\left(f - \frac{m}{T_s}\right) \right\} G(f)$$

$$= \frac{1}{T_s} G(f) \sum_{m=-\infty}^{\infty} \left[\delta\left(f - \frac{m}{T_s}\right) + \mu M'\left(f - \frac{m}{T_s}\right) \right] \tag{3}$$

For a rectangular pulse $g(t)$ of duration $T = 0.45$s and with $AT = 1$, we have

$$G(f) = AT \operatorname{sinc}(fT)$$
$$= \operatorname{sinc}(0.45 f)$$

For $m'(t) = \cos(2\pi fmt)$ and $f_m = 0.25$ Hz, we have

$$M'(f) = \frac{1}{2}[\delta(f - 0.25) + \delta(f + 0.25)]$$

For $T_s = 1s$, the ideally sampled spectrum is

$$S_\delta(f) = \sum_{m=-\infty}^{\infty} [\delta(f - m) + \mu M'(f - m)] \tag{4}$$

which is plotted in Fig. 2(c).

The actual sampled spectrum is defined by

$$S(f) = \sum_{m=-\infty}^{\infty} \operatorname{sinc}(0.45 f)[\delta(f - m) + \mu M'(f - m)] \tag{5}$$

which is plotted in Fig. 1(b).

Continued on next slide

Problem 5-12 continued

(b) The ideal reconstruction filter would retain the centre 3 delta functions of $S(f)$. With no aperture effect, the two outer delta functions would have amplitude $\mu/2$. The aperture effect distorts the reconstructed signal by attenuating the high-frequency portion of the message signal.

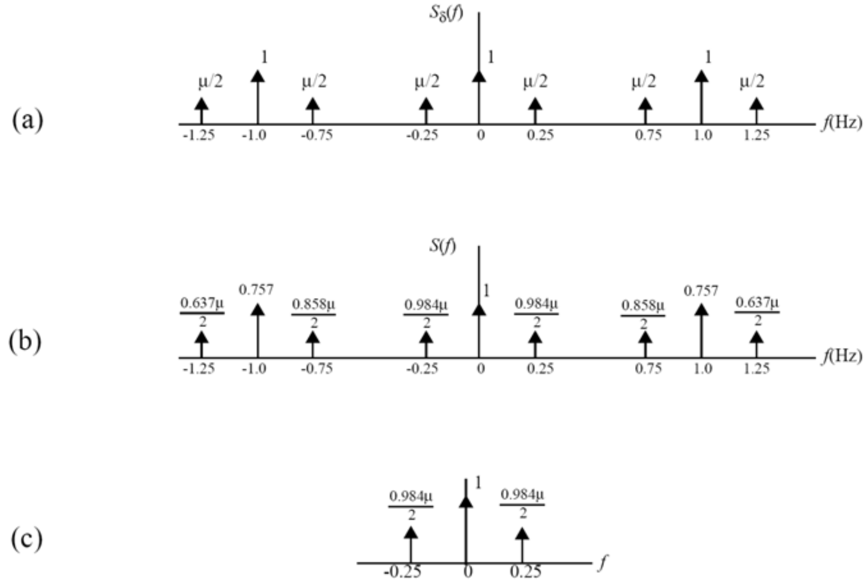


Figure 1

Problem 5.13

At $f = 1/2T_s$, which corresponds to the highest frequency component of the message signal for a sampling rate equal to the Nyquist rate, we find from Eq. (5.17) that the amplitude response of the equalizer normalized to that at zero frequency is defined by

$$\frac{1}{\text{sinc}(0.5T/T_s)} = \frac{(\pi/2)(T/T_s)}{\sin[(\pi/2)(T/T_s)]} \qquad (1)$$

where the ratio T/T_s is equal to the duty cycle. In Fig. 1, Eq. (1) is plotted as a function of T/T_s. Ideally, the graph should be equal to one for all values of T/T_s, as indicated by the dashed horizontal line in Fig. 1. For a duty cycle of 25 percent, it is approximately equal to 1.04, which exceeds the ideal case by about 4%.

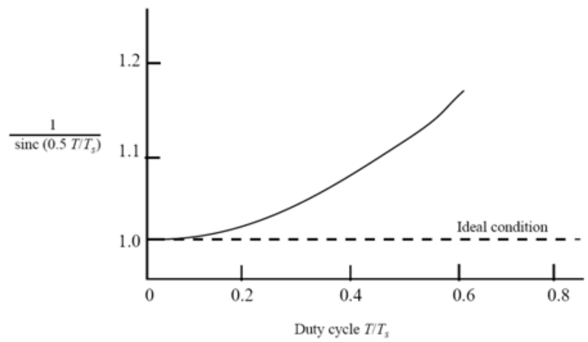


Figure 1

Problem 5.14

(a) The Nyquist rate for $s_1(t)$ and $s_2(t)$ is 160 Hz. Therefore, the factor $\frac{2400}{2^R}$ must be greater than 160, and the maximum R is 3.

(b) With $R = 3$, we may use the following signal format displayed in Fig. 1 to multiplex the signals $s_1(t)$ and $s_2(t)$ into a new signal, and then multiplex $s_3(t)$ and $s_4(t)$ and $s_5(t)$ including markers for synchronization.

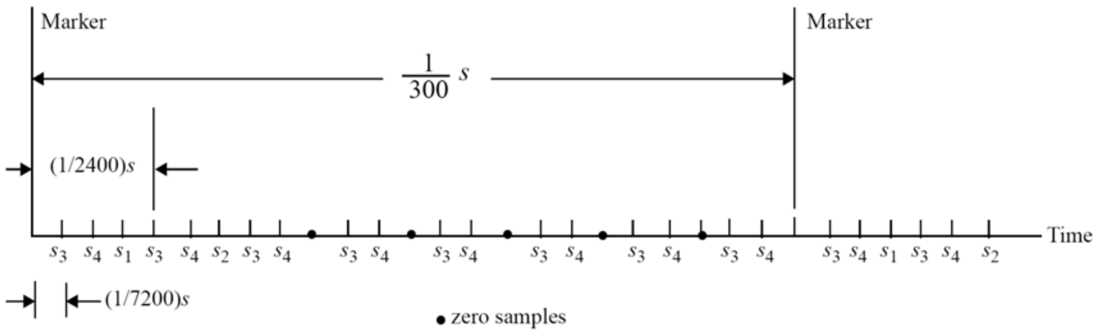


Figure 1

Based on the signal format shown in Fig. 1, we may develop the multiplexing system shown in Fig. 2.

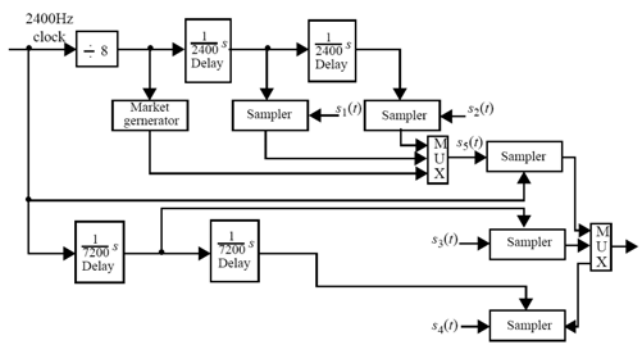
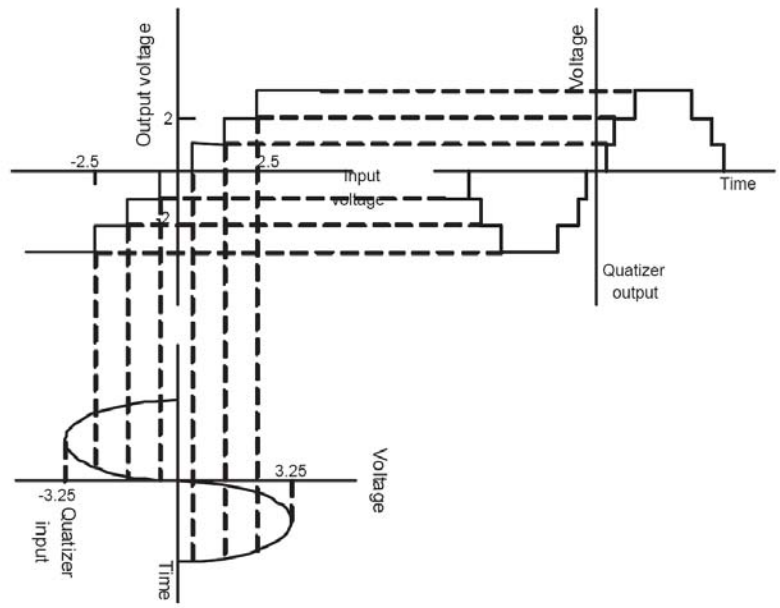
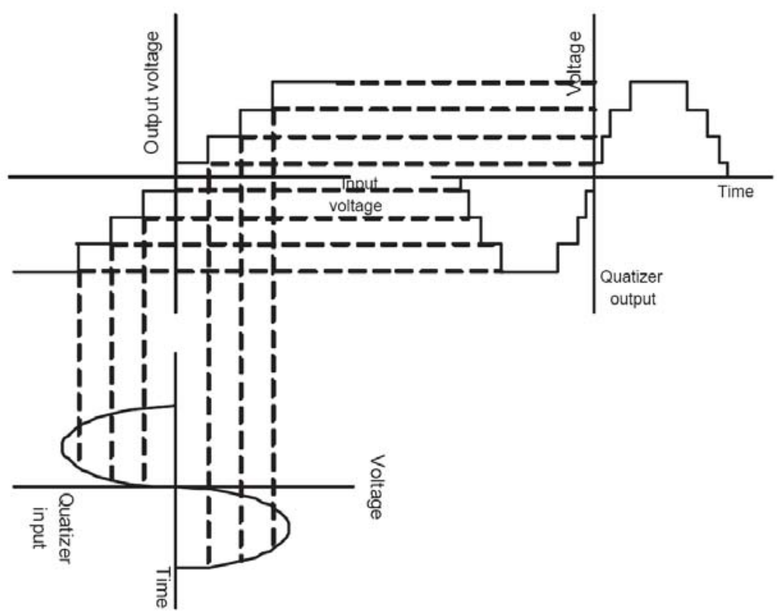


Figure 2

Problem 5.15



Problem:(5.15a)



Problem:(5.15b)

Problem 5.16

(a) An alternating sequence of 1's and 0's

On-off signaling: The signal $g(t)$ consists of a periodic train of rectangular pulses with pulse duration $T = T_0/2$, where T_0 is the period.

Bipolar return-to-zero signaling: The signal $g(t)$ consists of a periodic train of pulses of duration T and of alternating polarity.

(b) A long sequence of 1's followed by a long sequence of 0's

On-off signaling: The signal $g(t)$ consists of a unit step function defined for negative time, tl is, $u(-t)$.

Bipolar return-to-zero signaling: The signal $g(t)$ consists of pulses of alternating polari followed by a long period of zero volts.

(c) An alternating sequence of 1's followed by a single 0 and then a long sequence of 1's

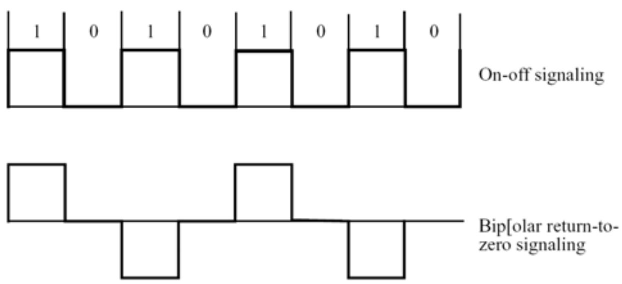
On-off signaling: The signal $g(t)$ consists of a dc component minus a rectangular pulse (of t same amplitude as the dc component).

Bipolar return-to-signal signaling: The signal $g(t)$ consists of two identically perio sequences of pulses separated by a period of zero volts.

The line codes just described are plotted in Fig. 1.

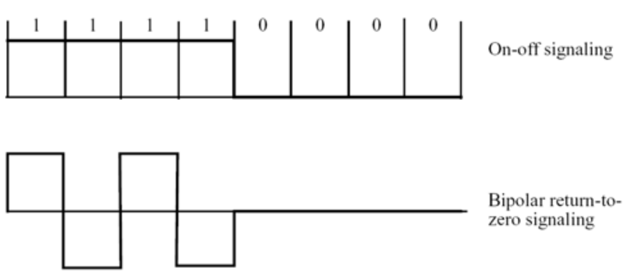
(a) An alternating sequence of 1's and 0's

(a)



(b) A long sequence of l's followed by a long sequence of 0's

(b)



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Problem 5-16 continued

(c) A long sequence of 1's followed by a single 0 and then a long sequence of 1's

(c)

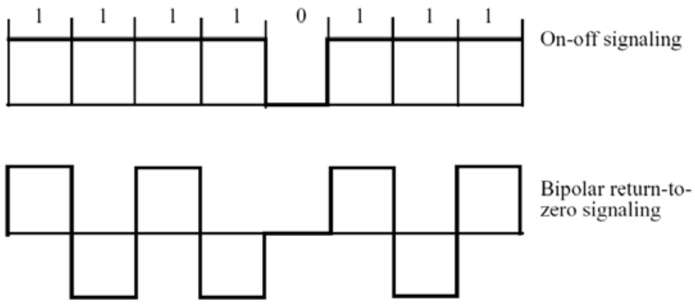


Figure 1

Problem 5.17

The quantizer has the following input-output curve plotted in Fig. 1

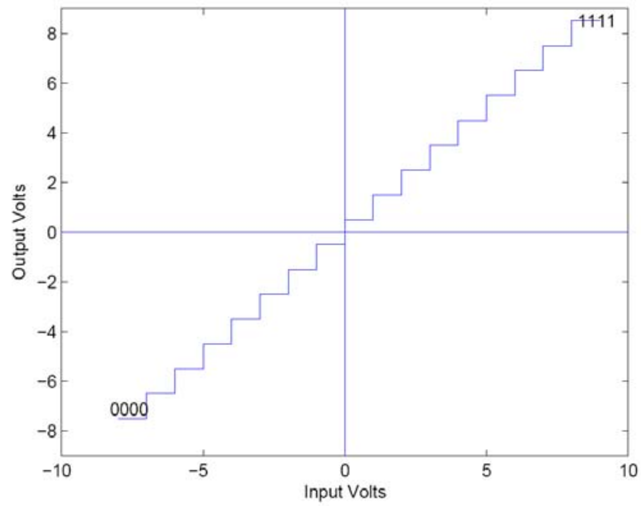


Figure 1

At the sampling instants we have:

t	$m(t)$	code
-3/8	$-3\sqrt{2}$	0011
-1/8	$-3\sqrt{2}$	0011
+1/8	$3\sqrt{2}$	1100
+3/8	$3\sqrt{2}$	1100

And the coded waveform is (assuming on-off signaling):

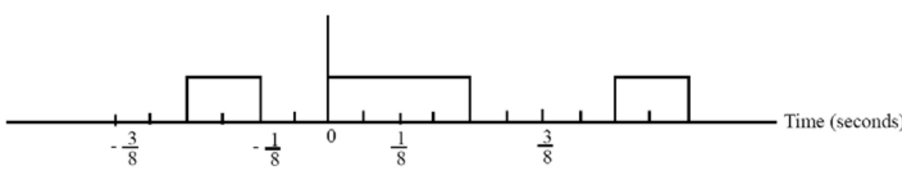


Figure 2

Problem 5.18

We are given
- Audio signal bandwidth, $W = 15$ kHz
- Number of uniform quantization levels = 512 levels
- Encoding : binary

(a) The Nyquist rate is $2W = 30$ kHz.

(b) To accommodate 512 quantization levels, we require a binary code with B bits, which would have to satisfy the following requirement:

$2^B = 512$

Hence, $B = 9$. The sampling period $T_s = 1/30$ milliseconds must be divided into 9 bits. The minimum sampling rate is therefore

$30 \times 9 = 270$ kilobits/second

$= 0.27$ megabits/second

Problem 5.19

(a) We are given
- Video bandwidth = 4.5 MHz
- Sampling rate = 15% in excess of the Nyquist rate
- Uniform quantization using 1024 levels
- Binary encoding

(b) The Nyquist rate is 2 x 4.5 = 9 MHz.
Actual sampling rate = 9 x 1.15 = 10.35 MHz
The sampling period is therefore

$$T_s = \frac{1}{10.35}\mu s$$

This sampling rate must be divided into B bits, where
$2^B = 1024$
Hence, $B = 10$. The bit duration is therefore

$$\frac{T_s}{10} = \frac{1}{103.5}\mu s$$

The permissible bit rate is therefore 103.5 megabits/s.

Problem 5.20

The transmitted code words, representing the PCM waveform

t/T_b	code
1	001
2	010
3	011
4	100
5	101
6	110

Accordingly, the sampled analog signal from which these code words are derived is shown in Fig. 1.

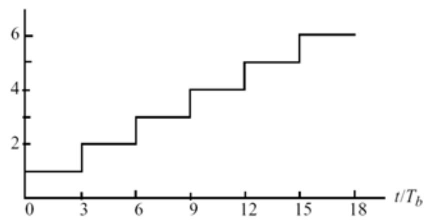


Figure 1

Problem 5.21

The modulating wave is

$m(t) = A_m \cos(2\pi f_m t)$

The slope of $m(t)$ is given by

$$\frac{dm(t)}{dt} = -2\pi f_m A_m \sin(2\pi f_m t)$$

The maximum slope of $m(t)$ is therefore equal to $2\pi f_m A_m$.

The maximum average slope of the approximating signal $m_a(t)$ produced by the delta modulator is δ/T_s, where δ is the step size and T_s is the sampling period. The limiting value of A_m is therefore given by

$$2\pi f_m A_m > \frac{\delta}{T_s}$$

or

$$A_m > \frac{\delta}{2\pi f_m T_s}$$

Assuming a load of 1 ohm, the transmitted power is $A_m^2/2$. Therefore, the maximum power that may be transmitted without slope-overload distortion is equal to $8^2/(\delta\pi^2 f_m^2 T_s^2)$.

Problem 5.22

Sampling rate = 64 kHz
Voice signal bandwidth = W = 3.1 kHz
Maximum signal amplitude A_{max} = 10 volts

(a) To avoid slope overload, we must satisfy the following requirement (see Problem 5.21)

$$A_{max} < \frac{\Delta}{2\pi W T_s}$$

Solving for the step size Δ, we write

$$\Delta > \frac{1}{2\pi W T_s A_{max}} = \frac{f_s}{2\pi W A_{max}} \qquad (1)$$

Substituting the given values into Eq. (1) yields

$$\Delta > \frac{64}{2\pi \times 3.1 \times 10}$$

or

$$\Delta > 0.33 \quad \text{volts}$$

Effectively, provided that the step size Δ is 0.33 volt, then slope-overload distortion is avoided.

(b) Let $\in(t)$ denote the granular noise, viewed as a function of time t. The average power of granular noise (analogous to quantization noise in PCM), is defined by

$$P_g = \frac{2}{\Delta}\int_{-\Delta/2}^{\Delta/2} \in^2 d\in$$

$$= \frac{2}{\Delta}\left[\frac{\in^3}{3}\right]_{\in=-\Delta/2}^{\Delta/2}$$

$$= \frac{\Delta^2}{3}$$

With Δ set at 0.33 volt, the average power of granular noise is therefore 0.03 watts (assuming that the power is calculated for a load of 1 ohm).

(c) The minimum channel bandwidth needed to transmit the DM encoded signal is the inverse of the sampling rate, that is, 64 kHz.

Problem 5.23

The values calculated in parts (a), (b) and (c) of Problem 5.22 also hold for a sinusoidal signal of peak amplitude 10 volts and frequency 3.1 kHz.

Problem 5.24

The transmitting prediction filter operates on exact samples of the signal while the receiving prediction filter operates on quantized samples. Hence, unlike the DPCM system described in Section 5.8, the prediction filters in the transmitter and receiver of Fig. 5.26 operate on different signals.

Problem 5.25

(a) In theory, any physical signal (exemplified by audio and video signals) has a spectrum that gradually decreases towards zero. From Fourier transform theory, we know that any signal cannot simultaneously have finite duration and finite bandwidth. Therefore, theoretically speaking, given a physical signal of finite duration, the band of frequencies occupied by that signal is infinitely large. Accordingly, when the signal is sampled in accordance with the Nyquist sampling theorem, there will always be some distortion produced by sampling the signal due to the aliasing phenomenon.

(b) In practice, however, we usually limit the sampling rate to some finite value, depending on the application of interest. For example, for telephonic communication, it has been found experimentally that 3.1 kHz is considered to be adequate for describing the "effective" bandwidth of a voice signal, be that for a male or female. Thus, choosing a rate of 8 kHz is considered to be adequate for the uniform sampling of a voice signal in telephonic communication. In reality, there is some distortion produced by the sampling process, but for all practical purposes, the distortion is not significant enough to be perceived by a human listener. Indeed, it is for this reason that a sampling rate of 8 kHz is the universally accepted standard for the sampling of voice signals transmitted over a telephone line.

Similar remarks apply to the sampling of video signals; naturally, the sampling rate used for video signals is much higher than 8 kHz,

Problem 5.26

Let $2W$ denote the bandwidth of a narrowband signal with carrier frequency f_c. The in-phase and quadrature components of this signal are both low-pass signals with a common bandwidth of W. According to the sampling theorem, there is no information loss if the in-phase and quadrature components are sampled at a rate higher than $2W$. For the problem at hand, we have

$f_c \quad = 100$ kHz

$2W = 10$ kHz

Hence, $W = 5$ kHz, and the minimum rate at which it is permissible to sample the in-phase and quadrature components is 10 kHz.

From the sampling theorem, we also know that a physical waveform can be represented over the interval $-\infty < t < \infty$ by

$$g(t) = \sum_{n=-\infty}^{\infty} a_n \phi_n(t) \qquad (1)$$

where $\{\phi_n(t)\}$ is a set of orthogonal functions defined as

$$\phi_n(t) = \frac{\sin\{\pi f_s(t - n/f_s)\}}{\pi f_s(t - n/f_s)}$$

where n is an integer and f_s is the sampling frequency. If $g(t)$ is a low-pass signal limited to W Hz, and $f_s \geq 2W$, then the coefficient a_n can be shown to equal $g(n/f_s)$. That is, for $f_s \geq 2W$, the orthogonal coefficients are simply the values of the waveform that are obtained when the waveform is sampled every $1/f_s$ second.

As already mentioned, the narrowband signal is two-dimensional, consisting of in-phase and quadrature components. In light of Eq. (1), we may represent them as follows, respectively:

$$g_I(t) = \sum_{n=-\infty}^{\infty} g_I(n/f_s)\phi_n(t)$$

$$g_Q(t) = \sum_{n=-\infty}^{\infty} g_Q(n/f_s)\phi_n(t)$$

Hence, given the in-phase samples $g_I\left(\frac{n}{f_s}\right)$ and quadrature samples $g_Q\left(\frac{n}{f_s}\right)$, we may reconstruct the narrowband signal $g(t)$ as follows:

$$g(t) = g_I(t)\cos(2\pi f_c t) - g_Q(t)\sin(2\pi f_c t)$$

$$= \sum_{n=-\infty}^{\infty}\left[g_I\left(\frac{n}{f_s}\right)\cos(2\pi f_c t) - g_Q\left(\frac{n}{f_s}\right)\sin(2\pi f_c t)\right]\phi_n(t)$$

where $f_c = 100$ kHz and $f_s \geq 10$ kHz, and where the same set of orthonormal basis functions is used for reconstructing both the in-phase and quadrature components.

Problem 5.27

(a) The commutator at the output of the bipolar chopper switches between the direct path and inverted path at the frequency f_s. In effect, every $1/f_s$ seconds, the output of the chopper consists of the input $x(t)$ -- via the direct path -- for $1/2f_s$ seconds followed by the inverted version of $x(t)$ -- via the inverted path -- for the remaining $1/2f_s$ seconds of the commutation period. For one period of the commutation process, we may thus write

$$y(t) = \begin{cases} x(t) & \text{for} \quad 0 \le t \le 1/(2f_s) \\ -x(t) & \text{for } 1/(2f_s) \le t \le 1/f_s \end{cases} \tag{1}$$

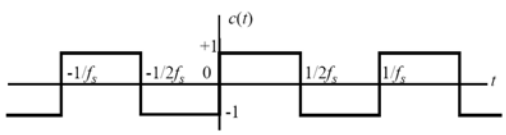
Equation (1) repeats itself every $1/f_s$ seconds.

(b) Equation (1) may be equivalently expressed as follows:

$$y(t) = c(t)x(t) \tag{2}$$

where $c(t)$ consists of the square wave (see Fig. 1)

$$c(t) = \begin{cases} 1+ & \text{for} \quad 0 \le t \le 1/(2f_s) \\ -1 & \text{for } 1/(2f_s) \le t \le 1/f_s \end{cases} \tag{3}$$



By inspection, we may make three observations from Fig. 1:

(i) The dc component of $c(t)$ is zero.

(ii) The Fourier series representation of $c(t)$ consists of sine components with a fundamental frequency f_s.

(iii) The even harmonic components of $c(t)$ are all zero.

Accordingly, we may represent $c(t)$ by the Fourier series:

$$c(t) = b_1 \sin(2\pi f_s t) + b_3 \sin(6\pi f_s t) + b_5 \sin(10\pi f_s t) + \dots \tag{4}$$

where b_n is defined by

$$\begin{aligned} b_n &= f_s \int_0^{1/f_s} c(t) \sin(2\pi n f_s t) dt \\ &= f_s \int_0^{1/2f_s} \sin(2\pi n f_s t) dt - f_s \int_{-1/2f_s}^{1/f_s} \sin(2\pi n f_s t) dt \\ &= \frac{-1}{2\pi n}[\cos(2\pi n f_s t)]_{t=0}^{1/2f_s} + \frac{1}{2\pi n}[\cos(2\pi n f_s t)]_{t=(1/2f_s)}^{1/f_s} \\ &= -\frac{1}{2\pi n}(\cos(n\pi) - 1) + \frac{1}{2\pi n}(\cos(n\pi) - \cos(n\pi)) \end{aligned}$$

$$= \begin{cases} \dfrac{2}{\pi n} & \text{for } n = 1, 3, 5, \dots \\ 0 & \text{for } n = 0, 2, 4, \dots \end{cases} \tag{5}$$

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Problem 5-27 continued

We may thus express the Fourier series of the commutation function $c(t)$ as

$$c(t) = \frac{2}{\pi}\sin(2\pi f_s t) + \frac{2}{3\pi}\sin(6\pi f_s t) + \frac{2}{5\pi}\sin(10\pi f_s t) + \dots \quad (6)$$

Using Eq. (6) in (2) yields

$$y(t) = \frac{2}{\pi}\sin(2\pi f_s t)x(t) + \frac{2}{3\pi}\sin(6\pi f_s t)x(t) + \frac{2}{5\pi}\sin(10\pi f_s t) + \dots \quad (7)$$

The Fourier transform of $y(t)$ is therefore defined by

$$Y(f) = \frac{1}{j\pi}[X(f-f_s) - X(f+f_s)]$$

$$+ \frac{1}{j3\pi}[X(f-3f_s) - X(f+3f_s)]$$

$$+ \frac{1}{j5\pi}[X(f-5f_s) - X(f+5f_s)] + \dots \quad (8)$$

where $X(f)$ is the Fourier transform of the input $x(t)$.

Figure 2 displays the relationship between the two Fourier transforms: $X(f)$ and $Y(f)$. Note that $X(f)$ can only be recovered from $Y(f)$ only through a band-pass filter with bandwidth $2W$ centered on f_s.

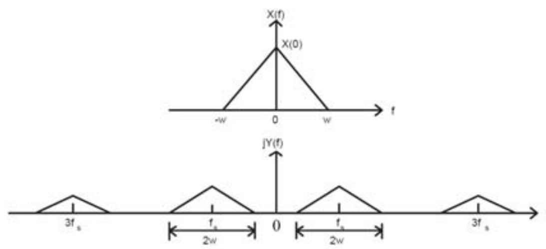

Figure 2

Problem 5.28

(a) Consider a periodic waveform $x(t)$ whose Fourier transform is defined by

$$X(f) = \sum_{k=-m}^{m} c_k \delta(f - kf_0) \tag{1}$$

where f_0 is the fundamental frequency of $x(t)$. In effect, we are assuming that $x(t)$ is the result of prefiltering a periodic signal with period $1/f_0$ and all harmonic components in excess of the mth component have been suppressed. The highest frequency of $x(t)$ is therefore mf_0.

Suppose now $x(t)$ is purposely sampled at the rate

$$f_s = (1-a)f_0 \tag{2}$$

where $0 < a < 1$. The sampling rate f_s is clearly less than the Nyquist rate $2mf_0$, hence the possibility of aliasing. From Eq. (5.2) in the text, recall that the Fourier transform of the sampled version of $x(t)$ is defined by

$$\begin{aligned} \frac{1}{f_s} X_\delta(f) &= \sum_{i=-\infty}^{\infty} X(f - if_s) \\ &= \sum_{i=-\infty}^{\infty} X(f - if_0 + aif_0) \end{aligned} \tag{3}$$

Substituting Eq. (1) into (3) yields

$$\frac{1}{f_s} X_\delta(f) = \sum_{i=-\infty}^{\infty} \sum_{k=-m}^{\infty} c_k \delta(f - (i+k)f_0 + aif_0) \tag{4}$$

To proceed further with this equation, we will use *induction* to solve Problem 5.28.

(i) Let $m = 1$, for which Eq. (1) reads as

$$X(f) = c_0\delta(f) + c_1[\delta(f - f_0) + \delta(f + f_0)] \tag{5}$$

This spectrum represents a sinusoidal wave of amplitude $2c_1$, superimposed on a dc bias of c_0; see Fig. 1(a). For this case, Eq. (4) simplifies to

$$\begin{aligned} \frac{1}{f_s} X_\delta(f) &= \sum_{i=-\infty}^{\infty} \sum_{k=-1}^{\infty} c_k \delta(f - (i+k)f_0 + af_0) \\ &= \sum_{i=-\infty}^{\infty} [c_0\delta(f - if_0 + aif_0) \\ &\quad + c_i\delta(f - (i+1)f_0 + aif_0) \\ &\quad + c_i\delta(f - (i-1)f_0 + aif_0)] \end{aligned} \tag{6}$$

Evaluating Eq. (5) yields the sampled spectrum depicted in Fig. 1(b).

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Problem 5-28 continued

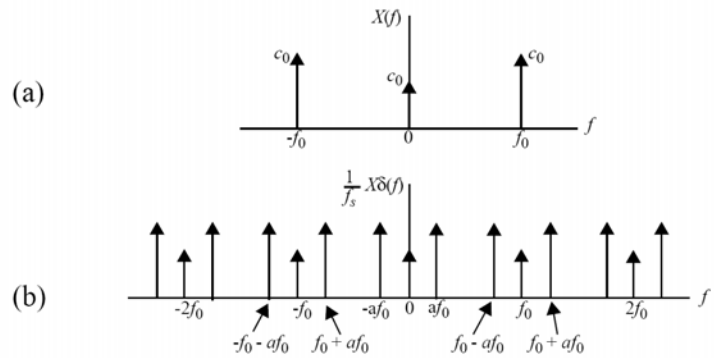

Figure 1

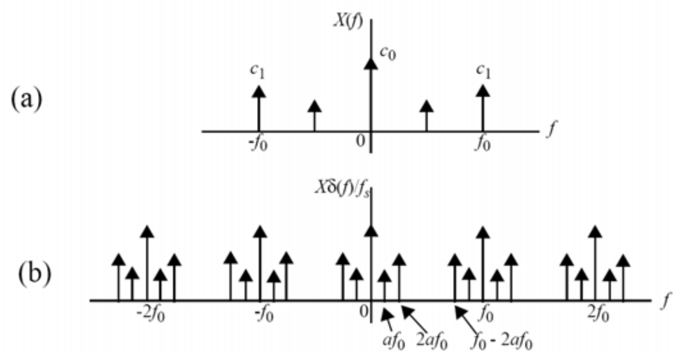

Figure 2

(ii) Next, let $m = 2$, for which we deduce that the relationship between the original spectrum $X(f)$ and the sampled spectrum $X_\delta(f)/f_s$ is pictured as shown in Fig. 2. The results displayed here follow from the evaluation of Eq. (4) for $m = 2$.

Based on the results depicted in Figs. 1 and 2, we may draw the following conclusions:

- The part of the spectrum $X_\delta(f)/f_s$ centered on the origin $f = 0$ is a compressed version of the original spectrum $X(f)$.
- The original spectrum $X(f)$ can be recovered from $X_\delta(f)/f_s$ by using a low-pass filter, provided there is no spectral overlap. In both figures, there is no spectral overlap. For this to be so, in Fig. 1(b) with $m = 1$ we must choose

$$(f_0 - af_0) > af_0$$

or

$$a < \frac{1}{2} \qquad (7)$$

In the case of Fig. 2(b) with $m = 2$, we must choose

$$(f_0 - 2af_0) > 2af_0$$

or

$$a < \frac{1}{4} \qquad (8)$$

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Problem 5-28 continued

Generalizing these two results, we may say that spectral overlap in the sampled spectrum $X_\delta(f)/f_s$ is avoided provided that we choose

$$a < \frac{1}{2m}$$

However, the choice of $1/2m$ does not leave any room for the design of a realizable low-pass reconstruction filter. This last provision is made by choosing

$$a < \frac{1}{2M+1} \qquad (9)$$

- From Fourier transform theory, we recall that spectral compression in the frequency domain corresponds to signal expansion in the frequency domain. We therefore conclude that provided the choice of parameter a satisfies Eq. (9), then we may use the scheme described in Fig. 5.28 to expand the time display of a periodic waveform with highest frequency component mf_0 and do so with a realizable reconstruction filter, provided that parameter a satisfies the condition of Eq. (9).

Problem 5.29

Consider Fig. 1(a) that shows the mirror rotating counter clockwise about the horizontal axis at a rate of $2\pi f$ radians per second. At a given time t, the angular position of the position of the narrow horizontal strip on the television screen as seen in the mirror forms an angle of $2\pi ft$ with respect to the coordinate axes. The position of the narrow strip relative to the origin as seen in the mirror is described by

$$x(T_s) = \exp(j2\pi f T_s)$$

which is the sampled version of the complex exponential

$$x(t) = \exp(j2\pi ft)$$

(a) If there is exactly one revolution of the mirror between frames on the television screen, then the rotation speed of the mirror matches the sampling rate of the video signal. In this situation, the horizontal strip on the television screen does *not* appear to be rotating, as illustrated in Fig. 1(a).

(b) If however the mirror rotates at an angle less than π radians between television frames, then the rotation of the narrow strip as seen in the mirror appears like a left-to-right motion (i.e., backwards), as illustrated in Fig. 1(c). This situation implies that

$$2\pi f T_s < \pi$$

That is, with $T_s = 1/60$ seconds, the rotation rate of the mirror defined by $w = 2\pi f$ is

$$w < 60\pi \text{ radians/second}$$

which is one half of the television's sampling rate. If the rotation rate of the mirror satisfies this condition, then no aliasing occurs and the rotation of the mirror is visually consistent with the left-to-right motion.

On the other hand, if the mirror rotates between π and 2π radians between television frames, then the rotation of the mirror appears to be visually inconsistent with linear motion, as illustrated in Fig. 1(d). This inconsistent situation occurs when

$$\pi < 2\pi f T_s < 2\pi$$

or, with $T_s = 1/60$ seconds,

$$30 < f < 60 \text{ hertz}$$

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Problem 5-29 continued

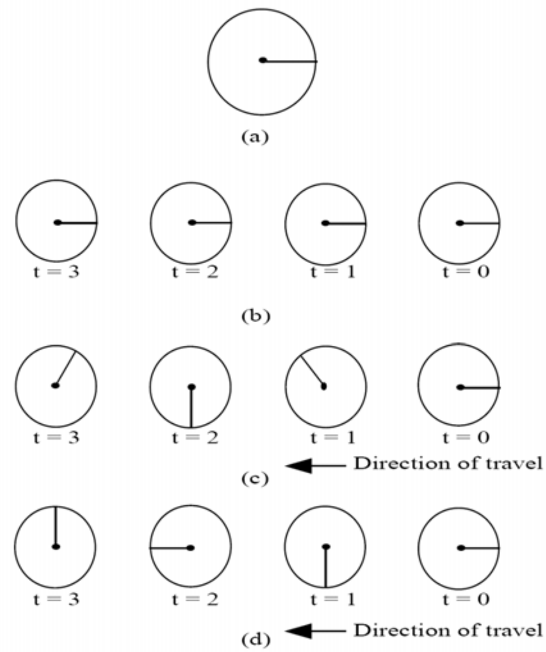

Figure 1

Problem 5.30

The first-order hold corresponds to extrapolating into the future with a straight line, as shown in Fig.1.

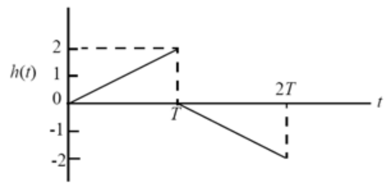

Figure 1

Specifically, the impulse response of the first-order hold may be expressed as

$$h(t) = \begin{cases} (t+T)/T & \text{for} \quad 0 \le t \le T \\ -(t-T)/T & \text{for} \quad T \le t \le 2T \\ 0 & \text{elsewhere} \end{cases} \tag{1}$$

Equivalently, we may express $h(t)$ as

$$h(t) = u(t) + \frac{t}{T}u(t) - 2u(t-T)$$
$$-2\frac{t-T}{T}u(t-T) + u(t-2T) + \frac{t-2T}{T}u(t-2T) \tag{2}$$

where $u(t)$ is the unit step function.

(a) Taking the Fourier transform of Eq. (2) and using the Fourier-transform pairs of Table A6.2, we may therefore express the frequency response of the first-order hold as

$$H(f) = \frac{1}{j2\pi f} + \frac{1}{T(j2\pi f)^2} - \frac{2}{j2\pi f}\exp(-j2\pi fT)$$
$$-\frac{2}{T(j2\pi f)^2} + \frac{1}{j2\pi f}\exp(-j4\pi fT) + \frac{1}{T(j2\pi f)^2}\exp(-j4\pi fT)$$

which, after collecting and simplifying terms, yields

$$H(f) = T(1 + j2\pi fT)\left(\frac{1-\exp(-j2\pi fT)}{j2(\pi fT)}\right)^2 \tag{3}$$

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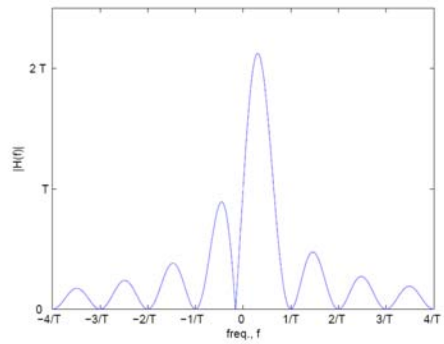
Problem 5-30 continued

(b) Figure 2 shows the magnitude and phase responses of the first-order hold.



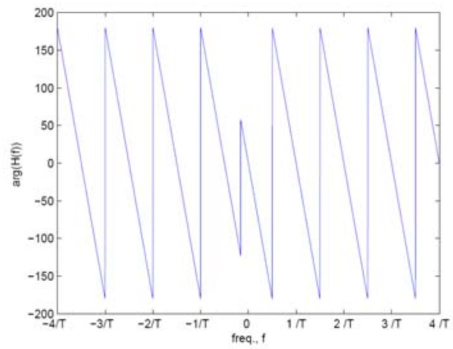


Figure 2

(c) For perfect reconstruction of the original analog signal, we need an equalizer whose transfer function is the inverse of $H(f)$ of Eq. (3), as shown by

$$H_{\mathrm{eq}}(f) = \frac{1}{H(f)}$$

$$= \frac{1}{T(1+j2\pi fT)}\left(\frac{j2\pi fT}{1-\exp(-j2\pi fT)}\right)^2 \tag{4}$$

For a duty cycle $(T/T_s) = 0.1$, the use of Eq. (4) yields

$$H_{\mathrm{eq}}(f_s) = \frac{1}{T}(0.8732 + 0.0589)$$

(d) For the sinusoidal input

$x(t) = \cos(50t)$

and $f_s = 100$Hz and T - 0.01, Fig. 3(c) shows the response produced by the first-order hold. Part (b) of the figure shows the corresponding response of the sample-and-hold filter. Comparing these two parts of Fig. 3, we may make the following observations:

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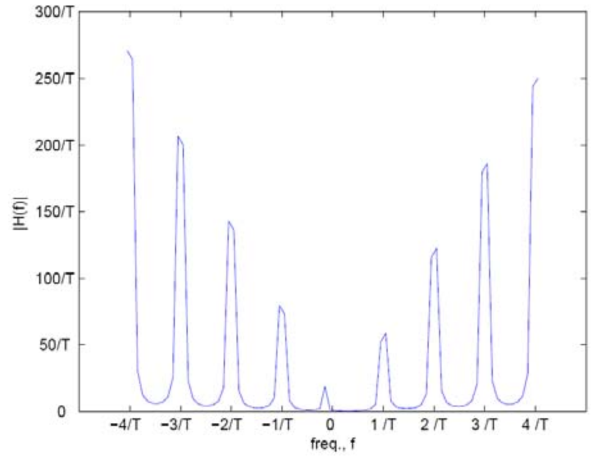
Problem 5-30 continued



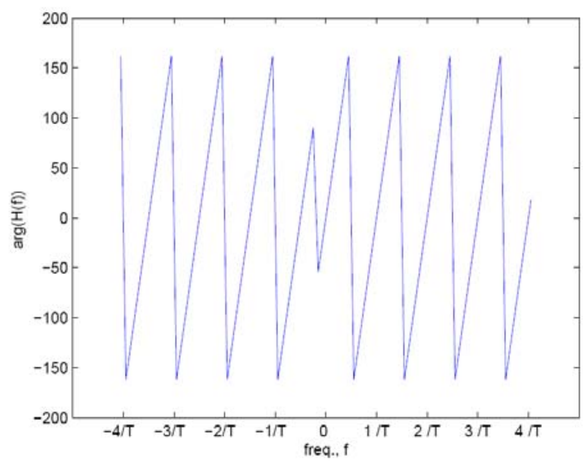


Figure 3

Problem 5.31

(a) Starting with the Fourier-transform pair

$$\exp(-\pi t^2) \rightleftharpoons \exp(-\pi f^2) \tag{1}$$

and applying the differentiation property of the Fourier transform to Eq. (1), we write

$$\frac{d}{dt}\exp(-\pi t^2) \rightleftharpoons j2\pi f \exp(-\pi f^2)$$

or, equivalently

$$-2\pi t \exp(-\pi t^2) \rightleftharpoons j2\pi f \exp(-\pi f^2) \tag{2}$$

Multiplying the left-hand side of Eq. (1) by A and invoking the linearity property of the Fourier transform, we go on to write

$$-2\pi t A \exp(-\pi t^2) \rightleftharpoons j2\pi f A \exp(-\pi f^2)$$

Simplifying terms:

$$tA\exp(-\pi t^2) \rightleftharpoons jfA\exp(-\pi f^2) \tag{3}$$

Finally, applying the dilation property of the Fourier transform to Eq. (3). we get

$$A\left(\frac{t}{\tau}\right)\exp\left(-\pi\left(\frac{t}{\tau}\right)^2\right) \rightleftharpoons -j\tau f A\exp(-\pi f^2\tau^2) \tag{4}$$

The left-hand side of this transform pair is recognized as the time function (see Eq. (5.39))

$$v(t) = A\left(\frac{t}{\tau}\right)\exp\left(-\pi\left(\frac{t}{\tau}\right)^2\right) \tag{5}$$

From Fig. 5.22, we see that the maximum value of $v(t)$ is +1. To find this maximum, we differentiate $v(t)$ with respect to time t and set the result equal to zero, obtaining

$$\frac{A}{\tau}\exp\left(-\pi\left(\frac{t}{\tau}\right)^2\right) - A\left(\frac{t}{\tau}\right)(2\pi t/\tau)\exp(-\pi t/\tau^2) = 0$$

Cancelling common terms and solving for t_{max}/τ, we get

$$\frac{t_{max}}{\tau} = \left(\frac{1}{2\pi}\right)^{1/2} \tag{6}$$

Using this value in Eq. (5):

$$v(t_{max}) = A\left(\frac{1}{2\pi}\right)^{1/2}\exp\left(-\frac{1}{2}\right)$$

With $v(t_{max}) = 1$, it follows that

$$A = (2\pi)^{1/2}\exp\left(\frac{1}{2}\right) = 4.1327$$

(b) The formula used to plot the spectrum of Fig. 5.23 is defined by the Fourier transform on the right-hand side of Eq. (4), that is,

$$V(f) = -j2\pi f A\exp(-\pi f^2\tau^2) \tag{7}$$

Problem 6.1

The pulse shape $p(t)$ of a baseband binary PAM system is defined by

$$p(t) = \text{sinc}\left(\frac{t}{T_b}\right)$$

where T_b is the bit duration of the input binary data. The amplitude levels at the pulse generator output are +1 V or -1 V, depending on whether the binary symbol at the input is 1 or 0, respectively. Sketch the waveform at the output of the receiving filter in response to the input data 001101001.

Solution

For the input data sequence 001101001, the waveform at the output receiving filter consists of the positive sinc pulse +sinc(t/T_b) every time symbol 1 is transmitted and the negative sinc pulse -sinc(t/T_b) every time symbol 0 is transmitted. Moreover, there will be no intersymbol interference present in this waveform because the sinc pulse for a particularly symbol goes through zero whenever another symbol is transmitted.

Problem 6.2

Show that for positive frequencies, the area under the normalized raised-cosine curve of $P(f)/(\sqrt{E}/2B_0)$ versus f/B_0 is equal to unity for all values of the roll-off factor in the range $0 < \alpha \leq 1$. A similar statement holds for negative frequencies.

Solution

For $\alpha = 0$, the normalized raised-cosine curve reduces to the idealized Nyquist channel, for which the area under this curve for the frequencies is immediately seen to be unity. For nonzero values of α in the range $0 < \alpha < 1$, the raised-cosine curve is odd-symmetric about the value $P(f)/(\sqrt{E}2B_0) = 0.5$. Consequently, the area under this normalized curve remains equal to unity for positive frequencies.

Problem 6.3

Given that $P(f)$ is the Fourier transform of a pulse-like function $p(t)$, we may state the following theorem:[1]

The pulse $p(t)$ decreases asymptotically with time as $1/t^{k+1}$ provided that the following two conditions hold:

1. The first k-1 derivatives of the Fourier transform $P(f)$ with respect to frequency f are all continuous.
2. The kth derivative of $P(f)$ is discontinuous.

Demonstrate the validity of this theorem for the three different values of α plotted in Fig. 6.3(a).

Solution

Consider first the idealized Nyquist channel for which $\alpha = 0$. With the brick-wall characteristic of this limiting case, it is immediately apparent that the Fourier transform $P(f)$ has *no* continuous derivatives with respect to f. Hence, according to the theorem, the inverse Fourier transform $p(t)$ decreases asymptotically as $1/|t|$; this is confirmed by the formula of Eq. (6.14), where the numerator ranges between -1 and +1, whereas the denominator is proportional to t.

Consider next the case of a raised-cosine pulse $p(t)$ defined in Eq. (6.19), rewritten here as

$$p(t) = \sqrt{E}\ \frac{\sin(2\pi B_0 t)}{2\pi B_0 t}\left(\frac{\cos(2\pi\alpha B_0 t)}{1 - 16\alpha^2 B_0^2 t^2}\right)$$

In this case, we readily see that $p(t)$ decreases asymptotically as $1/|t|^3$, for $0 < \alpha \leq 1$. Examining the two plots shown in Fig. 6.3(a), we see that the first derivative of $P(f)$ for this range of values of α is continuous, but the second derivative is discontinuous. Here again validity of the theorem is established.

1. For a detailed discussion of this theorem, see Gitlin, Hayes and Weinstein (1992), p.258.

Problem 6.4

Equation (6.17) defines the raised-cosine pulse spectrum $P(f)$ as real-valued and therefore zero delay. In practice, every transmission system experiences some finite delay. To accommodate this practicality, we may associate with $P(f)$ a linear phase characteristic over the frequency band $0 \le |f| \le 2B_0 - f_1$.

(a) Show that this modification of $P(f)$ introduces a finite delay into its inverse Fourier transform, namely, the pulse shape $p(t)$.

(b) According to Eq. (6.19), $p(t)$ represents a non-causal time response. The delay introduced into $p(t)$ through the modification of $P(f)$ has also a beneficial effect, tending to make $p(t)$ essentially causal. For this to happen however, the delay must not be less than a certain value dependent on the roll-off factor α. Suggest suitable values for the delay for $\alpha = 0$, 1/2, and 1.

Solution

(a) Let the linear phase characteristic appended to $P(f)$ be

$$\theta(f) = 2\pi f\tau$$

where τ is delay to be determined. Then, the modified raised-cosine pulse spectrum is defined by

$$P_{\text{modified}}(f) = P(f)e^{-j\theta(f)}$$
$$= P(f)e^{-j2\pi f\tau}$$

Invoking the time-shifting property, we therefore have

$$p_{\text{modified}}(t) = p(t-\tau)$$

where $p(t)$ is defined by Eq. (6.19).

(b) For $p_{\text{modified}}(t)$ to be causal, it has to be zero for $t < 0$.

From Fig. 6.3(b) in the text, we deduce that we may essentially set

(i) $\tau = 5$s for $\alpha = 0$

(ii) $\tau = 3$s for $\alpha = 1/2$

(iii) $\tau = 2.5$s for $\alpha = 1$

Increasing α corresponds to increasing transmission bandwidth B_T. We therefore find that as the transmission bandwidth B_T is increased, the necessary delay τ is progressively reduced, which is in accord with the inverse relationship that exists between behaviors of a function in the time- and frequency-domains.

(c) The slope of $\theta(f)$ with respect to f is

$$\frac{\partial\theta(f)}{\partial f} = 2\pi\tau$$

Hence,

(i) slope = -10π for $\alpha = 0$

(ii) slope = -6π for $\alpha = 1/2$

(iii) slope = -5π for $\alpha = 1$

Problem 6.5

Starting with the formula of Eq. (6.24) and using the definition of Eq. (6.26), demonstrate the property of Eq. (6.25).

Solution

Using $f' = f - B_0$ in Eq. (6.24), we may express the second line of Eq. (6.24) in the text for positive frequencies

$$P_{\mathrm{v}}(f') = \frac{\sqrt{E}}{4B_0}\left\{1 + \cos\left[\frac{\pi(f' + B_0 - f_1)}{2(B_0 - f_1)}\right]\right\}$$

$$= \frac{\sqrt{E}}{4B_0}\left\{1 + \cos\left[\frac{\pi}{2} + \frac{\pi f'}{2(B_0 - f_1)}\right]\right\}$$

$$= \frac{\sqrt{E}}{4B_0}\left\{1 - \sin\left[\frac{\pi(f')}{2(B_0 - f_1)}\right]\right\} \qquad \text{for } f_1 - B_0 \le f' \le 0 \qquad (1)$$

Similarly, we may express the third line of Eq. (6.24) as

$$P_{\mathrm{v}}(f') = \frac{\sqrt{E}}{4B_0}\left\{\sin\left(\left(\left[\frac{\pi(f')}{2(B_0 - f_1)}\right]\right) - 1\right)\right\} \qquad \text{for } 0 \le f' \le B_0 - f_1 \qquad (2)$$

From Eqs. (1) and (2), we readily see that

$P_{\mathrm{v}}(-f') = P_{\mathrm{v}}(f')$

which is the desired property.

Problem 6.6

Assume the following perfect conditions:

- The residual distortion in the data transmission system is zero.
- The pulse shaping is partitioned equally between the transmitter-channel combination and the receiver.
- The transversal equalizer is infinitely long.

(a) Find the corresponding value of the equalizer's transfer function in terms of infinite the overall pulse spectrum $P(f)$.

(b) For the roll-off factor $\alpha = 1$, demonstrate that a transversal equalizer of length 6 would essentially satisfy the perfect condition found in part (a) of the problem.

Solution

(a) With the pulse-shaping shared equally between the transmit filter-channel combination and receive filter, we may use an equalizer of transfer function $P^{1/2}(f)$ to realize the receive filter, where $P(f)$ is the raised cosine-pulse spectrum.

(b) For a roll-off factor $\alpha = 0$, $P(f)$ reduces to the idealized brick-wall function

$$P(f) = \begin{cases} \frac{\sqrt{E}}{2B_0}, & \text{for } B_0 < f < B_0 \\ 0, & \text{otherwise} \end{cases}$$

which defines the Nyquist channel. In light of the transfer function of the equalizer (used to realize the receive filter) is defined by

$$P(f) = \begin{cases} \frac{E^{1/4}}{(2B_0)^{1/2}}, & \text{for } B_0 < f < B_0 \\ 0, & \text{otherwise} \end{cases}$$

Correspondingly, the impulse response of the equalizer is required to pass through an infinite number of time instants at $t = \pm 1/(2B_0)$, $\pm 1/B_0$, $\pm 3/(2B_0)$,.... We may satisfy this idealized requirement by using an equalizer of infinite length. Such an equalizer would have an infinite number of adjustable parameters $\ldots W_N, \ldots, W_{-1}, W_0, W_1, \ldots, W_N$ that can be used to satisfy the zero-forcing basis of Eq. (6.43) of the text. In practice, however, the idealized impulse response of the channel reduces effectively to zero at some large enough time t, which, in turn, means that an equalizer of large enough length can be used to satisfy the idealized Nyquist channel.

Note: In the first printing of the book, the following correction in the first line of part (b) of Problem 6.6 should be made: - Roll-off factor $\alpha = 0$.

Problem 6.7

Since $P(f)$ is an even real-valued function, its inverse Fourier transform may be simplified to the formula

$$p(t) = 2\int_0^\infty P(f)\cos(2\pi ft)df \qquad (1)$$

The $P(f)$ is itself defined by Eq. (6.17) which is reproduced here in the following form (ignoring the scaling factor $\sqrt{E}$ for convenience of presentation)

$$P(f) = \begin{cases} \dfrac{1}{2B_0}, & 0 < |f| f_1 \\ \dfrac{1}{4B_0}\left\{1 + \cos\left[\dfrac{\pi(|f| - f_1)}{2B_0 - 2f_1}\right]\right\}, & f_1 < f < 2B_0 - f_1 \\ 0, & |f| > 2B_0 - f_1 \end{cases} \qquad (2)$$

Hence, using Eq. (2) in (1) and recognizing that $\alpha = (B_0 - f_1)/B_0$, we may write

$$p(t) = \frac{1}{B_0}\int_0^{f_1}\cos(2\pi ft)df + \frac{1}{2B_0}\int_{f_1}^{2B_0-f_1}\left[1 + \cos\left(\frac{\pi(f - f_1)}{2B_0\alpha}\right)\right]\cos(2\pi ft)df$$

$$= \left[\frac{\sin(2\pi ft)}{2\pi B_0 t}\right] + \left[\frac{\sin(2\pi ft)}{4\pi B_0 t}\right]_{f_1}^{2B_0 - f_1}$$

$$+ \frac{1}{4}B_0\left[\frac{\sin\left(2\pi ft + \dfrac{\pi(f - f_1)}{2B_0\alpha}\right)}{2\pi t + \pi/2B_0\alpha}\right]_{f_1}^{2B_0 - f_1} + \frac{1}{4B_0}\left[\frac{\sin\left(2\pi ft - \dfrac{\pi(f - f_1)}{2B_0\alpha}\right)}{2\pi t + \pi/2B_0\alpha}\right]_{f_1}^{2B_0 - f_1}$$

$$= \frac{\sin(2\pi f_1 t)}{4\pi B_0 t} + \frac{\sin[2\pi t(2B_0 - f_1)]}{4\pi B_0 t}$$

$$-\frac{1}{4B_0}\ \frac{\sin(2\pi f_1 t) + \sin[2\pi t(2B_0 - f_1)]}{2\pi t - \pi/2B_0\alpha} + \frac{\sin(2\pi f_1 t) + \sin[2\pi t(2B_0 - f_1 t)]}{2\pi t - \pi/2B_0\alpha}$$

$$= \frac{1}{B_0}[\sin(2\pi f_1 t) + \sin[2\pi t(2B_0 - f_1)]]\left[\frac{1}{4Wt} - \frac{\pi t}{(2\pi t)^2 - (\pi/2B_0\alpha)^2}\right]$$

$$= \frac{1}{B_0}[\sin(2\pi B_0 t)\cos(2\pi\alpha B_0)]\left[\frac{-\pi/(2B_0\alpha)^2}{4\pi t[(2\pi t)^2 - \pi/(2B_0\alpha)^2]}\right]$$

$$= \mathrm{sinc}(2B_0 t)\cos(2\pi\alpha B_0 t)\left[\frac{1}{1 - 16\alpha^2 B_0{}^2 t^2}\right] \qquad (3)$$

Equation (3) is a reproduction of Eq. (6.19), except for the scaling factor $\sqrt{E}$ which we ignored in Eq. (2) for convenience of presentation.

Problem 6.8

Starting with Eq. (3) in the solution to Problem 6.7, reproduced here for $0 < \alpha \le 1$:

$$p(t) = \text{sinc}(2B_0 t)\left(\frac{\cos(2\pi\alpha B_0 t)}{1 - 16\alpha^2 B_0^2 t^2}\right)$$

For $\alpha = 1$, this formula reduces to

$$p(t) = \text{sinc}(2B_0 t)\left(\frac{\cos(2\pi B_0 t)}{1 - 16B_0^2 t^2}\right) \tag{1}$$

Next, using the trigonometric identity

$$\sin(A)\cos(A) = \frac{1}{2}\sin(2A)$$

and the definition of the sinc function

$$\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

we may go on to write

$$\begin{aligned}\text{sinc}(2B_0 t)\cos(2\pi B_0 t) &= \frac{\sin(2\pi B_0 t)\cos(2\pi B_0 t)}{2\pi B_0 t} \\ &= \frac{\sin(4\pi B_0 t)}{4\pi B_0 t} \\ &= \text{sinc}(4B_0 t) \qquad (2)\end{aligned}$$

Accordingly, using Eq. (2) in (1), we get

$$p(t) = \frac{\text{sinc}(4B_0 t)}{1 - 16B_0^2 t^2}$$

which is the desired result, except for the scaling factor $\sqrt{E}$.

Problem 6.9

The bandwidth B of a raised cosine pulse spectrum is $2B_0 - f_1$, where $B_0 = 1/2T_b$ and $f_1 = B_0(1 - a)$. Thus $B = B_0(1 + \alpha)$. For a data rate of 56 kilobits per second, $B_0 = 28$ kHz.

(a) $\alpha = 0.25$,
$B = 28$ kHz x 1.25 = 35 kHz

(b) $\alpha = 0.5$,
$B = 28$ kHz x 1.5 = 42 kHz

(c) $\alpha = 0.75$,
$B = 28$ x 1.75 = 49 kHz

(d) $\alpha = 1.0$,
$B = 28$ x 2 = 56 kHz

Problem 6.10

The raised cosine pulse bandwidth $B_T = 2B_0 - f_1$, where $B_0 = 1/2T_b$. For this channel, $B_T = 75$ kHz. For the given bit duration, $B_0 = 50$ kHz. Then,

$$\begin{aligned} f_1 &= 2B_0 - B_T \\ &= 25 \text{ kHz} \\ \alpha &= 1 - f_1/B_T \\ &= 0.5 \end{aligned}$$

The design parameters of the required raised-cosine pulse spectrum are $f_1 = 25$ kHz and $\alpha = 0.5$.

Problem 6.11

The transmission bandwidth B_T is related to the excess bandwidth f_v by the formula (see Eqs. (6.21) and (6.22))

$$B_T = B_0 + f_v$$

where $B_0 = 1/(2T_b)$. We may therefore express the bit rate $1/T_b$ as a function of the excess bandwidth f_v as follows:

$$\frac{1}{T_b} = 2(B_T - f_v) \qquad (1)$$

From Eq. (1), we see that the bit rate $1/T_b$ decreases linearly with the excess bandwidth f_v for a fixed channel bandwidth B_T. Specifically, with B_T = 3 kHz, the bit rate versus excess bandwidth graph takes the form shown in Fig. 1. Note that the excess bandwidth f_v attains its largest value when the roll-off factor α equals unity, in which case f_v = 3 kHz.

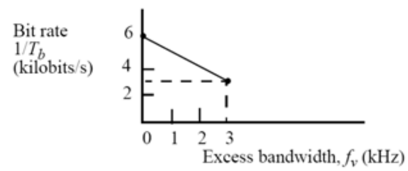


Figure 1

Problem 6.12

We are given the following specifications:

$$B_T = 3 \text{ kHz}$$

$$\frac{1}{T_b} = 4.5 \text{ kilobits/s}$$

(a) The transmission bandwidth is related to the roll-off factor by the formula (see Eq. (6.21))

$$B_T = B_0(1+\alpha) \qquad (1)$$

where

$$B_0 = 1/(2T_b)$$

Therefore, with $(1/T_b) = 4.5$ kilobits/s, we have

$$B_0 = 2.25 \text{ kHz}$$

Hence, solving Eq. (1) for the roll-off factor, we get

$$\alpha = \frac{B_T}{B_0} - 1$$

$$= \frac{3}{2.25} - 1$$

$$= \frac{1}{3}$$

(b) The excess bandwidth is defined (see Eq. (6.22))

$$f_v = \alpha B_0$$

$$= \frac{1}{3} \times 2.25$$

$$= 0.75 \text{ kHz}$$

Problem 6.13

According to Eq. (6.30), the pulse-shaping criterion for zero-intersymbol interference is embodied in the relation

$$\sum_{m=-\infty}^{\infty} P\left(f - \frac{m}{T}\right) = \text{constant} \tag{1}$$

where $P(f)$ is pulse-shaping spectrum and $1/T$ is the signaling rate.

(a) The pulse-shaping spectrum of Fig. 6.13(a) is defined by

$$P(f) = \begin{cases} \sqrt{E}/(2B_0) & \text{for} \quad f = 0 \\ \dfrac{\sqrt{E}}{2B_0}\left(1 - \dfrac{f}{B_0}\right) & \text{for} \quad 0 < f < B_0 \\ 0 & \text{for} \quad f = B_0 \end{cases} \tag{2}$$

Substituting Eq. (2) into (1) leads to the following condition on the signaling rate

$$\frac{1}{T} = \frac{B_0}{2}$$

or, equivalently,

$$B_0 = 2/T \tag{3}$$

(b) The pulse-shaping spectrum of Fig. 6.12(b) is defined by

$$P(f) = \begin{cases} \sqrt{E}/(2B_0) & \text{for } 0 \le |f| < f_1 \\ \dfrac{\sqrt{E}}{2B_0}\left(1 - \dfrac{f - f_1}{B_0 - f_1}\right) & \text{for } f_1 < f < B_0 \\ 0 & \text{for } f > B_0 \end{cases} \tag{4}$$

Substituting Eq. (3) into (1) leads to the following condition on the signaling rate

$$\frac{1}{T} = \frac{1}{2}(f_1 + B_0)$$

Equivalently, for a given f_1, we require that

$$B_0 = \frac{2}{T} - f_1 \tag{5}$$

(c) Among the four pulse-shaping spectra described in Figs. 6.2(a), 6.3(a), 6.12(a) and 6.12(b) the prescriptions defined in Fig. 6.3(a) corresponding to the roll-off factor $\alpha = 1/2$ and $\alpha = 1$ are the preferred choices in practice for the following reasons:

- Mathematical simplicity and therefore relative ease of practical realization.
- Improved signaling rate compared to the prescriptions described in Fig.s 6.12(c) and 6.12(b).

Problem 6.14

The transmission bandwidth is maintained at the value

$B_T = 3 \text{ kHz}$

In using an 8-level PAM system, the signaling rate is raised to

$$\frac{1}{T} = (\log_2 8) \times \left(\frac{1}{T_b}\right), \qquad T_b = \text{bit duration}$$

$$= 3 \times 4.5$$

$$= 13.5 \text{ kilobits/s}$$

However, the symbol rate is maintained at 4.5×10^3 symbols/s. Hence, as in Problem 6.12,

(a) The roll-off factor remains at $\alpha = 1/3$.

(b) The excess bandwidth remains at $f_v = 0.75$ kHz.

Problem 6.15

The codeword consists of $\log_2(128) = 7$ bits. With an additional bit added for synchronization, the overall codeword consists of 8 bits. The method of data transmission is quaternary (i.e., 4-level) PAM, and the roll-off factor $\alpha = 1$.

(a) For binary PAM, the signaling rate is defined by (see Eqs. (6.13) and (6.21))

$$\frac{1}{T_b} = \frac{2B_T}{1+\alpha} \tag{1}$$

For $\alpha = 1$ and $B_T = 13$ kHz, the use of Eq. (1) yields

$$\frac{1}{T_b} = \frac{2 \times 13}{1+1}$$
$$= 13 \text{ kilobits/s}$$

The signaling rate of the quaternary PAM system is therefore

$$\frac{1}{T} = \frac{\log_2 4}{T_b}$$
$$= 2 \times 13 \text{ kilosymbols/s}$$

(b) Each element of the overall codeword of the PCM signal must fit into the bit duration

$$T_b = \frac{1}{13 \times 10^3} \text{seconds}$$
$$= 77 \ \mu\text{s}$$

With each code-word consisting of 8 bits, the code-word occupies the duration

$$T_s = 8T_b$$
$$= 8 \times 77 = 616 \ \mu\text{s}$$

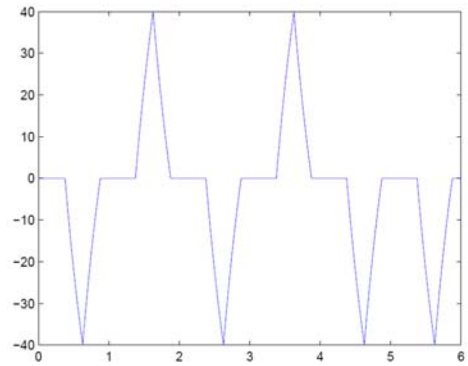
The sampling rate applied to the analog signal is therefore

$$f_s = \frac{1}{T_s}$$
$$= \frac{10^6}{616} \text{Hz}$$
$$= 162 \text{ kHz}$$

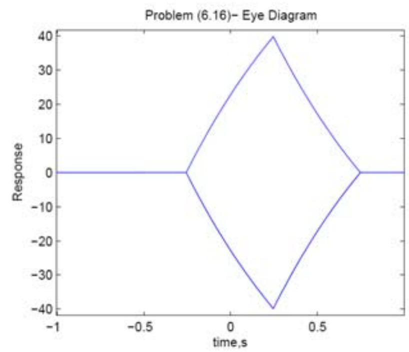
The highest frequency component of the analog signal is therefore

$$W = \frac{f_s}{2} = 81 \text{ kHz}$$

Problem 6.16

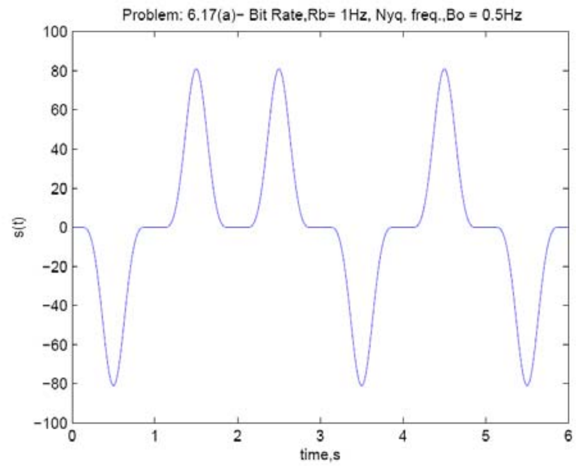


40
30
20
10
0
−10
−20
−30
−40
0
1
2
3
4
5
6

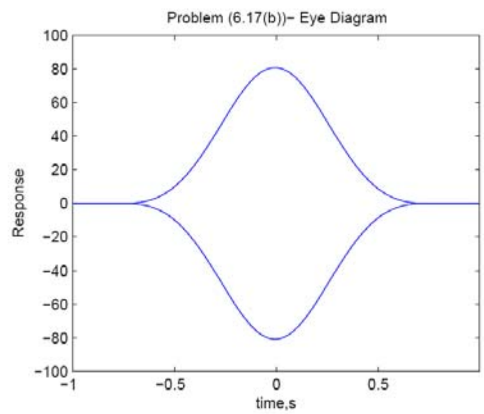


Problem (6.16)− Eye Diagram
Response
time,s
40
30
20
10
0
−10
−20
−30
−40
−1
−0.5
0
0.5

Problem 6.17



Problem: 6.17(a)– Bit Rate,Rb= 1Hz, Nyq. freq.,Bo = 0.5Hz
s(t)
time,s



Problem (6.17(b))– Eye Diagram
Response
time,s

Problem 6.18

(a) The impulse response of the data-transmission system is defined by (see Fig. 1)

$c_n = \{0.0, 0.15, 0.68, -0.22, 0.08\}$

Using a three-tap transversal filter for zero-forcing equalization, we write in accordance with Eq. (6.43):

$$\begin{bmatrix} 0.68 & 0.15 & 0.0 \\ -0.22 & 0.68 & 0.15 \\ 0.08 & -0.22 & 0.68 \end{bmatrix} \begin{bmatrix} w_{-1} \\ w_0 \\ w_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad (1)$$

In Eq. (1), we have set $\sqrt{E} = 1$ to simplify the presentation. Solving this simultaneous system of three equations, we obtain the tap-weight (parameter) vector,

$$\mathbf{w} = \begin{bmatrix} w_{-1} \\ w_0 \\ w_1 \end{bmatrix} = \begin{bmatrix} 0.68 & 0.15 & 0.0 \\ -0.22 & 0.68 & 0.15 \\ 0.08 & -0.22 & 0.68 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -0.2825 \\ 1.2805 \\ 0.4475 \end{bmatrix} \quad (2)$$

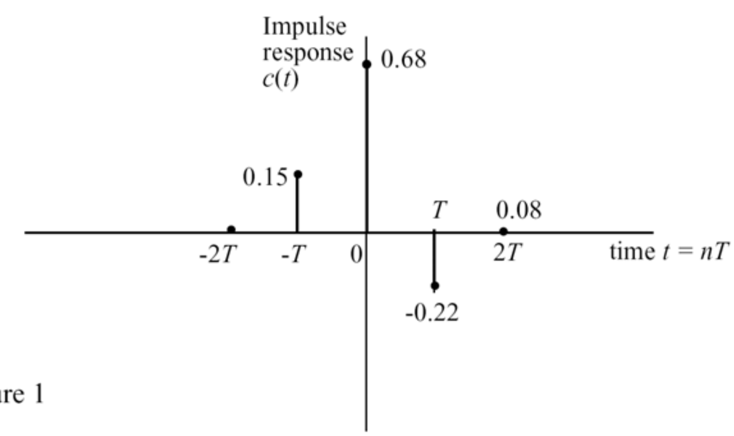


Figure 1

Continued on next slide

Problem 6-18 continued

(b) The residual intersymbol interference produced at the equalizer output is given by

$$\mathbf{z} = \mathbf{c}\mathbf{w} \tag{3}$$

where

$$\mathbf{c} = \begin{bmatrix} 0.15 & 0.0 & 0.0 \\ 0.68 & 0.15 & 0.0 \\ -0.22 & 0.68 & 0.15 \\ 0.08 & -0.22 & 0.68 \\ 0.0 & 0.08 & -0.22 \end{bmatrix} \tag{4}$$

Therefore, using Eqs. (4) and (2) in (3), we get the residual interference vector

$$\mathbf{z} = \begin{bmatrix} -0.0424 \\ 0 \\ 1 \\ 0 \\ 0.004 \end{bmatrix} \tag{5}$$

(c) From Eq. (5), we see that the largest contribution to the residual interference is

Problem 6.19

In this problem, the transversal zero-forcing equalizer has five adjustable weights. As in Problem 6.18, the unequalized impulse response is defined by

$c_n = \{0.0, 0.15, 0.68, -0.22, 0.08\}$

Accordingly, application of Eq. (6.43) yields (again setting $\sqrt{E} = 1$ to simplify the presentation)

$$\underbrace{\begin{bmatrix} 0.68 & 0.15 & 0.0 & 0.0 & 0.0 \\ -0.22 & 0.68 & 0.15 & 0.0 & 0.0 \\ 0.08 & -0.22 & 0.68 & 0.15 & 0.0 \\ 0.0 & 0.08 & -0.22 & 0.68 & 0.15 \\ 0.0 & 0.0 & 0.08 & -0.22 & 0.68 \end{bmatrix}}_{\mathbf{c}} \underbrace{\begin{bmatrix} w_{-2} \\ w_{-1} \\ w_0 \\ w_1 \\ w_2 \end{bmatrix}}_{\mathbf{w}} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad (1)$$

Solving this system of five simultaneous equations for the tap-weight vector, we get

$$\mathbf{w} = \begin{bmatrix} -0.0581 \\ -0.2635 \\ 1.2800 \\ 0.4465 \\ -0.0061 \end{bmatrix} \quad (2)$$

Comparing Eq. (1) of this problem with Eq. (1) of the previous problem, we see some basic differences and therefore consequences:

(i) Unlike Problem 6.18, the 5-by-5 metric **c** in Eq. (1) has a row (namely, the third row) which completely describes the unequalized impulse response of the data-transmission system.

(ii) As a consequence of point (i), the 5-by-1 parameter vector **w** produces complete equalization of the system; that is, unlike Problem 6.18, there is no residual intersymbol interference left after equalization.

(iii) The zero residual interference is the result of using a five-tap equalizer which has sufficient degrees of freedom to force each element of the impulse response $\{c_n\}$ down to the desired value of zero.

Problem 6.20

(a) When the two-level sequence embodying

$$a_k = \begin{cases} +1 \text{ if symbol } b_k \text{ is } 1 \\ -1 \text{ if symbol } b_k \text{ is } -1 \end{cases} \tag{1}$$

is applied to the duobinary conversion filter, the sequence is converted into a three-level output defined by

$$c_k = a_k + a_{k-1} \tag{2}$$

The three levels of c_k are -2, 0, and +2. One effect of transforming Eq. (1) into Eq. (2) is to produce correlated three-level sequence c_k from an uncorrelated two-level sequence a_k.

The overall transfer function of the duobinary conversion filter is therefore defined by

$$\begin{aligned} H(f) &= H_{\text{Nyquist}}(f)[1 + \exp(-j2\pi f T_b)] \\ &= H_{\text{Nyquist}}(f)[\exp(j\pi f T_b) + \exp(-j\pi f T_b)]\exp(-j\pi f T_b) \\ &= 2H_{\text{Nyquist}}(f)\cos(\pi f T_b)\exp(-j\pi f T_b) \end{aligned} \tag{3}$$

For an ideal Nyquist channel, $B_0 = 1/2T_b$. Ignoring the scaling factor $1/T_b$, we may therefore write

$$H_{\text{Nyquist}}(f) = \begin{cases} 1, & |f| \le 1/2T_b \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

Substituting Eq. (4) into (3), we obtain

$$H(f) = \begin{cases} 2\cos(\pi f T_b)\exp(-j\pi f T_b), & |f| \le 1/2T_b \\ 0, & \text{otherwise} \end{cases} \tag{5}$$

(b) From the first line of Eq. (3) and the defining Eq. (4), we find that the impulse response of the duobinary conversion filter is

$$\begin{aligned} h(t) &= \frac{\sin(\pi t/T_b)}{\pi t/T_b} + \frac{\sin[\pi(t-T_b)/T_b]}{\pi(t-T_b)/T_b} \\ &= \frac{\sin(\pi t/T_b)}{\pi t/T_b} + \frac{\sin[(\pi t/T_b)-\pi]}{\pi(t-T_b)/T_b} \\ &= \frac{\sin(\pi t/T_b)}{\pi t/T_b} - \frac{\sin(\pi t/T_b)}{\pi(t-T_b)/T_b} \\ &= \frac{T_b^2 \sin(\pi t/T_b)}{\pi t(T_b - t)} \end{aligned} \tag{6}$$

(c) The original sequence may be detected from the duobinary-coded sequence using decision feedback, as shown by

$$\hat{a}_k = c_k - \hat{a}_{k-1} \tag{7}$$

A major drawback of this detection rule is that for the current detection $\hat{a}_k$ to be correct, the previous detection $\hat{a}_{k-1}$ has to be correct. If this requirement is not satisfied, we have error propagation.

Problem 6.21

To overcome the error-propagation problem experienced in Problem 6.20, we use precoding before the duobinary coding, as shown in Fig. 6.14. The precoder is defined by

$$d_k = b_k \oplus d_{k-1} \tag{1}$$

where the symbol $\oplus$ denotes modulo-two addition (i.e., EXCLUSIVE OR) According to Eq. (1), we have

$$d_k = \begin{cases} \text{symbol } 1 & \text{if either } b_k \text{ or } d_{k-1} \text{ is } 1 \\ \text{symbol } 0 & \text{otherwise} \end{cases}$$

As before, the pulse-amplitude modulator output is therefore defined by $a_k = \pm 1$. Applying this sequence to the duobinary conversion filter, we get

$$c_k = a_k + a_{k-1} \tag{2}$$

Note that unlike the linear operation of duobinary coding of Eq. (2), the precoding of Eq. (1) is nonlinear.

The combined use of Eqs. (1) and (2) yields

$$c_k = \begin{cases} 0 & \text{if the original data symbol } b_k \text{ is } 1 \\ \pm 2 & \text{if } b_k \text{ is } 0 \end{cases} \tag{3}$$

From Eq. (3), we therefore deduce the following decision rule for detecting the original data sequence b_k from c_k, as follows:

If $|c_k| < 1$, say symbol b_k is 1

If $|c_k| > 1$, say symbol b_k is 0 (4)

which can be realized by using a rectifier followed by a threshold device.

The solutions to parts (a), (b) and (c) of the problem in response to the input sequence 0010110 are presented in Table 1.

Table 1: Illustrating Example 3 on Duobinary Coding

Binary sequence $\{b_k\}$		0	0	1	0	1	1	0
Precoded sequence $\{d_k\}$	1	1	1	0	0	1	0	0
Two-level sequence $\{a_k\}$	+1	+1	+2	-1	-1	+1	-1	-1
Duobinary coder output $\{c_k\}$		+2	+2	0	-2	0	0	-2
Binary sequence obtained by applying decision rule of Eq. (7.76)		0	0	1	0	1	1	0

Problem 6.22

(a) For the modified duobinary conversion filter shown in Fig. 6.15, we have

$$c_k = a_k - a_{k-2} \tag{1}$$

Here again, we find that a three-level sequence is generated. Specifically, for $a_k = \pm 1$, we find from Eq. (1) that c_k has three possible values: 2, 0, +2.

The overall transfer function of the modified duobinary conversion filter shown i Fig. 6.15 is therefore given by

$$\begin{aligned} H(f) &= H_{\text{Nyquist}}(f)[1 - \exp(-j4\pi f T_b)] \\ &= H_{\text{Nyquist}}(f)[\exp(j2\pi f T_b) - \exp(-j2\pi f T_b)]\exp(-j2\pi f T_b) \\ &= 2jH_{\text{Nyquist}}(f)\sin(2\pi f T_b)\exp(-j\pi f T_b) \end{aligned} \tag{2}$$

With

$$H_{\text{Nyquist}}(f) = \begin{cases} 1 & \text{for } |f| \le 1/2T_b \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

we may therefore express $H(f)$ as

$$H(f) = \begin{cases} 2j\sin(2\pi f T_b)\exp(-j2\pi f T_b) & \text{for } |f| \le 1/2T_b \\ 0 & \text{elsewhere} \end{cases} \tag{4}$$

which is the form of a half-cycle sine function.

(b) The corresponding impulse response of the modified duobinary conversion filter follows from the first line of Eq. (2); specifically,

$$\begin{aligned} h(t) &= \frac{\sin(\pi t/T_b)}{\pi t/T_b} - \frac{\sin[\pi(t-2T_b)/T_b]}{\pi(t-2T_b)/T_b} \\ &= \frac{\sin(\pi t/T_b)}{\pi t/T_b} - \frac{\sin[\pi(t/T_b) - 2\pi]}{\pi(t-2T_b)/T_b} \\ &= \frac{\sin(\pi t/T_b)}{\pi t/T_b} - \frac{\sin[\pi t/T_b]}{\pi(t-2T_b)/T_b} \\ &= \frac{2T_b^2\sin(\pi t/T_b)}{\pi t(2T_b - t)} \end{aligned} \tag{5}$$

(c) With the precoder in place at the front end of the modified duobinary conversion filter as shown in Fig. 6.15, we have

$$d_k = b_k \oplus d_{k-1} \tag{6}$$

where b_k is the incoming binary sequence and d_k is the precoder output.

Assuming the use of a polar representation for the precoded sequence d_k, we find that the original data sequence b_k may be detected from the encoded sequence c_k by disregarding the polarity; specifically,

If $|c_k| > 1$, say symbol b_k is 1

If $|c_k| < 1$, say symbol b_k is 0 (7)

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Problem 6-22 continued

(d) The virtues of modified duobinary coding are two-fold:

- In the absence of channel noise, the detected binary sequence b_k is exactly the same as the original data sequence b_k; this statement also applies to the duobinary coding with precoding.

- The use of Eq. (6) requires the addition of two extra bits to the precoded sequence b_k in accordance with Eq. (6). The composition of the decoded sequence $\hat{b}_k$ using Eq. (7) is invariant to the selection made for these two additional bits.

Note: In the first printing of the book, the delay element of the precoder in Fig. 6.15 should read $2T_b$ to be consistent with Eq. 7.

Problem 6.23

From Eq. (4) in the solution to Problem 6.22 we see that the transfer function of the modified duobinary conversion filter (shown in Fig. 6.15) is zero at $f = 0$. Hence, unlike the ordinary duobinary conversion filter, the modified duobinary conversion filter can be used to handle single-sideband transmission of data.

Specifically, Fig. 1(a) depicts the proposed data transmission system. The transmitter consists of two functional blocks:

- Modified duobinary conversion filter, which transforms the incoming binary data into a new format whose spectrum has low-frequency content around the origin.
- Single sideband modulator, which upconverts the transformed data to the desired band occupied by the lower or upper sideband of the modulated wave.

Correspondingly, the receiver consists of two functional blocks (see Fig. 1(b))

- Single sideband demodulator.
- Detector, consisting of a rectifier followed by decision device, for recovering the original binary data stream.

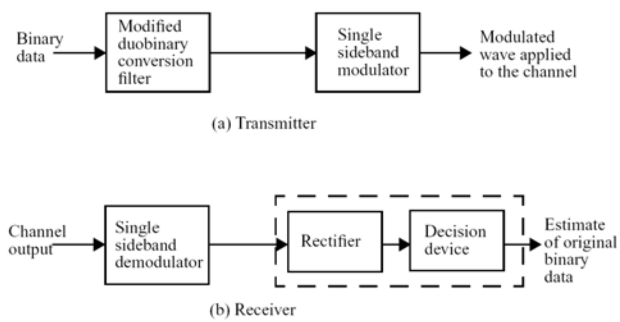

Figure 1

Problem 7.1

Invoking the band-pass assumption, show that

$$\int_0^{T_b} \sin(2\pi f_c t)\cos(2\pi f_c t)dt \approx 0$$

regardless of how the bit duration T_b is exactly related to f_c so long as $f_c >> 1/T_b$.

Solution

Let

$$I(T_b) = \int_0^{T_b} \sin(2\pi f_c t)\cos(2\pi f_c t)dt$$

Using the trigonometric identity

$$\sin(A)\cos(A) = \frac{1}{2}\sin(2A)$$

we may express $I(T_b)$ as

$$\begin{aligned} I(T_b) &= \frac{1}{2}\int_0^{T_b} \sin(4\pi f_c t)dt \\ &= \frac{1}{2} \cdot \frac{1}{4\pi f_c}\cos(4\pi f_c t)\Big|_{t=0}^{T_b} \\ &= \frac{1}{8\pi f_c}[\cos(4\pi f_c T_b) - 1] \end{aligned}$$

So long as $f_c > \frac{1}{T_b}$, we may set $\cos(4\pi f_c T_b) \approx 1$, in which case, $I(T_b) \approx 0$, thereby obtaining the desired result.

Problem 7.2

Show that Eq. (7.8) is *invariant* with respect to the carrier phase ϕ_c (i.e., it holds for all ϕ_c).

Solution

Assuming a carrier phase ϕ_c, the carrier is itself written as $\cos(2\pi f_c t + \phi_c)$. Then Eq. (7.7) modifies to

$$E_b = \int_0^{T_b} |s(t)|^2 dt$$

$$= \frac{1}{T_b}\int_0^{T_b} |b(t)|^2 + \frac{1}{T_b}\int_0^{T_b} |b(t)|^2 \cos(4\pi f_c t + 2\phi_c) dt$$

where we have made use of the trigonometric identity

$$\cos^2\theta = \frac{1}{2}(\cos(2\theta))$$

Hence, with $|b(t)|^2$ remaining essentially constant over one complete cycle of $\cos(4\pi f_c t + 2\phi_c)$, we have

$$\int_0^{T_b} |b(t)|^2 \cos(4\pi f_c t + \phi_c) dt \approx 0 \text{ for all } \phi_c$$

Correspondingly, we may write

$$E_b = \int_0^{T_b} |b(t)|^2 \quad \text{for all } \phi_c.$$

Problem 7.3

Although QPSK and OQPSK signals have different waveforms, their magnitude spectra are identical; but their phase spectra differ by a nonlinear phase component. Justify the validity of this two-fold statement.

Solution

In QPSK, the modulated signal is defined by (see Eq. (7l15))

$$s_{\text{QPSK}}(t) = \sqrt{\frac{2E}{T}}\cos\left[(2i-1)\frac{\pi}{4}\right]\cos(2\pi f_c t) - \sqrt{\frac{2E}{T}}\sin\left[(2i-1)\frac{\pi}{4}\right]\sin(2\pi f_c t) \qquad (1)$$

where $0 \leq t \leq T$ the index $i = 1,2,3,4$, depending on which particular dibit is sent. For a specific index i, the in-phase component of $S_{\text{QPSK}}(t)$ is therefore

$$s_{I,\text{QPSK}}(t) = \sqrt{\frac{2E}{T}}\cos\left[(2i-1)\frac{\pi}{4}\right], \qquad 0 \leq t \leq T \qquad (2a)$$

and its quadrature component is

$$s_{Q,\text{QPSK}}(t) = \sqrt{\frac{2E}{T}}\sin\left[(2i-1)\frac{\pi}{4}\right], \qquad 0 \leq t \leq T \qquad (2b)$$

In OQPSK, the in-phase component is left intact but the quadrature component is delayed by $T/2$ (half symbol period). Accordingly, for the same index i in QPSK, we may express the in-phase component of OQPSK as

$$s_{I,\text{OQPSK}}(t) = \sqrt{\frac{2E}{T}}\cos\left[(2i-1)\frac{\pi}{4}\right], \qquad 0 \leq t \leq T \qquad (3a)$$

and its quadrature component as

$$s_{Q,\text{OQPSK}}(t) = \sqrt{\frac{2E}{T}}\sin\left[(2i-1)\frac{\pi}{4}\right], \qquad \frac{T}{2} \leq t \leq \frac{3}{2}T \qquad (3b)$$

Let $b_I(t)$ denote a rectangular pulse of duration T, representing the in-phase component of the QPSK signal and $b_Q(t)$ denote the corresponding quadrature component. Then, in light of Eqs. (2) and (3), we may express the complex envelope of QPSK as

$$\tilde{s}_{\text{QPSK}}(t) = b_I(t) + jb_Q(t), \qquad 0 \leq t \leq T \qquad (4)$$

and

$$\tilde{s}_{\text{OQPSK}}(t) = b_I(t) + jb_Q\left(t - \frac{T}{2}\right), \qquad 0 \leq t \leq T \qquad (5)$$

Applying the Fourier transform to Eqs. (4) and (5), we correspondingly have

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Problem 7-3 continued

$$\tilde{s}_{\text{QPSK}}(f) = B_I(f) + jB_Q(f) \tag{6}$$

and

$$\begin{aligned}\tilde{s}_{\text{OQPSK}}(f) &= B_I(f) + jB_Q(f)\exp(-j\pi f\tau) \\ &= B_I(f) + jB_Q(f)[\cos(\pi f\tau) j\sin(\pi f\tau)] \\ &-[B_I(f)B_Q(f)\sin(\pi f\tau)] + jB_Q(f)\cos(\pi f\tau)\end{aligned} \tag{7}$$

From Eqs. (6) and (7), it therefore follows that for the QPSK

$$\left|\tilde{s}_{\text{QPSK}}(f)\right|^2 = B_I^2(f) + B_Q^2(f) \tag{8a}$$

and

$$\arg[\tilde{s}_{\text{QPSK}}(f)] = \tan^{-1}\left(\frac{B_Q(f)}{B_I(f)}\right) \tag{8b}$$

Similarly, for the OQPSK

$$\begin{aligned}\tilde{s}_{\text{OQPSK}}(f)\Big|^2 &= [B_I(f) \ - \ B_Q\sin(\pi f\tau)]^2 + [B_Q(f)\cos(\pi fT)]^2 \\ &= B_I^2(f) + B_Q^2(f) - 2B_I(f)B_Q(f)\sin(\pi fT)\end{aligned} \tag{9a}$$

and

$$\arg[\tilde{s}_{\text{OQPSK}}(f)] = \tan^{-1}\left[\frac{B_Q(f)\cos(\pi fT)}{B_I(f) - B_Q\sin(\pi fT)}\right] \tag{9b}$$

For a square wave input, we typically find that the cross-product term $2B_I(f)B_Q(f)\sin(\pi fT)$ is small compared to the composite term $B_I^2(f) + B_Q^2(f)$. Accordingly, from Eqs. (8a) and (9a), it follows that for all practical purposes, the magnitude spectra $\left|S_{\text{QPSK}}(f)\right|$ and $\left|S_{\text{OQPSK}}(f)\right|$ are identical. In direct contrast, however, from Eqs. (8b) and (9b), we find that the corresponding phase spectra are not only different but the difference between them is a nonlinear function of frequency f.

Note: In the problem statement, the following correction should be made:
The term "linear phase component" is replaced by "nonlinear phase component".

Problem 7.4

Show that the modulation process involved in generating Sunde's BFSK is nonlinear.

Solution

Let

$$f_1 = f_c + \frac{1}{2T_b}, \quad \text{for symbol 1}$$

and

$$f_2 = f_c + \frac{1}{2T_b}, \quad \text{for symbol 0}$$

where f_c is the unmodulated carrier frequency. We may therefore express the instantaneous frequency of Sunde's BFSK signal as

$$f_i(t) = f_c + k\frac{1}{2T_b}, \qquad 0 \leq t \leq T_b \tag{1}$$

where

$$k = \begin{cases} +1 \text{ for symbol } 1 \\ -1 \text{ for symbol } 0 \end{cases}$$

Correspondingly, we may define the BFSK signal itself as

$$\begin{aligned} s(t) &= \sqrt{\frac{2E_b}{T_b}}\cos[2\pi f_i t] \\ &= \sqrt{\frac{2E_b}{T_b}}\cos\left(2\pi f_c t + \frac{\pi k}{T_b}t\right) \\ &= \sqrt{\frac{2E_b}{T_b}}\cos(2\pi f_c t)\cos\left(\frac{\pi k}{T_b}t\right) - \sqrt{\frac{2E_b}{T_b}}\sin(2\pi f_c t)\sin\left(\frac{\pi k}{T_b}t\right) \\ &= \sqrt{\frac{2E_b}{T_b}}\cos(2\pi f_c t)\cos\left(\pm\frac{\pi}{T_b}t\right) - \sqrt{\frac{2E_b}{T_b}}\sin(2\pi f_c t)\sin\left(\pm\frac{\pi}{T_b}t\right) \end{aligned} \tag{2}$$

Recognizing that

$$\cos(-A) = \cos A$$

and

$$\sin(-A) = -\sin A$$

we may rewrite Eq. (2) in the new form

$$s(t) = \sqrt{\frac{2E_b}{T_b}}\cos(2\pi f_c t)\cos\left(\frac{\pi}{T_b}t\right) \mp \sqrt{\frac{2E_b}{T_b}}\sin(2\pi f_c t)\sin\left(\frac{\pi}{T_b}t\right) \tag{3}$$

where $0 \leq t \leq T_b$; the minus sign corresponds to symbol 0 and the plus sign corresponds to symbol 1. Equation (3) reveals the following two characteristics of Sunde's BFSK:

(i) The in-phase component of $s(t)$ is independent of the incoming binary data stream.

(ii) The incoming binary data stream only affects the quadrature component.

It is because of property (ii) that we may go on to state that Sunde's BFSK is nonlinear.

Problem 7.5

To summarize matters, we may say that MSK is an OQPSK where the symbols in the in-phase and quadrature components (on a dibit-by-dibit basis) are weighted by the basic pulse function

$$p(t) = \sin\left(\frac{\pi t}{2T_b}\right)\text{rect}\left(\frac{t}{2T_b} - \frac{1}{2}\right)$$

where T_b is the bit duration, and rect(t) is the rectangular function of unit duration and unit amplitude. Justify this summary.

Solution

With $f_0 = \frac{1}{4T_b}$, it follows that

$$\cos(2\pi f_0 t) = \cos(\pi t/2T_b) = \sin\left[(\pi t/2T_b) + \frac{\pi}{2}\right] = \sin\left(\pi\left(\frac{t}{2T_b} + \frac{1}{2}\right)\right)$$

and

$$\sin(2\pi f_0 t) = \sin(\pi t/2T_b)$$

Following Eqs. (7.29) and (7.30), we next note that the binary waves $a_1(t)$ and $a_2(t)$, constituting the MSK signal, are extracted from the incoming binary data stream through demultiplexing and offsetting in a manner similar to OQPSK. Since $a_1(t)$ and $a_2(t)$ are themselves weighted by the sinusoidal functions $\cos(2\pi f_0 t)$ and $\sin(2\pi f_0 t)$, we may go on to state that the in-phase and quadrature components of the MSK signal are weighted (on a dibit-by-dibit basis) by the basic pulse function

$$p(t) = \sin\left(\frac{\pi t}{2T_b}\right)\text{rect}\left(\frac{t}{2T_b} - \frac{1}{2}\right)$$

Problem 7.6

The sequence 11011100 is applied to an MSK modulator. Assuming that the angle $\theta(t)$ of the MSK signal is zero at time $t = 0$, plot the trellis diagram that displays the evolution of $\theta(t)$ over the eight binary symbols of the input sequence.

Solution

Evolution of the phase $\theta(t)$ of the MSK signal produced by the sequence 11011100 is displayed in Fig. 1.

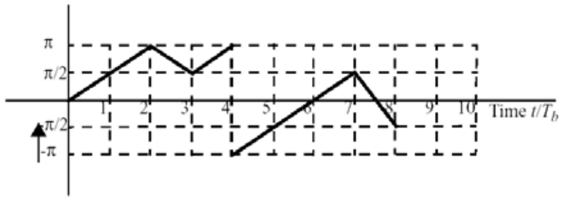


Figure 1

Problem 7.7

The process of angle modulation involved in the generation of an MSK signal is linear. Justify this assertion.

Solution

We first recognize from Problem 7.5 that MSK is an OQPSK signal with only a basic difference:

(i) In OQPSK, the weighting applied to the in-phase and quadrature components of the modulated signal (on a dibit-by-dibit basis) is in the form of a rectangular function. On the other hand, in MSK, the corresponding weighting functions are sinusoidal.

(ii) The OQPSK is the result of a linear modulation process.

In light of these two points, we may therefore state that the angle modulation process involved in generating MSK is a linear process.

Problem 7.8

A simple way of demodulating an MSK signal is to use a frequency discriminator, which was discussed in Chapter 4 on angle modulation. Justify this use and specify the linear input-output characteristic of the discriminator.

Solution

The MSK signal is basically an FSK signal, as shown by

$$s(t) = \sqrt{\frac{2E_b}{T_b}}\cos(2\pi f_c t + \theta(t))$$

where

$$\theta(t) = \pm\frac{\pi t}{2T_b}$$

The plus sign corresponds to symbol 1 and the minus sign corresponds to symbol 0.

We may therefore demodulate $s(t)$ by using a frequency discriminator whose input-output characteristic is described in Fig. 1

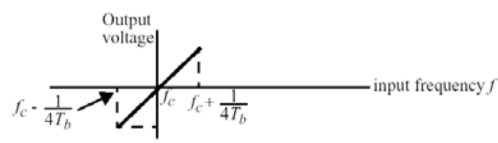


Figure 1

Problem 7.9

Starting with Eq. (7.41), prove the orthogonality property of Eq. (7.42) that characterizes M-ary FSK.

Solution

From Eq. (7.41), we have

$$s_i(t) = \sqrt{\frac{2E}{T}}\cos\left[\frac{\pi}{T}(n+i)t\right] \qquad \begin{array}{l} i = 0, 1, \ldots, M\text{-}1 \\ 0 \le t \le T \end{array}$$

Applying Eq. (7.42), we therefore have

$$\int_0^T s_i(t)s_j(t)dt = \frac{2E}{T}\int_0^T \cos\left[\frac{\pi}{T}(n+i)t\right]\cos\left[\frac{\pi}{T}(n+j)t\right]dt$$

$$= \frac{E}{T}\int_0^T \left\{\cos\left[\frac{\pi}{T}(2n+i+j)t\right] + \cos\left[\frac{\pi}{T}(i-j)t\right]\right\}dt \qquad (1)$$

Let the integer $k = 2n + i + j$, and $i - j = l$ for $i \neq j$. We may then rewrite Eq. (1) as

$$\int_0^T s_i(t)s_j(t)dt = \frac{E}{T}\int_0^T \left\{\cos\left(\frac{\pi}{T}kt\right) + \cos\left(\frac{\pi}{T}lt\right)\right\}dt$$

$$= \frac{E}{T}\left[\frac{T}{k\pi}\sin\left(\frac{\pi}{T}kt\right) + \frac{T}{l\pi}\sin\left(\frac{\pi}{T}lt\right)\right]_{t=0}^{T}$$

$$= 0 \text{ for all integer } k \text{ and } l$$

which is the desired result.

Problem 7.10
Justify Eqs. (7.47) and (7.49).

Solution
Starting with Eq. (7.47), we write

$$\mathbf{s}_1 = \frac{2}{T_b}\sqrt{E_b} \int_0^{T_b} \cos^2(2\pi f_c t)\,dt$$

$$= \frac{1}{T_b}\sqrt{E_b}\int_0^{T_b} [\cos^2(4\pi f_c t) + 1]\,dt \qquad (1)$$

For $f_c = n/T_b$ for some integer n, Eq. (1) takes the form

$$\mathbf{s}_1 = \frac{\sqrt{E_b}}{T_b} \int_0^{T_b} \left[\cos\left(\frac{4\pi n}{T_b}t\right) + 1\right]dt$$

$$= \frac{\sqrt{E_b}}{T_b}\left[\frac{T_b}{4\pi n}\sin\left(\frac{4\pi n}{T_b}t\right) + t\right]_0^{T_b}$$

$$= \frac{\sqrt{E_b}}{T_b}\left[\frac{T_b}{4\pi n}\sin(4\pi n) + T_b\right], \quad n \text{ integer}$$

$$= \sqrt{E_b}$$

Similarly, for symbol 0, we have

$$\mathbf{s}_2 = -\sqrt{E_b}$$

Problem 7.11

(a) The transmission bandwidth of the BASK signal is effectively defined by

$$B_T = \frac{2}{T_b}$$

where T_b is the bit duration. With $T_b = 1\ \mu s$, we therefore have

$$B_T = \frac{2}{10^{-6}} \text{ Hz}$$
$$= 2 \text{ MHz}$$

(b)

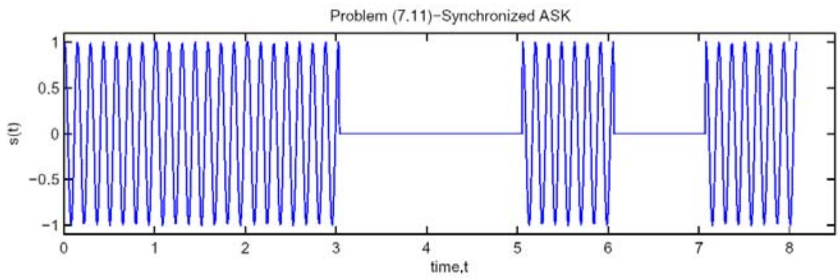


Figure 1

In the waveform plotted in Fig. 1, time t is measured in microseconds.

Problem 7.12

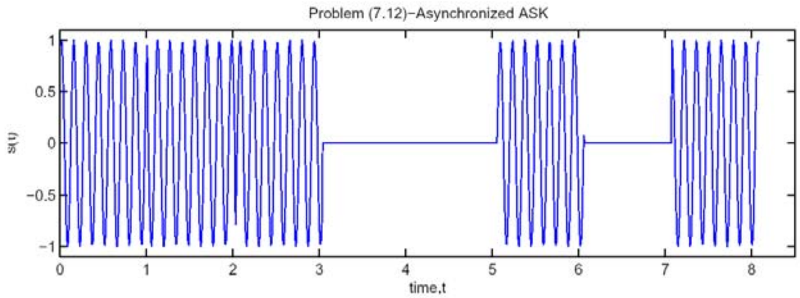


Figure 1, where time t is measured in microseconds

Comparing the BASK waveform plotted in Fig. 1 of this solution with that of the BASK signal considered in Problem 7.11, we see that continuity in time is not maintained in Fig. 1 of the solution to Problem 7.12, when a succession of 1s is transmitted.

Problem 7.13

(a)

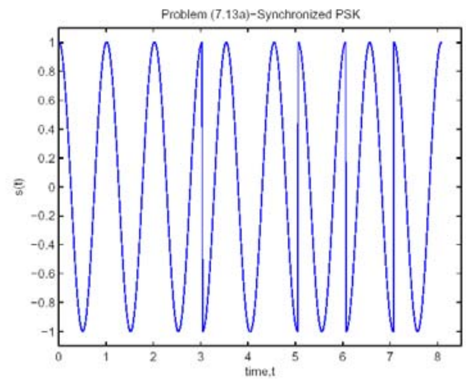


Figure 1

(b)

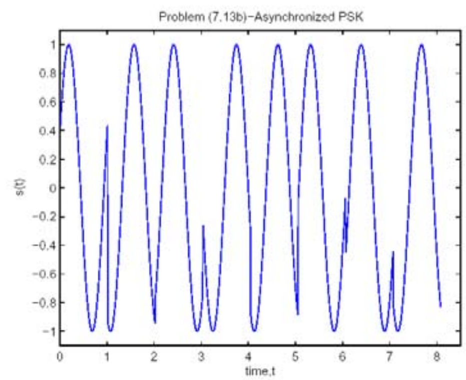


Figure 2

<u>Notes</u>:

(i) In both Figures 1 and 2, time t is measured in microseconds.

(ii) For clarity of presentation, the carrier frequency in both figures has been scaled down from 7 MHZ to 1 MHZ.

(iii) In Fig. 1 of the solution, there is synchronism between the carrier phase and the times at which the incoming data switch for symbol 1 or 0 or vice versa. No such synchronism exists in Fig. 2.

Problem 7.14

(a) The transmission bandwidth of the QPSK signal is

$$B_T = \frac{2}{T} = \frac{2}{2T_b} = \frac{1}{T_b}$$

where T is the symbol (dibit) duration and T_b is the bit duration. With $T_b = 1\mu s$, it follows therefore that

$$B_T = \frac{1}{10^{-6}} \text{ Hz}$$
$$= 1 \text{ MHz}$$

(b)

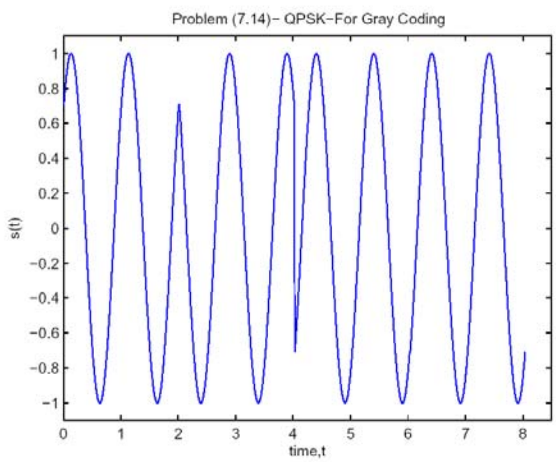


Figure 1

Notes:

(i) Time t in Fig. 1 is measured in microseconds.

(ii) For clarify of presentation, we have plotted the QPSK waveform using a carrier of 1 MHz instead of 6 MHz.

(iii) Synchronism between the timing waveform representing the incoming binary data stream and the clock responsible for generating the carrier is assumed.

Problem 7.15

(a) The transmission bandwidth of OQPSK is exactly the same as that of QPSK, which, for the problem at hand, is 1 MHz.

(b)

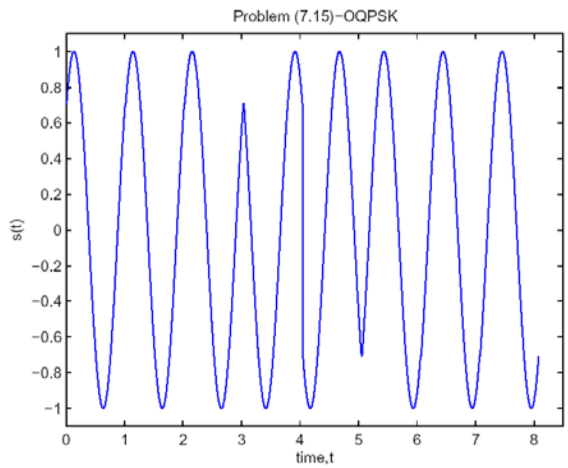


Figure 1

In plotting the OQPSK waveform in Fig. 1, we have followed the same notes made in the solution to Problem 7.14 on QPSK.

Problem 7.16

(a) The transmission bandwidth of Sunde's BFSK is greater than that of the corresponding BPSK. This means that for the problem at hand, it will be greater than 1 MHz. In particular, examining the spectrum shown in Fig. 7.12, we see that the main lobe occupies a bandwidth of 3 Hz for the bit duration $T_b = 1\text{s}$. Therefore, scaling this result for $T_b = 1\mu\text{s}$, we may say that the corresponding transmission bandwidth is

$$B_T = 3 \text{ MHz}$$

which is 50% greater than that of the corresponding BPSK.

(b)

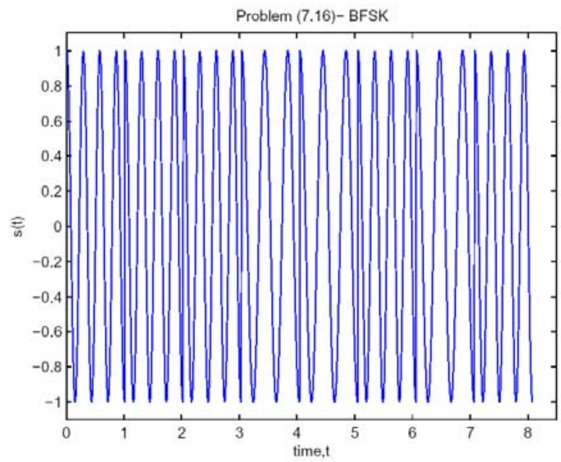


Figure 1

Notes:
In plotting the BFSK waveform in Fig. 1, we have followed the same notes outlined in the solution to Problem 7.14.

Problem 7.17

(a) Examining the continuous phase FSK waveform plotted in Fig. 7.1(c), we observe the following two points (assuming that time t is measured in seconds):

(i) The carrier for symbol 00 occupies 3 complete cycles. Therefore,

$$f_2 = \frac{1}{(2 \text{ seconds})/(3 \text{ cycles})} = 1.5 \text{ Hz}$$

(ii) The carrier for symbol 11 occupies 5 complete cycles. Therefore,

$$f_1 = \frac{1}{(2 \text{ seconds})/(5 \text{ cycles})} = 2.5 \text{ Hz}$$

Hence, the frequency excursion is

$$\begin{aligned} \delta f &= f_1 - f_2 \\ &= 2.5 - 1.5 = 1 \text{ Hz} \end{aligned}$$

(b) The frequency parameter f_0 is defined by (see Eq. (7.34))

$$\begin{aligned} f_0 &= \frac{1}{4T_b} \\ &= \frac{1}{4 \times 1 \ \mu\text{s}} = 0.25 \text{ MHz} \end{aligned}$$

Problem 7.18

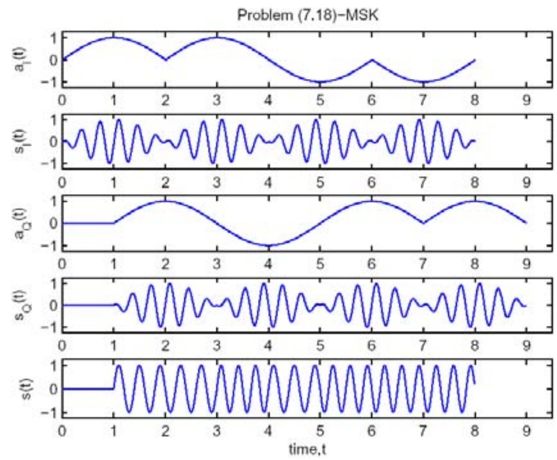


Figure 1

In plotting the MSK waveform and its constituents shown in Fig. 1, the following two points should be noted:

(i) Time t is measured in microseconds.

(ii) Synchronism is assumed between the timing waveform responsible for generating the incoming binary sequences (and therefore the constituent sequences $s_I(t)$ and $s_Q(t)$) and the clock responsible for generating the carrier.

Problem 7.19

The bit duration is

$$T_b = \frac{1}{20 \times 10^3} \text{ seconds}$$
$$= 50 \ \mu\text{s}$$

The carrier frequency is

$$f_c = 50 \ \text{MHz}$$

From Eq. (7.19), the frequency excursion is

$$\delta f = \frac{1}{2T_b}$$
$$= \frac{1}{2 \times 50 \times 10^{-6}} \text{ Hz} = 10 \ \text{kHz}$$

From Eqs. (7.21) and (7.22), we have

$$f_1 = f_c + \frac{\delta f}{2}$$
$$= 50 \ \text{MHz} + 5 \ \text{kHz}$$
$$= 50.005 \ \text{MHz}$$
$$f_2 = f_c - \frac{\delta f}{2}$$
$$= 50 \ \text{MHz} - 5 \ \text{kHz}$$
$$= 49.995 \ \text{MHz}$$

(a) The instantaneous frequency of the MSK signal is therefore

$$f_i(t) = \begin{cases} 50.005 \ \text{MHz} & \text{for symbol } 1 \\ 49.995 \ \text{MHz} & \text{for symbol } 0 \end{cases}$$

Specifically, $f_i(t)$ alternates between these two values.

(b) When the incoming data sequence consists of all 1s, we have

$$f_i(t) = 50.005 \ \text{MHz} \quad \text{for all time } t$$

Problem 7.20

Extraction of the bit-timing may proceed as follows:

(i) Given the MSK signal $s(t)$, a band-pass analyzer is used to extract the in-phase component $s_I(t)$ and quadrature component $s_Q(t)$.

(ii) From the first line of Eq. (7.31), and Eqs. (7.33) and (7.34), we have

$$r(t) = \frac{s_Q(t)}{s_I(t)} = -\tan\left(\frac{\pi t}{2T_b}\right) = -\tan(\theta(t))$$

which depends on the bit duration T_b alone.

(iii) From Eq. (7.32), we recall that whenever two successive binary symbols in the original data stream are the same, then $\theta(t)$ is negative and therefore the ratio $r(t)$ is positive. On the other hand, from Eq. (7.33), we recall that whenever two successive binary symbols are different, then $\theta(t)$ is positive and therefore the ratio $r(t)$ is negative.

Hence, by observing the zero-crossings of the waveform obtained from $r(t) = [s_Q(t)/s_I(t)]$, it should be possible to extract the timing waveform.

Problem 7.21

The envelope of BFSK is constant with time, whereas the envelope of BASK is variable. Accordingly, the noncoherent receiver of Fig. 7.18 for BFSK offers the following practical advantages over the noncoherent receiver of Fig. 7.17 for BASK:

(i) Reduced sensitivity to nonlinear transmission.

(ii) Improved performance in the presence of channel noise and interference.

Problem 7.22

For the noncoherent receiver of Fig. 7.29 to offer an identical performance to the noncoherent receiver of Fig. 7.18, the following conditions must be satisfied:

(i) The bit-timing circuitry of both receivers must be equally accurate.

(ii) The common bandwidth of the band-pass filter must occupy at the minimum the main spectral lobe of the incoming BFSK signal. As such, a reasonably good choice for this bandwidth is the reciprocal of $2T_b$, where T_b is the bit duration.

(iii) With one band-pass filtered centred on f_1 and the other centred on f_2, the frequencies f_1 and f_2 must be separated from each other by at least $1/(2T_b)$.

These three conditions do not guarantee the exact equivalence of the two noncoherent receivers of Figs. 7.18 and 7.19, but, for all practical purposes, would assure identical performance.

Problem 7.23

(a) The transmission bandwidth of DSK signal is the same as that of the corresponding BPSK. Therefore, for a bit duration $T_b = 1\mu s$, the bandwidth is

$$B_T = \frac{2}{T_b} = 2 \times 10^{+6} \text{ Hz} = 2 \text{ MHz}$$

(b) In plotting the DPSK waveform shown in Fig. 1, we have followed three points:

(i) Time t is measured in microseconds.

(ii) For clarity of presentation, a carrier frequency $f_c = 1$ MHz has been used in place of $f_c = 6$ MHz.

(iii) Synchronism is assumed between the timing circuitry responsible for line encoding the incoming binary data stream and the clock responsible for generating the carrier.

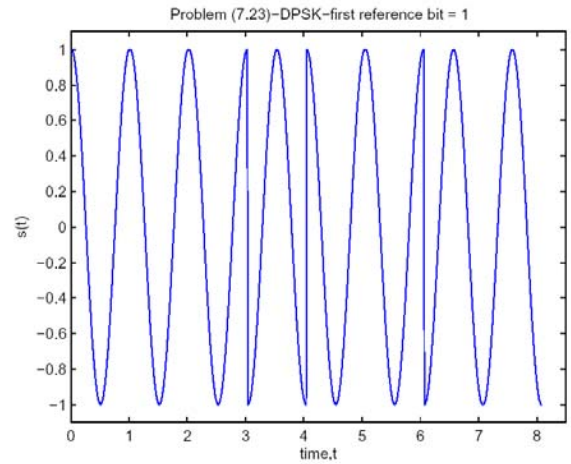


Figure 1

(c) Decoding in the receiver is first accomplished by multiplying the received signal by $\cos(2\pi f_c t)$ and then low-pass filtering. Next, the low-pass filter output is applied to a DPSK decoder. Thus, starting with a reference bit 1 and assuming perfect transmission (i.e., zero channel noise), the receiver output is the same as the original binary sequence, namely, 11100101.

Problem 7.24

(a) The noiseless PSK signal is given by

$$s(t) = A_c \cos[2\pi f_c t + k_p m(t)]$$
$$= A_c \cos(2\pi f_c t)\cos[k_p m(t)] - A_c \sin(2\pi f_c t)\sin[k_p m(t)]$$

Since $m(t) = \pm 1$, it follows that

$$\cos[k_p m(t)] = \cos(\pm k_p) = \cos(k_p)$$
$$\sin[k_p m(t)] = \sin(\pm k_p) = \pm\sin(k_p) = m(t)\sin(k_p)$$

Therefore,

$$s(t) = A_c \cos(k_p)\cos(2\pi f_c t) - A_c m(t)\sin(k_p)\sin(2\pi f_c t) \quad (1)$$

The VCO output is

$$r(t) = A_v \sin[2\pi f_c t + \theta(t)]$$

The multiplier output in the phase-locked loop is therefore

$$r(t)s(t) = \frac{1}{2}A_c A_v \cos(k_p)\{\sin[\theta(t)] + \sin[4\pi f_c t + \theta(t)]\}$$
$$-\frac{1}{2}A_c A_v m(t)\sin(k_p)\{\cos(\theta(t)) + \cos[4\pi f_c t + \theta(t)]\}$$

The loop filter removes the double-frequency components, producing the output

$$e(t) = \frac{1}{2}A_c A_v \cos(k_p)\sin[\theta(t)] - \frac{1}{2}A_c A_v m(t)\sin(k_p)\cos[\theta(t)]$$

Note that if $k_p = \pi/2$, (i.e., the carrier is fully deviated), there would be no carrier component for the PLL to track.

(b) Since the error signal tends to drive the loop into lock (i.e., $\theta(t)$ approaches zero), the loop filter output reduces to

$$e(t) = -\frac{1}{2}A_c A_v \sin(k_p)m(t)$$

which is proportional to the desired data signal $m(t)$. Hence, the phase-locked loop may be used to recover the original message $m(t)$.

Problem 7.25

(a) The correlation coefficient of the signals $s_0(t)$ and $s_1(t)$ is

$$\rho = \frac{\int_0^{T_b} s_0(t)s_1(t)dt}{\left[\int_0^{T_b} s_0^2(t)dt\right]^{1/2}\left[\int_0^{T_b} s_1^2(t)dt\right]^{1/2}}$$

$$= \frac{A_c^2\int_0^{T_b}\cos\left[2\pi\left(f_c+\frac{1}{2}\Delta f\right)t\right]\cos\left[2\pi\left(f_c-\frac{1}{2}\Delta f\right)t\right]}{\left[\frac{1}{2}A_c^2T_b\right]^{1/2}\left[\frac{1}{2}A_c^2T_b\right]^{1/2}}$$

$$= \frac{1}{T_b}\int_0^{T_b}[\cos(2\pi\Delta ft)+\cos(4\pi f_ct)]dt$$

$$= \frac{1}{2\pi T_b}\left[\frac{\sin(2\pi\Delta fT_b)}{\Delta f}+\frac{\sin(4\pi f_cT_b)}{2f_c}\right] \qquad (1)$$

Since $f_c >> \Delta f$, then we may ignore the second term in Eq. (1), obtaining

$$\rho \approx \frac{\sin(2\pi\Delta fT_b)}{2\pi T_b\Delta f} = \text{sinc}(2\Delta fT_b)$$

(b) The dependence of ρ on Δf is as shown in Fig. 1. The two signals $s_0(t)$ and $s_1(t)$ are orthogonal when $\rho = 0$. Therefore, the minimum value of Δf for which they are orthogonal is $1/2T_b$.
$s_0(t)$ and $s_1(t)$ are orthogonal when $\rho = 0$. Therefore, the minimum value of Δf for which they are orthogonal is $1/2T_b$.

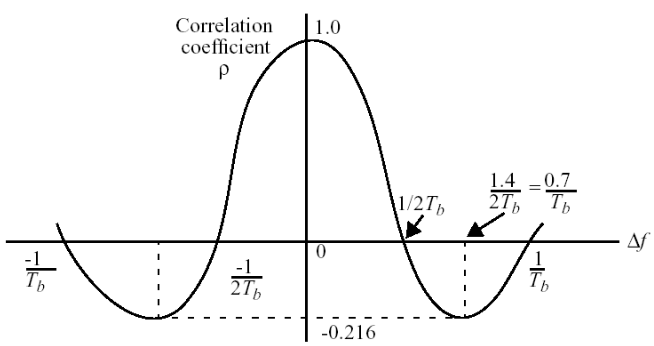


Figure 1

Problem 7.26

(a) The given binary FSK signal is defined by

$$s_{\text{FSK}}(t) = \begin{cases} \sqrt{\dfrac{2E_b}{T_b}}\cos(2\pi f_1 t + \theta_1) & \text{for symbol } 0 \\ \sqrt{\dfrac{2E_b}{T_b}}\cos(2\pi f_2 t + \theta_2) & \text{for symbol } 1 \end{cases} \tag{1}$$

Equation (1) may be expressed in the equivalent form

$$s_{\text{FSK}}(t) = s_1(t) + s_2(t) \tag{2}$$

where

$$s_1(t) = \begin{cases} \sqrt{\dfrac{2E_b}{T_b}}\cos(2\pi f_1 t + \theta_1) & \text{for symbol } 0 \\ 0 & \text{for symbol } 1 \end{cases} \tag{3}$$

and

$$s_2(t) = \begin{cases} 0 & \text{for symbol } 0 \\ \sqrt{\dfrac{2E_b}{T_b}}\cos(2\pi f_2 t + \theta_2) & \text{for symbol } 1 \end{cases} \tag{4}$$

The digitally modulated signals $s_1(t)$ and $s_2(t)$ are recognized as two complementary BASK signals, operating in parallel. In light of Eqs. (1) through (4), we may construct the two-transmitter equivalence depicted in Fig. 1.

(a)

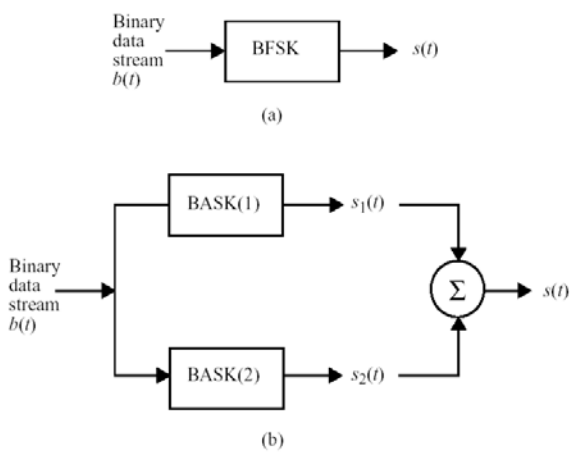
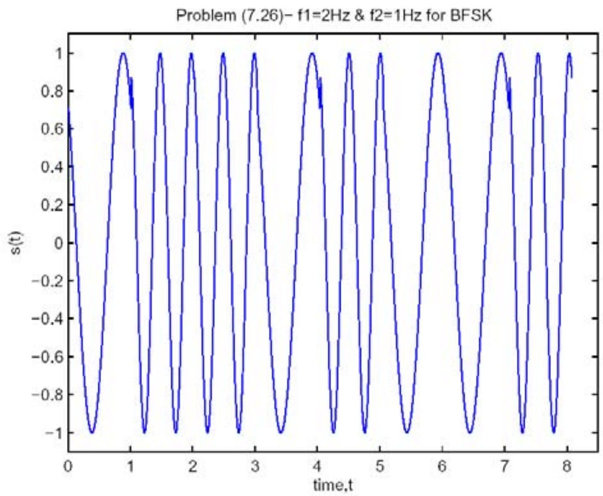


Figure 1

Continued on next slide

Problem 7-26 continued

(b)



Problem 7.27

From the description of minimum-shift keying presented in Section 7.4, we recall the following:

- The transmission of symbol 1 increases the phase of the MSK signal by $\pi/2$ radians.
- The transmission of symbol 0 decreases the phase of the MSK signal by $\pi/2$ radians.

Accordingly, we may justify the entries listed in Table 7.4 as follows:

(a) When $\theta(0) = 0$, the transmission of symbol 0 yields

$$\theta(T_b) = -\pi/2 \text{ radians}$$

(b) When $\theta(0) = \pi$ radians, the transmission of symbol 1 yields

$$\theta(T_b) = \pi + \pi/2 = 3\pi/2 \text{ radians}$$

which, in modulo-2π arithmetic, is equivalent to

$$\theta(T_b) = 3\pi/2 - 2\pi = -\pi/2 \text{ radians}$$

(c) When $\theta(0) = \pi$ radians, the transmission of symbol 0 yields

$$\theta(T_b) = \pi - (\pi/2) = +\pi/2 \text{ radians}$$

(d) When $\theta(0) = 0$, the transmission of symbol 1 yields

$$\theta(T_b) = 0 + \pi/2 = +\pi/2 \text{ radians}$$

Problem 7.28

The idea of quadrature multiplexing rests on the following premise: Two signals can be transmitted over a common channel, provided that two conditions are satisfied:
(i) The two signals are orthogonal to each other.
(ii) They both occupy the same bandwidth.
This principle is satisfied by quadriphase-shift keying (QPSK), as demonstrated next.

Consider the QPSK signal defined by

$$s(t) = \begin{cases} \sqrt{\frac{2E}{T}}\cos(2\pi f_c t), & \text{dibit } 00 \\ -\sqrt{\frac{2E}{T}}\sin((2\pi f_c t),) & \text{dibit } 01 \\ \sqrt{\frac{2E}{T}}\cos(2\pi f_c t + \pi), & \text{dibit } 11 \\ -\sqrt{\frac{2E}{T}}\cos(2\pi f_c t + \pi), & \text{dibit } 10 \end{cases} \tag{1}$$

This signal can be decomposed into the sum of two BPSK signals, defined as follows:

$$s_1(t) = \begin{cases} \sqrt{\frac{2E}{T}}\cos(2\pi f_c t), & \text{dibit } 00 \\ \sqrt{\frac{2E}{T}}\cos(2\pi f_c t + \pi), & \text{dibit } 11 \end{cases} \tag{2}$$

and

$$s_2(t) = \begin{cases} -\sqrt{\frac{2E}{T}}\sin(2\pi f_c t), & \text{dibit } 01 \\ -\sqrt{\frac{2E}{T}}\sin(2\pi f_c t + \pi), & \text{dibit } 10 \end{cases} \tag{3}$$

In light of Eqs. (1) through (3), we may write

$$s(t) = s_1(t) + s_2(t) \tag{4}$$

which means that $s_1(t)$ and $s_2(t)$ can be transmitted simultaneously on a common channel and be detected separately at the receiver. This statement is justified on two accounts:

Continued on next slide

Problem 7.28 continued

(i) Both $s_1(t)$ and $s_2(t)$ occupy exactly the same bandwidth, as their magnitude spectra are identical.

(ii) They are orthogonal over the symbol period T, as shown by

$$\int_0^T s_1(t)s_2(t) = \int_0^T \sqrt{\frac{2E}{T}}\cos(2\pi f_c t)\left(-\sqrt{\frac{2E}{T}}\right)\sin(2\pi f_c t)dt$$

$$= -\frac{E}{T}\int_0^T \sin 4(\pi f_c t)dt$$

which is zero by the band-pass assumption, provided that the carrier frequency f_c is high enough.

The assertion embodied in Eq. (4) holds for any clockwise or counterclockwise rotation of the QPSK constellation defined in Eq. (1).

Consider next the 8-PSK defined by

$$s'(t) = \begin{cases} \sqrt{\frac{2E}{T}}\cos(2\pi f_c t), & \text{symbol } 000 \\ \sqrt{\frac{2E}{T}}\cos\left(2\pi f_c t + \frac{\pi}{4}\right), & \text{symbol } 001 \\ \sqrt{\frac{2E}{T}}\cos\left(2\pi f_c t + \frac{\pi}{2}\right), & \text{symbol } 101 \\ \sqrt{\frac{2E}{T}}\cos\left(2\pi f_c t + \frac{3\pi}{4}\right), & \text{symbol } 111 \\ \sqrt{\frac{2E}{T}}\cos(2\pi f_c t + \pi), & \text{symbol } 011 \\ \sqrt{\frac{2E}{T}}\cos\left(2\pi f_c t + \frac{5\pi}{4}\right), & \text{symbol } 010 \\ \sqrt{\frac{2E}{T}}\cos\left(2\pi f_c t + \frac{3\pi}{2}\right), & \text{symbol } 110 \\ \sqrt{\frac{2E}{T}}\cos\left(2\pi f_c t + \frac{7\pi}{4}\right), & \text{symbol } 100 \end{cases} \quad (5)$$

Following what we did with the QPSK signal of Eq. (1), we may decompose the 8-PSK of Eq. (5) as follows:

$s(t) = s'_1(t) + s'_2(t)$

whose constituents are defined by

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$$s'_1(t) = \begin{cases} \sqrt{\frac{2E}{T}}\cos(2\pi f_c t), & \text{symbol } 000 \\ \sqrt{\frac{2E}{T}}\cos\left(2\pi f_c t + \frac{\pi}{2}\right), & \text{symbol } 101 \\ \sqrt{\frac{2E}{T}}\cos(2\pi f_c t + \pi), & \text{symbol } 011 \\ \sqrt{\frac{2E}{T}}\cos\left(2\pi f_c t + \frac{3\pi}{2}\right), & \text{symbol } 110 \end{cases} \tag{6}$$

and

$$s'_2(t) = \begin{cases} \sqrt{\frac{2E}{T}}\cos\left(2\pi f_c t + \frac{\pi}{4}\right), & \text{symbol } 001 \\ \sqrt{\frac{2E}{T}}\cos\left(2\pi f_c t + \frac{3\pi}{4}\right), & \text{symbol } 111 \\ \sqrt{\frac{2E}{T}}\cos\left(2\pi f_c t + \frac{5\pi}{4}\right), & \text{symbol } 010 \\ \sqrt{\frac{2E}{T}}\cos\left(2\pi f_c t + \frac{7\pi}{4}\right), & \text{symbol } 100 \end{cases} \tag{7}$$

Basically, the signal $s'_1(t)$ is a rewrite of the QPSK signal $s(t)$ of Eq. (1). The signal $s'_2(t)$ is a rotated version of $s(t)$. The two constituent QPSK signals $s'_1(t)$ and $s'_2(t)$ satisfy the common bandwidth requirement (i). However, they fail to satisfy requirement (ii). To demonstrate this failure, let us test the first components of $s'_1(t)$ and $s'_2(t)$ for orthogonality by writing

$$\begin{aligned} \int_0^T s'_1(t)s'_2(t) &= \int_0^T \sqrt{\frac{2E}{T}}\cos(2\pi f_c t)\cdot\sqrt{\frac{2E}{T}}\cos\left(2\pi f_c t + \frac{\pi}{4}\right)dt \\ &= \frac{2E}{T}\int_0^T \cos(2\pi f_c t)\cos\left(2\pi f_c t + \frac{\pi}{4}\right)dt \\ &= \frac{E}{T}\int_0^T \left[\cos\left(\frac{\pi}{4}\right) + \cos\left(4\pi f_c t + \frac{\pi}{4}\right)\right]dt \\ &= \frac{E}{T}\cdot\frac{T}{\sqrt{2}} + \frac{E}{T}\int_0^T \cos\left(4\pi f_c t + \frac{\pi}{4}\right)dt \end{aligned} \tag{8}$$

The integral term of Eq. (8) may be set equal to zero under the band-pass assumption, provided that the carrier frequency f_c is high enough. But the first term, namely, $E/\sqrt{2}$ is nonzero. We therefore conclude that the orthogonality requirement is violated by the two QPSK signals $s'_1(t)$ and $s'_2(t)$. Hence, The "conquer and divide" approach theorem cannot be exploited beyond the QPSK signal.

Problem 7.29

To simplify the presentation, hereafter we concentrate on the complex envelope (i.e., complex baseband signal) of the QPSK signal, and likewise for the OQPSK signal. Otherwise, the phase spectra of the QPSK and OQPSK signals would become dominated by the contribution of the carrier, which complicates the graphical plots.

Figure 1 plots the phase spectrum of the QPSK signal with a square wave applied to each of the *I*- and *Q*-channels. The phase spectrum has impulses spaced uniformly at the symbol rate, corresponding to the phase discontinuities that occur at the symbol rate.

Figure 2 plots the phase spectrum of the corresponding OQPSK spectrum, with the same square wave applied to each of the *I*- and *Q*-channels. The phase spectrum of Fig. 2 is similar to that of Fig. 1 for the QPSK in that both of them consist of a series of impulses. However, in Fig. 2 the impulses are shifted in frequency as well as amplitude. Moreover, the impulses in Fig. 2 are spaced by twice the symbol rate, because every second harmonic is cancelled out.

The phase spectra plotted in figs. 1 and 2 depend on the symbol rate of the incoming square wave and the way in which the square wave is positioned with respect to the origin (i.e., time $t = 0$).

Finally, Fig. 3 plots the phase difference between the QPSK and OQPSK. From this figure we readily see that this phase difference is a nonlinear function of frequency.

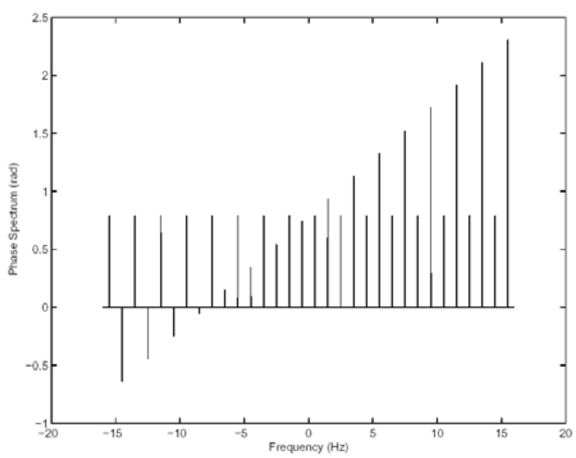
<u>Note</u>

The last sentence in the statement of Problem 7.29 should be corrected as follows:

"Hence, justify the assertion made in Drill Problem 7.3 that these two methods differ by a nonlinear phase component."

Also, add the following:

Hint: Use the complex envelope for the representation of QPSK and OQPSK signals.



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Problem 7-29 continued

Phase spectrum of QPSK with square wave in each of *I* and *Q*-channels

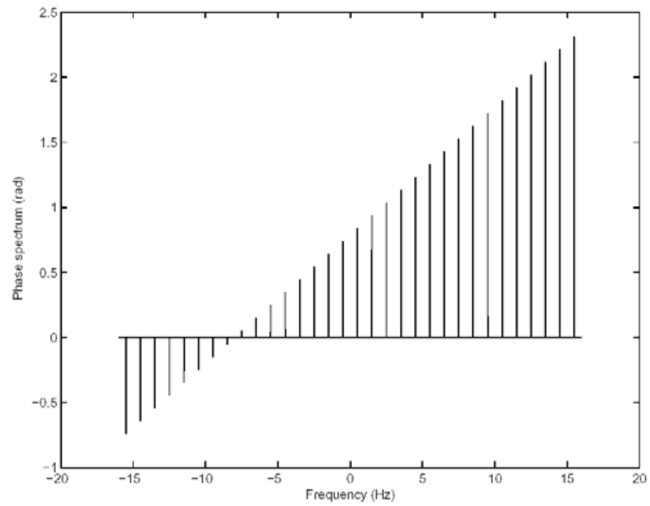

Figure 1

Phase spectrum of OQPSK with square wave in each of *I* and *Q*-channels

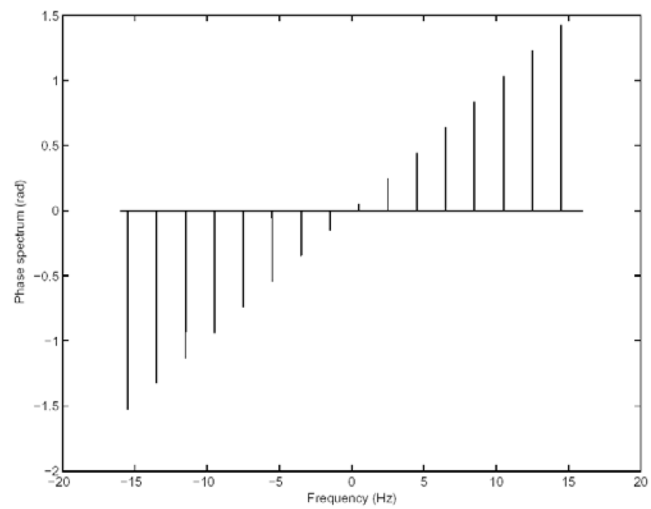

Figure 2

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Problem 7-29 continued

The phase difference spectrum

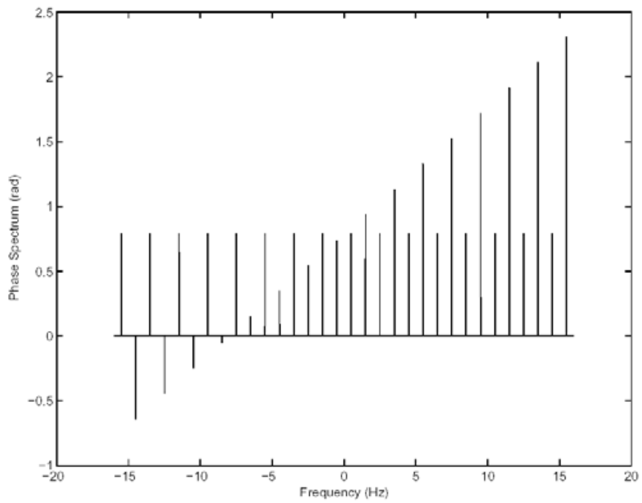

Figure 3

Problem 7.30

(a) Running the Matlab script provided in Appendix 7, the plots shown in Fig. 1 are obtained. The top plot of the figure shows the time-domain version of the bandpass signal. The carrier appears to show a small amount of amplitude modulation but this is due to the sampling process; if the sampling rate is increased by a factor of four, this amplitude modulation disappears as we would expect with rectangular pulse-shaping. The bottom plot of Fig. 1 shows the frequency-domain version of the bandpass signal. The plot is in the form of a (sin *x)/x* spectrum that is centered at the carrier frequency of 10 Hz, and the first null is offset by the bit rate of 1 Hz. The spectrum is not perfectly symmetric about the carrier due to aliasing, which affects the higher frequency components.

(b) We modify the provided Matlab script by inserting the statement

```
b = bIp + j*bQp;
```

and modifying the two statements

```
subplot(2,1,1), plot(t,real(b));       % time display
[spec,freq] = spectrum(b,nFFT,nFFT/4,nFFT/2,Fs);
```

With these changes, we obtain the plots shown in Fig. 2. The top plot of the figure shows the time-domain sequence of random data with rectangular pulse shaping. The bottom plot shows the (sin *x*)/*x* magnitude spectrum centered at the origin. In part (a), distortion of both the time-domain and frequency domain signals was noted due to the limitations of the sampling rate. In part (b), these distortions are much less evident. Consequently, if we simulate signals at complex baseband, then we may use much lower sampling rates (and thus less computational requirements) than for bandpass signals and obtain the same accuracy.

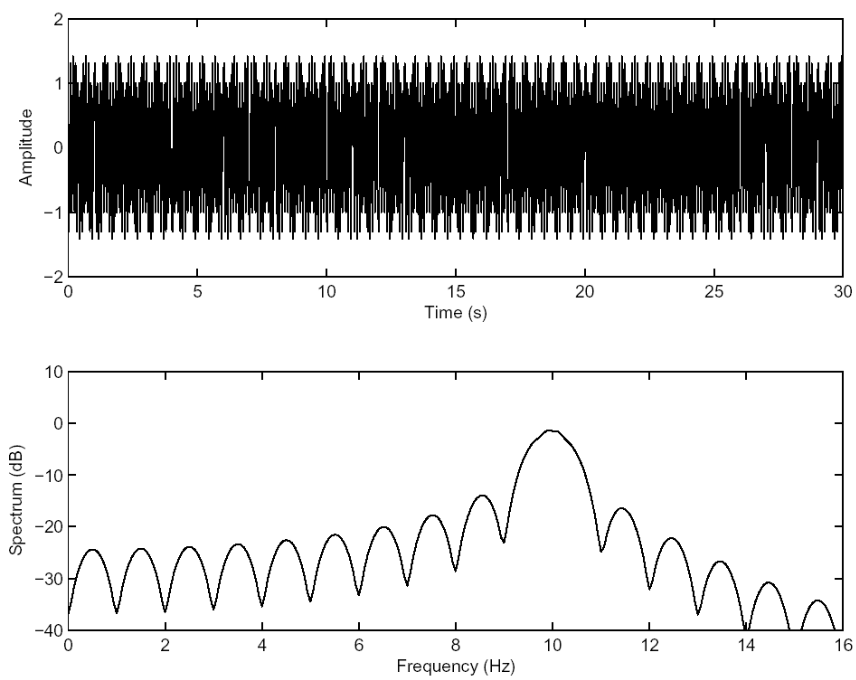

Figure 1

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Problem 7-30 continued

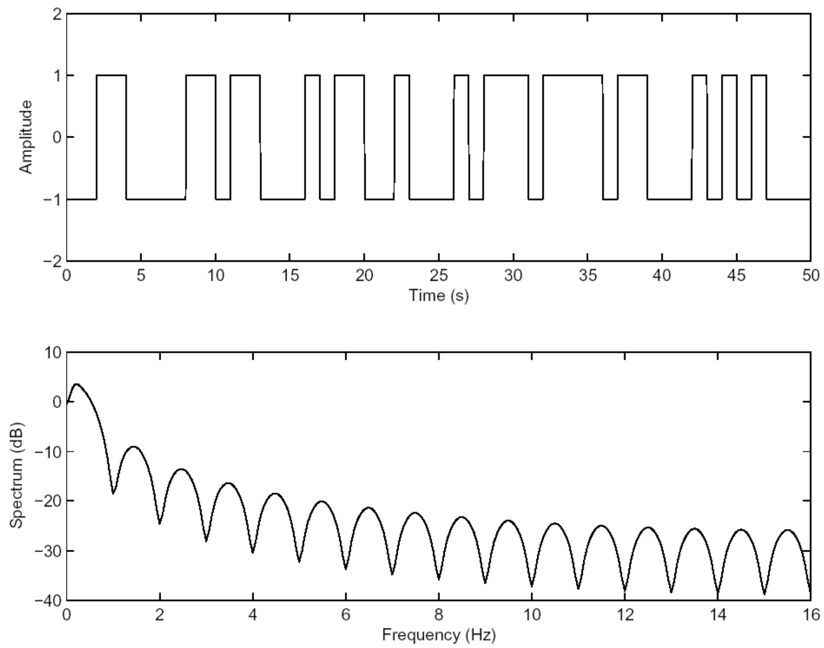


Figure 2

Problem 7.31

We modify the script of Problem 7.30(a) by replacing

```
Pulse Shape = ones(1,Fs); % rectangular pulse shape
With the lines
B0   = 0.5;                          % (Hz)
t    = [-2.001: 1/Fs : +2.001]        % time scale for pulse shape
rcos = sinc(4*B0*t) ./ (1-16*B0^2*t.^2);  % from Eq.(6.20)
Pulse Shape = rcos;
```

Doing so, we obtain the graphs plotted in Fig. 1. The top graph of the figure shows the time-domain version of the bandpass signal, including the amplitude modulation that occurs with raised cosine pulse-shaping. The bottom graph of the figure shows the raised cosine spectrum of the transmitted signal. Presence of the effects of aliasing is evident in the plot due to the spurious signal present at 0 Hz in the magnitude spectrum.

If we make changes similar to those of Problem 7.30(b), then we obtain the plots shown in Fig. 2. The top graph of the figure shows the baseband *I*-channel consisting of a random data stream with raised cosine pulse shaping. The bottom graph of the figure shows the magnitude spectrum of the complex baseband signal. There is no evidence of significant aliasing effects in this figure. The effects of aliasing are less evident in the raised-cosine case, because the spectrum is much more constrained than it is with rectangular pulse-shaping.

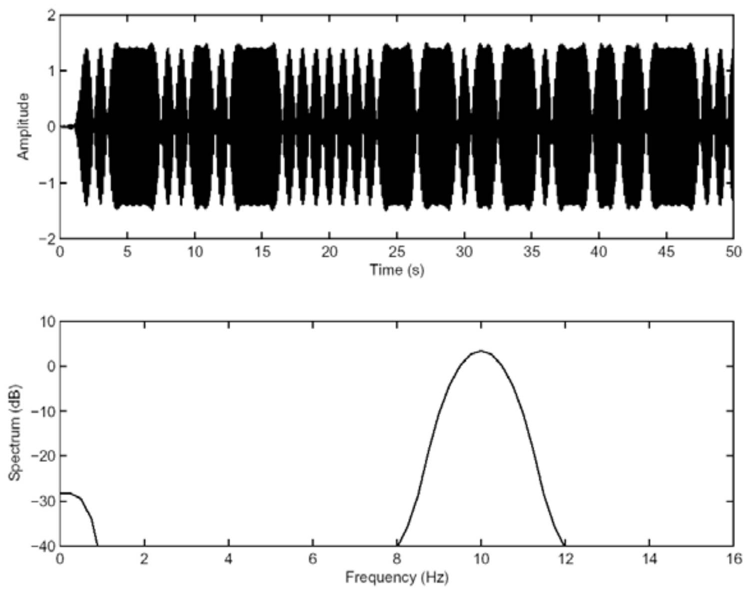


Figure 1

Continued on next slide

Problem 7-31 continued

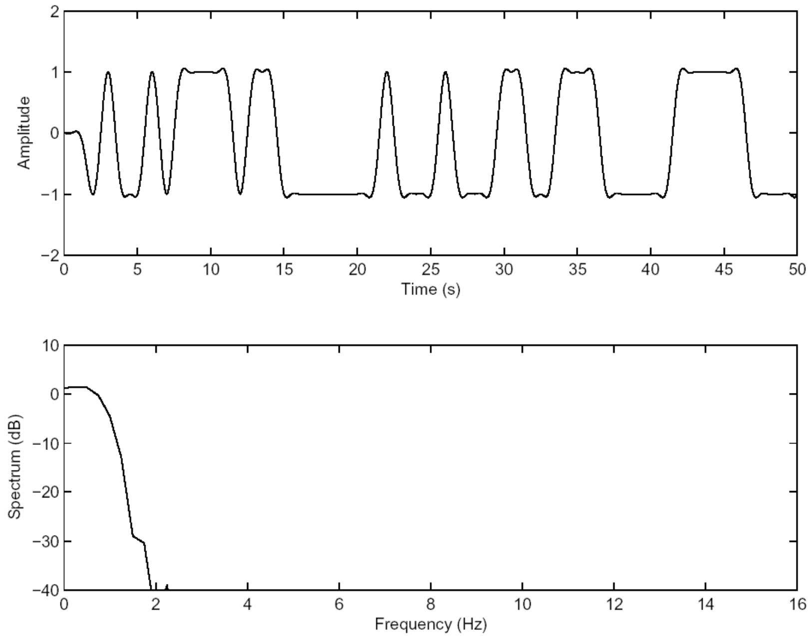


Figure 2

Problem 8.1 An information packet contains 200 bits. This packet is transmitted over a communications channel where the probability of error for each bit is 10^{-3}. What is the probability that the packet is received error-free?

Solution

Recognizing that the number of errors has a binomial distribution over the sequence of 200 bits, let x represent the number of errors with p = 0.001 and n = 200. Then the probability of no errors is

$$\begin{aligned} \mathbf{P}\left[\mathrm{x}=0\right] &= (1-p)^n \\ &= (1-.001)^{200} \\ &= .999^{200} \\ &= 0.82 \end{aligned}$$

Problem 8.2 Suppose the packet of the Problem 8.1 includes an error-correcting code that can correct up to three errors located anywhere in the packet. What is the probability that a particular packet is received in error in this case?

Solution

The probability of a packet error is equal to the probability of more than three bit errors. This is equivalent to 1 minus the probability of 0, 1, 2, or 3 errors:

$$
\begin{aligned}
1-\mathbf{P}[x \le 3] &= 1-\left(\mathbf{P}[x=0]+\mathbf{P}[x=1]+\mathbf{P}[x=2]+\mathbf{P}[x=3]\right) \\
&= 1-(1-p)^n - \binom{n}{1} p(1-p)^{n-1} - \binom{n}{2} p^2 (1-p)^{n-2} - \binom{n}{3} p^3 (1-p)^{n-3} \\
&= 1-(1-p)^{n-3}\left[(1-p)^3 + np(1-p)^2 + \frac{n(n-1)}{2} p^2 (1-p) + \frac{n(n-1)(n-2)}{6} p^3\right] \\
&= 5.5\times 10^{-5}
\end{aligned}
$$

Problem 8.3 Continuing with Example 8.6, find the following conditional probabilities: $\mathbf{P}[X=0|Y=1]$ and $\mathbf{P}[X=1|Y=0]$.

Solution

From Bayes' Rule

$$\mathbf{P}[X=0|Y=1] = \frac{\mathbf{P}[Y=1|X=0]\mathbf{P}[X=0]}{\mathbf{P}[Y=1]} = \frac{pp_0}{pp_0 + (1-p)p_1}$$

$$\mathbf{P}[X=1|Y=0] = \frac{\mathbf{P}[Y=0|X=1]\mathbf{P}[X=1]}{\mathbf{P}[Y=0]} = \frac{pp_1}{pp_1 + (1-p)p_0}$$

Problem 8.4 Consider a binary symmetric channel for which the conditional probability of error $p = 10^{-4}$, and symbols 0 and 1 occur with equal probability. Calculate the following probabilities:

a) The probability of receiving symbol 0.
b) The probability of receiving symbol 1.
c) The probability that symbol 0 was sent, given that symbol 0 is received
d) The probability that symbol 1 was sent, given that symbol 0 is received.

Solution

(a)

$$\begin{aligned}\mathbf{P}[Y=0] &= \mathbf{P}[Y=0 \mid X=0]\mathbf{P}[X=0] + \mathbf{P}[Y=0 \mid X=1]\mathbf{P}[X=1] \\ &= (1-p)p_0 + pp_1 \\ &= .9999\ \tfrac{1}{2} + .0001\ \tfrac{1}{2} \\ &= \tfrac{1}{2}\end{aligned}$$

(b)

$$\begin{aligned}\mathbf{P}[Y=1] &= 1-\mathbf{P}[Y=0] \\ &= \tfrac{1}{2}\end{aligned}$$

(c) From Eq.(8.30)

$$\begin{aligned}\mathbf{P}[X=0 \mid Y=0] &= \frac{(1-p)p_0}{(1-p)p_0 + pp_1} \\ &= \frac{(1-10^{-4})\tfrac{1}{2}}{(1-10^{-4})\tfrac{1}{2} + 10^{-4}\ \tfrac{1}{2}} \\ &= 1-10^{-4}\end{aligned}$$

(d) From Prob. 8.3

$$\begin{aligned}\mathbf{P}[X=1 \mid Y=0] &= \frac{pp_1}{pp_1 + (1-p)p_0} \\ &= \frac{10^{-4}\ \tfrac{1}{2}}{10^{-4}\ \tfrac{1}{2} + (1-10^{-4})\ \tfrac{1}{2}} \\ &= 10^{-4}\end{aligned}$$

Problem 8.5 Determine the mean and variance of a random variable that is uniformly distributed between *a* and *b*.

Solution

The mean of the uniform distribution is given by

$$\begin{aligned}\mu = \mathbf{E}[X] &= \int_{-\infty}^{\infty} x f_X(x)dx \\ &= \int_a^b x \frac{1}{b-a} dx \\ &= \left.\frac{x^2}{2(b-a)}\right|_a^b \\ &= \frac{b^2-a^2}{2(b-a)} \\ &= \frac{b+a}{2}\end{aligned}$$

The variance is given by

$$\begin{aligned}\mathbf{E}\left[(X-\mu)^2\right] &= \int_{-\infty}^{\infty} (x-\mu)^2 f_X(x)dx \\ &= \int_a^b \frac{(x-\mu)^2}{b-a} dx \\ &= \frac{1}{b-a} \frac{(b-\mu)^3}{3} - \frac{(a-\mu)^3}{3}\end{aligned}$$

If we substitute $\mu = \frac{b+a}{2}$ then

$$\begin{aligned}\mathbf{E}\left[(X-\mu)^2\right] &= \frac{1}{b-a}\left[\frac{(b-a)^3}{24} - \frac{(a-b)^3}{24}\right] \\ &= \frac{(b-a)^2}{12}\end{aligned}$$

Problem 8.6 Let X be a random variable and let $Y = (X-\mu_X)/\sigma_X$. What is the mean and variance of the random variable Y?

Solution

$$\mathbf{E}[Y] = \mathbf{E}\left[\frac{X-\mu_X}{\sigma_X}\right] = \frac{\mathbf{E}[X]-\mu_X}{\sigma_X} = \frac{0}{\sigma_X} = 0$$

$$\mathbf{E}(Y-\mu_Y)^2 = \mathbf{E}[Y^2] = \mathbf{E}\left(\frac{X-\mu_X}{\sigma_X}\right)^2$$

$$= \frac{\mathbf{E}(X-\mu_X)^2}{\sigma_X{}^2} = \frac{\sigma_X{}^2}{\sigma_X{}^2} = 1$$

Problem 8.7 What is the probability density function of the random variable *Y* of Example 8.8? Sketch this density function.

Solution

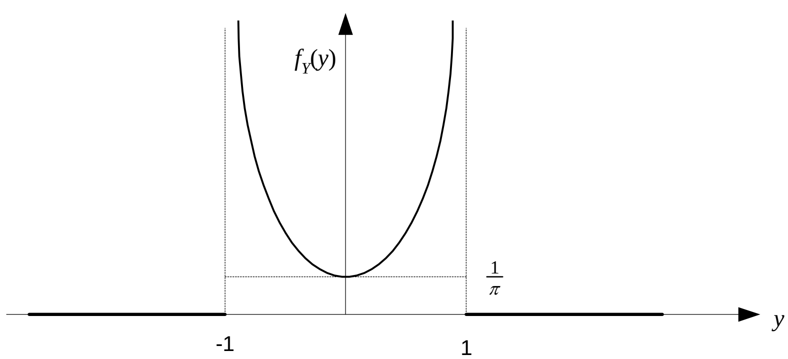
From Example 8.8, the distribution of *Y* is

$$F_Y(y) = \begin{cases} 0 & y < -1 \\ \dfrac{2\pi - 2\cos^{-1}(y)}{2\pi} & |y| < 1 \\ 1 & y > 1 \end{cases}$$

Thus, the density of *Y* is given by

$$\frac{dF_Y(y)}{dy} = \begin{cases} 0 & y < -1 \\ \dfrac{1}{\pi\sqrt{1-y^2}} & |y| < 1 \\ 0 & y > 1 \end{cases}$$

This density is sketched in the following figure.



Problem 8.8 Show that the mean and variance of a Gaussian random variable *X* with the density function given by Eq. (8.48) are μ_X and σ_X^2 .

Solution

Consider the difference $\mathbf{E}[X]-\mu_X$:

$$\mathbf{E}[X]-\mu_X = \int_{-\infty}^{\infty}\frac{(x-\mu_X)}{\sqrt{2\pi}\sigma_X}\exp\left\{-\frac{(x-\mu_X)^2}{2\sigma_X{}^2}\right\}dx$$

Let $y = x-\mu_X$ and substitute

$$\begin{aligned}\mathbf{E}[X]-\mu_X &= \int_{-\infty}^{\infty}\frac{y}{\sqrt{2\pi}\sigma_X}\exp\left(-y^2 / 2\sigma_X{}^2\right)dy\\ &= 0\end{aligned}$$

since integrand has odd symmetry. This implies $E[X]=\mu_X$. With this result

$$\begin{aligned}\mathrm{Var}(X) &= \mathbf{E}(x-\mu_X)^2\\ &= \int_{-\infty}^{\infty}\frac{(x-\mu_X)^2}{\sqrt{2\pi}\sigma_X}\exp\left\{\frac{-(x-\mu_X)^2}{2\sigma_X{}^2}\right\}dx\end{aligned}$$

In this case let

$$y = \frac{x-\mu_X}{\sigma_X}$$

and making the substitution, we obtain

$$\mathrm{Var}(X) = \sigma_X{}^2\int_{-\infty}^{\infty}\frac{y^2}{\sqrt{2\pi}}\exp\left\{\frac{-y^2}{2}\right\}dy$$

Recalling the integration-by-parts, i.e., $\int udv = uv - \int vdu$, let $u = y$ and $dv = y\exp\left(\frac{-y^2}{2}\right)dy$. Then

Continued on next slide

Problem 8.8 continued

$$\begin{aligned}\operatorname{Var}(X) &= \sigma_X{}^2 \frac{(-)y}{\sqrt{2\pi}} \exp\left(-y^2/2\right)\Big|_{-\infty}^{\infty} + \sigma_X{}^2 \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) dy \\ &= 0 + \sigma_X{}^2 \bullet 1 \\ &= \sigma_X{}^2\end{aligned}$$

where the second integral is one since it is integral of the normalized Gaussian probability density.

Problem 8.9 Show that for a Gaussian random variable X with mean μ_X and variance σ_X^2 the transformation $Y = (X - \mu_X)/\sigma_X$, converts X to a normalized Gaussian random variable.

Solution

Let $y = \dfrac{x-\mu_X}{\sigma_X}$. Then

$$\begin{aligned}\mathbf{E}[Y] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} y \exp\left(-y^2/2\right)dy \\ &= 0\end{aligned}$$

by the odd symmetry of the integrand. If $\mathbf{E}[Y] = 0$, then from the definition of Y, $\mathbf{E}[X] = \mu_X$. In a similar fashion

$$\begin{aligned}\mathbf{E}[Y^2] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} y^2 \exp\left(-\frac{y^2}{2}\right)dy \\ &= \frac{(-)y}{\sqrt{2\pi}}\cdot\exp\left\{-\frac{y^2}{2}\right\}\Bigg|_{-\infty}^{\infty} + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(-y^2/2\right)dy \\ &= 1\end{aligned}$$

where we use integration by parts as in Problem 8.8. This result implies

$$E\left(\frac{x-\mu_X}{\sigma_X}\right)^2 = 1$$

and hence $\mathbf{E}(x-\mu_X)^2 = \sigma_X^2$

Problem 8.10 Determine the mean and variance of the sum of five independent uniformly-distributed random variables on the interval from -1 to +1.

Solution

Let X_i be the individual uniformly distributed random variables for i = 1,..,5, and let Y be the random variable representing the sum:

$$Y = \sum_{i=1}^{5} X_i$$

Since X_i has zero mean and Var(X_i) = 1/3 (see Problem 8.5), we have

$$\mathbf{E}[Y] = \sum_{i=1}^{5} \mathbf{E}[X_i] = 0$$

and

$$\begin{aligned} \mathrm{Var}(Y) &= \mathbf{E}\left[(Y-\mu_Y)^2\right] = \mathbf{E}\left[Y^2\right] \\ &= \mathbf{E}\left[\left(\sum X_i\right)^2\right] \\ &= \sum_{i=1}^{5} \mathbf{E}\left[X_i^2\right] + \sum_{i \neq j} \mathbf{E}\left[X_i X_j\right] \end{aligned}$$

Since the X_i are independent, we may write this as

$$\begin{aligned} \mathrm{Var}(Y) &= 5\left(1/3\right) + \sum \mathbf{E}[X_i]\mathbf{E}[X_j] \\ &= 5/3 + 0 \\ &= 5/3 \end{aligned}$$

Problem 8.11 A random process is defined by the function

$$X(t,\theta) = A\cos(2\pi ft + \theta)$$

where A and f are constants, and θ is uniformly distributed over the interval 0 to 2π. Is X stationary to the first order?

Solution

Denote

$$Y = X(t_1,\theta) = A\cos(2\pi ft_1 + \theta)$$

for any t_1. From Problem 8.7, the distribution of Y and therefore of X for any t_1 is

$$F_{X(t_1)}(y) = \begin{cases} 0 & y < -A \\ \dfrac{2\pi - 2\cos^{-1}(y/A)}{2\pi} & |y| < A \\ 1 & y > A \end{cases}$$

Since the distribution is independent of t it is stationary to first order.

Problem 8.12 Show that a random process that is stationary to the second order is also stationary to the first order.

Solution

Let the distribution F be stationary to second order

$$F_{X(t_1)X(t_2)}(x_1,x_2)=F_{X(t_1+\tau)X(t_2+\tau)}(x_1,x_2)$$

Then,

$$\begin{aligned}F_{X(t_1)X(t_2)}(x_1,\infty)&=F_{X(t_1)}(x_1)\\&=F_{X(t_1+\tau)X(t_2+\tau)}(x_1,\infty)\\&=F_{X(t_1+\tau)}(x_1)\end{aligned}$$

Thus the first order distributions are stationary as well.

Problem 8.13 Let $X(t)$ be a random process defined by

$$X(t) = A\cos(2\pi ft)$$

where A is uniformly distributed between 0 and 1, and f is constant. Determine the autocorrelation function of X. Is X wide-sense stationary?

Solution

$$\begin{aligned}\mathbf{E}[X(t_1)X(t_2)] &= \mathbf{E}[A^2]\cos(2\pi ft_1)\cos(2\pi ft_2) \\ &= \mathbf{E}[A^2][\cos(2\pi f(t_1 - t_2)) + \cos 2\pi f(t_1 + t_2)]\end{aligned}$$

$$\mathbf{E}[A^2] = \int_0^1 x^2 dx = \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3}$$

Since the autocorrelation function depends on $t_1 + t_2$ as well as $t_1 - t_2$, the process is not wide-sense stationary.

Problem 8.14 A discrete-time random process $\{Y_n : n = \ldots,-1,0,1,2,\ldots\}$ is defined by

$$Y_n = \alpha_0 Z_n + \alpha_1 Z_{n-1}$$

where $\{Z_n\}$ is a random process with autocorrelation function $R_Z(n) = \sigma^2 \delta(n)$. What is the autocorrelation function $R_Y(n,m) = \mathbf{E}[Y_n Y_m]$? Is the process $\{Y_n\}$ wide-sense stationary?

Solution

We implicitly assume that Z_n is stationary and has a constant mean μ_Z. Then the mean of Y_n is given by

$$\begin{aligned}\mathbf{E}[Y_n] &= \alpha_0 \mathbf{E}[Z_n] + \alpha_1 \mathbf{E}[Z_{n-1}] \\ &= (\alpha_0 + \alpha_1)\mu_Z\end{aligned}$$

The autocorrelation of Y is given by

$$\begin{aligned}\mathbf{E}[Y_n Y_m] &= \mathbf{E}[(\alpha_0 Z_n + \alpha_0 Z_{n-1})(\alpha_0 Z_m + \alpha_1 Z_{m-1})] \\ &= \alpha_0^2 \mathbf{E}[Z_n Z_m] + \alpha_1 \alpha_0 \mathbf{E}[Z_n Z_{m-1}] + \alpha_0 \alpha_1 \mathbf{E}[Z_{n-1} Z_m] + \alpha_1^2 \mathbf{E}[Z_{m-1} Z_{n-1}] \\ &= \alpha_0^2 \sigma^2 \delta(n-m) + \alpha_1 \alpha_0 \sigma^2 \delta(m-1-n) + \alpha_0 \alpha_1 \sigma^2 \delta(n-1-m) + \alpha_1^2 \delta(m-1-(n-1)) \\ &= (\alpha_0^2 + \alpha_1^2)\sigma^2 \delta(n-m) + \alpha_0 \alpha_1 \sigma^2 [\delta(n-m-1) + \delta(m-n-1)]\end{aligned}$$

Since the autocorrelation only depends on the time difference *n-m*, the process is wide-sense stationary with

$$R_Y(n) = (\alpha_0^2 + \alpha_1^2)\sigma^2 \delta(n) + \alpha_0 \alpha_1 \sigma^2 (\delta(n-1) + \delta(n+1))$$

Problem 8.15 For the discrete-time process of Problem 8.14, use the discrete Fourier transform to approximate the corresponding spectrum. That is,

$$S_Y(k) = \sum_{n=0}^{N-1} R_Y(n) W^{kn}$$

If the sampling in the time domain is at n/T_s where $n = 0, 1, 2, \ldots, N$-1. What frequency does k correspond to?

Solution

Let $\beta_0 = (\alpha_0^2 + \alpha_1^2)\sigma^2$ and $\beta_1 = \alpha_0\alpha_1\sigma^2$. Then

$$\begin{aligned}
S_Y(k) &= \sum_{n=0}^{N-1} [\beta_0\delta(n) + \beta_1(\delta(n-1) + \delta(n+1))] W^{kn} \\
&= \beta_0 W^0 + \beta_1 (W^{-k} + W^{+k}) \\
&= \beta_0 + \beta_1 \left(e^{\frac{-j2\pi k}{N}} + e^{\frac{+j2\pi k}{N}} \right) \\
&= \beta_0 + 2\beta_1 \cos\left(\frac{2\pi k}{N}\right)
\end{aligned}$$

The term $S_Y(k)$ corresponds to frequency $\frac{kf_s}{N}$ where $f_S = \frac{1}{T_S}$.

Problem 8.16 Is the discrete-time process $\{Y_n: n = 1,2,\ldots\}$ defined by: $Y_0 = 0$ and

$$Y_{n+1} = \alpha Y_n + W_n,$$

a Gaussian process, if W_n is Gaussian?

Solution

(Proof by mathematical induction.) The first term $Y_1 = \alpha Y_0 + W_0$ is Gaussian since $Y_0 = 0$ and W_0 are Gaussian. The second term $Y_2 = \alpha Y_1 + W_1$ is Gaussian since Y_1 and W_1 are Gaussian. Assume Y_n is Gaussian. Then $Y_{n+1} = \alpha Y_n + W_n$ is Gaussian since Y_n and W_n are both Gaussian.

Problem 8.17 A discrete-time white noise process $\{W_n\}$ has an autocorrelation function given by $R_W(n) = N_0\delta(n)$.

(a) Using the discrete Fourier transform, determine the power spectral density of $\{W_n\}$.

(b) The white noise process is passed through a discrete-time filter having a discrete-frequency response

$$H(k) = \frac{1-(\alpha W^k)^N}{1-\alpha W^k}$$

where, for a N-point discrete Fourier transform, $W = \exp\{j2\pi/N\}$. What is the spectrum of the filter output?

Solution

The spectrum of the discrete white noise process is

$$\begin{aligned} S(k) &= \sum_{n=0}^{N-1} R(n) W^{nk} \\ &= \sum_{n=0}^{N-1} N_0 \delta(n) W^{nk} \\ &= N_0 \end{aligned}$$

The spectrum of the process after filtering is

$$\begin{aligned} S_Y(k) &= |H(k)|^2 S(k) \\ &= N_0 \left| \frac{1-(\alpha W^k)^N}{1-\alpha W^k} \right|^2 \end{aligned}$$

Problem 8.18 Consider a deck of 52 cards, divided into four different suits, with 13 cards in each suit ranging from the two up through the ace. Assume that all the cards are equally likely to be drawn.

(a) Suppose that a single card is drawn from a full deck. What is the probability that this card is the ace of diamonds? What is the probability that the single card drawn is an ace of any one of the four suits?

(b) Suppose that two cards are drawn from the full deck. What is the probability that the cards drawn are an ace and a king, not necessarily the same suit? What if they are of the same suit?

<u>Solution</u>

(a)

$$\mathbf{P}[\text{Ace of diamonds}] = \frac{1}{52}$$

$$\mathbf{P}[\text{Any ace}] = \frac{1}{13}$$

(b)

$$\begin{aligned}\mathbf{P}[\text{Ace and king}] &= \mathbf{P}[\text{Ace on first draw}]\mathbf{P}[\text{King on second}] + \mathbf{P}[\text{King on first draw}]\mathbf{P}[\text{Ace on seco}\\ &= \frac{1}{13}\times\frac{4}{51}+\frac{1}{13}\times\frac{4}{51}\\ &= \frac{8}{663}\end{aligned}$$

$$\begin{aligned}\mathbf{P}[\text{Ace and king of same suit}] &= \frac{1}{13}\times\frac{1}{51}+\frac{1}{13}\times\frac{1}{51}\\ &= \frac{1}{663}\end{aligned}$$

Problem 8.19 Suppose a player has one red die and one white die. How many outcomes are possible in the random experiment of tossing the two dice? Suppose the dice are indistinguishable, how many outcomes are possible?

Solution

The number of possible outcomes is $6 \times 6 = 36$, if distinguishable.
If the die are indistinguishable then the outcomes are

(11) (12)…(16)
(22)(23)…(26)
(33)(34)…(36)
(44)(45)(46)
(55)(56)
(66)

And the number of possible outcomes are 21.

Problem 8.20 Refer to Problem 8.19.
(a) What is the probability of throwing a red 5 and a white 2?
(b) If the dice are indistinguishable, what is the probability of throwing a sum of 7? If they are distinguishable, what is this probability?

Solution

(a) $\mathbf{P}\left[\text{Red 5 and white 2}\right] = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$

(b) The probability of the sum does not depend upon whether the die are distinguishable or not. If we consider the distinguishable case the possible outcomes are (1,6), (2,5), (3,4), (4,3), (5,2), and (6,1) so

$$\mathbf{P}\left[\text{sum of } 7\right] = \frac{6}{36} = \frac{1}{6}$$

Problem 8.21 Consider a random variable X that is uniformly distributed between the values of 0 and 1 with probability ¼ takes on the value 1 with probability ¼ and is uniformly distributed between values 1 and 2 with probability ½ . Determine the distribution function of the random variable X.

Solution

$$F_X(x) = \begin{cases} 0 & x \le 0 \\ x/4 & 0 < x < 1 \\ 1/2 & x = 1 \\ \frac{1}{2} + \frac{1}{2}(x-1) & 1 < x \le 2 \\ 1 & x > 2 \end{cases}$$

Problem 8.22 Consider a random variable X defined by the double-exponential density where a and b are constants.

$$f_X(x) = a\exp\left(-b|x|\right) \qquad -\infty < x < \infty$$

(a) Determine the relationship between a and b so that $f_X(x)$ is a probability density function.
(b) Determine the corresponding distribution function $F_X(x)$.
(c) Find the probability that the random variable X lies between 1 and 2.

Solution

(a)

$$\int_{-\infty}^{\infty} f_X(x)dx = 1 \quad \Rightarrow 2\int_0^{\infty} a\exp(-bx)dx = 1$$

$$-\frac{2a}{b}\exp(-bx)\Big|_0^{\infty} = 1$$

$$\Rightarrow \frac{2a}{b} = 1 \;\; or \quad b = 2a$$

(b)

$$F_X(x) = \int_{-\infty}^{x} a\exp\left(-b|s|\right)ds$$

$$= \begin{cases} -\dfrac{a}{b}\exp(-b(-s))\Big|_{-\infty}^{x} & -\infty < x < 0 \\ \dfrac{1}{2} + -\dfrac{a}{b}\exp(-bs)\Big|_0^{x} & 0 < x < \infty \end{cases}$$

$$= \begin{cases} \dfrac{a}{b}\exp(bs) & -\infty < x < 0 \\ \dfrac{1}{2} + \dfrac{a}{b} - \dfrac{a}{b}\exp(-bs) & 0 \le x < \infty \end{cases}$$

$$= \begin{cases} \dfrac{1}{2}\exp(bx) & -\infty < x < 0 \\ 1 - \dfrac{1}{2}\exp(-bx) & 0 \le x < \infty \end{cases}$$

(c) The probability that $1 \le X \le 2$ is

$$F_X(2) - F_X(1) = \frac{1}{2}\left[\exp(-b) - \exp(-2b)\right]$$

Problem 8.23 Show that the expression for the variance of a random variable can be expressed in terms of the first and second moments as

$$\mathrm{Var}(X) = \mathbf{E}\left[X^2\right] - \left(\mathbf{E}[X]\right)^2$$

Solution

$$\begin{aligned}
\mathrm{Var}(X) &= \mathbf{E}\left[(X - \mathbf{E}(X))^2\right] \\
&= \mathbf{E}\left(X^2 - 2X\mathbf{E}(X) + (\mathbf{E}[X])^2\right) \\
&= \mathbf{E}\left[X^2\right] - 2\mathbf{E}[X]\mathbf{E}[X] + (\mathbf{E}[X])^2 \\
&= \mathbf{E}\left[X^2\right] - (\mathbf{E}[X])^2
\end{aligned}$$

Problem 8.24 A random variable R is Rayleigh distributed with its probability density function given by

$$f_R(r) = \begin{cases} \frac{r}{b}\exp\left(-r^2/2b\right) & 0 \le r < \infty \\ 0 & otherwise \end{cases}$$

(a) Determine the corresponding distribution function

(b) Show that the mean of R is equal to $\sqrt{b\pi/2}$

(c) What is the mean-square value of R?

(d) What is the variance of R?

Solution

(a) The distribution of R is

$$\begin{aligned} F_R(r) &= \int_0^r f_R(s)ds \\ &= \int_0^r \frac{s}{b}\exp\left(-\frac{s^2}{2b}\right)ds \\ &= -\exp\left(-\frac{s^2}{2b}\right)\Bigg|_0^r \\ &= 1-\exp\left(-r^2\big/2b\right) \end{aligned}$$

(b) The mean value of R is

$$\begin{aligned} \mathbf{E}[R] &= \int_0^\infty s f_R(s)ds \\ &= \int_0^\infty \frac{s^2}{b}\exp\left(\frac{-s^2}{2b}\right)ds \\ &= \frac{1}{b}\sqrt{2\pi b}\left[\frac{1}{\sqrt{2\pi b}}\int_0^\infty s^2\exp\left(\frac{-s^2}{2b}\right)ds\right] \end{aligned}$$

The bracketed expression is equivalent to the evaluation of the half of the variance of a zero-mean Gaussian random variable which we know is b in this case, so

$$\mathbf{E}[R] = \frac{\sqrt{2\pi b}}{b}\frac{1}{2}(b) = \sqrt{\frac{\pi b}{2}}$$

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Problem 8.24 continued

(c) The second moment of R is

$$
\begin{aligned}
\mathbf{E}\left[R^2\right] &= \int_0^\infty s^2 f_R(s)ds \\
&= \int_0^\infty \frac{s^3}{b}\exp\left(\frac{-s^2}{2b}\right)ds \\
&= s^2 F_R(s)\Big|_0^\infty - \int_0^\infty 2s \;\; F_R(s)\, ds \\
&= s^2 F_R(s)\Big|_0^\infty - \int_0^\infty 2s\left(1-\exp\left(\frac{s^2}{2b}\right)\right)ds \\
&= s^2\left(F_R(s)-1\right)\Big|_0^\infty + 2b\int_0^\infty f_R(s)ds \\
&= 2b
\end{aligned}
$$

(d) The variance of R is

$$
\begin{aligned}
\mathrm{Var}(R) &= \mathbf{E}\left[R^2\right] - \left(\mathbf{E}[R]\right)^2 \\
&= 2b - \left(\sqrt{b\pi}/2\right)^2 \\
&= b\left(2 - \pi/2\right)
\end{aligned}
$$

Problem 8.25 Consider a uniformly distributed random variable Z, defined by

$$f_Z(z) = \begin{cases} \dfrac{1}{2\pi}, & 0 \le z < 2\pi \\ 0, & \text{otherwise} \end{cases}$$

The two random variables *X* and *Y* are related to *Z* by $X = \sin(Z)$ and $Y = \cos(Z)$.

(a) Determine the probability density functions of *X* and *Y*.
(b) Show that *X* and *Y* are uncorrelated random variables.
(c) Are *X* and *Y* statistically independent? Why?

<u>Solution</u>
(a) The distribution function of *X* is formally given by

$$F_X(x) = \begin{cases} 0 & x \le -1 \\ \mathbf{P}[-1 \le X \le x] & -1 < x < 1 \\ 1 & x \ge 1 \end{cases}$$

Analogous to Example 8.8, we have

$$\mathbf{P}(-1 \le X \le x) = \begin{cases} \mathbf{P}[\pi - \sin^{-1}(x) \le Z \le 2\pi + \sin^{-1}(x)] & -1 \le x \le 0 \\ \dfrac{1}{2} + \mathbf{P}[0 \le Z \le \sin^{-1}(x)] + \mathbf{P}[\pi - \sin^{-1}(x) \le Z \le \pi] & 0 \le x \le 1 \end{cases}$$

$$= \begin{cases} \dfrac{\pi + 2\sin^{-1}(x)}{2\pi} & -1 \le x \le 0 \\ \dfrac{1}{2} + \dfrac{2\sin^{-1}(x)}{2\pi} & 0 \le x \le 1 \end{cases}$$

$$= \frac{1}{2} + \frac{\sin^{-1}(x)}{\pi} \qquad -1 \le x \le 1$$

where the second line follows from the fact that the probability for a uniform random variable is proportional to the length of the interval. The distribution of *Y* follows from a similar argument (see Example 8.8).

Continued on next slide

Problem 8.25 continued

(b) To show *X* and *Y* are uncorrelated, consider

$$\begin{aligned}\mathbf{E}[XY] &= \mathbf{E}[\sin(Z)\cos(Z)] \\ &= \mathbf{E}\left[\frac{\sin(2Z)}{2}\right] \\ &= \frac{1}{4\pi}\int_0^{2\pi}\sin(2z)dz \\ &= -\frac{1}{8\pi}\cos(2z)\Big|_0^{2\pi} = 0\end{aligned}$$

Thus *X* and *Y* are uncorrelated.

(c) The random variables *X* and *Y* are not statistically independent since

$$\Pr[X|Y] \neq \Pr[X]$$

Problem 8.26 A Gaussian random variable has zero mean and a standard deviation of 10 V. A constant voltage of 5 V is added to this random variable.
(a) Determine the probability that a measurement of this composite signal yields a positive value.
(b) Determine the probability that the arithmetic mean of two independent measurements of this signal is positive.

Solution

(a) Let Z represent the initial Gaussian random variable and Y the composite random variable. Then

$$Y = 5 + Z$$

and the density function of Y is given by

$$f_y(y) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-(y-\mu)^2/2\sigma^2\right\}$$

where μ corresponds to a mean of 5V and σ corresponds to a standard deviation of 10V. The probability that Y is positive is

$$\begin{aligned} P[Y > 0] &= \frac{1}{\sqrt{2\pi}\sigma}\int_0^\infty \exp\left\{-(y-\mu)/2\sigma^2\right\}dy \\ &= \frac{1}{\sqrt{2\pi}}\int_{\frac{-\mu}{\sigma}}^\infty \exp\left(-s^2/2\right)ds \\ &= Q\left(\frac{-\mu}{\sigma}\right) \end{aligned}$$

where, in the second line, we have made the substitution

$$s = \frac{y-\mu}{\sigma}$$

Making the substitutions for μ and σ, we have **P**[Y>0] = Q(- ½). We note that in Fig. 8.11, the values of $Q(x)$ are not shown for negative x; to obtain a numerical result, we use the fact that $Q(-x) = 1 - Q(x)$. Consequently, $Q(-½) = 1 - 0.3 = 0.7$.

Continued on next slide

Problem 8.26 continued

(b) Let W represent the arithmetic mean of two measurements Y_1 and Y_2, that is

$$W = \frac{Y_1 + Y_2}{2}$$

It follows that W is a Gaussian random variable with $\mathbf{E}[W] = \mathbf{E}[Y] = 5$. The variance of W is given by

$$\begin{aligned}\mathrm{Var}(W) &= \mathbf{E}\left[(W - \mathbf{E}(W))^2\right] \\ &= \mathbf{E}\left[\left(\frac{Y_1 + Y_2}{2} - \frac{\mathbf{E}(Y_1) + \mathbf{E}(Y_2)}{2}\right)^2\right] \\ &= \frac{1}{4}\left(\mathbf{E}\left[(Y_1 - \mathbf{E}(Y_1))^2 + (Y_2 - \mathbf{E}(Y_2))^2 + 2(Y_1 - \mathbf{E}(Y_1))(Y_2 - \mathbf{E}(Y_2))\right]\right)\end{aligned}$$

The first two terms correspond to the variance of Y. The third term is zero because the measurements are independent. Making these substitutions, the variance of W reduces to

$$\mathrm{Var}[W] = \sigma^2/2$$

Using the result of part (a), we then have

$$\mathbf{P}[W > 0] = Q\left(\frac{-\mu}{\left(\sigma/\sqrt{2}\right)}\right) = Q\left(-\frac{1}{\sqrt{2}}\right)$$

Problem 8.27 Consider a random process defined by

$$X(t) = \sin(2\pi W t)$$

in which the frequency W is a random variable with the probability density function

$$f_W(w) = \begin{cases} \dfrac{1}{B} & 0 < w < B \\ 0 & \text{otherwise} \end{cases}$$

Show that $X(t)$ is nonstationary.

Solution

At time $t = 0$, $X(0)=0$ and the distribution of $X(0)$ is

$$F_{X(0)}(x) = \begin{cases} 0 & x < 0 \\ 1 & x \ge 0 \end{cases}$$

At time $t = 1$, $X(1) = \sin(2\pi w)$, and the distribution of X(1) is clearly not a step function so

$$F_{X(1)}(x) \ne F_{X(0)}(x)$$

And the process $X(t)$ is not first-order stationary, and hence nonstationary.

Problem 8.28 Consider the sinusoidal process

$$X(t) = A\cos(2\pi f_c t)$$

where the frequency is constant and the amplitude A is uniformly distributed:

$$f_A(a) = \begin{cases} 1 & 0 < a < 1 \\ 0 & \text{otherwise} \end{cases}$$

Determine whether or not this process is stationary in the strict sense.

Solution

At time $t = 0$, $X(0) = A$, and $F_{X(0)}(0)$ is uniformly distributed over 0 to 1.
At time $t = (4f_c)^{-1}$, $X(\,(4f_c)^{-1}) = 0$ and

$$F_{X\left(\frac{1}{4f_c}\right)}(x) = \delta(0)$$

Thus, $F_{X(0)}(x) \neq F_{X(1/4f_c)}(x)$ and the process $X(t)$ is not stationary to first order.

Hence not strictly stationary.

Problem 8.29 A random process is defined by

$$X(t) = A\cos(2\pi f_c t)$$

where A is a Gaussian random variable of zero mean and variance σ^2. This random process is applied to an ideal integrator, producing an output $Y(t)$ defined by

$$Y(t) = \int_0^t X(\tau)d\tau$$

(a) Determine the probability density function of the output at a particular time.
(b) Determine whether or not is stationary.

Solution

(a) The output process is given by

$$\begin{aligned} Y(t) &= \int_0^t X(\tau)d\tau \\ &= \int_0^t A\cos(2\pi f_c \tau)d\tau \\ &= \frac{A}{2\pi f_c}\sin(2\pi f_c t) \end{aligned}$$

At time t_0, it follows that $Y(t_0)$ is Gaussian with zero mean, and variance

$$\frac{\sigma^2}{(2\pi f_c)^2}\sin^2(2\pi f_c t_0)$$

(b) No, the process $Y(t)$ is not stationary as $F_{Y(t_0)} \neq F_{Y(t_1)}$ for all t_1 and t_0.

Problem 8.30 Prove the following two properties of the autocorrelation function $R_X(\tau)$ of a random process $X(t)$:

(a) If $X(t)$ contains a dc component equal to A, then $R_X(\tau)$ contains a constant component equal to A^2.

(b) If $X(t)$ contains a sinusoidal component, then $R_X(\tau)$ also contains a sinusoidal component of the same frequency.

Solution

(a) Let $Y(t) = X(t) - A$ and $Y(t)$ is a random process with zero *dc* component. Then

$$\mathbf{E}[X(t)] = A$$

and

$$\begin{aligned} R_X(\tau) &= \mathbf{E}[X(t)X(t+\tau)] \\ &= \mathbf{E}[((X(t)-A)+A)((X(t+\tau)-A)+A)] \\ &= \mathbf{E}[(X(t)-A)(X(t+\tau)-A)] + \mathbf{E}(X(t+\tau)-A)A + \mathbf{E}(X(t)A) + A^2 \\ &= R_Y(\tau) + 0 + 0 + A^2 \end{aligned}$$

And thus $R_X(\tau)$ has a constant component A^2.

(b) Let $X(t) = Y(t) + A\sin(2\pi f_c t)$ where $Y(t)$ does not contain a sinusoidal component of frequency f_c.

$$\begin{aligned} R_X(\tau) &= \mathbf{E}[X(t)X(t+\tau)] \\ &= \mathbf{E}[(Y(t)+A\sin(2\pi f_c t))(Y(t+\tau)+A\sin(2\pi f_c t)) + \mathbf{E}[A^2\sin(2\pi f_c t)\sin(2\pi f_c(t+\tau))]] \\ &= R_Y(\tau) + \ldots + \frac{A^2}{2}[\cos 2\pi f_c t + \cos 2\pi f_c(2t+\tau) + \theta] \\ &= R_Y(\tau) + \frac{A^2}{2}\cos(2\pi f_c \tau) \end{aligned}$$

And thus $R_X(\tau)$ has a sinusoidal component at f_c.

Problem 8.31 A discrete-time random process is defined by

$$Y_n = \alpha Y_{n-1} + W_n \qquad\qquad n = \ldots, -1, 0, +1, \ldots$$

where the zero-mean random process W_n is stationary with autocorrelation function $R_W(k) = \sigma^2\delta(k)$. What is the autocorrelation function $R_y(k)$ of Y_n? Is Y_n a wide-sense stationary process? Justify your answer.

Solution

We partially address the question of whether Y_n is wide-sense stationary (WSS) first by noting that

$$\begin{aligned}\mathbf{E}[Y_n] &= \mathbf{E}[\alpha Y_{n-1} + W_n] \\ &= \alpha\mathbf{E}[Y_{n-1}] + \mathbf{E}[W_n] \\ &= \alpha\mathbf{E}[Y_{n-1}]\end{aligned}$$

since $\mathbf{E}[W_n] = 0$. To be WSS, the mean of the process must be constant and consequently, we must have that $\mathbf{E}[Y_n] = 0$ for all n, to satisfy the above relationship.

We consider the autocorrelation of Y_n in steps. First note that $R_Y(0)$ is given by

$$R_Y(0) = \mathbf{E}[Y_nY_n] = \mathbf{E}[Y_n^2]$$

and that $R_Y(1)$ is

$$\begin{aligned}R_Y(1) &= \mathbf{E}[Y_nY_{n+1}] \\ &= \mathbf{E}[Y_n(\alpha Y_n + W_n)] \\ &= \alpha\mathbf{E}[Y_n^2] + \mathbf{E}[Y_nW_n]\end{aligned}$$

Although not explicitly stated in the problem, we assume that W_n is independent of Y_n, thus $\mathbf{E}[Y_nW_n] = \mathbf{E}[Y_n]\mathbf{E}[W_n] = 0$, and so

$$R_Y(1) = \alpha R_Y(0)$$

We prove the result for general positive k by assuming $R_Y(k) = \alpha^k R_Y(0)$ and then noting that

$$\begin{aligned}R_Y(k+1) &= \mathbf{E}[Y_nY_{n+k+1}] \\ &= \mathbf{E}[Y_n(\alpha Y_{n+k} + W_{n+k})] \\ &= \alpha\mathbf{E}[Y_nY_{n+k}] + \mathbf{E}[Y_nW_{n+k}]\end{aligned}$$

Continued on next slide

Problem 8.31 continued

To evaluate this last expression, we note that, since

$$\begin{aligned} Y_n &= \alpha Y_{n-1} + W_n \\ &= \alpha^2 Y_{n-2} + \alpha W_{n-1} + W_n \\ &= \alpha^3 Y_{n-3} + \alpha^2 W_{n-2} + \alpha W_{n-1} + W_n \\ &= \cdots \end{aligned}$$

we see that Y_n only depends on W_k for $k \le n$. Thus $\mathbf{E}[Y_n W_{n+k}] = 0$. Thus, for positive k, we have

$$\begin{aligned} R_Y(k+1) &= \alpha R_Y(k) \\ &= \alpha^{k+1} R_Y(0) \end{aligned}$$

Using a similar argument, a corresponding result can be shown for negative k. Combining the results, we have

$$R_Y(k) = \alpha^{|k|} R_Y(0)$$

Since the autocorrelation only depends on the time difference k, and the process is wide-sense stationary.

Problem 8.32 Find the power spectral density of the process that has the autocorrelation function

$$R_X(\tau) = \begin{cases} \sigma^2\left(1-|\tau|^2\right) & |\tau|<1 \\ 0 & \text{otherwise} \end{cases}$$

Solution

The Wiener-Khintchine relations imply the power spectral density is given by the Fourier transform of $R_X(\tau)$, which is (see Appendix 6)

$$S_X(f) = \sigma^2 \text{sinc}^2(f)$$

Problem 8.33. A random pulse has amplitude A and duration T but starts at an arbitrary time t_0. That is, the random process is defined as

$$X(t) = A\mathrm{rect}(t + t_0)$$

where rect(t) is defined in Section 2.9. The random variable t_0 is assumed to be uniformly distributed over [0,T] with density

$$f_{t_0}(s) = \begin{cases} \frac{1}{T} & 0 \le s \le T \\ 0 & \text{otherwise} \end{cases}$$

(a) What is the autocorrelation function of the random process $X(t)$?
(b) What is the spectrum of the random process $X(t)$?

Solution

First note that the process $X(t)$ is not stationary. This may be demonstrated by computing the mean of $X(t)$ for which we use the fact that

$$f_X(x) = \int_{-\infty}^{\infty} f_X(x \mid s) f_{t_0}(s) ds$$

combined with the fact that

$$\begin{aligned} \mathbf{E}\left[X(t) \mid t_0\right] &= \int_{-\infty}^{\infty} x f_X(x \mid t_0) dx \\ &= \begin{cases} A & t_0 \le t \le t_0 + T \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Consequently, we have

$$\begin{aligned} \mathbf{E}\left[X(t)\right] &= \int_{-\infty}^{\infty} \mathbf{E}[X(t) \mid s] f_{t_0}(s) ds \\ &= \begin{cases} 0 & t < 0 \\ At/T & 0 \le t \le T \\ A(2 - t/T) & T < t \le 2T \\ 0 & t > 2T \end{cases} \end{aligned}$$

Thus the mean of the process is dependent on t, and the process is nonstationary.

Continued on next slide

Problem 8.33 continued

We take a similar approach to compute the autocorrelation function. First we break the situation into a number of cases:

i) For any $t < 0$, $s < 0$, $t > 2T$, or $s > 2T$, we have that

$$\mathbf{E}\left[X(t)X(s)\right] = 0$$

ii) For $0 \le t < s \le 2T$, we first assume t_0 is known

$$\mathbf{E}\left[X(t)X(s) \mid t_0\right] = \begin{cases} A^2 & t > t_0,\ \ s < t_0 + T,\ \ 0 < t_0 < T \\ 0 & \text{otherwise} \end{cases}$$
$$= \begin{cases} A^2 & \max(s - T, 0) < t_0 < \min(t, T) \\ 0 & \text{otherwise} \end{cases}$$

Evaluating the unconditional expectation, we have

$$\mathbf{E}\left[X(t)X(s)\right] = \int_{-\infty}^{\infty} \mathbf{E}\left[X(t)X(s) \mid w\right] f_{t_0}(w)dw$$
$$= \int_{\max(0, s-T)}^{\min(t,T)} A^2 \left(\frac{1}{T}\right) dw$$
$$= \frac{A^2}{T} \max\left\{\left\{\min(t,T) - \max(0, s-T)\right\}, 0\right\}$$

where the second maximum takes care of the case where the lower limit on the integral is greater than the upper limit.

iii) For $0 \le s < t \le 2T$, we use a similar argument to obtain

$$\mathbf{E}\left[X(t)X(s) \mid t_0\right] = \begin{cases} A^2 & \max(t - T, 0) < t_0 < \min(s, T) \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mathbf{E}\left[X(t)X(s)\right] = \frac{A^2}{T} \max\left\{\left\{\min(s,T) - \max(0, t-T)\right\}, 0\right\}$$

Continued on next slide

Problem 8.33 continued

Combining all of these results we have the autocorrelation is given by

$$\mathbf{E}\left[X(t)X(s)\right]=\begin{cases}\dfrac{A^2}{T}\max\left\{\left\{\min(t,T)-\max(0,s-T)\right\},0\right\} & 0\le t<s\le 2T\\ \dfrac{A^2}{T}\max\left\{\left\{\min(s,T)-\max(0,t-T)\right\},0\right\} & 0\le s<t\le 2T\\ 0 & \text{otherwise}\end{cases}$$

This result depends upon both *t* and *s*, not just t-s, as one would expect for a non-stationary process.

Problem 8.34 Given that a stationary random process $X(t)$ has an autocorrelation function $R_X(\tau)$ and a power spectral density $S_X(f)$, show that:

(a) The autocorrelation function of $dX(t)/dt$, the first derivative of $X(t)$ is equal to the negative of the second derivative of $R_X(\tau)$.

(b) The power spectral density of $dX(t)/dt$ is equal to $4\pi^2 f^2 S_X(f)$.

Hint: Use the results of Problem 2.24.

Solution

(a) Let $Y(t) = \frac{dX}{dt}(t)$, and from the Wiener-Khintchine relations, we know the autocorrelation of $Y(t)$ is the inverse Fourier transform of the power spectral density of Y. Using the results of part (b),

$$\begin{aligned} R_Y(f) &= \mathbf{F}^{-1}\left[S_Y(f)\right] \\ &= \mathbf{F}^{-1}\left[4\pi^2 f^2 S_X(f)\right] \\ &= -\mathbf{F}^{-1}\left[(j2\pi f)^2 S_X(f)\right] \end{aligned}$$

from the differential properties of the Fourier transform, we know that differentiation in the time domain corresponds to multiplication by $j2\pi f$ in the frequency domain. Consequently, we conclude that

$$\begin{aligned} R_Y(f) &= -\mathbf{F}^{-1}\left[(j2\pi f)^2 S_X(f)\right] \\ &= -\frac{d^2}{d\tau^2} R_X(\tau) \end{aligned}$$

Continued on next slide

Problem 8.34 continued

(b) Let $Y(t) = \frac{dX}{dt}(t)$, then the spectrum of *Y*(*t*) is given by (see Section 8.8)

$$S_Y(f) = \lim_{T\to\infty} \frac{1}{2T} \mathbf{E}\left[\left|H_T^Y(f)\right|^2\right]$$

where $H_T^Y(f)$ is the Fourier transform of *Y*(*t*) from –*T* to +*T*. By the properties of Fourier transforms $H_T^Y(f) = (j2\pi f)H_T^X(f)$ so we have

$$\begin{aligned} S_Y(f) &= \lim_{T\to\infty} \frac{1}{2T} \mathbf{E}\left[\left|H_T^Y(f)\right|^2\right] \\ &= \lim_{T\to\infty} \frac{1}{2T} \mathbf{E}\left[\left|(j2\pi f)H_T^X(f)\right|^2\right] \\ &= \lim_{T\to\infty} \frac{1}{2T} \left(4\pi^2 f^2\right) \mathbf{E}\left[\left|H_T^X(f)\right|^2\right] \\ &= 4\pi^2 f^2 S_X(f) \end{aligned}$$

Note that the expectation occurs at a particular value of *f*; frequency plays the role of an index into a family of random variables.

Problem 8.35 Consider a wide-sense stationary process $X(t)$ having the power spectral density $S_X(f)$ shown in Fig. 8.26. Find the autocorrelation function $R_X(\tau)$ of the process $X(t)$.

Solution

The Wiener-Khintchine relations imply the autocorrelation is given by the inverse Fourier transform of the power spectral density, thus

$$\begin{aligned} R(\tau) &= \int_{-\infty}^{\infty} S_X(f)\exp(j2\pi ft)df \\ &= \int_0^1 (1-f)\cos(2\pi ft)df \end{aligned}$$

where we have used the symmetry properties of the spectrum to obtain the second line. Integrating by parts, we obtain

$$\begin{aligned} R_X(\tau) &= (1-f)\frac{\sin(2\pi f\tau)}{2\pi\tau}\bigg|_0^1 + \int_0^1 \frac{\sin(2\pi f\tau)}{2\pi\tau}df \\ &= 0 + \frac{-\cos(2\pi f\tau)}{(2\pi\tau)^2}\bigg|_0^1 \\ &= \frac{1-\cos(2\pi\tau)}{(2\pi\tau)^2} \end{aligned}$$

Using the half-angle formula $\sin^2(\theta) = ½(1-\cos(2\theta))$, this result simplifies to

$$\begin{aligned} R_X(\tau) &= \frac{2\sin^2(\pi\tau)}{(2\pi\tau)^2} \\ &= \tfrac{1}{2}\operatorname{sinc}^2(\tau) \end{aligned}$$

where $\operatorname{sinc}(x) = \sin(\pi x)/\pi x$.

Problem 8.36 The power spectral density of a random process $X(t)$ is shown in Fig. 8.27.

(a) Determine and sketch the autocorrelation function $R_X(\tau)$ of the $X(t)$.
(b) What is the dc power contained in $X(t)$?
(c) What is the ac power contained in $X(t)$?
(d) What sampling rates will give uncorrelated samples of X(t)? Are the samples statistically independent?

Solution

(a) Using the results of Problem 8.35, and the linear properties of the Fourier transform

$$R(\tau) = 1 + \tfrac{1}{2}\,\mathrm{sinc}^2(f_0\tau)$$

(b) The *dc* power is given by power centered on the origin

$$\begin{aligned} dc \text{ power} &= \lim_{\delta\to 0}\int_{-\varepsilon}^{\varepsilon} S_X(f)\,df \\ &= \lim_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon} \delta(f)\,df \\ &= 1 \end{aligned}$$

(c) The *ac* power is the total power minus the dc power

$$\begin{aligned} ac\ \ power &= R_X(0) - dc \text{ power} \\ &= R_X(0) - 1 \\ &= \tfrac{1}{2} \end{aligned}$$

(d) The correlation function $R_X(\tau)$ is zero if samples are spaced at multiples of $1/f_0$.

Problem 8.37 Consider the two linear filters shown in cascade as in Fig. 8.28. Let $X(t)$ be a stationary process with autocorrelation function $R_X(\tau)$. The random process appearing at the first filter output is $V(t)$ and that at the second filter output is $Y(t)$.

(a) Find the autocorrelation function of $V(t)$.

(b) Find the autocorrelation function of $Y(t)$.

Solution

Expressing the first filtering operation in the frequency domain, we have

$$V(f) = H_1(f)X(f)$$

where $H_1(f)$ is the Fourier transform of $h_1(t)$. From Eq. (8.87) it follows that the spectrum of $V(t)$ is

$$S_V(f) = \left|H_1(f)\right|^2 S_X(f)$$

By analogy, we have

$$S_Y(f) = \left|H_2(f)\right|^2 S_V(f)$$

$$= \left|H_2(f)\right|^2 \left|H_1(f)\right|^2 S_X(f)$$

Consequently, apply the convolution properties of the Fourier transform, we have

$$R_Y(\tau) = g_2(t) * g_1(t) * R_X(f)$$

where * denotes convolution; $g_2(t)$ and $g_1(t)$ are the inverse Fourier transforms of $|H_2(f)|^2$ and $|H_1(f)|^2$, respectively.

Problem 8.38 The power spectral density of a narrowband random process $X(t)$ is as shown in Fig. 8.29. Find the power spectral densities of the in-phase and quadrature components of $X(t)$, assuming f_c = 5 Hz.

Solution

From Section 8.11, the power spectral densities of the in-phase and quadrature components are given by

$$S_{N_1}(f) = S_{N_0}(f) = \begin{cases} S(f + f_c) + S(f - f_c) & |f| < B \\ 0 & 0 \geq B \end{cases}$$

Evaluating this expression for Fig. 8.29, we obtain

$$S_{N_I}(f) = S_{N_Q}(f) = \begin{cases} 1 - \frac{|f|}{2} & 1 < |f| < 2 \\ \left(2 - 3\frac{|f|}{2}\right) & 0 < |f| < 1 \\ 0 & otherwise \end{cases}$$

Problem 8.39 Assume the narrow-band process $X(t)$ described in Problem 8.38 is Gaussian with zero mean and variance σ_X^2.

(a) Calculate σ_X^2.
(b) Determine the joint probability density function of the random variables Y and Z obtained by observing the in-phase and quadrature components of $X(t)$ at some fixed time.

Solution

(a) The variance is given by

$$\begin{aligned}\sigma_X^2 &= R(0) = \int_{-\infty}^{\infty} S(f)df \\ &= 2\left(\frac{1}{2}b_1h_1 + \frac{1}{2}b_2h_2\right) \\ &= 2\left(\frac{1}{2}\cdot 2\cdot 1 + \frac{1}{2}\cdot 1\cdot 1\right) \\ &= 3\ watts\end{aligned}$$

(b) The random variables Y and Z have zero mean, are Gaussian and have variance σ_X^2. If Y and Z are independent, the joint density is given by

$$\begin{aligned}f_{Y,Z}(Y,Z) &= \frac{1}{\sqrt{2\pi}\sigma_X}\exp\left(-y^2/2\sigma_X^2\right)\cdot\frac{1}{\sqrt{2\pi}\sigma_X}\exp\left(-z^2/2\sigma_X^2\right) \\ &= \frac{1}{2\pi\sigma_X^2}\exp\left(-\frac{y^2+z^2}{2\sigma_X^2}\right)\end{aligned}$$

Problem 8.40 Find the probability that the last two digits of the cube of a natural number (1, 2, 3, …) will be 01.

Solution

Let a natural number be represented by concatenation *xy* where *y* represents last two digits and *x* represents the other digits. For example, the number 1345 has $y = 45$ and $x = 13$. Then

$$(xy)^3 = (x00 + y)^3 = \left(x^3 000000\right) + 3\left(x^2 0000\right)y + 3(x00)y^2 + y^3$$

where we have used the binomial expansion of $(a+b)^3$. The last digits of the first three terms on the right are clearly 00. Consequently, it is the last two digits of y^3 which determines the last two digits of $(xy)^3$. Checking the cube of all two digit numbers for 00 to 99, we find that: (a) y^3 ends in 1, only if *y* ends in 1; and (b) only the number $(01)^3$ gives 01 as the last two digits. From this counting argument, the probability is 0.01.

Problem 8.41 Consider the random experiment of selecting a number uniformly distributed over the range {1, 2, 3, …, 120}. Let A, B, and C be the events that the selected number is a multiple of 3, 4, and 6, respectively.

a) What is the probability of event A, i.e. **P**[A]?
b) What is **P**[B]?
c) What is $\mathbf{P}[A \cap B]$?
d) What is $\mathbf{P}[A \cup B]$?
e) What is $\mathbf{P}[A \cap C]$?

Solution

(a) From a counting argument, $\mathbf{P}(A) = \frac{40}{120} = \frac{1}{3}$

(b) $$\mathbf{P}(B) = \frac{30}{120} = \frac{1}{4}$$

(c) $$\mathbf{P}(A \cap B) = \frac{12}{120} = \frac{1}{10}$$

(d)
$$\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B)$$
$$= 1/3 + 1/4 - 1/10 = \frac{20 + 15 - 6}{60} = \frac{29}{60}$$

(e) $$\mathbf{P}(A \cap C) = \mathbf{P}(C) = \frac{20}{120} = \frac{1}{6}$$

Problem 8.42 A message consists of ten "0"s and "1"s.

a) How many such messages are there?
b) How many such messages are there that contain exactly four "1"s?
c) Suppose the 10th bit is not independent of the others but is chosen such that the modulo-2 sum of all the bits is zero. This is referred to as an even parity sequence. How many such even parity sequences are there?
d) If this ten-bit even-parity sequence is transmitted over a channel that has a probability of error p for each bit. What is the probability that the received sequence contains an undetected error?

Solution

(a) A message corresponds to a binary number of length 10, there are thus 2^{10} possibilities.

(b) The number of messages with four "1"s is

$$\binom{10}{4} = \frac{10!}{4!6!} = \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2} = 10 \times 3 \times 7 = 210$$

(c) Since there are only 9 independent bits in this case, the number of such message is 2^9.

(d) The probability of an undetected error corresponds to the probability of 2, 4, 6, 8, or 10 errors. The received message corresponds to a Bernoulli sequence, so the corresponding error probabilities are given by the binomial distribution and is

$$\binom{10}{2} p^2 (1-p)^8 + \binom{10}{4} p^4 (1-p)^6 + \binom{10}{6} p^6 (1-p)^4 + \binom{10}{8} p^8 (1-p)^2 + \binom{10}{10} p^{10}$$

Problem 8.43 The probability that an event occurs at least once in four independent trials is equal to 0.59. What is the probability of occurrence of the event in one trial, if the probabilities are equal in all trials?

Solution

The probability that the event occurs on a least one trial is 1 minus the probability that the event does not occur at all. Let *p* be the probability of occurrence on a single trial, so 1-*p* is the probability of not occurring on a single trial. Then

$$\mathbf{P}\left[\text{at least one occurence}\right] = 1-\left(1-p\right)^4$$
$$0.59 = 1-(1-p)^4$$

Solving for *p* gives a probability on a single trial of 0.20.

Problem 8.44 The arrival times of two signals at a receiver are uniformly distributed over the interval [0,T]. The receiver will be jammed if the time difference in the arrivals is less than τ. Find the probability that the receiver will be jammed.

Solution

Let X and Y be random variables representing the arrival times of the two signals. The probability density functions of the random variables are

$$f_X(x) = \begin{cases} \dfrac{1}{T} & 0 < x < T \\ 0, & \text{otherwise} \end{cases}$$

and $f_Y(y)$ is similarly defined. Then the probability that the time difference between arrivals is less than τ is given by

$$\begin{aligned} \mathbf{P}\big[|X-Y| < \tau\big] &= \mathbf{P}\big[|X-Y| < \tau \mid X > Y\big]\mathbf{P}\big[X > Y\big] + \mathbf{P}\big[|X-Y| < \tau \mid Y > X\big]\mathbf{P}\big[Y > X\big] \\ &= \mathbf{P}\big[|X-Y| < \tau \mid X > Y\big] \end{aligned}$$

where the second line follows from the symmetry between the random variables X and Y, namely, $\mathbf{P}[X > Y] = \mathbf{P}[Y > X]$. If we only consider the case $X > Y$, then we have the conditions: $0 < X < T$ and $0 < Y < X < \tau+Y$. Combining these conditions we have $Y < X < \min(T, \tau+Y)$. Consequently,

$$\mathbf{P}\big[|X-Y| < \tau\big] = \int_0^T \int_y^{\min(T,\tau+y)} f_X(x) f_Y(y) dx\, dy$$

$$= \int_0^T \int_y^{\min(T,\tau+y)} \left(\frac{1}{T}\right)^2 dx\, dy$$

$$= \frac{1}{T^2}\int_0^T \{\min(T, \tau+y) - y\} dy$$

Combining the two terms of the integrand,

$$P\big[|X-Y| < \tau\big] = \frac{1}{T^2}\int_0^T \min(T-y, \tau) dy$$

$$= \frac{1}{T^2}\min\left(Ty - \frac{y^2}{2}, \tau y\right)\Bigg|_0^T$$

$$= \min\left(\frac{1}{2}, \frac{\tau}{T}\right)$$

Problem 8.45 A telegraph system (an early version of digital communications) transmits either a dot or dash signal. Assume the transmission properties are such that 2/5 of the dots and 1/3 of the dashes are received incorrectly. Suppose the ratio of transmitted dots to transmitted dashes is 5 to 3. What is the probability that a received signal as the transmitted if:

a) The received signal is a dot?
b) The received signal is a dash?

Solution

(a) Let X represent the transmitted signal and Y represent the received signal. Then by application of Bayes' rule

$$\mathbf{P}(Y = \text{dot}) = \mathbf{P}(X = \text{dot} \mid \text{No error})\mathbf{P}(\text{No dot error}) + \mathbf{P}(X = \text{dash} \mid \text{error})\mathbf{P}(\text{dash error})$$
$$= \tfrac{5}{8}\left(\tfrac{3}{5}\right) + \left(\tfrac{3}{8}\right)\left(\tfrac{1}{3}\right)$$
$$= \tfrac{3}{8} + \tfrac{1}{8} = \tfrac{1}{2}$$

(b) Similarly,

$$\mathbf{P}[Y = \text{dash}] = \mathbf{P}[X = \text{dash} \mid \text{no error}]\mathbf{P}(\text{no dash error}) + \mathbf{P}(X = \text{dot})\mathbf{P}[\text{dot error}]$$
$$= \tfrac{3}{8} \cdot \tfrac{2}{3} + \left[\tfrac{5}{8}\right]\tfrac{2}{5}$$
$$= \tfrac{2}{8} + \tfrac{2}{8} = \tfrac{1}{2}$$

Problem 8.46 Four radio signals are emitted successively. The probability of reception for each of them is independent of the reception of the others and equal, respectively, 0.1, 0.2, 0.3 and 0.4. Find the probability that k signals will be received where k = 1, 2, 3, 4.

Solution

For one successful reception, the probability is given by the sum of the probabilities of the four mutually exclusive cases

$$\begin{aligned} P = & \; p_1(1-p_2)(1-p_3)(1-p_4)+ \\ & (1-p_1)p_2(1-p_3)(1-p_4)+ \\ & (1-p_1)(1-p_2)p_3(1-p_4)+ \\ & (1-p_1)(1-p_2)(1-p_3)p_4 \\ = & \; .1\cdot.8\cdot.7\cdot.6+.9\cdot.2\cdot.7\cdot.6+.9\cdot.8\cdot.3\cdot.6+.9\cdot.8\cdot.7\cdot.4 \\ = & \; 0.4404 \end{aligned}$$

For k = 2, there six mutually exclusive cases

$$\begin{aligned} P = & \; p_1 p_2(1-p_3)(1-p_4)+ \\ & p_1(1-p_2)p_3(1-p_4)+ \\ & p_1(1-p_2)(1-p_3)p_4+ \\ & (1-p_1)p_2 p_3(1-p_4)+ \\ & (1-p_1)p_2(1-p_3)p_4+ \\ & (1-p_1)(1-p_2)p_3 p_4 \\ = & \; .1\cdot.2\cdot.7\cdot.6+.1\cdot.8\cdot.3\cdot.6+.1\cdot.8\cdot.7\cdot.4+.9\cdot.2\cdot.3\cdot.6+.9\cdot.2\cdot.7\cdot.4+.9\cdot.8\cdot.3\cdot.4 \\ = & \; 0.2144 \end{aligned}$$

For k =3 there are four mutually exclusive cases

$$\begin{aligned} P = & \; p_1 p_2 p_3(1-p_4)+ \\ & p_1(1-p_2)p_3 p_4+ \\ & p_1 p_2(1-p_3)p_4+ \\ & (1-p_1)p_2 p_3 p_4 \\ = & \; .1\cdot.2\cdot.3\cdot.6+.1\cdot.8\cdot.3\cdot.4+.1\cdot.2\cdot.7\cdot.4+.9\cdot.2\cdot.3\cdot.4 \\ = & \; 0.0404 \end{aligned}$$

For k = 4 there is only one term

$$\begin{aligned} P = & \; p_1 p_2 p_3 p_4 \\ = & \; .1\cdot.2\cdot.3\cdot.4 \\ = & \; 0.0024 \end{aligned}$$

Problem 8.47 In a computer-communication network, the arrival time τ between messages is modeled with an exponential distribution function, having the density

$$f_T(\tau) = \begin{cases} \frac{1}{\lambda} e^{-\lambda\tau} & \tau \geq 0 \\ 0 & otherwise \end{cases}$$

a) What is the mean time between messages with this distribution?
b) What is the variance in this time between messages?

Solution (Typo in problem statement, should read $f_T(\tau)=(1/\lambda)\exp(-\tau/\lambda)$ for $\tau>0$)
(a) The mean time between messages is

$$\begin{aligned} \mathbf{E}[T] &= \int_0^\infty \tau f_T(\tau)d\tau \\ &= \int_0^\infty \frac{\tau}{\lambda} \exp(-\tau/\lambda)d\tau \\ &= -\tau \exp(-\tau/\lambda)\Big|_0^\infty + \int_0^\infty \exp(-\tau/\lambda)d\tau \\ &= 0 - \lambda \exp(-\tau/\lambda)\Big|_0^\infty \\ &= \lambda \end{aligned}$$

where the third line follows by integration by parts.

(b) To compute the variance, we first determine the second moment of T

$$\begin{aligned} \mathbf{E}[T^2] &= \int_0^\infty \tau^2 f_T(\tau)d\tau \\ &= \int_0^\infty \frac{\tau^2}{\lambda} \exp(-\tau/\lambda)d\tau \\ &= -\tau^2 \exp(-\tau/\lambda)\Big|_0^\infty + 2\int_0^\infty \tau \exp(-\tau/\lambda)d\tau \\ &= 0 + 2\lambda\mathbf{E}[T] \\ &= 2\lambda^2 \end{aligned}$$

Continued on next slide

Problem 8.47 continued

The variance is then given by the difference of the second moment and the first moment squared (see Problem 8.23)

$$\begin{aligned}\mathrm{Var}(T) &= \mathbf{E}\left[T^2\right] - \left(\mathbf{E}\left[T\right]\right)^2 \\ &= 2\lambda^2 - \lambda^2 \\ &= \lambda^2\end{aligned}$$

Problem 8.48 If X has a density $f_X(x)$, find the density of Y where

a) $Y = aX + b$ for constants a and b.

b) $Y = X^2$.

c) $Y = \sqrt{X}$, assuming X is a non-negative random variable.

Solution

(a) If $Y = aX + b$, using the results of Section 8.3 for $Y = g(X)$

$$\begin{aligned} f_Y(y) &= f_X\left(g^{-1}(y)\right)\left|\frac{dg^{-1}(y)}{dy}\right| \\ &= f_X\left(\frac{y-b}{a}\right)\frac{1}{a} \end{aligned}$$

(b) If $Y = X^2$, then

$$f_Y(y) = \left(f_X\left(-\sqrt{y}\right) + f_X\left(+\sqrt{y}\right)\right)\left(\frac{1}{2\sqrt{y}}\right)$$

(c) If $Y = \sqrt{X}$, then we must assume X is positive valued so, this is a one-to-one mapping and

$$f_Y(y) = f_X\left(y^2\right)\cdot 2y$$

Problem 8.49 Let X and Y be independent random variables with densities $f_X(x)$ and $f_Y(y)$, respectively. Show that the random variable $Z = X+Y$ has a density given by

$$f_Z(z) = \int_{-\infty}^{z} f_Y(z-s)\, f_X(s)\, ds$$

Hint: $\mathbf{P}[Z \le z] = \mathbf{P}[X \le z, Y \le z - X]$

Solution (*Typo in problem statement - should be "positive" independent random variables*)

Using the hint, we have that $F_Z(z) = \mathbf{P}[Z \le z]$ and

$$F_Z(z) = \int_{-\infty}^{z} \int_{-\infty}^{z-x} f_X(x) f_Y(y)\, dy\, dx$$

To differentiate this result with respect to z, we use the fact that if

$$g(z) = \int_{a}^{b} h(x,z)\, dx$$

then

$$\frac{\partial g(z)}{dz} = \int_{a}^{b} \frac{\partial}{\partial z} h(x,z)\, dx + h(b,z)\frac{db}{dz} - h(a,z)\frac{da}{dz} \qquad (1)$$

Inspecting $F_Z(z)$, we identify $h(x,z)$

$$h(x,z) = \int_{-\infty}^{z-x} f_X(x) f_Y(y)\, dy$$

and $a = -\infty$ and $b = z$. We then obtain

$$\begin{aligned} f_Z(z) &= \frac{d}{dz} F_Z(z) \\ &= \int_{-\infty}^{z} \left[\frac{d}{dz} \int_{-\infty}^{z-x} f_X(x) f_Y(y)\, dy \right] dx + \int_{-\infty}^{z-z} f_X(z) f_Y(y)\, dy \frac{dz}{dz} - \int_{-\infty}^{z-(-\infty)} f_X(-\infty) f_Y(y)\, dy \cdot 0 \\ &= \int_{-\infty}^{z} \left[\frac{d}{dz} \int_{-\infty}^{z-x} f_X(x) f_Y(y)\, dy \right] dx \end{aligned}$$

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Problem 8-49 continued

where the second term of the second line is zero since the random variables are positive, and the third term is zero due to the factor zero. Applying the differentiation rule a second time, we obtain

$$f_Z(z) = \int_{-\infty}^{z} \left[0 + f_X(x) f_Y(z-x) \frac{d(z-x)}{dz} - f_X(x) f_Y(-\infty) \frac{d(-\infty)}{dz} \right] dx$$
$$= \int_{-\infty}^{z} f_X(x) f_Y(z-x) dx$$

which is the desired result.

An alternative solution is the following: we note that

$$\begin{aligned} \mathbf{P}[Z \le z \mid X = x] &= \mathbf{P}[X + Y \le z \mid X = x] \\ &= \mathbf{P}[x + Y \le z \mid X = x] \\ &= \mathbf{P}[x + Y \le z] \\ &= \mathbf{P}[Y \le z - x] \end{aligned}$$

where the third equality follows from the independence of X and Y. By differentiating both sides with respect to z, we see that

$$f_{Z|X}(z \mid x) = f_Y(z-x)$$

By the properties of conditional densities

$$f_{Z,X}(z,x) = f_X(x) f_{Z|X}(z \mid x) = f_X(x) f_Y(z-x)$$

Integrating to form the marginal distribution, we have

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) dx$$

If Y is a positive random variable then $f_Y(z\text{-}x)$ is zero for $x > z$ and the desired result follows.

Problem 8.50 Find the spectral density $S_Z(f)$ if

$$Z(t) = X(t)Y(t)$$

where $X(t)$ and $Y(t)$ are independent zero-mean random processes with

$$R_X(\tau) = a_1 e^{-\alpha_1|\tau|} \quad \text{and} \quad R_Y(\tau) = a_2 e^{-\alpha_2|\tau|}.$$

Solution

The autocorrelation of $Z(t)$ is given by

$$\begin{aligned} R_Z(\tau) &= \mathbf{E}[Z(t)Z(t+\tau)] \\ &= \mathbf{E}[X(t)X(t+\tau)Y(t)Y(t+\tau)] \\ &= \mathbf{E}[X(t)X(t+\tau)]\mathbf{E}[Y(t)Y(t+\tau)] \\ &= R_X(\tau)R_Y(\tau) \end{aligned}$$

By the Wiener-Khintchine relations, the spectrum of $Z(t)$ is given by

$$\begin{aligned} S_Z(f) &= \mathbf{F}^{-1}[R_X(\tau)R_Y(\tau)] \\ &= \mathbf{F}^{-1}[a_1 a_2 \exp(-(\alpha_1+\alpha_2)|\tau|)] \\ &= \frac{2a_1a_2(\alpha_1+\alpha_2)}{(\alpha_1+\alpha_2)^2 + (2\pi f)^2} \end{aligned}$$

where the last line follows from the Fourier transform of the double-sided exponential (See Example 2.3).

Problem 8.51 Consider a random process $X(t)$ defined by

$$X(t) = \sin(2\pi f_c t)$$

where the frequency f_c is a random variable uniformly distributed over the interval $[0,W]$. Show that $X(t)$ is nonstationary. *Hint*: Examine specific sample functions of the random process $X(t)$ for, say, the frequencies $W/4$, $W/2$, and W.

Solution

To be stationary to first order implies that the mean value of the process $X(t)$ must be constant and independent of t. In this case,

$$\begin{aligned}\mathbf{E}[X(t)] &= \mathbf{E}[\sin(2\pi f_c t)] \\ &= \frac{1}{W}\int_0^W \sin(2\pi wt)\,dw \\ &= \left.\frac{-\cos(2\pi wt)}{2\pi Wt}\right|_0^W \\ &= \frac{1-\cos(2\pi Wt)}{2\pi Wt}\end{aligned}$$

This mean value clearly depends on t, and thus the process $X(t)$ is nonstationary.

Problem 8.52 The oscillators used in communication systems are not ideal but often suffer from a distortion known as phase noise. Such an oscillator may be modeled by the random process

$$Y(t) = A\cos\left(2\pi f_c t + \phi(t)\right)$$

where $\phi(t)$ is a slowly varying random process. Describe and justify the conditions on the random process $\phi(t)$ such that $Y(t)$ is wide-sense stationary.

Solution

The first condition for wide-sense stationary process is a constant mean. Consider $t = t_0$, then

$$\mathbf{E}\left[Y(t_0)\right] = \mathbf{E}\left[A\cos\left(2\pi f_c t_0 + \phi(t_0)\right)\right]$$

In general, the function cos θ takes from values -1 to +1 when θ varies from 0 to 2π. In this case θ corresponds to $2\pi f_c t_0 + \phi(t_0)$. If $\phi(t_0)$ varies only by a small amount then θ will be biased toward the point $2\pi f_c t_0 + \mathbf{E}[\phi(t_0)]$, and the mean value of $\mathbf{E}[Y(t_0)]$ will depend upon the choice of t_0. However, if $\phi(t_0)$ is uniformly distributed over [0, 2π] then $2\pi f_c t_0 + \phi(t_0)$ will be uniformly distributed over [0, 2π] when considered modulo 2π, and the mean $\mathbf{E}[Y(t_0)]$ will be zero and will not depend upon t_0.

Thus the first requirement is that $\phi(t)$ must be uniformly distributed over $[0,2\pi]$ for all t.

The second condition for a wide-sense stationary $Y(t)$ is that the autocorrelation depends only upon the time difference

$$\begin{aligned}\mathbf{E}\left[Y(t_1)Y(t_2)\right] &= \mathbf{E}\left[A\cos\left(2\pi f_c t_1 + \phi(t_1)\right)A\cos\left(2\pi f_c t_2 + \phi(t_2)\right)\right] \\ &= \frac{A^2}{2}\mathbf{E}\left[\cos\left(2\pi f_c (t_1 + t_2) + \phi(t_1) + \phi(t_2)\right) + \cos\left(2\pi f_c (t_1 - t_2) + \phi(t_1) - \phi(t_2)\right)\right]\end{aligned}$$

where we have used the relation $\cos A \cos B = \frac{1}{2}\left(\cos(A+B) + \cos(A-B)\right)$. In general, this correlation does not depend solely on the time difference t_2-t_1. However, if we assume:

We first note that if $\phi(t_1)$ and $\phi(t_2)$ are both uniformly distributed over [0,2π] then so is $\psi = \phi(t_1) + \phi(t_2)$ (modulo 2π), and

$$\begin{aligned}\mathbf{E}\left[\cos\left(2\pi f_c (t_1 + t_2) + \psi\right)\right] &= \frac{1}{2\pi}\int_0^{2\pi} \cos(2\pi f_c (t_1 + t_2) + \psi)\, d\psi \\ &= 0\end{aligned} \qquad (1)$$

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Problem 8.52 continued

We consider next the term $R_Y(t_1,t_2)=\mathbf{E}\left[\cos\left(2\pi f_c(t_1-t_2)+\phi(t_1)-\phi(t_2)\right)\right]$ and three special cases:

(a) if $\Delta t = t_1\text{-}t_2$ is small then $\phi(t_1)\approx\phi(t_2)$ since $\phi(t)$ is a slowly varying process, and

$$R_Y(t_1,t_2)=\frac{A^2}{2}\cos\left(2\pi f_c(t_1-t_2)\right)$$

(b) if Δt is large then $\phi(t_1)$ and $\phi(t_2)$ should be approximately independent and $\phi(t_1)-\phi(t_2)$ would be approximately uniformly distributed over $[0,2\pi]$. In this case

$$R_Y(t_1,t_2)\approx 0$$

using the argument of Eq. (1).

(c) for intermediate values of Δt, we require that

$$\phi(t_1)-\phi(t_2)\approx g(t_1-t_2)$$

for some arbitrary function $g(t)$.

Under these conditions the random process $Y(t)$ will be wide-sense stationary.

Problem 8.53 A baseband signal is disturbed by a noise process $N(t)$ as shown by

$$X(t) = A\sin(0.3\pi t) + N(t)$$

where $N(t)$ is a stationary Gaussian process of zero mean and variance σ^2.

(a) What are the density functions of the random variables X_1 and X_2 where

$$X_1 = X(t)\big|_{t=1}$$

$$X_2 = X(t)\big|_{t=2}$$

(b) The noise process $N(t)$ has an autocorrelation function given by

$$R_N(\tau) = \sigma^2 \exp(-|\tau|)$$

What is the joint density function of X_1 and X_2, that is, $f_{X_1,X_2}(x_1, x_2)$?

Solution

(a) The random variable X_1 has a mean

$$\begin{aligned}\mathbf{E}[X(t_1)] &= \mathbf{E}[A\sin(0.3\pi) + N(t_1)] \\ &= A\sin(0.3\pi) + \mathbf{E}[N(t_1)] \\ &= A\sin(0.3\pi)\end{aligned}$$

Since X_1 is equal to $N(t_1)$ plus a constant, the variance of X_1 is the same as that of $N(t_1)$. In addition, since $N(t_1)$ is a Gaussian random variable, X_1 is also Gaussian with a density given by

$$f_{X_1}(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp\{-(x-\mu_1)/2\sigma^2\}$$

where $\mu_1 = \mathbf{E}[X(t_1)]$. By a similar argument, the density function of X_2 is

$$f_{X_2}(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp\{-(x-\mu_2)/2\sigma^2\}$$

where $\mu_2 = A\sin(0.6\pi)$.

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Problem 8-53 continued

(b) First note that since the mean of $X(t)$ is not constant, $X(t)$ is not a stationary random process. However, $X(t)$ is still a Gaussian random process, so the joint distribution of N Gaussian random variables may be written as Eq. (8.90). For the case of $N = 2$, this equation reduces to

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{2\pi|\Lambda|^{1/2}} \exp\left\{-(\mathbf{x}-\boldsymbol{\mu})\Lambda^{-1}(\mathbf{x}-\boldsymbol{\mu})^T / 2\right\}$$

where Λ is the 2x2 covariance matrix. Recall that $\text{cov}(X_1,X_2) = \mathbf{E}[(X_1-\mu_1)(X_2-\mu_2)]$, so that

$$\begin{aligned}\Lambda &= \begin{bmatrix} \text{cov}(X_1, X_1) & \text{cov}(X_1, X_2) \\ \text{cov}(X_2, X_1) & \text{cov}(X_2, X_2) \end{bmatrix} \\ &= \begin{bmatrix} R_N(0) & R_N(1) \\ R_N(1) & R_N(0) \end{bmatrix} \\ &= \begin{bmatrix} \sigma^2 & \sigma^2 \exp(-1) \\ \sigma^2 \exp(-1) & \sigma^2 \end{bmatrix}\end{aligned}$$

If we let $\rho = \exp(-1)$ then

$$|\Lambda| = \sigma^4(1-\rho^2)$$

and

$$\Lambda^{-1} = \frac{1}{\sigma^2(1-\rho^2)} \begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix}$$

Making these substitutions into the above expression, we obtain upon simplification

$$f_{X_1,X_2}(x_1,x_2) = \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} \exp\left\{-\frac{(x_1-\mu_1)^2 + (x_2-\mu_2)^2 - 2\rho(x_1-\mu_1)(x_2-\mu_2)}{2\sigma^2(1-\rho^2)}\right\}$$

Problem 9.1 In practice, we often cannot measure the signal by itself but must measure the signal plus noise. Explain how the SNR would be calculated in this case.

Solution

Let $r(t) = s(t) + n(t)$ be the received signal plus noise. Assuming the signal is independent of the noise, we have that the received power is

$$\begin{aligned}
R_0 &= \mathbf{E}\left[r^2(t)\right] \\
&= \mathbf{E}\left[\left(s(t)+n(t)\right)^2\right] \\
&= \mathbf{E}\left[s^2(t)\right] + 2\mathbf{E}\left[s(t)n(t)\right] + \mathbf{E}\left[n^2(t)\right] \\
&= \mathbf{E}\left[s^2(t)\right] + 2\mathbf{E}\left[s(t)\right]\mathbf{E}\left[n(t)\right] + \mathbf{E}\left[n^2(t)\right] \\
&= S + 0 + N
\end{aligned}$$

where S is the signal power and N is the average noise power. We then measure the noise alone

$$\begin{aligned}
R_1 &= \mathbf{E}\left[n^2(t)\right] \\
&= N
\end{aligned}$$

and the SNR is given by

$$\text{SNR} = \frac{R_0 - R_1}{R_1}$$

Problem 9.2 A DSC-SC modulated signal is transmitted over a noisy channel, having a noise spectral density $N_0/2$ of $2\text{x}10^{-17}$ watts per hertz. The message bandwidth is 4 kHz and the carrier frequency is 200 kHz. Assume the average received power of the signal is -80 dBm. Determine the post-detection signal-to-noise ratio of the receiver.

Solution

From Eq. (9.23), the post-detection SNR of DSB-SC is

$$\text{SNR}_{\text{post}}^{\text{DSB}} = \frac{A_c^2 P}{2N_0 W}$$

The average received power is $\frac{A_c^2 P}{2} = -80\ dBm = 10^{-11}$ watts. With a message bandwidth of 4 kHz, the post-detection SNR is

$$\text{SNR}_{\text{post}}^{\text{DSB}} = \frac{10^{-11}}{(4\times 10^{-17})4000} = 62.5 \sim 18.0\ dB$$

Problem 9.3. For the same received signal power, compare the post-detection SNRs of DSB-SC with coherent detection and envelope detection with k_a = 0.2 and 0.4. Assume the average message power is P = 1.

Solution

From Eq. (9.23), the post-detection SNR of DSB-SC with received power $\frac{{}^{DSB}A_c^{\,2}P}{2}$ is

$$SNR_{post}^{DSB} = \frac{{}^{DSB}A_c^2 P}{2N_0 W}$$

From Eq. (9.30), the post-detection SNR of AM with received power $\frac{{}^{AM}A_c^2}{2}\left(1+k_a^{\,2}P\right)$ is

$$SNR_{post}^{AM} = \frac{{}^{AM}A_c^2 k_a^2 P}{2N_0 W}$$

So, by equating the transmit powers for DSB-Sc and AM, we obtain

$$\frac{{}^{DSB}A_c^2 P}{2} = \frac{{}^{AM}A_c^2}{2}\left(1+k_a^{\,2}P\right)$$
$$\Rightarrow \frac{{}^{AM}A_c^2}{2} = \frac{{}^{DSB}A_c^2}{2} \quad \frac{P}{1+k_a^{\,2}P}$$

Substituting this result into the expression for the post-detection SNR of AM,

$$\mathrm{SNR}_{\mathrm{post}}^{\mathrm{AM}} = \frac{{}^{DSB}A_c^{\,2}P}{2N_0W}\left(\frac{k_a^{\,2}P}{1+k_a^{\,2}P}\right) = \mathrm{SNR}_{\mathrm{post}}^{\mathrm{DSB}}\Delta$$

Where the factor Δ is

$$\Delta = \frac{k_a^{\,2}P}{1+k_a^{\,2}P}$$

With k_a= 0.2 and P= 1, the AM SNR is a factor $\Delta = \frac{(.2)^2}{1.04} = .04$ less.

With k_a = 0.4 and P = 1, the AM SNR is a factor $\Delta = \frac{(.4)^2}{1+.16} = \frac{.16}{1.16} \approx 0.14$ less.

Problem 9.4. In practice, there is an arbitrary phase θ in Eq. (9.24). How will this affect the results of Section 9.5.2?

Solution

Envelope detection is insensitive to a phase offset.

Problem 9.5. The message signal of Problem 9.2 having a bandwidth W of 4 kHz is transmitted over the same noisy channel having a noise spectral density $N_0/2$ of $2\text{x}10^{-17}$ watts per hertz using single-sideband modulation. If the average received power of the signal is -80 dBm, what is the post-detection signal-to-noise ratio of the receiver? Compare the transmission bandwidth of the SSB receiver to that of the DSB-SC receiver.

Solution

From Eq. (9.23)

$$\text{SNR}_{\text{post}}^{\text{SSB}} = \frac{A_c^2 P}{2N_0 W}$$

with $\frac{A_c^2 P}{2} = -80 \text{ dBm}$, $W = 4 \text{ kHz}$, and $N_0 = 4 \times 10^{-17}$. The

$$\text{SNR}_{\text{post}}^{\text{SSB}} = 18 \, dB$$

The transmission bandwidth of SSB is 4 kHz, half of the 8 kHz used with DSB-SC.

Problem 9.6 The signal $m(t) = \cos(2000\pi t)$ is transmitted by means of frequency modulation. If the frequency sensitivity k_f is 2 kHz per volt, what is the Carson's rule bandwidth of the FM signal. If the pre-detection SNR is 17 dB, calculate the post-detection SNR. Assume the FM demodulator includes an ideal low-pass filter with bandwidth 3.1 kHz.

<u>Solution</u>

The Carson Rule bandwidth is $B_T = 2(k_f A + f_m) = 2(2(1) + 2) = 8\ kHz$. Then from Eq.(9.59),

$$\mathrm{SNR}_{\mathrm{post}}^{\mathrm{FM}} = \frac{3A_c^2 k_f^2 P}{2N_0 W^3} = \frac{A_c^2}{2N_0 B_T}\left(\frac{3k_f^2 P}{W^3} B_T\right)$$

We observed that the first factor is the pre-detection SNR, and we may write this as

$$\begin{aligned}\mathrm{SNR}_{\mathrm{post}}^{\mathrm{FM}} &= \mathrm{SNR}_{\mathrm{pre}}^{\mathrm{FM}}\left(\frac{3\cdot 2^2\ ½}{(3.1)^3}8\right) \\ &= \mathrm{SNR}_{\mathrm{pre}}^{\mathrm{FM}} \times 1.61 \\ &\sim 19.2\ \mathrm{dB}\end{aligned}$$

(There is an error in the answer given in the text.)

Problem 9.7 Compute the post-detection SNR in the lower channel for Example 9.2 and compare to the upper channel.

Solution

The SNR of lower channel is, from Eq. (9.59)

$$\mathrm{SNR}_{\mathrm{post}}^{\mathrm{FM}} = \frac{3A_c^{\,2}k_f^{\,2}(P/2)}{2N_0W^3}$$

where we have assumed that half the power is in the lower channel. Using the approximation to Carson's Rule $B_T = 2\left(k_f P^{1/2} + D\right) \approx 2k_f P^{1/2} = 200\ \mathrm{kHz}$, that is, $k_f^2 P = B_T^2 / 4$ this expression becomes

$$\begin{aligned}\mathrm{SNR}_{\mathrm{post}}^{\mathrm{FM}} &= \frac{A_c^{\,2}}{2N_0B_T}\frac{3}{2}\frac{\left(B_T/2\right)^2}{W} \\ &= \mathrm{SNR}_{\mathrm{pre}}^{\mathrm{FM}}\frac{3}{8}\left(\frac{B_T}{W}\right)^3\end{aligned}$$

With a pre-detection SNR of 12 dB, we determine the post-detection SNR as follows

$$\begin{aligned}\mathrm{SNR}_{\mathrm{post}}^{\mathrm{FM}} &= \mathrm{SNR}_{\mathrm{pre}}^{\mathrm{FM}}\frac{3}{8}\left(\frac{200}{19}\right)^3 \\ &= 10^{12/10}\times 0.375\times(10.53)^3 \\ &= 6.94\times 10^3 \\ &\sim 38.4\ \mathrm{dB}\end{aligned}$$

(*The answer in the text for the lower channel is off by factor 0.5 or 3 dB.*) For the upper channel, Example 9.2 indicates this result should be scaled by 2/52 and

$$\begin{aligned}\mathrm{SNR}_{\mathrm{post}}^{\mathrm{FM}} &= \mathrm{SNR}_{\mathrm{pre}}^{\mathrm{FM}}\frac{3}{8}\left(\frac{200}{19}\right)^3 \times \frac{2}{52} \\ &\sim 24.3\ \mathrm{dB}\end{aligned}$$

So the upper channel is $10\log_{10}(52/2) \approx 14.1$ dB worse than lower channel.

Problem 9.8 An FM system has a pre-detection SNR of 15 dB. If the transmission bandwidth is 30 MHz and the message bandwidth is 6 MHz, what is the post-detection SNR? Suppose the system includes pre-emphasis and de-emphasis filters as described by Eqs. (9.63) and (9.64). What is the post-detection SNR if the $f_{3\text{dB}}$ of the de-emphasis filter is 800 kHz?

Solution

From Eq. (9.59), (see Problem 9.7), the post-detection SNR without pre-emphasis is

$$\begin{aligned}\text{SNR}_{\text{post}}^{\text{FM}} &= \text{SNR}_{\text{pre}}^{\text{FM}} \frac{3}{4}\left(\frac{B_T}{W}\right)^3 \\ &\sim 15\text{ dB} + 19.7\text{ dB} \\ &= 34.7\text{ dB}\end{aligned}$$

From Eq. (9.65), the pre-emphasis improvement is

$$\begin{aligned}I &= \frac{(6/0.8)^3}{3\left[(6/0.8) - \tan^{-1}(6/0.8)\right]} \\ &= 23.2 \\ &\sim 13.6\text{ dB}\end{aligned}$$

With this improvement the post-detection SNR with pre-emphasis is 48.3 dB.

Problem 9.9 A sample function

$$x(t) = A_c \cos(2\pi f_c t) + w(t)$$

is applied to a low-pass *RC* filter. The amplitude A_c and frequency f_c of the sinusoidal component are constant, and $w(t)$ is white noise of zero mean and power spectral density $N_0/2$. Find an expression for the output signal-to-noise ratio with the sinusoidal component of $x(t)$ regarded as the signal of interest.

Solution

The noise variance is proportional to the noise bandwidth of the filter so from Example 8.16,

$$\mathbf{E}\left[n^2(t)\right] = B_N N_0 = \frac{1}{4RC} N_0$$

and the signal power is $A_c^2/2$ fir a sinusoid, so the signal-to-noise ratio is given by

$$SNR = \frac{A_c^{\,2}}{2\left(\dfrac{N_0}{4RC}\right)} = \frac{2A_c^{\,2}RC}{N_0}$$

Problem 9.10 A DSC-SC modulated signal is transmitted over a noisy channel, with the power spectral density of the noise as shown in Fig. 9.19. The message bandwidth is 4 kHz and the carrier frequency is 200 kHz. Assume the average received power of the signal is -80 dBm, determine the output signal-to-noise ratio of the receiver.

Solution

From Fig. 9.19, the noise power spectral density ate 200 kHz is approximately $5\text{x}10^{-19}$ W/Hz. Using this value for $N_0/2$ (we are assuming the noise spectral density is approximately flat across a bandwidth of 4 kHz), the post-detection SNR is given by

$$\begin{aligned}\text{SNR} &= \frac{A_c^2 P}{2N_0 W} \\ &= \frac{10^{-11}}{4\times10^3\times5\times10^{-19}} \\ &= 5\times10^3 \\ &\sim 37\text{ dB}\end{aligned}$$

where we have used the fact that the received power is -80 dBm implies that $A_c^2 P/2 = 10^{-11}$ watts .

Problem 9.11 Derive an expression for the post-detection signal-to-noise ratio of the coherent receiver of Fig. 9.6, assuming that the modulated signal $s(t)$ is produced by sinusoidal modulating wave

$$m(t) = A_m \cos(2\pi f_m t)$$

Perform your calculation for the following two receiver types:

(a) Coherent DSB-SC receiver

(b) Coherent SSB receiver.

Assume the message bandwidth is f_m. Evaluate these expressions if the received signal strength is 100 picowatts, the noise spectral density is 10^{-15} watts per hertz, and f_m is 3 kHz.

Solution

(a) The post-detection SNR of the DSB detector is

$$\mathrm{SNR}^{DSB} = \frac{A_c^2 P}{2N_0 W} = \frac{A_c^2 A_m^2}{4N_0 f_m}$$

(b) The post-detection SNR of the SSB detector is

$$\mathrm{SNR}^{SSB} = \frac{A_c^2 P}{4N_0 W} = \frac{A_c^2 A_m^2}{8N_0 f_m}$$

Although the SNR of the SSB system is half of the DSB-SC SNR, note that the SSB system only transmits half as much power.

Problem 9.12 Evaluate the autocorrelation function of the in-phase and quadrature components of narrowband noise at the coherent detector input for the DSB-SC system. Assume the band-pass noise spectral density is $S_N(f) = N_0/2$ for $|f-f_c| < B_T$.

Solution

From Eg. (8.98), the in-phase power spectral density is (see Section 8.11)

$$S_{N_I}(f) = S_{N_Q}(f)$$
$$= \begin{cases} S_N(f - f_c) + S_N(f + f_c) & |f| < B_T/2 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} N_0 & |f| < B_T/2 \\ 0 & \text{otherwise} \end{cases}$$

From Example 8.13, the autocorrelation function corresponding to this power spectral density is

$$R_{N_Q}(\tau) = R_{N_I}(\tau) = N_0 B_T \operatorname{sinc}(B_T \tau)$$

Problem 9.13 Assume a message signal $m(t)$ has power spectral density

$$S_M(f) = \begin{cases} a\dfrac{|f|}{W} & |f| \le W \\ 0 & \text{otherwise} \end{cases}$$

where a and W are constants. Find the expression for post-detection SNR of the receiver when

(a) The signal is transmitted by DSB-SC.
(b) The signal is transmitted by envelope modulation with $k_a = 0.3$.
(c) The signal is transmitted with frequency modulation with $k_f = 500$ hertz per volt.

Assume that white Gaussian noise of zero mean and power spectral density $N_0/2$ is added to the signal at the receiver input.

Solution

(a) with DSB-SC modulation and detection, the post-detection SNR is given by

$$SNR^{DSB} = \frac{A_c^2 P}{2N_0 W}$$

For the given message spectrum, the power is

$$\begin{aligned} P &= \int_{-\infty}^{\infty} S_M(f)df \\ &= 2\int_0^W a\frac{f}{W}df \\ &= aW \end{aligned}$$

where we have used the even-symmetry of the message spectrum on the second line. Consequently, the post-detection SNR is

$$SNR^{DSB} = \frac{A_c^2 a}{2N_0}$$

(b) for envelope detection with $k_a = 0.3$, the post-detection SNR is

Continued on next slide

Problem 9.13 continued

$$\text{SNR}^{\text{AM}} = \frac{A_c^2 k_a^2 P}{2N_0 W}$$

$$= \frac{A_c^2 a}{2N_0} k_a^2$$

$$= 0.09 \frac{A_c^2 a}{2N_0}$$

(c) for frequency modulation and detection with k_f = 500 Hz/V, the post-detection SNR is

$$\text{SNR}^{\text{FM}} = \frac{3A_c^2 k_f^2 P}{2N_0 W^3}$$

$$= \frac{A_c^2 a}{2N_0} 3\left(\frac{k_f}{W}\right)^2$$

Problem 9.14 A 10 kilowatt transmitter amplitude modulates a carrier with a tone $m(t) = \sin(2000\pi t)$, using 50 percent modulation. Propagation losses between the transmitter and the receiver attenuate the signal by 90 dB. The receiver has a front-end noise N_0 = -113 dBW/Hz and includes a bandpass filter $B_T = 2W$ = 10 kHz. What is the post-detection signal-to-noise ratio, assuming the receiver uses an envelope detector?

Solution

If the output of a 10 kW transmitter is attenuated by 90 dB through propagation, then the received signal level R is

$$\begin{aligned} R &= 10^4 \times 10^{-90/10} \\ &= 10^{-5} \text{ watts} \end{aligned} \qquad (1)$$

For an amplitude modulated signal, this received power corresponds to

$$R = \frac{A_c^2}{2}\left(1 + k_a^2 P\right) \qquad (2)$$

From Eq. (9.30), the post-detection SNR of an AM receiver using envelope detection is

$$\text{SNR}_{\text{post}}^{\text{AM}} = \frac{A_C{}^2 k_a{}^2 P}{2N_0 W}$$

Substituting for k_a, P, and $A_c^2/2$ (obtained from Eq. (2)), we find

$$\begin{aligned} \text{SNR}_{\text{post}}^{\text{AM}} &= \frac{R}{1+k_a^2 P}\frac{k_a^2 P}{N_0 W} \\ &= \frac{10^{-5}}{1+0.25\times 0.5} \times \frac{0.25\times 0.5}{(5\times 10^{-12})(5\times 10^3)} \\ &= 44.4 \end{aligned}$$

where k_a = 0.5 and P = 0.5.

Problem 9.15 The average noise power per unit bandwidth measured at the front end of an AM receiver is 10^{-6} watts per Hz. The modulating signal is sinusoidal, with a carrier power of 80 watts and a sideband power of 10 watts per sideband. The message bandwidth is 4 kHz. Assuming the use of an envelope detector in the receiver, determine the output signal-to-noise ratio of the system. By how many decibels is this system inferior to DSB-SC modulation system?

Solution

For this AM system, the carrier power is 80 watts, that is,

$$\frac{A_C^{\ 2}}{2} = 80 \text{ watts} \qquad (1)$$

and the total sideband power is 20 watts, that is,

$$\frac{A_C^{\ 2}}{2} k_a^{\ 2} P = 20 \text{ watts} \qquad (2)$$

Comparing Eq.s (1) and (2), we determine that $k_a^2 P = \frac{1}{4}$. Consequently, that post-detection SNR of the AM system is

$$\begin{aligned} SNR_{post}^{AM} &= \frac{A_C^{\ 2} k_a^{\ 2} P}{2 N_0 W} \\ &= \frac{20}{10^{-6} \times 4000} \\ &= 5000 \\ &\sim 37 dB \end{aligned}$$

For the corresponding DSB system the post detection SNR is given by

$$\begin{aligned} \text{SNR}_{\text{post}}^{\text{DSB}} &= \frac{1 + k_a^2 P}{k_a^2 P} \quad \text{SNR}_{\text{post}}^{\text{AM}} \\ &= \frac{1 + \frac{1}{4}}{\frac{1}{4}} \\ &= 5 \times \text{SNR}_{\text{post}}^{\text{AM}} \\ &\sim 7 dB \text{ higher} \end{aligned}$$

Problem 9.16 An AM receiver, operating with a sinusoidal modulating wave and 80% modulation, has a post-detection signal-to-noise ratio of 30 dB. What is the corresponding pre-detection signal-to-noise ratio?

Solution

We are given that $k_a = 0.80$, and for sinusoidal modulation $P = 0.5$. A post-detection SNR of 30 dB corresponds to an absolute SNR of 1000. From Eq.(9.30),

$$SNR_{post}^{AM} = \frac{A_C{}^2}{2}\frac{k_a{}^2 P}{N_0 W}$$

$$1000 = \frac{A_c^2}{2N_0 W}(0.8)^2 0.5$$

Re-arranging this equation, we obtain

$$\frac{A_c^2}{2N_0 W} = 3125$$

From Eq. (9.26) the pre-detection SNR is given by

$$\begin{aligned} SNR_{pre}^{AM} &= \frac{A_C{}^2\left(1 + k_a{}^2 P\right)}{2N_0 B_T} \\ &= \frac{A_c^2}{2N_0(2W)}\left(1 + k_a^2 P\right) \\ &= \frac{3125}{2}\left(1 + (0.8)^2 0.5\right) \\ &= 2062.5 \end{aligned}$$

where we have assumed that $B_T = 2W$. This pre-detection SNR is equivalent to approximately 36 dB.

Problem 9.17. The signal $m(t) = \cos(400\pi t)$ is transmitted via FM. There is an ideal band-pass filter passing $100 \le |f| \le 300$ at the discriminator output. Calculate the post-detection SNR given that k_f = 1 kHz per volt, and the pre-detection SNR is 500. Use Carson's rule to estimate the pre-detection bandwidth.

Solution

We begin by estimating the Carson's rule bandwidth

$$\begin{aligned} B_T &= 2(k_f A + f_m) \\ &= 2(1000(1) + 200) \\ &= 2400 \text{ Hz} \end{aligned}$$

We are given that the pre-detection SNR is 500. From Section 9.7 this implies

$$\begin{aligned} SNR_{pre}^{FM} &= \frac{A_c^2}{2N_0 B_T} \\ 500 &= \frac{A_c^2}{2N_0}\frac{1}{2400} \end{aligned}$$

Re-arranging this equation, we obtain

$$\frac{A_c^2}{2N_0} = 1.2 \times 10^6 \text{ Hz}$$

The nuance in this problem is that the post-detection filter is not ideal with unity gain from 0 to W and zero for higher frequencies. Consequently, we must re-evaluate the post-detection noise using Eq. (9.58)

$$\begin{aligned} \text{Avg. post-detection noise power} &= \frac{N_0}{A_c^2}\left[\int_{-300}^{-100} f^2 df + \int_{100}^{300} f^2 df\right] \\ &= \frac{2N_0}{3A_c^2}\left[300^3 - 100^3\right] \\ &= \frac{2N_0}{3A_c^2} 2.6 \times 10^7 \end{aligned}$$

The post-detection SNR then becomes

Continued on next slide

Problem 9.17 continued

$$\begin{aligned}\mathrm{SNR}_{\mathrm{post}}^{\mathrm{FM}} &= \frac{3A_c^2 k_f^2 P}{2N_0\left(2.6\times10^7\right)} \\ &= 3\left(\frac{A_c^2}{2N_0}\right)\frac{k_f^2 P}{2.6\times10^7} \\ &= 3\left(1.2\times10^6\right)\frac{(1000)^2\, 0.5}{2.6\times10^7} \\ &= 69230.8\end{aligned}$$

where we have used the fact that k_f = 1000 Hz/V and P = 0.5 watts. In decibels, the post-detection SNR is 48.4 dB.

Problem 9.18. Suppose that the spectrum of a modulating signal occupies the frequency band $f_1 \le |f| \le f_2$. To accommodate this signal, the receiver of an FM system (without pre-emphasis) uses an ideal band-pass filter connected to the output of the frequency discriminator; the filter passes frequencies in the interval $f_1 \le |f| \le f_2$. Determine the output signal-to-noise ratio and figure of merit of the system in the presence of additive white noise at the receiver input.

Solution

Since the post detection filter is no longer an ideal brickwall filter, we must revert to Eq. (9.58) to compute the post-detection noise power. For this scenario (similar to Problem 9.17)

$$\begin{aligned}\text{Avg. post - detection noise power} &= \frac{N_0}{A_c^2}\left[\int_{-f_2}^{-f_1} f^2 df + \int_{f_1}^{f_2} f^2 df\right] \\ &= \frac{2N_0}{3A_c^2}\left[f_2^3 - f_1^3\right]\end{aligned}$$

Since the average output power is still $k_f^2 P$, the post detection SNR is given by

$$SNR_{post}^{FM} = \frac{3A_c^2 k_f^2 P}{2N_0\left(f_2^3 - f_1^3\right)}$$

For comparison purposes, the reference SNR is

$$SNR_{ref} = \frac{A_c^2}{2N_0(f_2 - f_1)}$$

The corresponding figure of merit is

$$\begin{aligned}\text{Figure of merit} &= \frac{\text{SNR}_{\text{post}}^{\text{FM}}}{\text{SNR}_{\text{ref}}} \\ &= \frac{3A_c^2 k_f^2 P}{2N_0\left(f_2^3 - f_1^3\right)} \Bigg/ \frac{A_c^2}{2N_0(f_2 - f_1)} \\ &= \frac{3k_f^2 P}{f_2^2 + f_2 f_1 + f_1^2}\end{aligned}$$

Problem 9.19. An FM system, operating at a pre-detection SNR of 14 dB, requires a post-detection SNR of 30 dB, and has a message power of 1 watt and bandwidth of 50 kHz. Using Carson's rule, estimate what the transmission bandwidth of the system must be. Suppose this system includes pre-emphasis and de-emphasis network with f_{3dB} of 10 kHz. What transmission bandwidth is required in this case?

Solution

We are given the pre-detection SNR of 14 dB (~25.1), so

$$SNR_{pre}^{FM} = \frac{A_c^2}{2N_0 B_T} = 25.1$$

and the post-detection SNR of 30 dB (~1000), so

$$SNR_{post}^{FM} = \frac{3A_c^2 k_f^2 P}{2N_0 W^3} = 1000$$

Combining these two expressions, we obtain

$$\frac{SNR_{post}^{FM}}{SNR_{pre}^{FM}} = \frac{3k_f^2 P B_T}{W^3} = 39.8$$

Approximating the Carson's rule for general modulation $B_T = 2\left(k_f P^{1/2} + W\right) \approx 2k_f P^{1/2}$, and if we replace $k_f^2 P$ with $B_T^2 / 4$ in this last equation, we obtain

$$\frac{SNR_{post}^{FM}}{SNR_{pre}^{FM}} \approx \frac{3B_T^3}{4W^3} = 39.8$$

Upon substituting W = 50 kHz, this last equation yields B_T = 187.9 kHz.

Problem 9.20. Assume that the narrowband noise $n(t)$ is Gaussian and its power spectral density $S_N(f)$ is symmetric about the midband frequency f_c. Show that the in-phase and quadrature components of $n(t)$ are statistically independent.

<u>Solution</u>

The narrowband noise $n(t)$ can be expressed as:

$$\begin{aligned} n(t) &= n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t) \\ &= \mathrm{Re}\left[z(t)e^{j2\pi f_c t} \right] \end{aligned},$$

where $n_I(t)$ and $n_Q(t)$ are in-phase and quadrature components of $n(t)$, respectively. The term $z(t)$ is called the complex envelope of $n(t)$. The noise $n(t)$ has the power spectral density $S_N(f)$ that may be represented as shown below

We shall denote $R_{nn}(\tau)$, $R_{n_I n_I}(\tau)$ and $R_{n_Q n_Q}(\tau)$ as autocorrelation functions of $n(t)$, $n_I(t)$ and $n_Q(t)$, respectively. Then

$$\begin{aligned} R_{nn}(\tau) &= \mathbf{E}\left[n(t)n(t+\tau) \right] \\ &= \mathbf{E}\left\{ \left[n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t) \right] \cdot \left[n_I(t+\tau)\cos(2\pi f_c(t+\tau)) - n_Q(t+\tau)\sin(2\pi f_c(t+\tau)) \right] \right\} \\ &= \frac{1}{2}\left[R_{n_I n_I}(\tau) + R_{n_Q n_Q}(\tau) \right]\cos(2\pi f_c \tau) + \frac{1}{2}\left[R_{n_I n_I}(\tau) - R_{n_Q n_Q}(\tau) \right]\cos(2\pi f_c(2t+\tau)) \\ &\quad - \frac{1}{2}\left[R_{n_Q n_I}(\tau) - R_{n_I n_Q}(\tau) \right]\sin(2\pi f_c \tau) - \frac{1}{2}\left[R_{n_Q n_I}(\tau) + R_{n_I n_Q}(\tau) \right]\sin(2\pi f_c(2t+\tau)) \end{aligned}$$

Since $n(t)$ is stationary, the right-hand side of the above equation must be independent of t, this implies

$$R_{n_I n_I}(\tau) = R_{n_Q n_Q}(\tau) \qquad (1)$$

$$R_{n_I n_Q}(\tau) = -R_{n_Q n_I}(\tau) \qquad (2)$$

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Problem 9.20 continued

Substituting the above two equations into the expression for $R_{nn}(\tau)$, we have

$$R_{nn}(\tau) = R_{n_I n_I}(\tau)\cos(2\pi f_c \tau) - R_{n_Q n_I}(\tau)\sin(2\pi f_c \tau) \qquad (3)$$

The autocorrelation function of the complex envelope $z(t) = n_I(t) + jn_Q(t)$ is

$$\begin{aligned} R_{zz}(\tau) &= E\left[z^*(t)z(t+\tau)\right] \\ &= 2R_{n_I n_I}(\tau) + j2R_{n_Q n_I}(\tau) \end{aligned} \qquad (4)$$

From the bandpass to low-pass transformation of Section 3.8, the spectrum of the complex envelope z is given bye

$$S_Z(f) = \begin{cases} S_N\left(f + f_c\right) & f > -f_c \\ 0 & \text{otherwise} \end{cases}$$

Since $S_N(f)$ is symmetric about f_c, $S_Z(f)$ is symmetric about $f = 0$. Consequently, the inverse Fourier transform of $S_Z(f) = R_{zz}(\tau)$ must be real. Since $R_{zz}(\tau)$ is real valued, based on Eq. (4), we have

$$R_{n_Q n_I}(\tau) = 0,$$

which means the in-phase and quadrature components of $n(t)$ are uncorrelated. Since the in-phase and quadrature components are also Gaussian, this implies that they are also statistically independent.

Problem 9.21. Suppose that the receiver bandpass-filter magnitude response $|H_{BP}(f)|$ has symmetry about $\pm f_c$ and noise bandwidth B_T. From the properties of narrowband noise described in Section 8.11, what is the spectral density $S_N(f)$ of the in-phase and quadrature components of the narrowband noise $n(t)$ at the output of the filter? Show that the autocorrelation of $n(t)$ is

$$R_N(\tau) = \rho(\tau)\cos(2\pi f_c \tau)$$

where $\rho(\tau) = \mathbf{F}^{-1}\left[S_N(f)\right]$; justify the approximation $\rho(\tau) \approx 1$ for $|\tau| < 1/B_T$.

Solution

Let the noise spectral density of the bandpass process be SH(f) then

$$S_H(f) = \frac{N_0}{2}|H_{BP}(f)|^2$$

From Section 8.11, the power spectral densities of the in-phase and quadrature components are given by

$$S_N(f) = \begin{cases} S_H(f-f_c) + S_H(f+f_c), & |f| \le B_T/2 \\ 0, & \text{otherwise} \end{cases}.$$

Since the spectrum $S_H(f)$ is symmetric about f_c, , the spectral density of the in-phase and quadrature components is

$$S_N(f) = \begin{cases} |H_{BP}(f-f_c)|^2 N_0 & |f| < B_T/2 \\ 0 & \text{otherwise} \end{cases} \qquad (1)$$

Note that if $|H_{BP}(f)|$ is symmetric about f_c then $|H_{BP}(f-f_c)|$ will be symmetric about 0. Consequently, the power spectral densities of the in-phase and quadrature components are symmetric about the origin. This implies that the corresponding autocorrelation functions are real valued (since they are related by the inverse Fourier transform). In Problem 9.20, we shown that if the autocorrelation function of the in-phase component is real valued then autocorrelation of $n(t)$ is $R_N(\tau) = R_{n_I n_I}(\tau)\cos(2\pi f_c \tau)$. If we denote

$$\rho(\tau) = R_{n_I n_I}(\tau) = \mathbf{F}^{-1}\left[S_N(f)\right] = N_O \mathbf{F}^{-1}\left[|H_{BP}(f-f_c)|^2\right]$$

Continued on next slide

Problem 9.21 continued

then the autocorrelation of the bandpass noise is

$$R_N(\tau) = \rho(\tau)\cos(2\pi f_c t)$$

For $|\tau| \ll 1/B_T$ (*there is a typo in the text*), we have

$$\begin{aligned}\rho(\tau) &= \int_{-\infty}^{\infty} S_N(f)\exp(-j2\pi f\tau)\,df \\ &= \int_{0}^{\infty} S_N(f)\cos(2\pi f\tau)\,df\end{aligned}$$

due to the real even-symmetric nature of $S_N(f)$. If the signal has noise bandwidth B_T then

$$\begin{aligned}\rho(\tau) &\approx \int_{0}^{B_T} S_N(f)\cos(2\pi f\tau)\,df \\ &\approx \int_{0}^{B_T} S_N(f)\cos(0)\,df \\ &= \int_{0}^{B_T} S_N(f)\,df \\ &= \text{a constant}\end{aligned}$$

where the second line follows from the assumption that $|\tau| \ll 1/B_T$. With suitable scaling the constant can be set to one.

Problem 9.22. Assume that, in the DSB-SC demodulator of Fig. 9.6, there is a phase error ϕ in the synchronized oscillator such that its output is $\cos(2\pi f_c t + \phi)$. Find an expression for the coherent detector output and show that the post-detection SNR is reduced by the factor $\cos^2 \phi$.

Solution

The signal at the input to the coherent detector of Fig. 9.6 is $x(t)$ where

$$\begin{aligned} x(t) &= s(t) + n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t) \\ &= A_c m(t)\cos(2\pi f_c t) + n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t) \end{aligned}$$

The output of mixer2 in Fig. 9.6 is

$$\begin{aligned} v(t) &= x(t)\cos(2\pi f_c t + \phi) \\ &= \left[A_c m(t) + n_I(t)\right]\cos(2\pi f_c t)\cos(2\pi f_c t + \phi) - n_Q(t)\sin(2\pi f_c t)\cos(2\pi f_c t + \phi) \\ &= \frac{1}{2}\left[A_c m(t) + n_I(t)\right]\cos\phi + \frac{1}{2}n_Q(t)\sin\phi + \frac{1}{2}\left[A_c m(t) + n_I(t)\right]\cos(4\pi f_c t + \phi) - \frac{1}{2}n_Q(t)\sin(4\pi f_c t + \phi) \end{aligned}$$

With the higher frequency components will be eliminated by the low pass filter, the received message at the output of the low-pass filter is

$$y(t) = \frac{1}{2}A_c m(t)\cos\phi + \frac{1}{2}n_I(t)\cos\phi + \frac{1}{2}n_Q(t)\sin\phi$$

To compute the post-detection SNR we note that the average output message power in this last expression is

$$\frac{1}{4}A_c^2 P\cos^2\phi$$

and the average output noise power is

$$\frac{1}{4}\cdot 2N_0 W\cos^2\phi + \frac{1}{4}\cdot 2N_0 W\sin^2\phi = \frac{1}{4}\cdot 2N_0 W$$

where $\mathbf{E}\left[n_I^2(t)\right] = \mathbf{E}\left[n_Q^2(t)\right] = N_0 W$. Consequently, the post-detection SNR is

$$\text{SNR} = \frac{1/4 A_c^2 P\cos^2\phi}{1/4\cdot 2N_0 W} = \frac{A_c^2 P\cos^2\phi}{2N_0 W}$$

Compared with (9.23), the above post-detection SNR is reduced by a factor of $\cos^2\phi$.

Problem 9.23. In a receiver using coherent detection, the sinusoidal wave generated by the local oscillator suffers from a phase error $\theta(t)$ with respect to the carrier wave $\cos(2\pi f_c t)$. Assuming that $\theta(t)$ is a zero-mean Gaussian process of variance σ_θ^2 and that most of the time the maximum value of $\theta(t)$ is small compared to unity, find the mean-square error of the receiver output for DSB-SC modulation. The mean-square error is defined as the expected value of the squared difference between the receiver output and message signal component of a synchronous receiver output.

Solution

Based on the solution of Problem 9.22, we have the DSB-SC demodulator output is

$$y(t) = \frac{1}{2}A_c m(t)\cos\left[\theta(t)\right] + \frac{1}{2}n_I(t)\cos\left[\theta(t)\right] + \frac{1}{2}n_Q(t)\sin\left[\theta(t)\right]$$

Recall from Section 9. that the output of a synchronous receiver is

$$\frac{1}{2}A_c m(t) + \frac{1}{2}n_I(t)$$

The mean-square error (MSE) is defined by

$$\text{MSE} = \mathbf{E}\left[\left(y(t) - \frac{1}{2}A_c m(t)\right)^2\right]$$

Substituting the above expression for $y(t)$, the mean-square error is

$$\begin{aligned}\text{MSE} &= \mathbf{E}\left[\left[\frac{1}{2}A_c m(t)\left[\cos\left(\theta(t)\right) - 1\right] + \frac{1}{2}n_I(t)\cos\left(\theta(t)\right) + \frac{1}{2}n_Q(t)\sin\left(\theta(t)\right)\right]^2\right] \\ &= \frac{A_c^2}{4}\mathbf{E}\left[m^2(t)\left[\cos\left(\theta(t)\right) - 1\right]^2\right] + \frac{1}{4}\mathbf{E}\left[n_I^2(t)\cos^2\left(\theta(t)\right)\right] + \frac{1}{4}\mathbf{E}\left[n_Q^2(t)\sin^2\left(\theta(t)\right)\right]\end{aligned}$$

where we have used the independence of $m(t)$, $n_I(t)$, $n_Q(t)$, and $\theta(t)$ and the fact that $\mathbf{E}\left[n_I(t)\right] = \mathbf{E}\left[n_Q(t)\right] = 0$ to eliminate the cross terms.

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Problem 9.23 continued

$$\begin{aligned}\text{MSE} &= \frac{A_c^2}{4}\mathbf{E}\left[m^2(t)\right]\mathbf{E}\left[\left(1-\cos\left(\theta(t)\right)\right)^2\right]+\frac{1}{4}\mathbf{E}\left[n_I^2(t)\right]\mathbf{E}\left[\cos^2\left(\theta(t)\right)\right]+\frac{1}{4}\mathbf{E}\left[n_Q^2(t)\right]\mathbf{E}\left[\sin^2(\theta(t)\right] \\ &= \frac{A_c^2 P}{4}\mathbf{E}\left[\left(1-\cos\left(\theta(t)\right)\right)^2\right]+\frac{1}{4}N_0W\mathbf{E}\left[\cos^2\left(\theta(t)\right)\right]+\frac{1}{4}N_0W\mathbf{E}\left[\sin^2(\theta(t)\right] \\ &= \frac{A_c^2 P}{4}\mathbf{E}\left[\left(1-\cos\left(\theta(t)\right)\right)^2\right]+\frac{N_0W}{2}\end{aligned}$$

where we have used the equivalences of $\mathbf{E}[m^2(t)] = P$, and $\mathbf{E}\left[n_I^2(t)\right]=\mathbf{E}\left[n_Q^2(t)\right]=2N_0W$.
The last line uses the fact that $\cos^2(\theta(t))+\sin^2(\theta(t)) = 1$. If we now use the relation that $1-\cos A = 2\sin^2(A/2)$, this expression becomes

$$\text{MSE} = A_c^2 P\mathbf{E}\left[\sin^4\left(\frac{\theta(t)}{2}\right)\right]+\frac{N_0W}{2}$$

Since the maximum value of $\theta(t) << 1$, $\sin(\theta(t)) \approx \theta(t)$ and we have

$$\begin{aligned}\text{MSE} &\approx A_c^2 P\mathbf{E}\left[\left(\frac{\theta(t)}{2}\right)^4\right]+\frac{N_0W}{2} \\ &= \frac{3}{16}A_c^2 P\sigma_\theta^4+\frac{N_0W}{2}\end{aligned}$$

where we have used the fact that if θ is a zero-mean Gaussian random variable then

$$\mathbf{E}\left[\theta^4\right]=3\left(\mathbf{E}\left[\theta^2\right]\right)^2=3\sigma_\theta^4$$

The mean square error is therefore $\frac{3}{16}A_c^2 P\sigma_\theta^4+\frac{1}{2}N_0W$.

Problem 9.24. Equation (9.59) is the FM post-detection noise for an ideal low-pass filter. Find the post-detection noise for an FM signal when the post-detection filter is a second-order low-pass filter with magnitude response

$$|H(f)| = \frac{1}{\left(1+\left(f/W\right)^4\right)^{1/2}}$$

Assume $\left|H_{BP}(f+f_c)\right|^2 \approx 1$ for $|f| < B_T/2$ and $B_T >> 2W$.

Solution

We modify Eq. (9.58) to include the effects of a non-ideal post-detection filter in order to estimate the average post-detection noise power:

$$\begin{aligned}\frac{N_0}{A_c^2}\int_{-W}^{W} f^2 \,|H_{BP}(f)|^2\, df &= \frac{N_0}{A_c^2}\int_{-W}^{W} f^2 \cdot \frac{1}{1+(f/W)^4}\,df \\ &= \frac{2N_0}{A_c^2}\int_0^W f^2 \cdot \frac{1}{1+(f/W)^4}\,df\end{aligned}$$

This can be evaluated by a partial fraction expansion of the integrand but for simplicity, we appeal to the formula:

$$\int \frac{x^2 dx}{a+bx^4} = \frac{1}{4bk}\left[\frac{1}{2}\log\frac{x^2-2kx+2k^2}{x^2+2kx+2k^2} + \tan^{-1}\frac{2kx}{2k^2-x^2}\right], \quad ab>0,\; k=\sqrt[4]{\frac{a}{2b}}$$

Using this result, we get the average post-detection noise power is

$$\text{Avg. post-detection noise power } = \frac{2N_0}{A_c^2}\cdot\frac{W^3}{4\sqrt{2}}\left[\log\frac{2-\sqrt{2}}{2+\sqrt{2}} + \pi\right] = 0.42\frac{N_0 W^3}{A_c^2}$$

Problem 9.25. Consider a communication system with a transmission loss of 100 dB and a noise density of 10^{-14} W/Hz at the receiver input. If the average message power is $P = 1$ watt and the bandwidth is 10 kHz, find the average transmitter power (in kilowatts) required for a post-detection SNR of 40 dB or better when the modulation is:

(a) AM with $k_a = 1$; repeat the calculation for $k_a = 0.1$.
(b) FM with $k_f = 10$, 50 and 100 kHz per volt.

In the FM case, check for threshold limitations by confirming that the pre-detection SNR is greater that 12 dB.

Solution

(a) In the AM case, the post detection SNR is given by

$$\mathrm{SNR}_{\mathrm{post}}^{\mathrm{AM}} = \frac{A_c^2 k_a^2 P}{2N_o W}$$

$$10^4 = \frac{A_c^2 k_a^2 (1)}{2(2\times 10^{-14})(10^4)}$$

$$\frac{A_c^2 k_a^2}{2} = 2\times 10^{-6}$$

where an SNR of 40 dB corresponds to 10^4 absolute and $N_0/2 = 10^{-14}$ W/Hz. For the different values of k_a

$$k_a = 1 \Rightarrow \quad A_c^2 = 4\times 10^{-6}$$

$$k_a = 0.1 \Rightarrow \quad A_c^2 = 4\times 10^{-4}$$

Average modulated signal power at the input of the detector is $\frac{1}{2}A_c^2(1+k_a^2 P)$.

$$k_a = 1 \Rightarrow \quad \frac{1}{2}A_c^2(1+k_a^2 P) = 4\times 10^{-6}$$

$$k_a = 0.1 \Rightarrow \quad \frac{1}{2}A_c^2(1+k_a^2 P) = 2.02\times 10^{-4}$$

The transmitted power is 100dB (10^{10}) greater than the received signal power so

$k_a = 1 \Rightarrow$ transmitted power = $4\times 10^4 = 40$ kW

$k_a = 0.1 \Rightarrow$ transmitted power = $2.02\times 10^6 = 2020$ kW

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Problem 9.25 continued

(b) In the FM case, the post detection SNR is

$$\text{SNR}_{\text{post}}^{\text{FM}} = \frac{3A_c^2 k_f^2 P}{2N_o W^3}$$

$$10^4 = \frac{3A_c^2 k_f^2 (1)}{2(2\times 10^{-14})(10^4)^3}$$

$$\frac{A_c^2 k_f^2}{2} = 0.667\times 10^2$$

For the different values of k_a

$$k_f = 10\,\text{kHz/V} \Rightarrow \frac{\text{A}_\text{c}^2}{2} = 0.667\times 10^{-6}$$

$$k_f = 50\,\text{kHz/V} \Rightarrow \frac{\text{A}_\text{c}^2}{2} = 26.667\times 10^{-9}$$

$$k_f = 100\,\text{kHz/V} \Rightarrow \frac{\text{A}_\text{c}^2}{2} = 0.667\times 10^{-8}$$

The transmitted power is 100dB (10^{10}) greater than the received signal power so

k_f = 10 kHz/V $\Rightarrow$ transmitted power = 0.667x10^4 W = 6.67 kW
k_f = 50 kHz/V $\Rightarrow$ transmitted power = 26.667x10^1 W = 0.27 kW
k_f = 100 kHz/V $\Rightarrow$ transmitted power = 0.667x10^2 W = 0.07 kW

To check the pre-detection SNR, we note that it is given by :

$$SNR_{pre}^{FM} = \frac{A_c^2}{2N_0 B_T} = \frac{A_c^2}{4N_0(k_f P^{1/2} + W)}$$

where from Carson's rule $B_T = 2(k_f P^{1/2} + W)$. From the above $A_c^2 = \frac{4\times 10^2}{3k_f^2}$, so

$$SNR_{pre}^{FM} = \frac{4\times 10^2}{3k_f^2 \times 4N_0(k_f P^{1/2} + W)} = \frac{10^2}{3k_f^2 \times 2\times 10^{-14}(k_f + 10^4)}$$

For the different values of k_f, the pre-detection SNR is

$$k_f = 10kHz \Rightarrow \quad SNR_{pre}^{FM} = 10^4/12 = 29dB > 12dB$$

$$k_f = 50kHz \Rightarrow \quad SNR_{pre}^{FM} = 11.11 = 10.45dB < 12dB$$

$$k_f = 100kHz \Rightarrow \quad SNR_{pre}^{FM} = 1.515 = 1.8dB < 12dB$$

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Problem 9.25 continued

Therefore, for k_f = 50 kHz and 100 kHz, the pre-detection *SNR* is too low and the transmitter power would have to be increased by 1.55 dB and 10.2 dB, respectively.

Problem 9.26 In this experiment we investigate the performance of amplitude modulation in noise. The MatLab script for this AM experiment is provided in Appendix 8 and simulates envelope modulation by a sine wave with a modulation index of 0.3, adds noise, and then envelope detects the message. Using this script:

(a) Plot the envelope modulated signal.

(b) Using the supporting function "spectra", plot its spectrum.

(c) Plot the envelope detected signal before low-pass filtering.

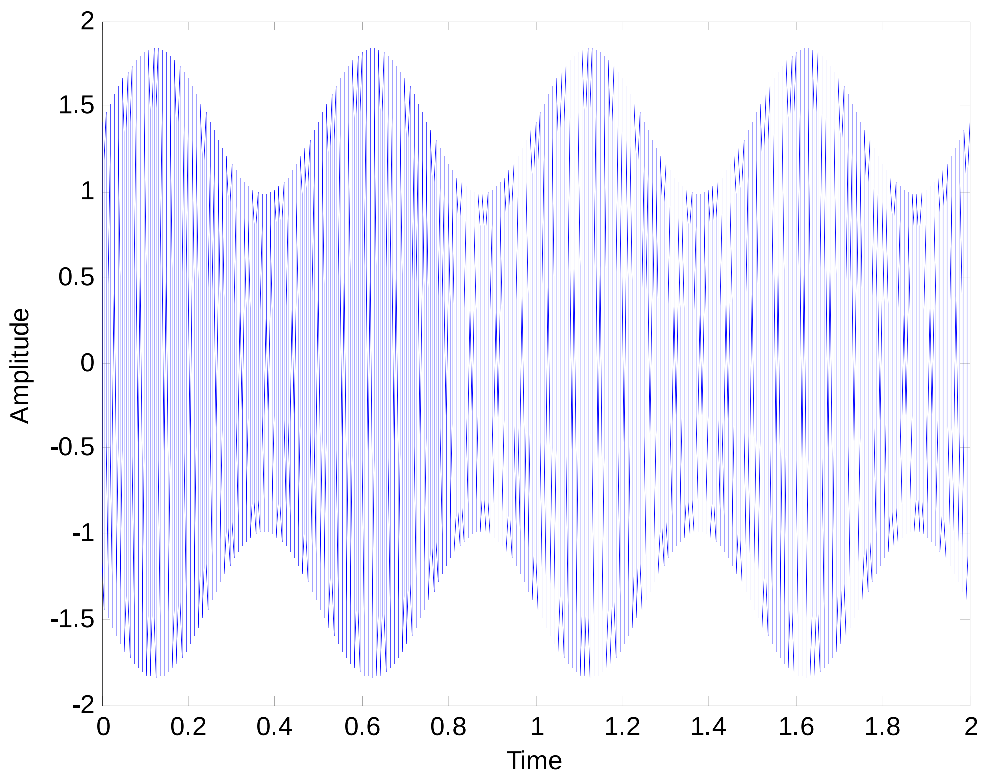
(d) Compare the post-detection SNR to theory.

Using the Matlab script given in Appendix 7 we obtain the following plots

(a) By inserting the statements

plot(t,AM)
xlabel('Time')
ylabel('Amplitude')

at the end of Modulator section of the code, we obtain the following plot of the envelope modulated signal:



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Problem 9.26 continued

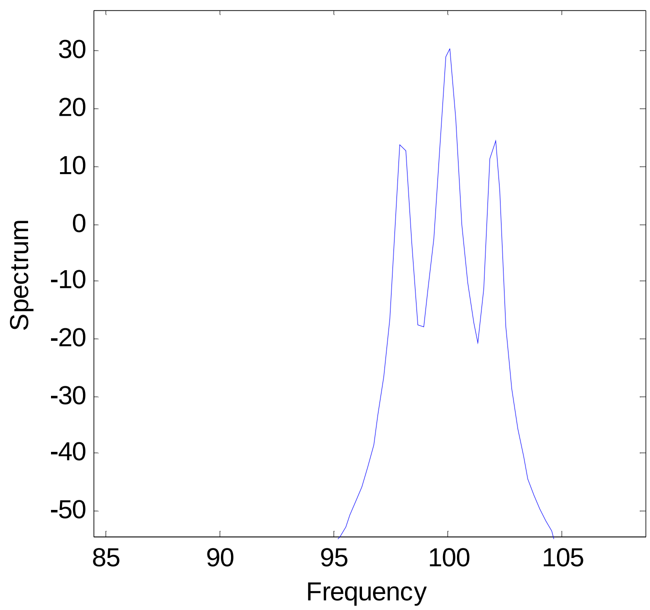
(b) The provided script simulates 2 seconds of the AM signal. Since the modulating signal is only 2 Hz, this is not a sufficient signal length to accurately estimate the spectrum. We extend the simulation to 200 seconds by modifying the statement

t = [0:1/Fs:200];

To plot the spectrum, we insert the following statements after the AM section

```
[P,F] = spectrum(AM,4096,0,4096,Fs);
plot(F,10*log10(P(:,1)))
xlabel('Frequency')
ylabel('Spectrum')
```

We use the large FFT size of 4096 to provide sufficient frequency resolution. (The resolution is *Fs* (1000 Hz) divided by the FFT size. We plot the spectrum of decibels because it more clearly shows the sideband components. With a linear plot, and this low modulation index, the sideband components would be difficult to see. The following figure enlarges the plot around the carrier frequency of 100 Hz.



(c) To plot the envelope-detected signal before low-pass filtering, we insert the statements (Decrease the time duration to 2 seconds to speed up processing for this part.)

```
plot(AM_rec)
xlabel('Time samples')
ylabel('Amplitude')
```

The following plot is obtained and illustrates the tracking of the envelope detector.

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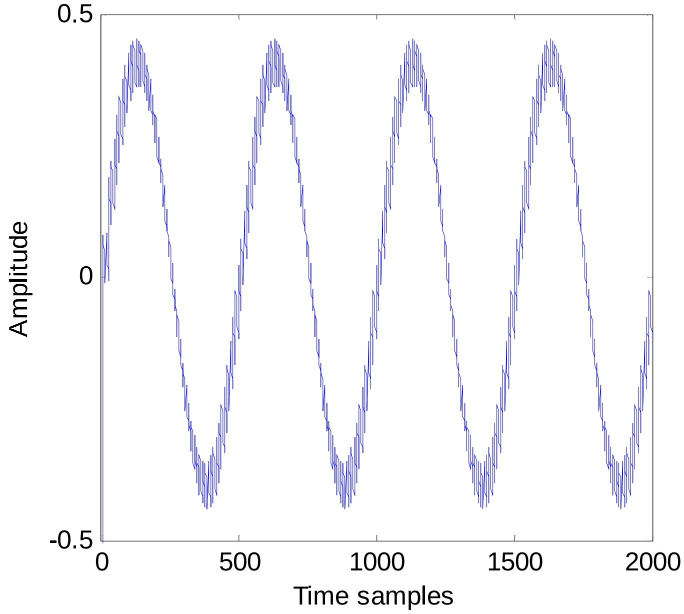
Problem 9.26 continued



(d) To compare the simulated post detection SNR to theory. Create a loop around the main body of the simulation by adding the following statements

```
for kk = 1:15
        SNRdBr = 10 + 2*kk
            ….
        PreSNR(kk) = 20*log10(std(RxAM)/std(RxAMn-RxAM));
        No(kk) = 2*sigma^2/Fs;
        ….
        SNRdBpost(kk) = 10*log10(C/error);
        W = 50; P = 0.5;
        Theory(kk) = 10*log10 ( A^2*ka^2*0.5 / (2*No(kk)*W));
end

plot(PreSNR, SNRdBpost)
hold on,
plot(PreSNR, Theory,'g');
```

The results are shown in the following chart.

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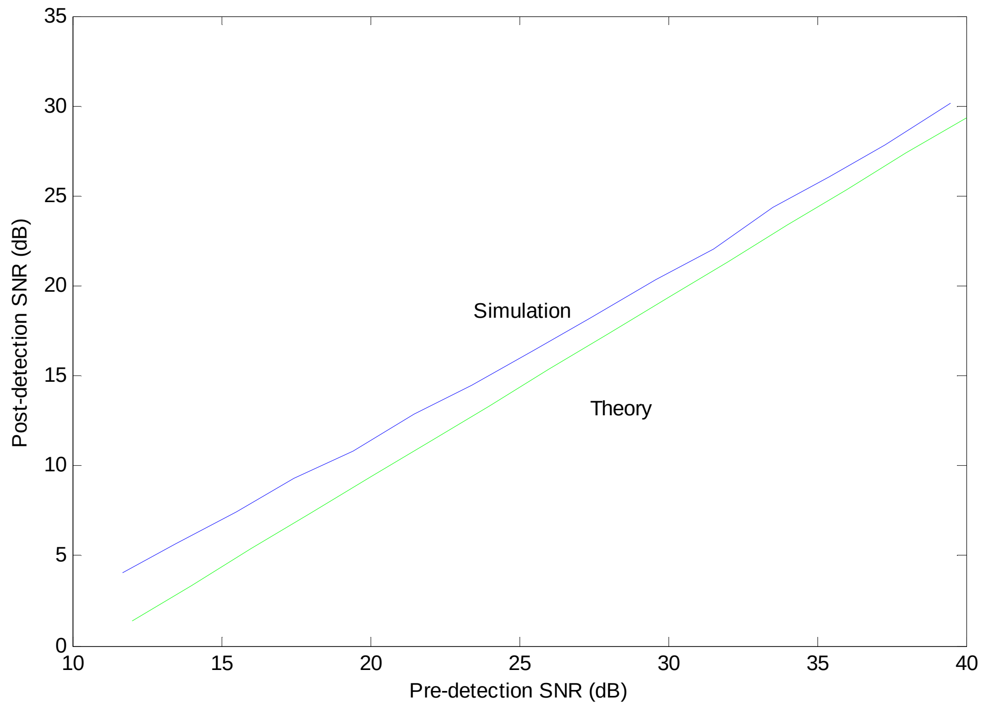
Problem 9.26 continued



These results indicate that the simulation is performing slightly better than theory? Why? As an exercise try adjusting either the frequency of the message tone or the decay of the envelope detector and compare the results.

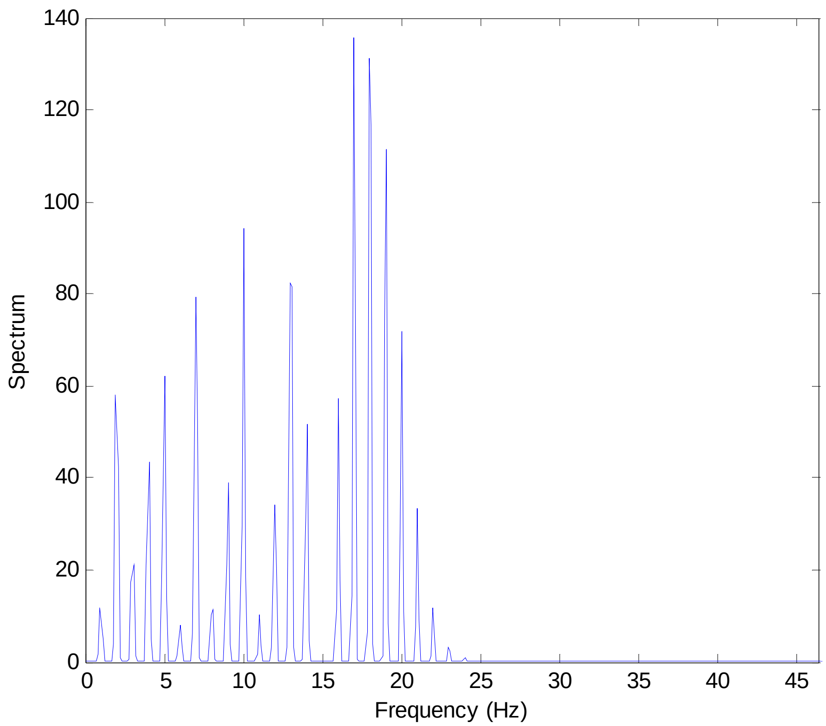
Problem 9.27. In this computer experiment, we investigate the performance of FM in noise. Using the Matlab script for the FM experiment provided in Appendix 8:
(a) Plot the spectrum of the baseband FM phasor.
(b) Plot the spectrum of the band-pass FM plus noise.
(c) Plot the spectrum of the detected signal prior to low-pass filtering.
(d) Plot the spectrum of the detected signal after low pass filtering.
(e) Compare pre-detection and post-detection SNRs for an FM receiver.

In the following parts (a) through (d), set the initial CNdB value to 13 dB in order to be operating above the FM threshold.

(a) By inserting the following statements after the definition of FM, we obtain the baseband spectrum

[P,F] = spectrum(FM,4096,0,4096,Fs);
plot(F,P(:,1))
xlabel('Frequency (Hz)')
ylabel('Spectrum')

An enlarged snapshot of the spectrum near 0 Hz is shown here. It shows the tones at the regular spacing that one would expect with FM tone modulation. Note that initial plot shows the "negative frequency" portion of the spectrum just below F_s = 500 Hz. This is due to the nature of the FFT and the sampling process.



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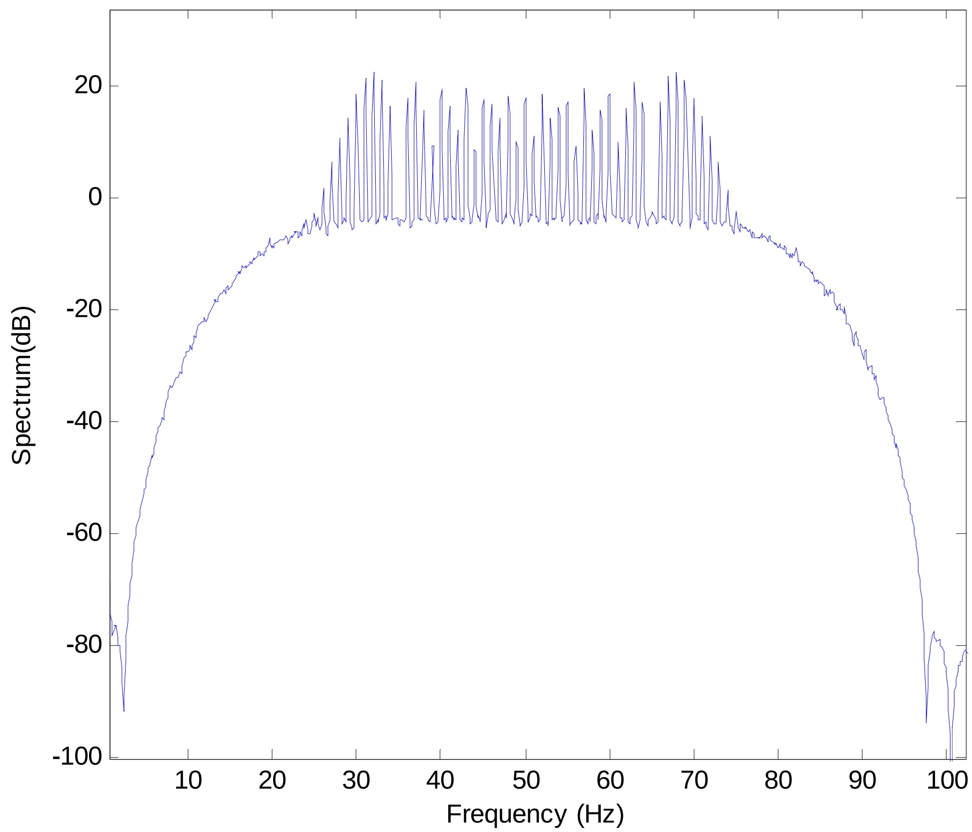
Problem 9.27 continued

(b) The spectrum of the bandpass FM plus noise is obtained by inserting the statements

*[P,F] = spectrum((FM+Noise).*Carrier,4096,0,4096,Fs);*
*plot(F,10*log10(P(:,1)))*
xlabel('Frequency (Hz)')
ylabel('Spectrum')

An expanded view of the result around the carrier frequency of 50 Hz is shown below. The spectrum has been plotted on a decibel scale to show both the FM tone spectrum and the noise pedestal.



(c) To plot the spectrum of the noisy signal before low-pass filtering, we insert the following statements in the FM discriminator function, prior to the low pass filter

[P,F] = spectrum(BBdec,1024,0,1024,Fsample/4)
*plot(F,10*log10(P(:,1)))*
xlabel('Frequency (Hz)')
ylabel('Spectrum(dB)')

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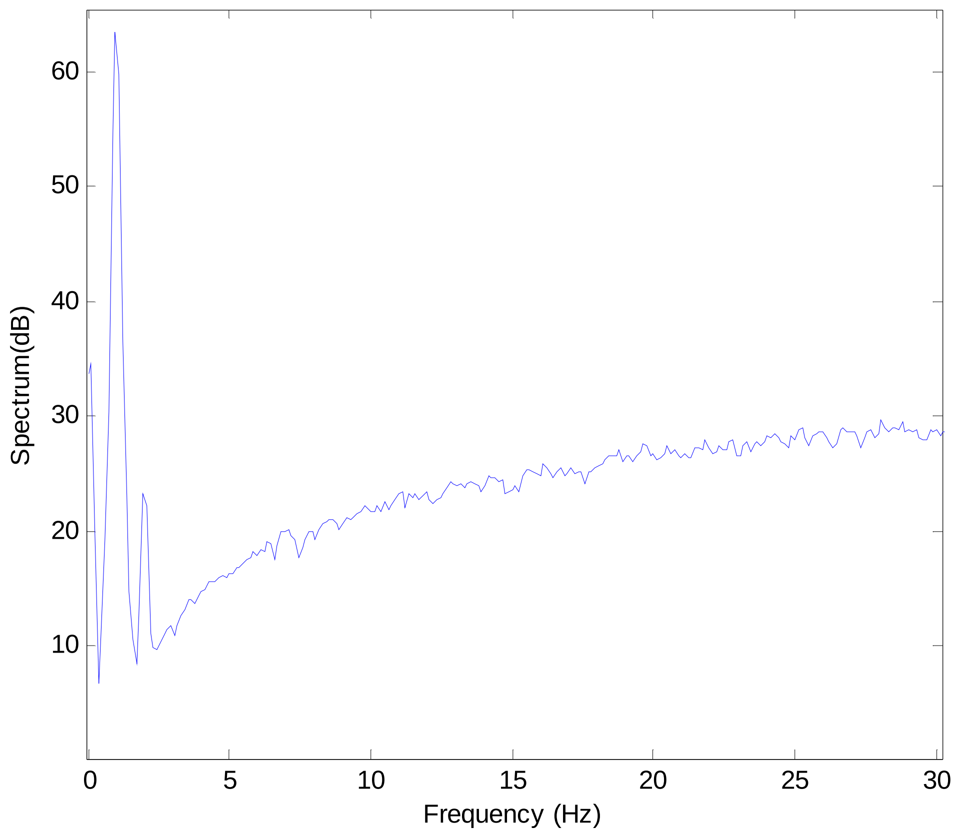
Problem 9.27 continued

The following plot is obtained when expanded near the origin. We plot the spectrum in decibels in order to show the noise and the non-flat nature of its spectrum more clearly. The decibel scale also illustrates some low-level distortion that has been introduced by the demodulation process as exhibited by the small second harmonic at 2 Hz and the low dc level.



(d) To plot the spectrum of the noisy signal before low-pass filtering, we insert the following statements in the FM discriminator function, after the low-pass filter

[P,F] = spectrum(Message,1024,0,1024,Fsample/4)
*plot(F,10*log10(P(:,1)))*
xlabel('Frequency (Hz)')
ylabel('Spectrum(dB)')

The following plot is obtained when expanded near the origin. Again we plot the spectrum in decibels in order to show the noise and, in this case, the effect of the low-pass filtering. The low-pass filtering does not affect the distortion introduced by the demodulator in the passband.

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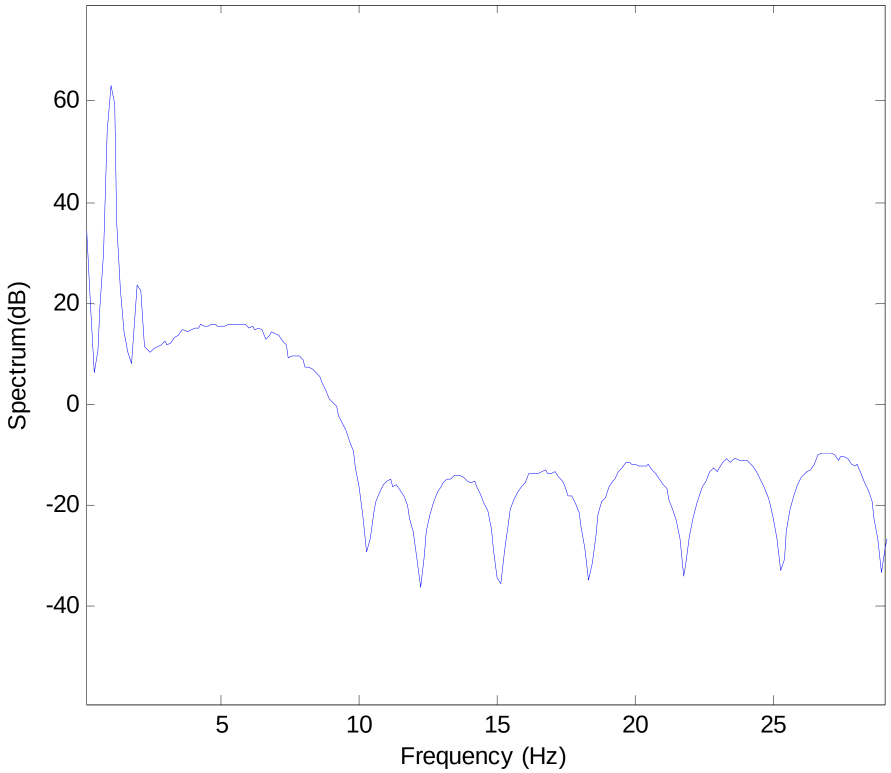
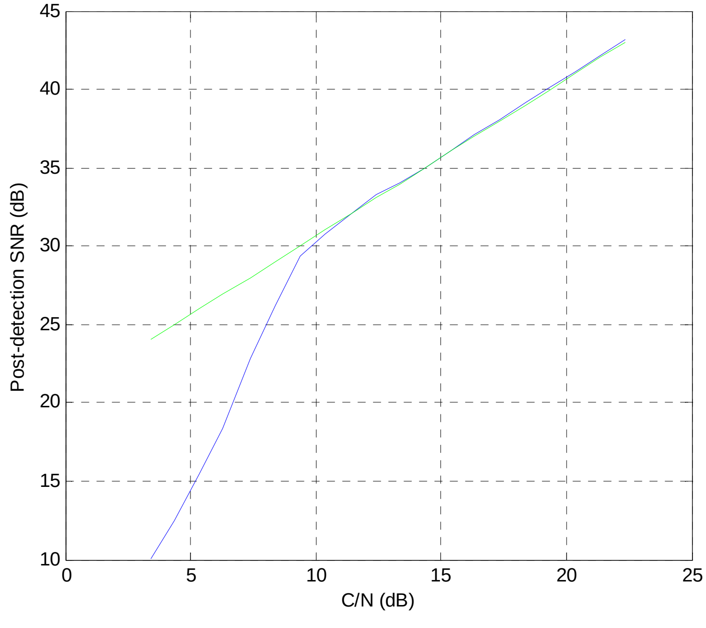
Problem 9.27 continued



(a) Running the code as provided produces the following comparison of the post-detection and pre-detection SNR.



Problem 10.1. Let H_0 be the event that a 0 is transmitted and let R_0 be the event that a 0 is received. Define H_1 and R_1, similarly for a 1. Express the BER in terms of the probability of these events when:
(a) The probability of a 1 error is the same as the probability of a 0 error.
(b) The probability of a 1 being transmitted is not the same as the probability of a 0 being transmitted.

Solution
In both cases, the probability of error may be expressed as

$$\mathbf{P}[error] = \mathbf{P}(R_0|H_1)\mathbf{P}(H_1) + \mathbf{P}(R_1|H_0)\mathbf{P}(H_0) \qquad (1)$$

(a) The BER is the same as the $\mathbf{P}[error]$ and with $\mathbf{P}(R_0|H_1) = \mathbf{P}(R_1/H_0) = p$ then

$$\mathbf{P}[error] = p[\mathbf{P}(H_1) + \mathbf{P}(H_0)] = p$$

since $\mathbf{P}(H_1)+\mathbf{P}(H_0) = 1$.

(b) With $\mathbf{P}(H_0) \neq \mathbf{P}(H_1)$, the answer is given by the general result of Eq. (1).

Problem 10.2. Suppose that in Eq. (10.4), $r(t)$ represents a complex baseband signal instead of a real signal. What would be the ideal choice for g(t) in this case? Justify your answer.

Solution

Inspecting the Schwarz inequality of Eq. (10.12), we see that equality is achieved with

$$g(T-t) = cs^{*}(t)$$

if $s(t)$ is complex.

Problem 10.3 If $g(t) = c \text{ rect}\left[\frac{\alpha(t-T/2)}{T}\right]$, determine c such g(t) satisfies Eq. (10.10) where α > 1.

Solution

From the definition of the rect(.) function,

$$g(t) = c \text{ rect}\left(\frac{\alpha(t-T/2)}{T}\right)$$
$$= \begin{cases} c & |t-T/2| < T/(2\alpha) \\ 0 & \text{otherwise} \end{cases}$$

Substituting this into Eq. (10.10)

$$T = \int_0^T |g(t)|^2 dt$$
$$= c^2 \int_{T/2-T/(2\alpha)}^{T/2+T/(2\alpha)} 1^2 dt$$
$$= c^2 T/\alpha$$

And so $c = \sqrt{\alpha}$.

Problem 10.4. Show that with on-off signaling, the probability of a Type II error in Eq.(10.23) is given by

$$P[Y > \gamma \mid H_0] = Q\left(\frac{\gamma}{\sigma}\right)$$

Solution

A Type II error probability is

$$P[Y > \gamma \mid H_0] = \frac{1}{\sqrt{2\pi}\sigma}\int_{\gamma}^{+\infty} \exp\left(-\frac{y^2}{2\sigma^2}\right)dy$$

Let $s = \dfrac{y}{\sigma}$, and then

$$P[Y > \gamma \mid H_0] = \frac{1}{\sqrt{2\pi}}\int_{\gamma/\sigma}^{+\infty} \exp\left(-\frac{s^2}{2}\right)ds = Q\left(\frac{\gamma}{\sigma}\right)$$

using the definition of the *Q*-function given in Section 8.4.

Problem 10.5 Prove the property of root-raised cosine pulse shape $p(t)$ given by Eq. (10.32), using the following steps:

(a) If $R(f)$ is the Fourier transform representation of $p(t)$, what is the Fourier transform representation of $p(t\text{-}lT)$?

(b) What is the Fourier transform of $q(\tau)=\int_{-\infty}^{\infty} p(\tau-t)\,p(t-lT)\,dt$? What spectral shape does it have?

(c) What $q(\tau)$? What is $q(kT)$?

Use these results to show that Eq. (10.32) holds.

Solution

(a) From the time-shifting property of Fourier transforms (see Section 2.2), we have that

$$\mathbf{F}\left[p(t-lT)\right]=R(f)\exp\left(-j2\pi flT\right)$$

(b) From the convolution property of Fourier transforms (See Section 2.2) we have that

$$\begin{aligned}Q(f)&=\mathbf{F}\left[q(\tau)\right]\\&=\mathbf{F}\left[p(t)\right]\mathbf{F}\left[p(t-lT)\right]\\&=R^2(f)\exp\left(-j2\pi flT\right)\end{aligned}$$

(c) Since $R(f)$ is the root-raised cosine spectrum, $R^2(f)$ is the raised cosine spectrum and so $q(\tau)$ corresponds to a raised cosine pulse. In particular, using the time-shifting property of inverse Fourier transforms

$$q(\tau)=m(\tau-lT)$$

where $m(\tau)$ is the raised cosine pulse shape. Using the properties of the raised cosine pulse shape (see Section 6.4)

$$\begin{aligned}q(kT)&=m(kT-lT)\\&=\begin{cases}1 & k=l\\ 0 & k\neq l\end{cases}\\&=\delta(k-l)\end{aligned}$$

and Eq. (10.32) holds.

Problem 10.6 Compare the transmission bandwidth required for binary PAM and BPSK modulation, if both signals have a data rate of 9600 bps and use root-raised cosine pulse spectrum with a roll-off factor of 0.5.

Solution

For BPSK modulation (bandpass signal), the transmission bandwidth is $B_T = 2\times\frac{1+\beta}{2T}$,

where β is the roll-off factor (0.5) and T is the symbol duration (1/9600 sec). Therefore, B_T = (1+0.5)×9600 = 14.4 kHz.

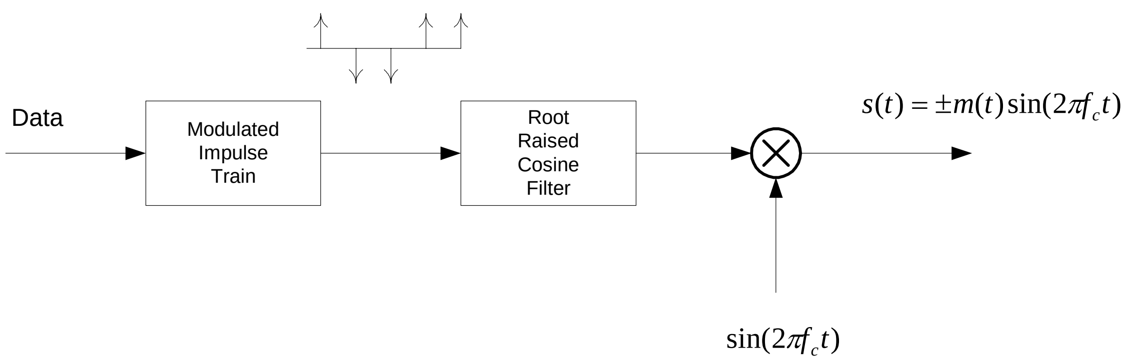
For binary PAM modulation (baseband signal), $B_T = \frac{1+\beta}{2T}$ = 7.2 kHz.

Problem 10.7 Sketch a block diagram of a transmission system including both transmitter and receiver for BPSK modulation with root-raised cosine pulse shaping.

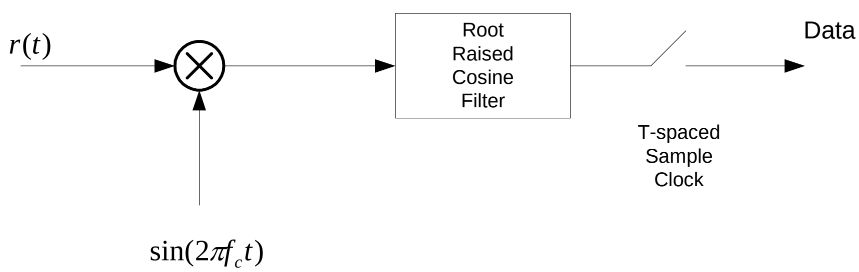
Solution

The BPSK transmitter with root-raised cosine pulse shaping is shown in (a), and the corresponding BPSK receiver is shown in (b).

(a)



(b)



Problem 10.8 Show that the integral of the high frequency term in Eq. (10.53) is approximately zero.

Solution

Consider the integral over the period from 0 to T of the high frequency term in Eq. (10.53):

$$\begin{aligned}\int_0^T \frac{A_C^{\,2}}{2}\cos\left(4\pi f_c t + 2\phi(t)\right) dt &= \frac{A_c^{\,2}}{8\pi f_c}\sin\left(4\pi f_c t + 2\phi(t)\right)\Big|_0^T \\ &= \frac{A_c^{\,2}}{8\pi f_c}\left[\sin\left(4\pi f_c T + 2\phi(T)\right) - \sin\left(2\phi(0)\right)\right] \\ &< \frac{A_c^2}{4\pi f_c}\end{aligned}$$

where the first line follows since $\phi(t)$ is constant over a symbol interval. By the bandpass assumption $f_c >> 1$, so this last line is small.

Problem 10.9. Use Eqs. (10.61), (10.64), and (10.66) to show that N_1 and N_2 are uncorrelated and therefore independent Gaussian random variables. Compute the variance of N_1-N_2.

Solution

The correlation of N_1 and N_2 is

$$\begin{aligned}
\mathbf{E}\left(N_1 N_2\right) &= \mathbf{E}\left[2\int_0^T\int_0^T w(s)w(t)\cos\left(2\pi f_1 t\right)\cos\left(2\pi f_2 s\right)dsdt\right] \\
&= 2\int_0^T\int_0^T \mathbf{E}\left[w(s)w(t)\right]\cos\left(2\pi f_1 t\right)\cos\left(2\pi f_2 s\right)dsdt \\
&= 2\frac{N_0}{2}\int\int \delta\left(t-s\right)\cos\left(2\pi f_1 t\right)\cos\left(2\pi f_2 s\right)ds\,dt \\
&= N_0\int_0^T \cos\left(2\pi f_1 t\right)\cos\left(2\pi f_2 t\right)dt \\
&= 0
\end{aligned}$$

where the last line follows from Eq.(10.61). Since N_1 and N_2 are uncorrelated

$$\begin{aligned}
\mathbf{E}\left[\left(N_1-N_2\right)^2\right] &= \mathbf{E}\left[\left(N_1\right)^2\right] + 2\mathbf{E}\left[N_1 N_2\right] + \mathbf{E}\left[\left(N_2\right)^2\right] \\
&= \mathbf{E}\left[\left(N_1\right)^2\right] + \mathbf{E}\left[\left(N_2\right)^2\right]
\end{aligned}$$

The variance of the N_1 term is

$$\begin{aligned}
\mathbf{E}\left(N_1 N_1\right) &= \mathbf{E}\left[2\int_0^T\int_0^T w(s)w(t)\cos\left(2\pi f_1 t\right)\cos\left(2\pi f_1 s\right)dsdt\right] \\
&= 2\int_0^T\int_0^T \mathbf{E}\left[w(s)w(t)\right]\cos\left(2\pi f_1 t\right)\cos\left(2\pi f_1 s\right)dsdt \\
&= 2\frac{N_0}{2}\int\int \delta\left(t-s\right)\cos\left(2\pi f_1 t\right)\cos\left(2\pi f_1 s\right)ds\,dt \\
&= N_0\int_0^T \cos^2\left(2\pi f_0 t\right)dt
\end{aligned}$$

Using the double angle formula $2\cos^2\theta = 1+2\cos\theta$, we have

$$\begin{aligned}
\mathbf{E}\left[\left(N_1\right)^2\right] &= \frac{N_0}{2}\int_0^T \left(1+\cos 4\pi ft\right)dt \\
&= \frac{N_0 T}{2}
\end{aligned}$$

The derivation of the variance of N_2 is similar and the combined variance is $N_0 T$.

Problem 10.10. Plot the BER performance of differential BPSK and compare the results to Fig. 10.16.

Solution

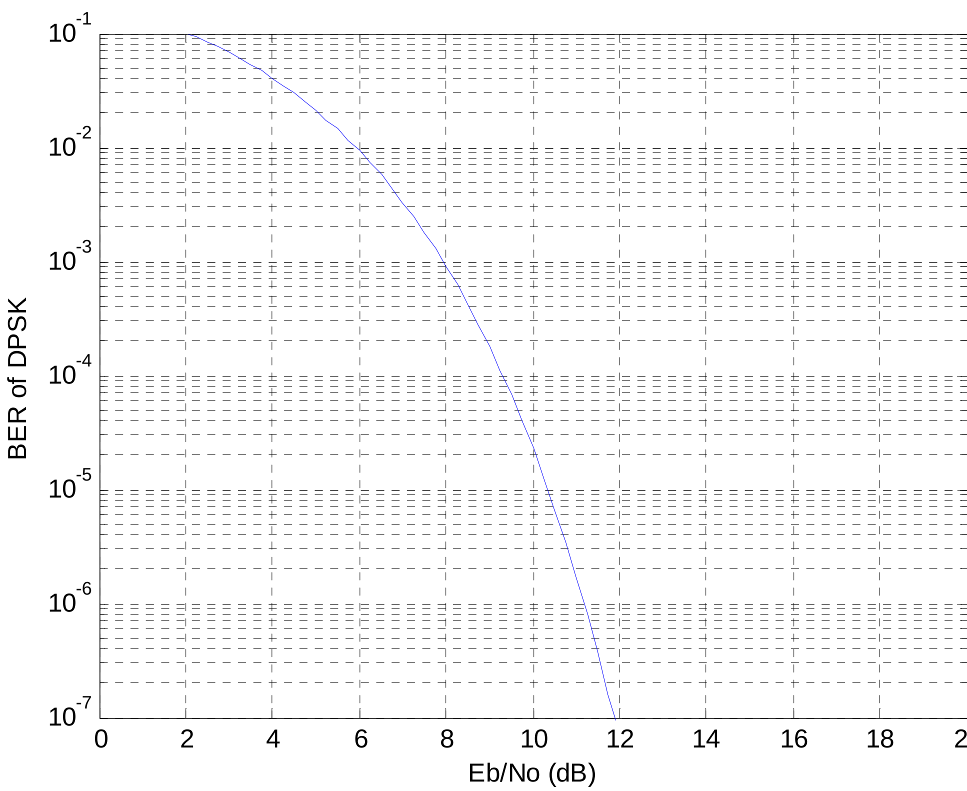
The bit error probability of differential BPSK is (Eq. (10.75))

$$P_e^{DPSK} = 0.5\exp\left(-\frac{E_b}{N_0}\right).$$

The following Matlab script plots this performance

```
EbNodB=[0:0.25:12];
EbNo = 10.^(EbNodB/10);
BER = 0.5*exp(-EbNo);
semilogy(EbNodB,BER)
grid
xlabel('Eb/No (dB)')
ylabel('BER of DPSK')
axis([0 20 1E-7 0.1])
```

This script produces the following plot.



The performance of DPSK is slightly worse than BPSK and QPSK. The relative loss with DPSK is less than 1 dB at E_b/N_0 of 8 dB and higher. The loss at lower E_b/N_0 ratios is greater.

Problem 10.11. A communication system that transmits single isolated pulses is subject to multipath such that, if the transmitted pulse is $p(t)$ of length T, the received signal is

$$s(t) = p(t) + \alpha\, p(t-\tau)$$

Assuming that α and τ are known, determine the optimum receiver filter for signal in the presence of white Gaussian noise of power spectral density $N_0/2$. What is the post-detection SNR at the output of this filter?

<u>Solution</u>

We first note that the pulse is non-zero over the interval $0 \le t \le T+\tau$. From Section 10.2 the appropriate linear receiver is

$$Y = \int_0^{T+\tau} g(T+\tau-u) r(u) du$$

and the optimum choice for $g(t)$ is

$$g(T+\tau-t) = c\left(p(t) + \alpha\, p(t-\tau)\right)$$

where c is chosen such that

$$\int_0^{T+\tau} |g(t)|^2\, dt = T+\tau$$

With this filtering arrangement, if follows from the modified Eq. (10.9) that

$$\mathbf{E}\left[N^2\right] = \frac{N_0\left(T+t\right)}{2}$$

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Problem 10.11 continued

The corresponding signal level S is

$$\begin{aligned} S &= c\int_0^{T+\tau} g(T-t)\big(p(t)+\alpha\, p(t+\tau)\big)\,dt \\ &= c\int_0^{T+\tau} \big(p(t)+\alpha\, p(t+\tau)\big)^2\,dt \\ &= T+\tau \end{aligned}$$

which follows from the normalization properties of c. The received signal to noise is then

$$\text{SNR} = \frac{S^2}{\mathbf{E}\left[N^2\right]} = \frac{T+\tau}{N_0/2}$$

Although the units on this expression may appear unusual, note that the units of N_0 are $(\text{volt})^2$/Hz = $(\text{volt})^2$-sec. The units of the numerator are also $(\text{volt})^2$-sec, although the $(\text{volt})^2$ has been suppressed. Consequently, the SNR is dimensionless, as it should be.

Problem 10.12. The impulse response corresponding to a root-raised cosine spectrum, normalized to satisfy Eq.(10.10), is given by

$$g(t) = \frac{4\alpha}{\pi} \frac{\cos\left[\frac{(1+\alpha)\pi t}{T}\right] + \frac{T}{4\alpha t}\sin\left[\frac{(1-\alpha)\pi t}{T}\right]}{1-\left(\frac{4\alpha t}{T}\right)^2}$$

where $T = 1/2B_0$ is the symbol period and α is the roll-off factor. Obtain a discrete-time representation of this impulse response by sampling it at $t = 0.1nT$ for integer n such that $-3T < t < 3T$. Numerically approximate match filtering (e.g. with Matlab) by performing the discrete-time convolution

$$q_k = 0.1\sum_{n=-60}^{60} g_n g_{k-n}$$

where $g_n = g(0.1nT)$. What is the value of $q_k = q(0.1kT)$ for $k = \pm 20, \pm 10$, and 0?

Solution

A Matlab script for this problem is shown below. Note the starting time of -3.01 is used to avoid divide-by-zero problems. Using the *filter* function is just one way the discrete convolution can be performed.

```
alpha = 0.5;
B0    = 0.5;
T     = 1/(2*B0);

t = [-3.01: 0.1 :3] * T;

%-- root raised cosine impulse response
g = cos( (1+alpha)*pi*t/T) + (T/4/alpha) ./ t .* sin( (1-
alpha)*pi*t/T);
g = g ./ (1 - (4*alpha*t/T).^2 );
g = 4*alpha/pi * g;

%--- discrete convolution -----
q = 0.1*filter(g,1, [g zeros(1,60)]);
tp = [-6.01:0.1:6] * T;
stem(tp,q)
xlabel('Time (T)')
ylabel('Amplitude')
axis([-4 4 -.2 1]), grid on
```

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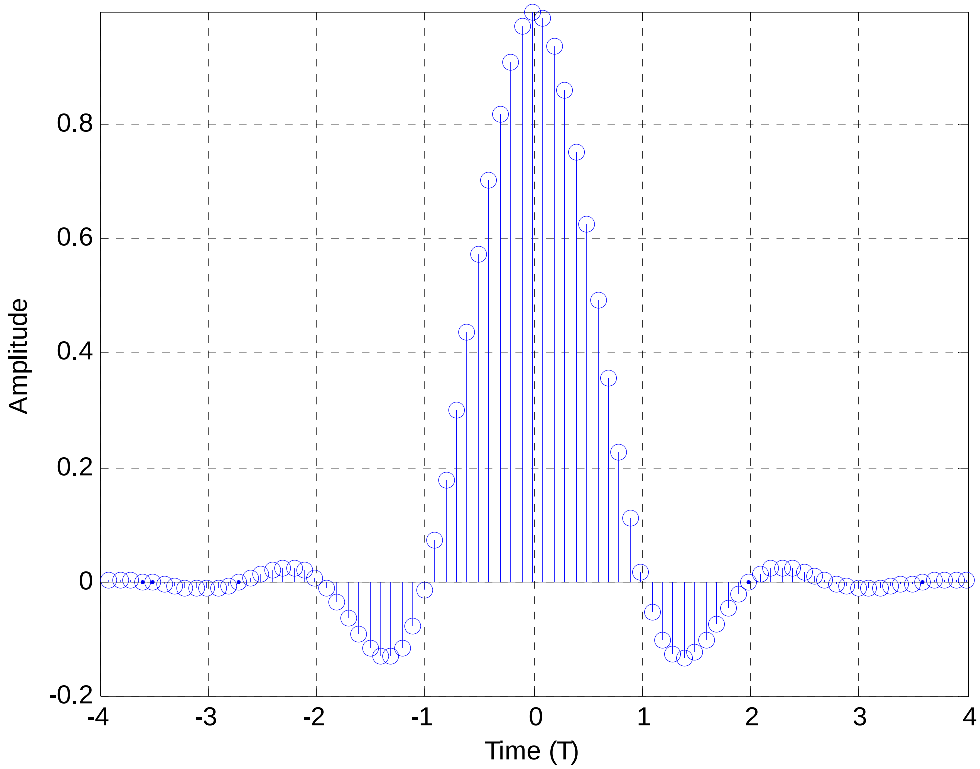
Problem 10.12 continued

The plot of q_k is shown below for $\alpha = 0.5$. At $k = \pm 20$ and ± 10, the amplitude is approximately zero. At $k = 0$ the amplitude is 1.



Problem 10.13. Determine the discrete-time autocorrelation function of the noise sequence $\{N_k\}$ defined by Eq. (10.34)

$$N_k = \int_{-\infty}^{\infty} p(kT - t)w(t)dt$$

where $w(t)$ is a white Gaussian noise process and the pulse $p(t)$ corresponds to a root-raised cosine spectrum. How are the noise samples corresponding to adjacent bit intervals related?

Solution

The autocorrelation function of the noise at samples spaced by T is

$$\begin{aligned}
R_N(n) &= \mathbf{E}\left[N_k N_{k+n}\right] \\
&= \mathbf{E}\left[\int_{-\infty}^{\infty} p(kT-t)w(t)dt \cdot \int_{-\infty}^{\infty} p((k+n)T-s)w(s)ds\right] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(kT-t)\,p((k+n)T-s)\mathbf{E}\left[w(t)w(s)\right]dtds \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(kT-t)\,p((k+n)T-s)\frac{N_0}{2}\delta(t-s)dtds
\end{aligned}$$

where we have interchanged integration and expectation on the third line, and the fourth line follows from the uncorrelated properties of the white noise. We next apply the sifting property of the delta function to obtain

$$\begin{aligned}
R_N(n) &= \int_{-\infty}^{\infty} p(kT-t)\,p((k+n)T-t)\frac{N_0}{2}dt \\
&= \frac{N_0}{2}\int_{-\infty}^{\infty} p(kT-t)\,p(t-(k+n)T)dt \\
&= \frac{N_0}{2}\delta(n)
\end{aligned}$$

where the second line follows from the even symmetry property of the raised cosine pulse, and third line follows from Eq. (10.32). Therefore, noise samples corresponding to adjacent bit intervals are not correlated.

Problem 10.14. Draw the Gray-encoded constellation (signal-space diagram) for 16-QAM and for 64-QAM. Can you suggest a constellation for 32-QAM?

Solution

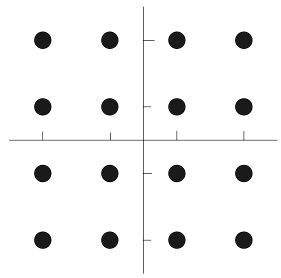
A general hint for Gray encoding is to

(a) first Gray encode two bits and assign one pair of the resulting encoding to each quadrant.

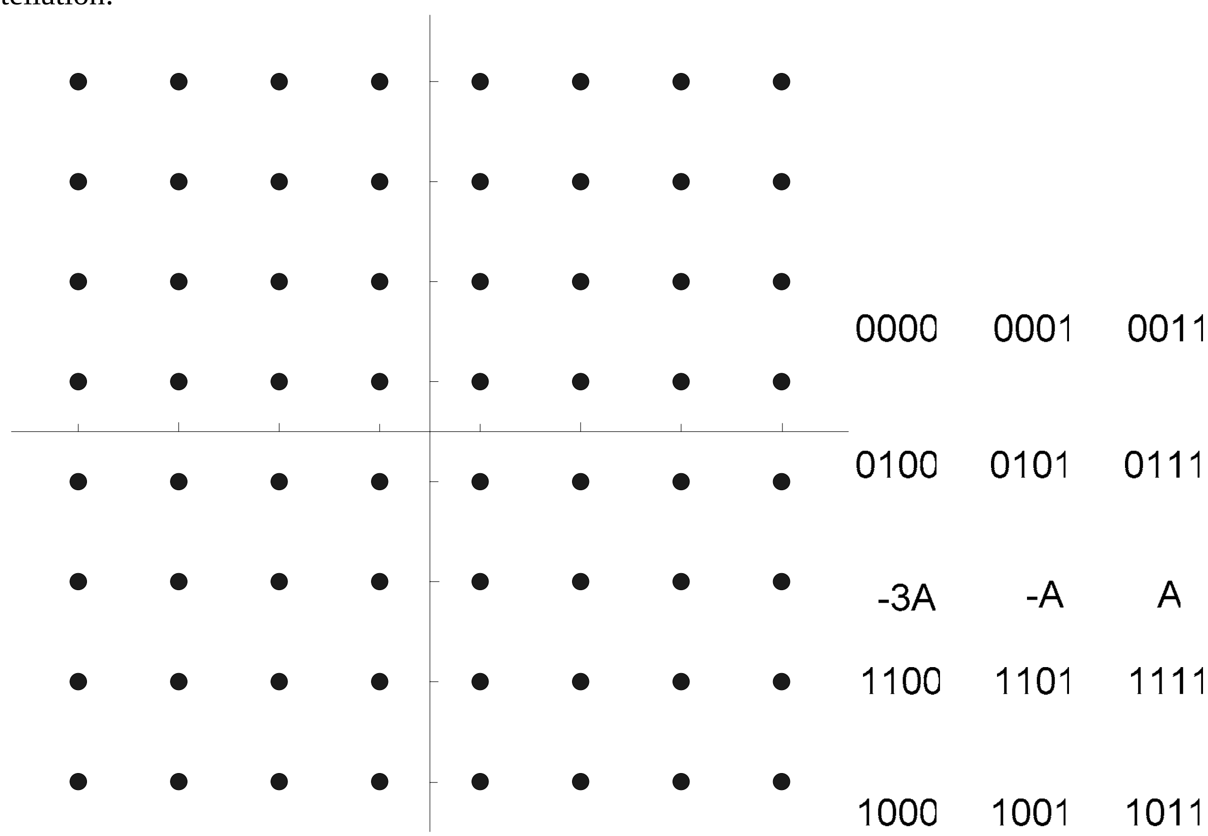
(b) Gray encode the remaining bits within one of the quadrants.

(c) obtain the Gray encodings for the remaining quadrants by reflecting the result across the in-phase and quadrature axes.

16-QAM constellation:



64-QAM constellation:



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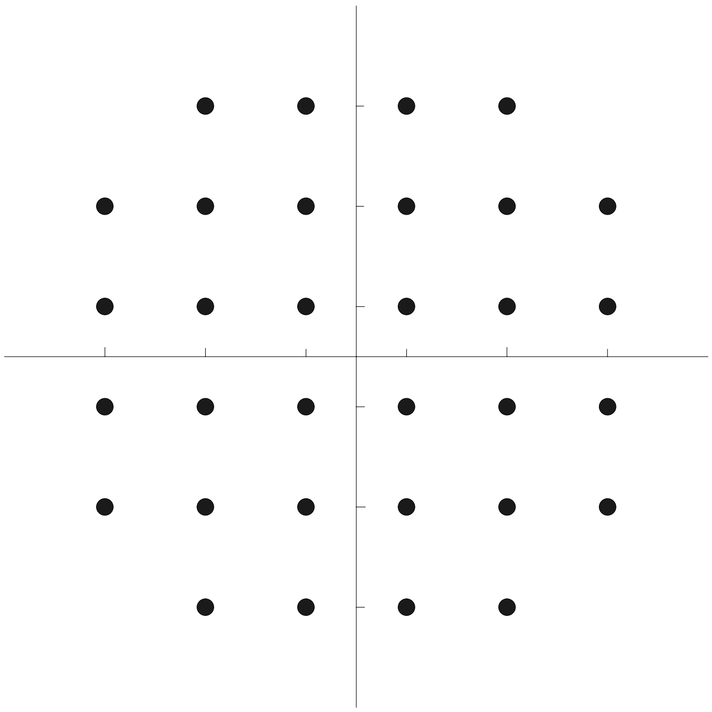
Problem 10.14 continued

32-QAM constellation: (There does not appear to be a Gray encoding for 32-QAM)



10000 10001 1

10101 00000 00001 0

10100 00100 00101 0

-5A -3A -A

11100 01100 01101 0

11101

01000 01001 0

11000 11001 1

Problem 10.15. Write the defining equation for a QAM-modulated signal. Based on the discussion of QPSK and multi-level PAM, draw the block diagram for a coherent QAM receiver.

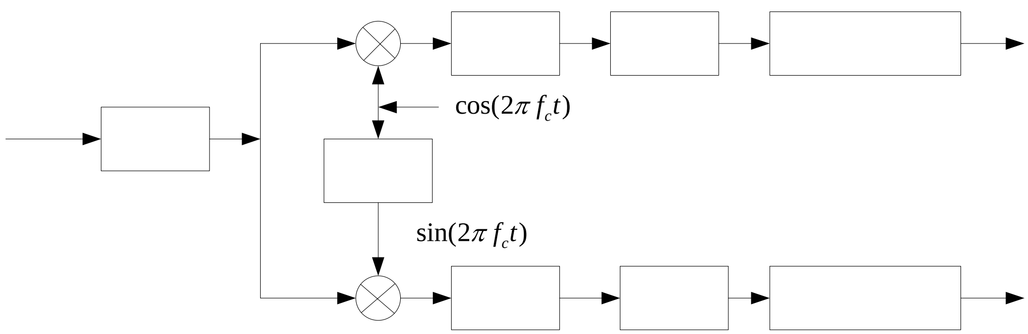
<u>Solution</u>

The QAM modulated signal can be defined as:

$$s(t) = \sum_k \left[b_{kI} h(t - kT)\cos(2\pi f_c t) + b_{kQ} h(t - kT)\sin(2\pi f_c t) \right],$$

where b_{kI}, b_{kQ} are different modulation levels on the *I* and *Q* channels, respectively. *T* is the QAM symbol duration, $h(t)$ is the pulse shape and is nonzero during $0 \le t < T$, and f_c is the carrier frequency.

The block diagram for a coherent QAM receiver is



Problem 10.16. Show that if T is a multiple of the period of f_c, then the terms $\sin(2\pi f_c t)$ and $\cos(2\pi f_c t)$ are orthogonal over the interval $\left[t_0, T+t_0\right]$.

Solution

$$\int_{t_0}^{T+t_0} \sin(2\pi f_c t)\cos(2\pi f_c t)dt = \int_{t_0}^{T+t_0} \frac{1}{2}\sin(4\pi f_c t)dt$$

$$= \frac{1}{8\pi f_c}\left[-\cos(4\pi f_c t)\right]\Big|_{t_0}^{T+t_0}$$

$$= -\frac{1}{8\pi f_c}\left[\cos(4\pi f_c(t_0+T)) - \cos(4\pi f_c t_0)\right]$$

$$= \frac{-1}{4\pi f_c}\sin(4\pi f_c t_0 + 2\pi f_c T)\cdot\sin(2\pi f_c T)$$

where we have used the equivalence $\cos A - \cos B = 2\sin[(A+B)/2]\sin[(B-A)/2)]$. If T is a multiple of the period of f_c, then $f_c T$ = integer, and $\sin(2\pi f_c T) = 0$.

Therefore, $\int_{t_0}^{t_0+T} \sin(2\pi f_c t)\cos(2\pi f_c t)dt = 0$. That is, $\sin(2\pi f_c t)$ and $\cos(2\pi f_c t)$ are orthogonal over the interval $[t_0, t_0+T]$.

Problem 10.17. For a rectangular pulse shape, by how much does null-to-null transmission bandwidth increase, if the transmission rate is increased by a factor of three?

Solution

Without loss of generality, consider the baseband BPSK signal:

$$s(t) = \sum_k b_k h(t - kT),$$

where T is the symbol duration, b_k = +1 or -1 for transmitted 1 or 0, respectively. The pulse $h(t)$ is rectangular,

$$h(t) = \text{rect}\left(\frac{t - T/2}{T}\right).$$

The Fourier transform $H(f)$ of $h(t)$ is

$$\begin{aligned} H(f) &= T\text{sinc}(fT) \cdot e^{-j2\pi fT/2} \\ &= T\,\frac{\sin(\pi fT)}{\pi fT} e^{-j\pi fT} \end{aligned}.$$

Inspecting a plot of the sinc function, we see the null-to-null transmission bandwidth of $H(f)$ is $B = 2/T$. When the transmission rate is increased by a factor three, we have the new symbol duration $T' = T/3$. The null-to-null bandwidth $B' = 2/T' = 3B$, increased by a factor of 3.

Problem 10.18. Under the bandpass assumptions, determine the conditions under which the two signals $\cos(2\pi f_0 t)$ and $\cos(2\pi f_1 t)$ are orthogonal over the interval from 0 to *T*.

Solution

For two signals to be orthogonal over the interval from 0 to *T*, they must satisfy

$$\int_0^T \cos(2\pi f_0 t)\cos(2\pi f_1 t)dt = 0 .$$

To verify this we perform the integration as follows:

$$\begin{aligned}\int_0^T \cos(2\pi f_0 t)\cos(2\pi f_1 t)dt &= \frac{1}{2}\int_0^T \left[\cos(2\pi(f_0+f_1)t)+\cos(2\pi(f_0-f_1)t)\right]dt \\ &= \frac{1}{4\pi(f_0+f_1)}\sin(2\pi(f_0+f_1))\Big|_0^T + \frac{1}{4\pi(f_0-f_1)}\sin(2\pi(f_0-f_1))\Big|_0^T \\ &= \frac{1}{4\pi(f_0+f_1)}\sin(2\pi(f_0+f_1)T) + \frac{1}{4\pi(f_0-f_1)}\sin(2\pi(f_0-f_1)T)\end{aligned}$$

By the bandpass assumption $(f_0+f_1) >> 1$ so the first term in the last line is negligible. For the second term to be zero it must satisfy

$$2\pi(f_0 - f_1)T = n\pi$$

where *n* is an integer. This implies that $(f_0 - f_1) = n/2T$.

Problem 10.19. Encode the sequence 1101 with a Hamming (7,4) block code.

Solution

Coded bit sequence $\mathbf{c} = \mathbf{x} \cdot \mathbf{G}$, where $\mathbf{G}$ is defined by (10.89).

$$\mathbf{c} = [1101] \cdot \begin{bmatrix} 1000101 \\ 0100111 \\ 0010110 \\ 0001011 \end{bmatrix}$$
$$= [1101001]$$

Problem 10.20. The Hamming (7,4) encoded sequence 1001000 was received. If the number of transmission errors is less than two, what was the transmitted sequence?

Solution

The syndrome of the received sequence is $\mathbf{S} = \mathbf{R}\cdot\mathbf{H}$ where **H** is defined by (10.92).

$$\mathbf{S} = \mathbf{R}\cdot\mathbf{H}$$

$$= [1001000]\cdot\begin{bmatrix} 101 \\ 111 \\ 110 \\ 011 \\ 100 \\ 010 \\ 001 \end{bmatrix}$$

$$= [110]$$

Based on Table 10.4, the error vector **E** = [0010000]. The transmitted sequence is $\mathbf{E} \oplus \mathbf{R}$ = [1011000].

Problem 10.21. A Hamming (15,11) block code is applied to a BPSK transmission scheme. Compare the block error rate performance of the uncoded and coded systems. Explain how this would differ if the modulation strategy was QPSK.

Solution

1) For the uncoded system, the probability of a bit error with BPSK is

$$P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

The probability of a block error with block length of 15 bits, assuming independent errors is:

$$P_b^{uncoded} = 1-(1-P_e)^{15}$$

2) For the coded system, with a (15,11) Hamming code, the probability of block error is

$$P_b^{coded} = 1-(1-P_e')^{15} - \binom{15}{1}(1-P_e')^{14}P_e',$$

where P_e' is the bit error probability of coded bit, since the code can correct a single bit error. The probability of bit error in this case is:

$$P_e' = Q\left(\sqrt{\frac{2E_c}{N_0}}\right),$$

where E_c is the coded bit energy, and E_c = 11/15E_b. Therefore

$$P_e' = Q\left(\sqrt{\frac{22E_b}{15N_0}}\right)$$

To compare the block error probabilities of uncoded and coded systems, we use Matlab to plot the block error rate curves for $P_b^{uncoded}$ and P_b^{coded} versus E_b/N_0 (dB), as shown below

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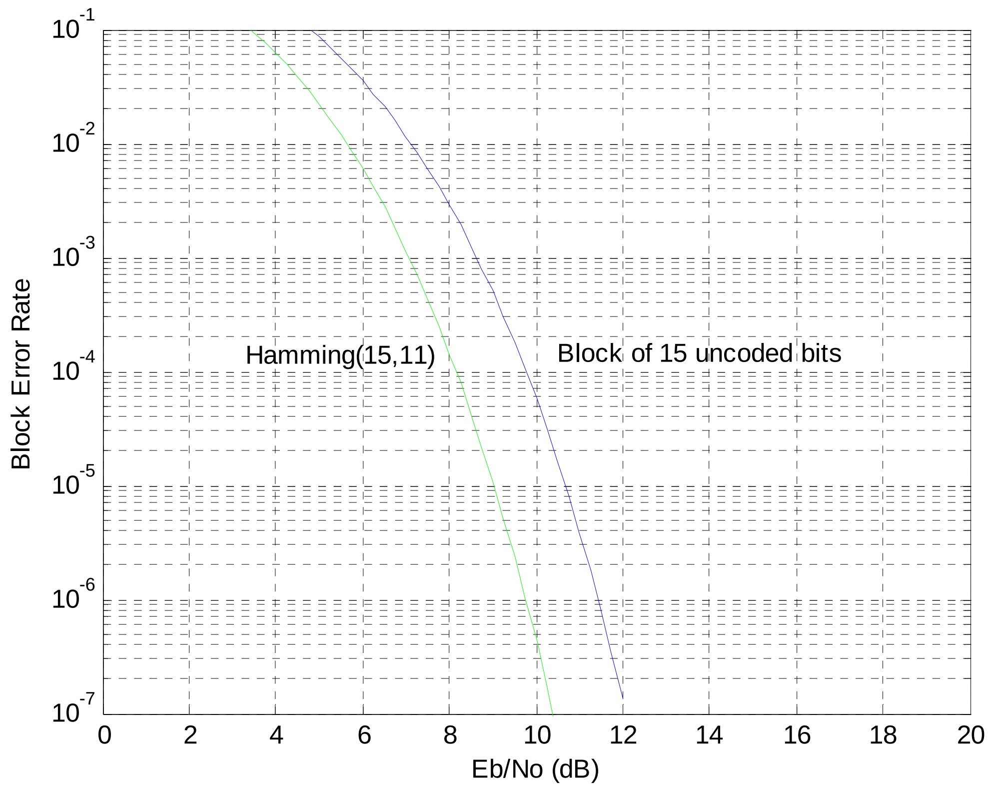
Problem 10.21 continued



The Matlab script that generates the above plot is

```
EbNodB=[0:0.25:12];
EbNo     = 10.^(EbNodB/10);
Pe       = 0.5*erfc(sqrt(EbNo));
Puncoded = 1 - (1-Pe).^15;
EcNo     = 11/15 * EbNo;
Peprime  = 0.5*erfc(sqrt(EcNo));
Pcoded   = 1 - (1-Peprime).^15 - 15*(1-Peprime).^14.*Peprime;
semilogy(EbNodB,Puncoded)
grid
xlabel('Eb/No (dB)')
ylabel('Block Error Rate')
axis([0 20 1E-7 0.1])
hold on, semilogy(EbNodB,Pcoded,'g'), hold off
```

3) Since for QPSK modulation, bit error probabilities of uncoded bits P_e and coded bits $P_e^{'}$ are unchanged compared with BPSK modulation, the block error probabilities of two systems are also the same as those of BPSK modulation.

Problem 10.22. Show that the choice $\gamma = \mu / 2$ minimizes the probability of error given by Eq. (10.26). Hint: The *Q*-function is continuously differentiable.

Solution

From (10.26), we have the average probability of error as:

$$P_e(\gamma) = \frac{1}{2} Q\left(\frac{\mu - \gamma}{\sigma}\right) + \frac{1}{2} Q\left(\frac{\gamma}{\sigma}\right)$$

Recall the definition of *Q*-function:

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{+\infty} \exp(-s^2/2) ds$$

$$(\text{let } u = -s)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-x} \exp(-u^2/2) du$$

So the derivative is given by

$$\frac{dQ(x)}{dx} = \frac{-1}{\sqrt{2\pi}} \exp(-x^2/2) \le 0$$

Substituting this result into the definition of $P_e(\gamma)$ we obtain

$$\frac{dP_e(\gamma)}{d\gamma} = \frac{1}{2} \cdot \frac{-1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{(\mu-\gamma)^2}{2\sigma^2}\right) \cdot \frac{-1}{\sigma} + \frac{1}{2} \cdot \frac{-1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{\gamma^2}{2\sigma^2}\right) \cdot \frac{1}{\sigma}$$

$$= \frac{1}{2\sqrt{2\pi}\sigma} \left\{ \exp\left(-\frac{(\mu-\gamma)^2}{2\sigma^2}\right) - \exp\left(-\frac{\gamma^2}{2\sigma^2}\right) \right\}$$

Setting $\frac{dP_e(\gamma)}{d\gamma} = 0$ implies

$$\exp\left(-\frac{(\mu-\gamma)^2}{2\sigma^2}\right) = \exp\left(-\frac{\gamma^2}{2\sigma^2}\right)$$

$$(\mu-\gamma)^2 = \gamma^2$$

$$\gamma = \mu/2$$

Continued on next slide

Problem 10.22 continued

Checking the second derivative, we have

$$\frac{d^2P_e(\gamma)}{d^2\gamma} = \frac{1}{2\sqrt{2\pi}\sigma}\left[\frac{2(\mu-\gamma)}{2\sigma^2}\cdot\exp\left(-\frac{(\mu-\gamma)^2}{2\sigma^2}\right)+\frac{2\gamma}{2\sigma^2}\exp\left(-\frac{\gamma^2}{2\sigma^2}\right)\right]$$

$$> 0$$

when $\gamma = \mu/2$. Therefore at $\gamma = \mu/2$, $P_e(\gamma)$ has a minimum value.

Problem 10.23. For *M*-ary PAM,

(a) Show that the formula for probability of error, namely,

$$P_e = 2\left(\frac{M-1}{M}\right)Q\left(\frac{A}{\sigma}\right)$$

holds for $M = 2$, 3, and 4. By mathematical induction, show that it holds for all *M*.

(b) Show the formula for average power, namely,

$$P = \frac{(M^2-1)A^2}{3}$$

holds for $M = 2$, and 3. Show it holds for all *M*.

<u>Solution</u>

(a) *M*-ary PAM with the separation between nearest neighbours as 2A. Assume that all *M* symbols are equally transmitted.

(i) For M=2, we have the result given in the text for binary PAM

$$\begin{aligned} P_e^{2PAM} &= Q\left(\frac{A}{\sigma}\right) \\ &= 2\frac{M-1}{M}Q\left(\frac{A}{\sigma}\right) \end{aligned}$$

for $M = 2$.

(ii) For $M = 3$, the constellation is: 

$$\begin{aligned} P_e &= \frac{1}{3}P\left[y > -A \,|\, (-2A) \text{ is transmitted}\right] + \frac{1}{3}P\left[y > A \text{ or } y < -A \,|\, 0 \text{ is transmitted}\right] \\ &\quad + \frac{1}{3}P\left[y < A \,|\, (2A) \text{ is transmitted}\right] \\ &= \frac{1}{3}\int_{-A}^{\infty}\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(y+2A)^2}{2\sigma^2}\right)dy + \frac{1}{3}\int_{A}^{\infty}\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{y^2}{2\sigma^2}\right)dy \\ &\quad + \frac{1}{3}\int_{-\infty}^{-A}\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{y^2}{2\sigma^2}\right)dy + \frac{1}{3}\int_{-\infty}^{A}\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(y-2A)^2}{2\sigma^2}\right)dy \\ &= \frac{4}{3}Q\left(\frac{A}{\sigma}\right) \end{aligned}$$

Continued on next slide

Problem 10.23 continued

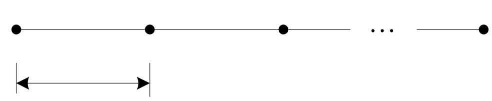
From the formula $P_e = \frac{2(M-1)}{M} Q\left(\frac{A}{\sigma}\right)$, when M=3, $P_e = \frac{4}{3} Q\left(\frac{A}{\sigma}\right)$. Thus the formula $P_e = \frac{2(M-1)}{M} Q\left(\frac{A}{\sigma}\right)$ holds for M = 3.

(iii) For M = 4, the constellation is:

$$
\begin{aligned}
P_e &= \frac{1}{4} P\left[y > -2A \,|\, (-3A) \text{ is transmitted}\right] + \frac{1}{4} P\left[y < -2A \text{ or } y > 0 \,|\, -A \text{ is transmitted}\right] \\
&\quad + \frac{1}{4} P\left[y < 0 \text{ or } y > 2A \,|\, +A \text{ is transmitted}\right] + \frac{1}{4} P\left[y < 2A \,|\, (3A) \text{ is transmitted}\right] \\
&= \frac{1}{4} \int_{-2A}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(y+3A)^2}{2\sigma^2}\right) dy \\
&\quad + 2\left[\frac{1}{4} \int_{0}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\left(y+A\right)^2}{2\sigma^2}\right) dy + \frac{1}{4} \int_{-\infty}^{-2A} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\left(y+A\right)^2}{2\sigma^2}\right) dy\right] \\
&\quad + \frac{1}{4} \int_{-\infty}^{A} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(y-3A)^2}{2\sigma^2}\right) dy \\
&= \frac{6}{4} Q\left(\frac{A}{\sigma}\right)
\end{aligned}
$$

where the factor 2 in the third last line, comes from the symmetry of the second and third terms of the first equation. From the formula $P_e = \frac{2(M-1)}{M} Q\left(\frac{A}{\sigma}\right)$, when M=4, $P_e = \frac{6}{4} Q\left(\frac{A}{\sigma}\right)$. Thus the formula $P_e = \frac{2(M-1)}{M} Q\left(\frac{A}{\sigma}\right)$ holds for M = 4.

(iv) Assume that the formula of P_e holds for (M-1)-ary PAM. By mathematical induction, we need to show it also holds for M-ary PAM. The (M-1)-ary PAM constellation may be illustrated as shown:



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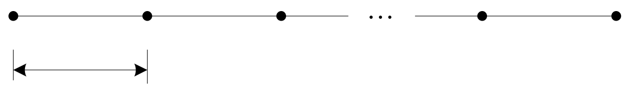
Problem 10.23 continued

By adding one point P2 on the (M-1)-ary PAM constellation, which has the distance 2A from point P1, we obtain M-ary PAM constellation as follows (in practice, the average or dc level may be adjusted as well but this has no effect on the symbol error rate):



Since error probabilities of P1 symbol on the (M-1)-ary PAM is the same as that of P2 point on the M-ary PAM, the error probability of M-ary PAM is

$$P_e^{M-ary} = \frac{M-1}{M} P_e^{(M-1)-ary} + \frac{1}{M} \cdot \text{symbol error prob. of P1 symbol on } M\text{-ary} \quad (1)$$

where 1/M is the probability that P1 is transmitted and (M-1)/M is the probability that one of the other constellation points is transmitted. The probability of error formula for (M-1)-ary PAM is given by

$$P_e^{(M-1)-ary} = 2\frac{(M-2)}{(M-1)} Q\left(\frac{A}{\sigma}\right). \quad (2)$$

2A

The symbol error rate of P1 symbol on M-ary PAM is

$$P_{P1} = \frac{1}{M} P\left[y < (\mu - A), \text{or } y > (\mu + A) \mid \text{P1 is transmitted}\right]$$

where μ is the signal level of P1 symbol.

$$\begin{aligned} P_{P1} &= \frac{1}{M}\int_{-\infty}^{\mu-A} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right) dy + \frac{1}{M}\int_{\mu+A}^{+\infty} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right) dy \\ &= \frac{2}{M} Q\left(\frac{A}{\sigma}\right) \end{aligned} \quad (3)$$

Substituting Eqs. (2) and (3) into (1), we obtain the symbol error probability of M-ary PAM

$$\begin{aligned} P_e^{M-PAM} &= \frac{M-1}{M} \cdot 2 \cdot \frac{M-2}{M-1} Q\left(\frac{A}{\sigma}\right) + \frac{2}{M} Q\left(\frac{A}{\sigma}\right) \\ &= 2\frac{M-1}{M} Q\left(\frac{A}{\sigma}\right) \end{aligned}.$$

Continued on next slide

Problem 10.23 continued

The formula holds for *M*-ary PAM. Therefore, by mathematical induction, the formula holds for all *M*.

(b) To compute the average symbol power we note:

i) For $M = 2$, the average symbol power is A^2 and the formula $P = \frac{(M^2-1)A^2}{3}$ holds for M=2.

ii) For $M = 3$, the average symbol energy is

$$P = \frac{1}{3}\left((2A)^2 + 0^2 + (2A)^2\right) = \frac{8}{3}A^2 .$$

The formula $P = \frac{(M^2-1)A^2}{3}$ holds for M=3.

iii) For general even *M*, the *M*-ary PAM constellation points are

$$\left\{-(M-1)A, \cdots, -3A, -A, A, 3A, \cdots, (M-1)A\right\} .$$

The average symbol energy is

$$\begin{aligned} P &= \frac{2\left[(M-1)^2 + (M-3)^2 + \cdots + 3^2 + 1\right]}{M} A^2 \\ &= \frac{2A^2}{M} \sum_{k=1}^{M/2} (2k-1)^2 \\ &= \frac{2A^2}{M} \left[2^2 \sum_{k=1}^{M/2} k^2 - 4\sum_{k=1}^{M/2} k + \sum_{k=1}^{M/2} 1 \right] \\ &= \frac{2A^2}{M} \left[4\frac{M(M/2+1)(M+1)}{2\cdot 6} - 4\frac{M(M/2+1)}{2\cdot 2} + \frac{M}{2} \right] \\ &= \frac{(M^2-1)A^2}{3} \end{aligned} .$$

where we have used the summation formulas of Appendix 6.

iv) For general odd *M*, the *M*-ary PAM constellation points are

$$\left\{-(M-1)A, \cdots, -2A, 0, 2A, \cdots (M-1)A\right\} .$$

Continued on next slide

Problem 10.23 continued

The average symbol energy is

$$
\begin{aligned}
P &= \frac{2\left[(M-1)^2 + (M-3)^2 + \cdots + 2^2\right]}{M} A^2 \\
&= \frac{2A^2}{M} 2^2 \left[\left(\frac{M-1}{2}\right)^2 + \left(\frac{M-3}{2}\right)^2 + \ldots + 1^2\right] \\
&= \frac{8A^2}{M} \sum_{k=1}^{(M-1)/2} k^2 \\
&= \frac{8A^2}{M} \frac{(M-1)(M+1)(M)}{2 \cdot 2 \cdot 6} \\
&= \frac{(M^2-1)A^2}{3}
\end{aligned}
$$

where the fourth line uses the summation formula found in Appendix 6.

Problem 10.24. Consider binary FSK transmission where $(f_1 - f_2)T$ is not an integer.

(a) What is the mean output of the upper correlator of Fig. 10.12, if a 1 is transmitted? What is the mean output of the lower correlator?

(b) Are the random variables N_1 and N_2 independent under these conditions? What is the variance of $N_1 - N_2$?

(c) Describe the properties of the random variable D of Fig. 10.12 in this case.

Solution:

(a) If a 1 is transmitted,

$$r(t) = A_c \cos(2\pi f_1 t) + n(t)$$

where $n(t)$ is a narrow band Gaussian noise. The output of the upper correlator is Y_1:

$$\begin{aligned} Y_1 &= \int_0^T r(t)\sqrt{2}\cos(2\pi f_1 t)dt \\ &= \int_0^T \sqrt{2}A_c \cos(2\pi f_1 t)\cos(2\pi f_1 t)dt + \int_0^T \sqrt{2}n(t)\cos(2\pi f_1 t)dt \\ &\cong \frac{1}{\sqrt{2}} A_c T + \int_0^T \sqrt{2}n(t)\cos(2\pi f_1 t)dt \end{aligned}$$

The expected value of Y_1 is $\mathbf{E}[Y_1] = \frac{1}{2} A_c T$, since $n(t)$ has zero mean.

The output of the lower correlator is Y_2:

$$\begin{aligned} Y_2 &= \int_0^T r(t)\sqrt{2}\cos(2\pi f_2 t)dt \\ &= \int_0^T \sqrt{2}A_c \cos(2\pi f_1 t)\cos(2\pi f_2 t)dt + \int_0^T \sqrt{2}n(t)\cos(2\pi f_2 t)dt \\ &= \frac{A_c}{\sqrt{2}}\int_0^T \cos(2\pi(f_1 + f_2)t)dt + \frac{A_c}{\sqrt{2}}\int_0^T \cos(2\pi(f_1 - f_2)t)dt + \sqrt{2}\int_0^T n(t)\cos(2\pi f_2 t)dt \\ &\cong \frac{A_c}{\sqrt{2}}\int_0^T \cos(2\pi(f_1 - f_2)t)dt + \int_0^T \sqrt{2}n(t)\cos(2\pi f_2 t)dt \end{aligned}$$

Continued on next slide

Problem 10.24 continued

where the first term of the third line is negligible due to the bandpass assumption. The expected value of Y_2 is

$$\begin{aligned}\mathbf{E}[Y_2] &= \frac{A_c}{\sqrt{2}}\int_0^T \cos(2\pi(f_1 - f_2)t)dt \\ &= \frac{A_c}{\sqrt{2}}\cdot\frac{1}{2\pi\left(f_1 - f_2\right)}\sin\left[2\pi\left(f_1 - f_2\right)t\right]\Big|_0^T \\ &= \frac{A_c}{2\sqrt{2}\pi\left(f_1 - f_2\right)}\sin(2\pi\left(f_1 - f_2\right)T)\end{aligned}$$

which clearly differs from the orthogonal case.

(b) The random variables N_1 and N_2 are given by

$$N_1 = \int_0^T \sqrt{2}n(t)\cos(2\pi f_1 t)dt$$

$$N_2 = \int_0^T \sqrt{2}n(t)\cos(2\pi f_2 t)dt$$

Since $n(t)$ is a Gaussian process, both N_1 and N_2 are Gaussian. To show N_1 and N_2 are correlated consider

$$\begin{aligned}\mathbf{E}[N_1 N_2] &= \mathbf{E}\left[\int_0^T n(t)\cos(2\pi f_1 t)dt\cdot\int_0^T n(\tau)\cos(2\pi f_2\tau)d\tau\right] \\ &= \int_0^T\int_0^T \mathbf{E}[n(t)n(\tau)]\cos(2\pi f_1 t)\cos(2\pi f_2\tau)dtd\tau \\ &= \int_0^T\int_0^T \frac{N_0}{2}\delta(t-\tau)\cos(2\pi f_1 t)\cos(2\pi f_2\tau)dtd\tau \\ &= \frac{N_0}{2}\int_0^T \cos(2\pi f_1 t)\cos(2\pi f_2 t)dt \\ &= \frac{N_0}{4}\int_0^T \left[\cos(2\pi(f_1 + f_2)t) + \cos(2\pi(f_1 - f_2)t)\right]dt \\ &= \frac{N_0}{4}\frac{\sin\left(2\pi(f_1 + f_2)t\right)}{2\pi(f_1 + f_2)}\Big|_0^T + \frac{\sin\left(2\pi(f_1 - f_2)t\right)}{2\pi(f_1 - f_2)}\Big|_0^T \\ &\cong \frac{N_0}{4}\operatorname{sinc}(2(f_1 - f_2)T)\end{aligned}$$

Continued on next slide

Problem 10.24 continued

where the first term of the second last line is assumed negligible due to the bandpass assumption. Since N_1 and N_2 are correlated, they are not independent. The variance of (N_1-N_2) is

$$\begin{aligned}\text{var}[N_1 - N_2] &= \text{var}[N_1] + \text{var}[N_2] - 2\mathbf{E}[N_1 N_2] \\ &= N_0 - \frac{N_0}{2}\text{sinc}\left(2(f_1 - f_2)T\right)\end{aligned}$$

(c) The random variable D is Gaussian with zero mean and variance var[N_1-N_2].

Problem 10.25. Show that the noise variance of the in-phase component $n_I(t)$ of the band-pass noise is the same as the band-pass noise $n(t)$ variance; that is, for a band-pass noise bandwidth B_N

$$\mathbf{E}\left[n_I^2(t)\right] = N_0 B_N$$

Solution

Recall the spectra of narrowband noise $n(t)$ and its in-phase component $n_I(t)$ shown in Figure 8.23. The variance of a random process $x(t) = R_x(0) = \int_{-\infty}^{\infty} X(f)df$, where $X(f)$ is the power spectral density of $x(t)$. Therefore,

$$\begin{aligned}
\text{Var}[n(t)] &= \mathbf{E}\left[n^2(t)\right] \\
&= \int_{-\infty}^{\infty} S_N(f)df \\
&= 2 \cdot \frac{N_0}{2} \cdot 2B \\
&= N_0 \cdot 2B
\end{aligned}$$

Where we have used the fact that for a bandpass signal $B_T = 2B$, that is twice the lowpass bandwidth. Similarly, the variance of the in-phase noise is

$$\begin{aligned}
\text{Var}[n_I(t)] &= \mathbf{E}\left[n_I^2(t)\right] \\
&= \int_{-\infty}^{\infty} S_{n_I}(f)df \\
&= N_0 \cdot 2B
\end{aligned}$$

Problem 10.26 In this problem, we investigate the effects when transmit and receive filters do not combine to form an ISI-free pulse shape. To be specific, data is transmitted at baseband using binary PAM with an exponential pulse shape $g(t)=\exp(-t/T)u(t)$ where T is the symbol period (see Example 2.2). The receiver detects the data using an integrate-and-dump detector.

(a) With data represented as ±1, what is magnitude of the signal component at the output of the detector.

(b) What is the worst case magnitude of the intersymbol interference at the output of the detector. (Assume the data stream has infinite length.) Using the value obtained in part (a) as a reference, by what percentage is the eye opening reduced by this interference.

(c) What is the rms magnitude of the intersymbol interference at the output of the detector? If this interference is treated as equivalent to noise, what is the equivalent signal-to-noise ratio at the output of the detector? Comment on how this would affect bit error rate performance of this system when there is also receiver noise present.

(Typo in problem statement, there should be minus sign in exponential.)

Solution

(a) For a data pulse

$$g(t) = A\exp(-t/T)u(t)$$

where A is the binary PAM symbol (±1). The desired output of an integrate-and-dump filter in the n^{th} symbol period is

$$\begin{aligned} G_n &= \int_{nT}^{(n+1)T} g(t-nT)dt \\ &= \int_0^T A_n \exp(-t/T)dt \\ &= A_n T\left(1-\exp(-1)\right) \end{aligned}$$

If the data is either ±1, then magnitude of the output is $T(1\text{-}e^{-1})$.

(b) In the n^{th} symbol period the received signal is

$$y(t) = \sum_{k=-\infty}^{\infty} A_n g(t-kT)$$

Continued on next slide

Problem 10.26 continued

The output of the detection filter in the n^{th} symbol period is

$$\begin{aligned}
Y_n &= \int_{nT}^{(n+1)T} y(t)dt \\
&= \int_{nT}^{(n+1)T} \sum_{k=-\infty}^{\infty} A_k \exp\left(-(t-kT)/T\right)dt \\
&= \int_{nT}^{(n+1)T} A_n \exp\left(-(t-nT)/T\right)dt + \sum_{k=1}^{\infty} \int_{nT}^{(n+1)T} A_{n-k} \exp\left\{-(t-(n-k)T)/T\right\} dt
\end{aligned}$$

where, due to the causality of the pulse shape, the symbols A_{n+1} and later due not cause intersymbol interference into symbol A_n. The first term in the above is the desired signal and the second term is the intersymbol interference. By letting $s = t - (n\text{-}k)T$, we can express this interference as

$$\begin{aligned}
J_n &= \sum_{k=1}^{\infty} \int_{kT}^{(k+1)T} A_{n-k} \exp\left(-t/T\right) dt \\
&= \sum_{k=1}^{\infty} A_{n-k} T\left(\exp(-k) - \exp(-(k+1))\right)
\end{aligned}$$

where each term in the summation corresponds to the interference caused by a previous symbol. For worst case interference we assume that all of the A_{n-k} have the same sign. Then this worst case interference is given by

$$\begin{aligned}
J_n &= \sum_{k=1}^{\infty} A_{n-k} T\left(\exp(-k) - \exp(-(k+1))\right) \\
&\le T\left(1-\exp(-1)\right) \sum_{k=1}^{\infty} \exp(-k)
\end{aligned}$$

To simplify the notation, we let $\alpha = \exp(-1)$. Then

$$\begin{aligned}
J_n^{\max} &= T\left(1-\alpha\right) \sum_{k=1}^{\infty} \alpha^k \\
&= T\left(1-\alpha\right) \frac{\alpha}{1-\alpha} \\
&= \alpha T
\end{aligned}$$

Comparing this worst case interference to the desired signal level G_n, the eye-opening is reduced by

$$\frac{J_n^{\max}}{G_n} \times 100 = \frac{T\alpha}{T\left(1-\alpha\right)} \times 100 = 58\%$$

Continued on next slide

Problem 10.26 continued

(c) From part (b), we found that k^{th} preceding symbol contributes an interference

$$I_n^k = A_{n-k}\left(1-\alpha\right)\alpha^k$$

The total interference is

$$\begin{aligned} J_n &= \sum_{k=1}^{\infty} I_n^k \\ &= \sum_{k=1}^{\infty} A_{n-k} T\left(1-\alpha\right)\alpha^k \end{aligned}$$

Since all symbol intervals are equivalent, we drop the subscript n on J_n. The mean value of this interference is $\mathbf{E}[J] = 0$ since $\mathbf{E}[A_{n-k}] = 0$. The variance of this interference is

$$\begin{aligned} Var(J) &= \mathbf{E}\left[J^2\right] \\ &= \sum_{k=1}^{\infty} \mathbf{E}\left[A_{n-k}^2\right] T^2\left(1-\alpha\right)^2 \alpha^{2k} \\ &= T^2(1-\alpha)^2 \frac{\alpha^2}{1-\alpha^2} \\ &= \left(\alpha T\right)^2 \frac{1-\alpha}{1+\alpha} \end{aligned}$$

where we have assumed the symbols are independent so that $\mathbf{E}[A_i A_j] = 0$ if $i \neq j$. The *rms* interference is given by the square root of the variance so

$$\begin{aligned} J_{rms} &= \alpha T \sqrt{\frac{1-\alpha}{1+\alpha}} \\ &= 0.25T \end{aligned}$$

which is clearly less than the worst case interference $J^{\max}$.

If we represent the signal power by S, the noise power by N, then the equivalent signal-to-noise ratio taking account of the intersymbol interference is

$$SNR = \frac{S}{N + J_{rms}^2}$$

The intersymbol interference will further degrade performance. In fact, if the worst case interference is large enough such that the eye closes, it will result in a lower limit on the bit error rate regardless of how little noise there is.

Problem 10.27. A BPSK signal is applied to a matched-filter receiver that lacks perfect phase synchronization with the transmitter. Specifically, it is supplied with a local carrier whose phase differs from that of the carrier used in the transmitter by ϕ radians.

(a) Determine the effect of the phase error ϕ on the average probability of error of this receiver.

(b) As a check on the formula derived in part (a), show that when the phase error is zero the formula reduces to the same form as in Eq. (10.44).

Solution

(a) With BPSK, assume the transmitted signal is (10.36):

$$s(t) = A_c \sum_{k=0}^{N} b_k h(t - kT)\cos(2\pi f_c t)\,,$$

where b_k = +1 for a 1 and b_k = -1 for a 0, $h(t)$ is the rectangular pulse $\text{rect}\left(\frac{t - T/2}{T}\right)$.

The received signal is

$$\begin{aligned} x(t) &= s(t) + n(t) \\ &= A_c \sum_{k=0}^{N} b_k h(t - kT)\cos(2\pi f_c t) + n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t) \end{aligned}$$

The receiver matched filter is the integrate-and-dump filter. The output for the k^{th} symbol after down-conversion with phase error ϕ and match filtering is:

$$\begin{aligned} Y_k &= \int_{(k-1)T}^{kT} x(t)\cos(2\pi f_c t + \phi)dt \\ &= \int_{(k-1)T}^{kT} \left[A_c b_k + n_I(t)\right]\cos(2\pi f_c t)\cos(2\pi f_c t + \phi)dt - \int_{(k-1)T}^{kT} n_Q(t)\sin(2\pi f_c t)\cos(2\pi f_c t + \phi)dt \\ &= \int_{(k-1)T}^{kT} \frac{1}{2}[A_c b_k + n_I(t)][\cos\phi + \cos(4\pi f_c t + \phi)]dt - \int_{(k-1)T}^{kT} \frac{1}{2} n_Q(t)[\sin(4\pi f_c t + \phi) + \sin(-\phi)]dt \\ &\cong \frac{T}{2} A_c b_k \cos\phi + \frac{1}{2}\int_{(k-1)T}^{kT} n_I(t)\cos\phi dt + \frac{1}{2}\int_{(k-1)T}^{kT} n_Q(t)\sin\phi dt \\ &= \frac{T}{2} A_c b_k \cos\phi + N_k \end{aligned}$$

where we define

$$N_k = \frac{1}{2}\int_{(k-1)T}^{kT} n_I(t)\cos\phi dt + \frac{1}{2}\int_{(k-1)T}^{kT} n_Q(t)\sin\phi dt$$

Continued on next slide

Problem 10.27 continued

The random variable N_k has zero mean and variance

$$\text{var}[N_k] = \cos^2\phi \frac{N_0 T}{4} + \sin^2\phi \frac{N_0 T}{4}$$
$$= \frac{N_0 T}{4} = \sigma^2$$

Let $\mu = \frac{T}{2} A_c \cos\phi$. Then the probability of bit error P_e is

$$P_e = \mathbf{P}[b_k = 1]\mathbf{P}[Y_k < 0 \mid b_k = 1] + \mathbf{P}[b_k = -1]\mathbf{P}[Y_k > 0 \mid b_k = -1]$$
$$= \frac{1}{2}\int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(y-\mu)^2}{\sigma^2}\right\} dy + \frac{1}{2}\int_{0}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(y+\mu)^2}{\sigma^2}\right\} dy$$
$$= Q\left(\frac{\mu}{\sigma}\right)$$

with $E_b = \frac{A_c^2 T}{2}, \mu = \frac{T}{2} A_c \cos\phi, \sigma = \frac{1}{2}\sqrt{N_0 T}$, we have $P_e = Q\left(\sqrt{\frac{2E_b \cos\phi}{N_0}}\right)$

(b) When the phase error ϕ=0, $P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$, as the same as Eq. (10.44).

Problem 10.28. A binary FSK system transmits data at the rate of 2.5 megabits per second. During the course of transmission, white Gaussian noise of zero mean and power spectral density 10^{-20} watts per hertz is added to the signal. In the absence of noise, the amplitude of the received signal is 1 μV across 50 ohm impedance. Determine the average probability of error assuming coherent detection of the binary FSK signal.

Solution

The average probability of error for coherent FSK is

$$P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

from Eq. (10.68). For this example, we have noise power spectral density is

$$N_0 = 2\times10^{-20}\,\text{watts}/\text{Hz}$$

and the energy per bit is

$$E_b = \frac{1}{2}\frac{A_c^2 T}{R},$$

In the text, we have nominally assumed the resistance is 1 ohm and omitted it. In this problem we use the resistance of R = 50 ohms. The symbol duration is $T = \frac{1}{2.5\times10^6}$ seconds and the amplitude of received signal is A_c = 1μV. Therefore,

$$\begin{aligned} E_b &= \frac{1}{2}\times\frac{1\times10^{-12}}{50}\times\frac{1}{2.5\times10^6} \\ &= 4\times10^{-21}\,\text{watts}/\text{Hz} \end{aligned}$$

Substituting the above values into the expression for P_e and we have the probability of error is

$$\begin{aligned} P_e &= Q(0.2) \\ &\cong 0.26 \end{aligned}$$

Problem 10.29. One of the simplest forms of forward error correction code is the repetition code. With an *N*-repetition code, the same bit is sent *N* times, and the decoder decides in favor of the bit that is detected on the majority of trials (assuming *N* is odd). For a BPSK transmission scheme, determine the BER performance of a 3-repetition code.

Solution

With 3-repetition code, the decoder will output the correct bit if there are one or fewer errors in the 3-bit code. Thus, assuming bit errors are independent, the bit error rate is

$$\begin{aligned} P_b^{coded} &= (1-P_e)^3 + \binom{3}{1} P_e (1-P_e)^2 \\ &= (1-P_e)^2 (1+2P_e) \end{aligned},$$

where P_e is the bit error rate of channel bit. With BPSK, the formula for bit error probability is

$$\begin{aligned} P_e &= Q\left(\sqrt{\frac{2E_c}{N_0}}\right) \\ &= Q\left(\sqrt{\frac{2E_b}{3N_0}}\right) \end{aligned},$$

since ratio of channel bit energy to information bit energy is given by $E_c = 1/3E_b$. Therefore, the bit error probability of the 3-repetition code is

$$P_b^{coded} = \left(1 - Q\left(\sqrt{\frac{2E_b}{3N_0}}\right)\right)^2 \left(1 + 2Q\left(\sqrt{\frac{2E_b}{3N_0}}\right)\right)$$

Problem 10.30 In this experiment, we simulate the performance of bipolar signalling in additive white Gaussian noise. The Matlab script included in Appendix 7 for this experiment:

- generates a random sequence with rectangular pulse shaping
- adds Gaussian noise
- detects the data with a simulated integrate-and-dump detector

With this Matlab script

(a) Compute the spectrum of the transmitted signal and compare to the theoretical.

(b) Explain the computation of the noise variance given an E_b/N_0 ratio.

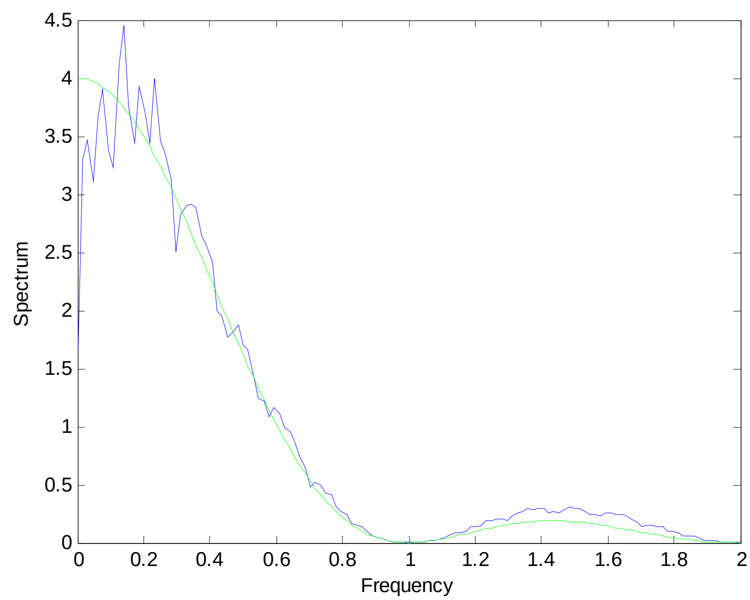
(c) Confirm the theoretically predicted bit error rate for E_b/N_0 from 0 to 10 dB.

Solution

(a) The provided script plots the simulated spectrum before noise is added. If we add the statement

*hold on, plot(F, abs(2*sinc(F)).^2,'g'), hold off*

at the same point, we obtain the following comparison graph. The two graphs agree reasonably well. There are two reasons for the differences observed with the simulated spectrum. The first is the relatively short random sequence used for generating the plot and the second is an aliasing effect.



Continued on next slide

Problem 10.30 continued

(b) The calculation of the noise variance in a discrete time simulation proceeds as follows. We are given the sampling rate F_s and the required E_b/N_0 to simulation. We then note that

$$\begin{aligned} E_b &= \int_{-\infty}^{\infty} |p(t)|^2 \, dt \\ &\approx \sum |p_k|^2 T_s \end{aligned} \qquad (1)$$

where $p(t)$ is the pulse shape, $\{p_k\}$ is its sample version and $T_s = 1/F_s$ is the sample interval. On the other hand, if generate noise of variance σ^2, due to Nyquist considerations this can only be distributed over a bandwidth Fs, thus the noise spectral density is

$$\frac{N_0}{2} = \frac{\sigma^2}{F_{s\ s}} \qquad (2)$$

Re-arranging Eq. (2) and substituting Eq. (1) and the knowns, we have

$$\begin{aligned} \sigma^2 &= \frac{N_0}{2} F_s \\ &= \left(\frac{F_s}{2}\right)\left(\frac{E_b}{N_0}\right)^{-1} E_b \\ &= \left(\frac{F_s}{2}\right)\left(\frac{E_b}{N_0}\right)^{-1} \sum |p_k|^2 T_s \\ &= \frac{1}{2}\left(\frac{E_b}{N_0}\right)^{-1} \sum |p_k|^2 \end{aligned}$$

which agrees with what is used in the script (except that in the script we have suppressed F_s and T_s, knowing they would cancel).

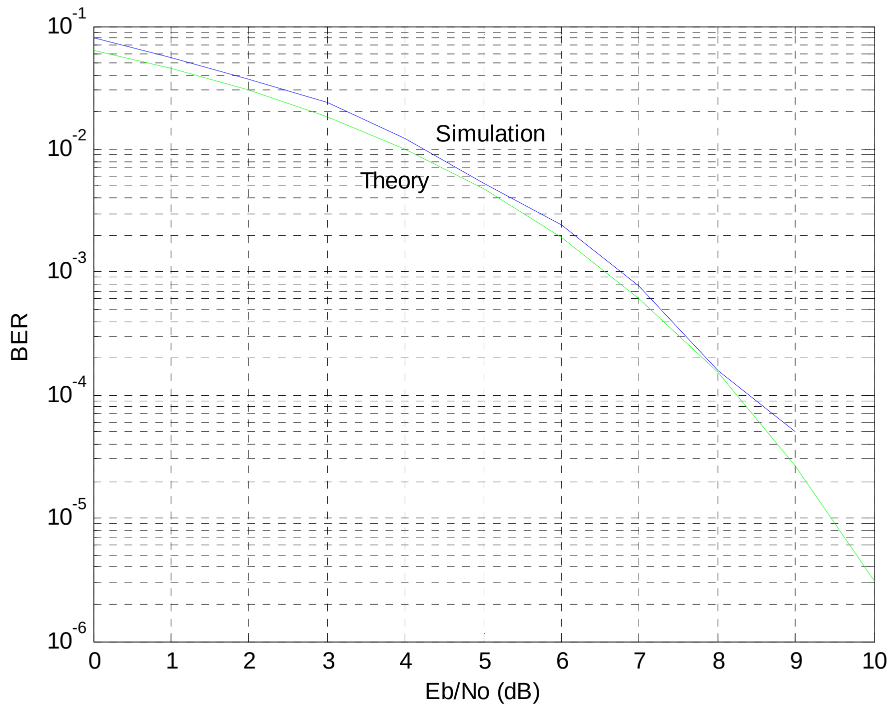
Continued on next slide

Problem 10.30 continued

(c) To compute the bit error rate for 0 to 10 dB, we add the following statements around the provided script

```
for kk = 0:10
        Eb_N0 = 10^(kk/10);
        Nbits = 100000; %  increase for higher Eb/N0
        …(provided script)
        BER(kk+1)   = Nerrs/Nbits
end
semilogy([0:10], BER)
xlabel('Eb/No (dB)')
ylabel('BER')
grid on
hold on, semilogy([0:10], 0.4*erfc(sqrt(10.^([0:10]/10))),'g')
```

The following plot is then produced by the Matlab script which shows good agreement between theory and simulation.



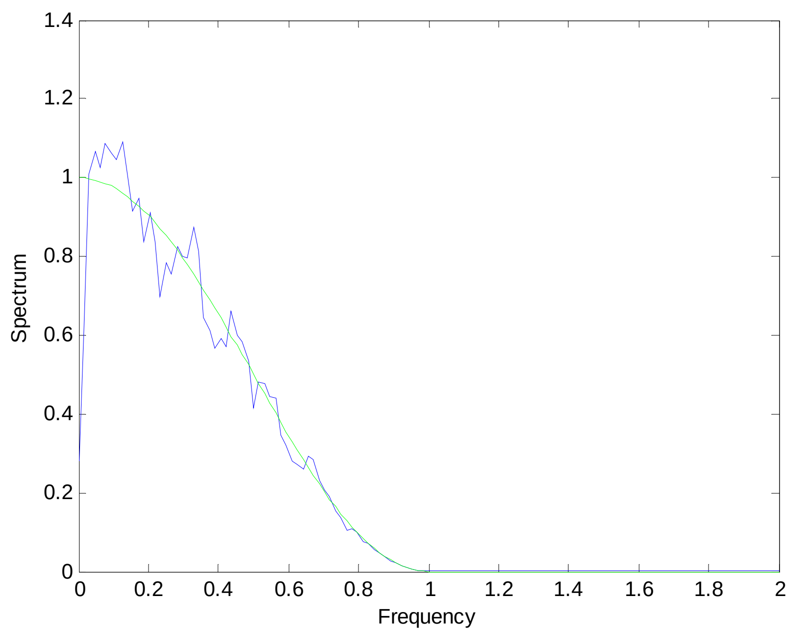
Problem 10.31 In this experiment, we simulate the performance of bipolar signalling in additive white Gaussian noise but with root-raised-cosine pulse shaping. A Matlab script is included in Appendix 7 for doing this. With this simulation:
(a) Compute the spectrum of the transmitted signal and compare to the theoretical. Also compare to the transmit spectrum with rectangular pulse shaping
(b) Plot the eye diagram of the received signal under no noise conditions. Explain the relationship of the eye opening to bit error rate performance.
(c) Confirm the theoretically predicted bit error rate for E_b/N_0 from 0 to 10 dB.

Solution

(a) We compare the spectra by inserting the following statements prior to noise being added to the signal

```
[P,F] = spectrum(S,256,0,Hanning(256),Fs);
plot(F,P(:,1));
midpt = floor(length(F)/2);
hold on, plot(F, abs([(1+cos(pi*F(1:midpt)))/2; 0*F(midpt+1:end)]),'g'), hold off
xlabel('Frequency'), ylabel('Spectrum')
```

The comparison plot is shown below.



(b) To plot the eye diagram we eliminate the noise by setting E_b/N_0 to a high value

```
Eb_N0 = 2000;
```

Then running the Matlab script produces the following eye diagram.

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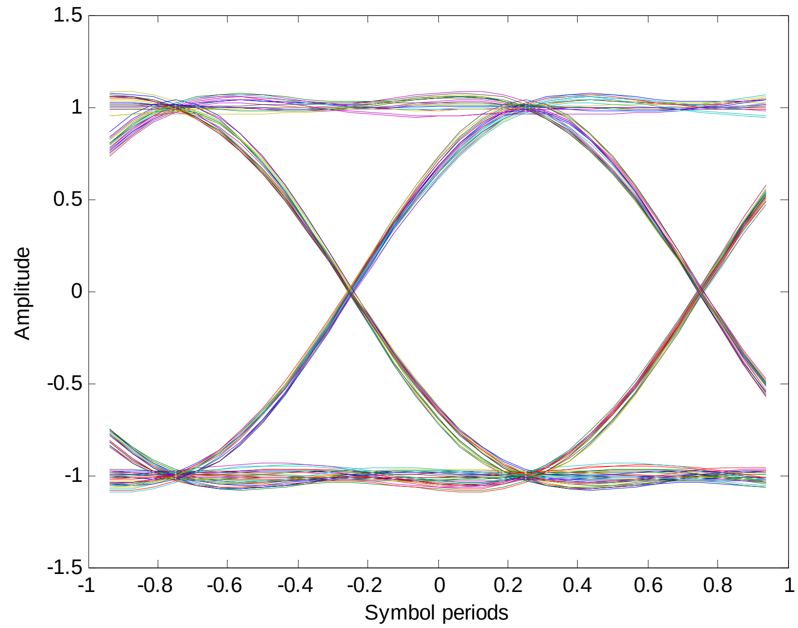
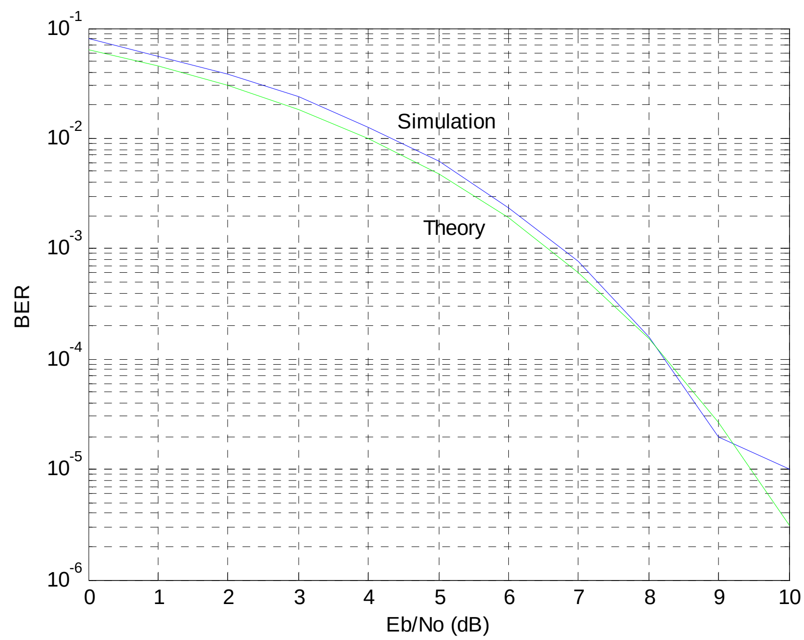
Problem 10.31 continued



(c) We simulate the bit error rate by commenting out the plotting statements and adding a set of statements similar to those used in Problem 10.30.



Problem 10.32 In this experiment, we simulate the effect of various mismatches in the communication system and their effect on performance. In particular, modify the MatLab scripts of the two preceding problems to:

(a) Simulate the performance of a system using rectangular pulse shaping at the transmitter and raised cosine pulse shaping at the receiver. Comment on the performance degradation.

(b) In the case of matched root-raised cosine filtering, include a complex phase rotation exp($j\theta$) in the channel. Plot the resulting eye diagram for θ being the equivalent of 5, 10, 20, and 45°. Compare to the case of 0°. Do likewise for the BER performance. What modification to the theoretical BER formula would accurately model this behaviour?

Solution

(a) We can create this mismatch by inserting the statements:

```
pulseTx = ones(1,Fs);

pulseRx = [  0.0064    0.0000   -0.0101    0.0000    0.0182   -0.0000   -0.0424 ...
             0.0000    0.2122    0.5000    0.6367    0.5000    0.2122   -0.0000 ...
            -0.0424    0.0000    0.0182   -0.0000   -0.0101    0.0000    0.0064 ];
Delay = floor((length(pulseTx)-1)/2 + (length(pulseRx)-1)/2 + 1);
Eb    = sum(pulseTx.^2);
```

And by modifying the statements

```
S       = filter(pulseTx,1,[b_delta zeros(1,Delay)]);
De = filter(pulseRx,1,R);
```

Then we obtain the performance shown below. Part of the loss seen is due to the filter mismatch but part of it is also due to a timing error; with the arrangement of the simulation the optimum sampling point for the data falls between the discrete samples. This sampling time loss could be recovered by interpolation.

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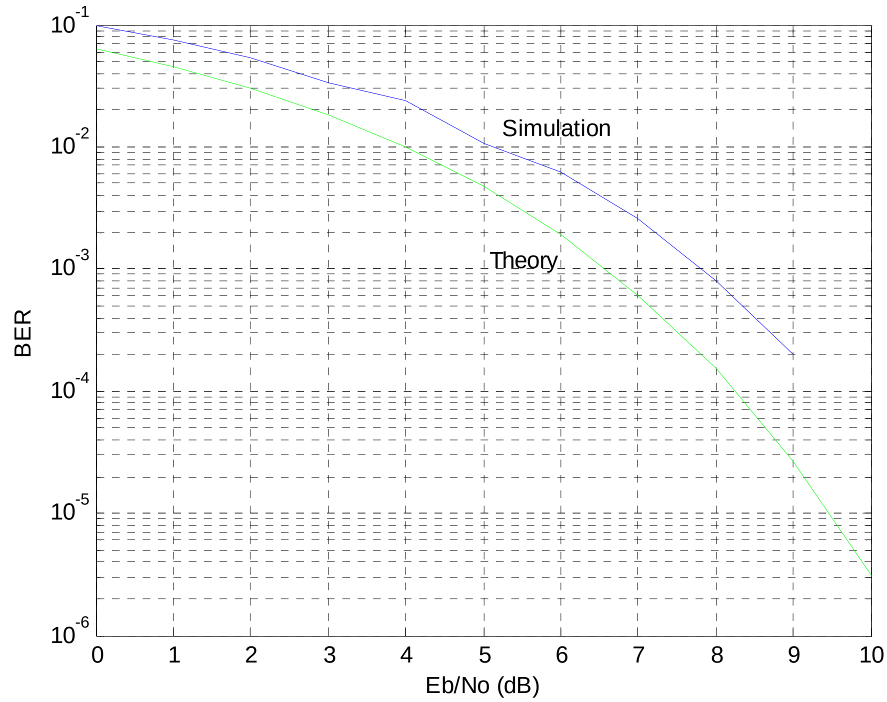
Problem 10.32 continued



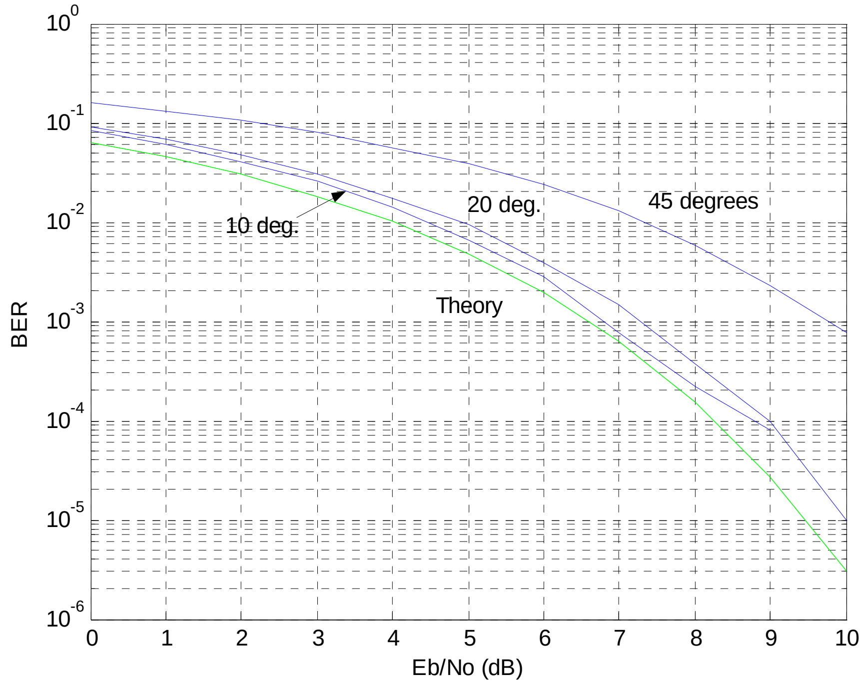
(b) Implementation of the phase rotation requires simulation of the complete complex baseband. To do this we must modify the channel portion of the simulation to the following

```
%--- add Gaussian noise ----
Noise = sqrt(N0/2)*(randn(size(S))+j*randn(size(S)));
R  = S + Noise;
R  = R*exp(j*10/180*pi);
R  = real(R);
```

Where we have now included the quadrature component of the noise. Note the receiver only uses the in-phase portion (real part) of the signal to characterize this degradation. The resulting performance for rotations of 10, 20 and 45° are shown below. Note that the 45° rotation results in a 3 dB loss in performance.

Continued on next slide

Problem 10.32 continued



Problem 11.1 What is the root-mean-square voltage across a 10 Mega-ohm resistor at room temperature if measured over a 1 GHz bandwidth? What is the available noise power?

Solution

Following Example 11.2, the available noise power is

$$\begin{aligned} P_N &= kTB_N \\ &= 1.38\times10^{-23}\times290\times10^{9} \\ &= 4\times10^{-12}\ \text{watts} \end{aligned}$$

The root-mean-square voltage across a 10 mega-ohm resistor is

$$\begin{aligned} V_{\text{rms}} &= \sqrt{P_N R} \\ &= \sqrt{4\times10^{-12}\times10^{7}} \\ &= 6.3\ \text{millivolts} \end{aligned}$$

Problem 11.2 What is the available noise power over 1 MHz due to shot noise from a junction diode that has a voltage differential of 0.7 volts and carries average current of 0.1 milliamperes, if the current source of the Norton equivalent circuit has a resistance of 250 ohms?

Solution

From Eq. (11.9), the saturation current for a junction diode is given by

$$\begin{aligned} I_S &= \frac{I}{\exp\left(\frac{qV}{RT}\right) - 1} \\ &= 1.8 \times 10^{-12} I \end{aligned}$$

Consequently, the noise contribution from the saturation current may be ignored. From Eq. (11.10) the expected current variance is then

$$\begin{aligned} \mathbf{E}\left[I_{shot}{}^2 \right] &= 2q(I + 2I_s)B_N \\ &\approx 2qIB_N \\ &= 2 \times \left(1.6 \times 10^{-19}\right) \times \left(0.1 \times 10^{-3}\right) \times 10^6 \\ &= 3.2 \times 10^{-17} \; Amp^2 \end{aligned}$$

The corresponding noise power with an equivalent resistance of 250 ohms is

$$\begin{aligned} P_N &= \mathbf{E}\left[I_{shot}{}^2\right] R \\ &= 8 \times 10^{-15} \text{ watts} \end{aligned}$$

Problem 11.3 An electronic device has a noise figure of 10 dB. What is the equivalent noise temperature?

<u>Solution</u>

From Eq. (11.7), the equivalent noise temperature is

$$\begin{aligned} T_e &= T_0(F-1) \\ &= 290(10-1) \\ &= 2610^o K \end{aligned}$$

Problem 11.4. The device of Problem 11.3 has a gain of 17 dB and is connected to a spectrum analyzer. If the input to the device has an equivalent temperature of 290°K and the spectrum analyzer is noiseless, express the measured power spectral density in dBm/Hz. If the spectrum analyzer has a noise figure of 25 dB, what is the measured power spectral density in this case?

Solution

For the device of Problem 11.3, the total output noise is, from Eq. (11.15),

$$\begin{aligned} N &= Gk(T+T_e)B_N \\ &= GkTFB_N \\ &= \left(10^{17/10}\right)\left(1.38\times10^{-23}\right)\left(290\right)\left(10^{10/10}\right)B_N \\ &= 2.0\times10^{-18}\,B_N \end{aligned}$$

The noise spectral density at the device output is approximately

$$\begin{aligned} \frac{N}{B_N} &= 2.0\times10^{-18} \text{ W/Hz} \\ &\sim \text{-147 dBm/Hz} \end{aligned}$$

If the spectrum analyzer has a noise figure of 25 dB, then we must use the results of the following section. Specifically, Eq. (11.21), to obtain the total noise figure of

$$\begin{aligned} F &= F_1 + \frac{F_2}{G_1} \\ &= \left(10^{10/10}\right) + \frac{10^{25/10}}{10^{17/10}} \\ &= 16.3 \\ &\sim 12.1\,\text{dB} \end{aligned}$$

Since the overall noise figure is increased 2.1 dB by the spectrum analyzer, the noise spectral density at the spectrum analyzer output is (assuming unity gain for the spectrum analyzer) -144.9 dBm/Hz. (*There is an error in the second answer given in the text.*)

Problem 11.5 A broadcast television receiver consists of an antenna with a noise temperature of 290°K and a pre-amplifier with a gain of 20 dB and a noise figure of 9 dB. A second-stage amplifier in the receiver provides another 20 dB of gain and has a noise figure of 20 dB. What is the noise figure of the overall system?

Solution

From Eq (11.21), after converting from decibels to absolute

$$
\begin{aligned}
F &= F_1 + \frac{F_2 - 1}{G_1} + \frac{F_3 - 1}{G_1 G_2} \\
&= 2 + \frac{7.94 - 1}{1} + \frac{99}{100} \\
&= 2 + 6.94 + .99 \\
&= 9.93
\end{aligned}
$$

Converting this result back to decibels, the overall noise figure is 9.97 dB.

Problem 11.6 A satellite antenna has a diameter of 4.6 meters and operates at 12 GHz. What is the antenna gain if the aperture efficiency is 60%? If the same antenna was used at 4 GHz what would be the corresponding gain?

Solution

From Eq (11.25), the antenna gain is

$$G = \frac{4\pi A_{eff}}{\lambda^2}$$

The effective area is given by Eq.(11.24)

$$\begin{aligned} A_{eff} &= \eta A \\ &= \eta \frac{\pi d^2}{4} \\ &= 9.97 \text{ m}^2 \end{aligned}$$

where the efficiency is 60% and the diameter is 4.6 meters. At 12 GHz, the wavelength $\lambda = c/f = 0.025$ meters. Consequently, the antenna gain is

$$\begin{aligned} G &= 2000458.7 \\ &\sim 53.0 \text{ dB} \end{aligned}$$

With a transmission frequency of 4 GHz, the wavelength $\lambda = 0.075$ m and the antenna gain is

$$\begin{aligned} G &= 22273.2 \\ &\sim 43.5 \text{ dB} \end{aligned}$$

Problem 11.7 A satellite at a distance of 40,000 kilometers transmits a signal at 12 GHz with an EIRP of 10 watts towards a 4.6 meter antenna that has an aperture efficiency of 60%. What is the received signal level at the antenna output?

<u>Solution</u>

From Eq.(11.32), the path loss due to free-space transmission of a 12 GHz signal over 40,000 kilometers is

$$\begin{aligned} L_P &= 20\log_{10}\left(\frac{4\pi r}{\lambda}\right) \\ &= 20\log_{10}\left(\frac{4\pi 40\times 10^6}{0.025}\right) \\ &\sim 206.1\,\text{dB} \end{aligned}$$

Substituting this result in Eq (11.29), the received power is

$$\begin{aligned} P_R &= EIRP - L_P + G_R \\ &= 10\text{ dBW} - 206.1\,\text{dB} + 53.0\text{ dB} \\ &= -143\,\text{dBW} \end{aligned}$$

where we have used the antenna gain of 53 dB from Problem 11.6.

Problem 11.8 The antenna of Problem 11.7 has a noise temperature of 70°K and is directly connected to a receiver with an equivalent noise temperature of 50°K and a gain of 60 dB. What is the system noise temperature? If the transmitted signal has a bandwidth of 100 kHz, what is the carrier-to-noise ratio? If the digital signal has a bit rate of 150 kbps, what is the E_b/N_0?

Solution

From Eq. (11.22), the combined system noise temperature is

$$\begin{aligned} T_s &= T_{ant} + \frac{T_{rx}}{1} \\ &= 70^o + 50^o \\ &= 120^o K \end{aligned}$$

where the electrical gain of the antenna is 1. For a bandwidth of 100 kHz the available noise power is

$$\begin{aligned} N &= kT_s B \\ &= 1.38 \times 10^{-23} \cdot 120 \cdot 10^5 \\ &= 1.66 \times 10^{-16} \text{ watts} \\ &\sim -157.8 \text{ dBW} \end{aligned}$$

Comparing to the result for Problem 11.7, we have that the C/N is 14.8 dB.

To convert the C/N_0 to an E_b/N_0, we use the formula

$$\frac{C}{N_0} = \frac{E_b}{N_0} \times R$$

where the bit rate R relates the energy per bit E_b to the power C. In decibels,

$$\left(\frac{C}{N_0}\right)_{dB-Hz} = \left(\frac{E_b}{N_0}\right)_{dB} + 10\log_{10} R$$

Continued on next slide

Problem 11.8 continued

Re-arranging, we have

$$\begin{aligned}\left(\frac{E_b}{N_0}\right)_{dB} &= \left(\frac{C}{N_0}\right)_{dB-Hz} - 10\log_{10} R \\ &= \left(\frac{C}{N}\right)_{dB} + 10\log B_N - 10\log_{10} R \\ &= 14.8 + 50 - 51.8 \\ &= 13\ \text{dB}\end{aligned}$$

Problem 11.9 Transmitting and receiving antennas for a 4 GHz signal are located on top of 20 meter towers separated by 2 kilometers. For free-space propagation, what is the maximum height permitted for an object located midway between the two towers?

Solution

The radius of the first Fresnel zone with $d_1 = d_2 = 1$ kilometer and $\lambda = c/f = 0.075$ m is

$$\begin{aligned} h &= \sqrt{\frac{\lambda d_1 d_2}{d_1 + d_2}} \\ &= \sqrt{\frac{(.075)(1000)(1000)}{2000}} \\ &= 6.1 \text{ m} \end{aligned}$$

Consequently, the maximum height of intermediate object is 20 m – 6.1m = 13.9 m, if we require free-space propagation conditions.

Problem 11.10 A measurement campaign indicates that the median path loss at 900 MHz in a suburban area may be modeled with a path-loss exponent of 2.9. What is the median path loss at a distance of 3 kilometers using this model? How does this loss compare to the free-space loss at the same distance?

Solution

From Eq (11.37), the free-space path loss at one meter, with a transmission frequency of 900 MHz, is

$$\beta_0 = \left(\frac{\lambda}{4\pi r_0}\right)^2 = \left(\frac{.333}{4\pi}\right)^2 = .000704$$

with r_0 = 1 m. From Eq.(11.37), the path loss with the terrestrial propagation model is

$$\begin{aligned}\frac{P_R}{P_T} &= \frac{\beta_0}{\left(r/r_0\right)^n} \\ &= \frac{.0007}{\left(3000\right)^{2.9}} \\ &= 5.8\times10^{-14} \\ &\sim\ -132\text{ dB}\end{aligned}$$

The free-space loss over the same distance is given by

$$\begin{aligned}\frac{P_R}{P_T} &= \left(\frac{\lambda}{4\pi r}\right)^2 \\ &= 7.8\times10^{-11} \\ &\sim -101.1\text{ dB}\end{aligned}$$

or 31 dB less.

Problem 11.11 Express the true median of the Rayleigh distribution as a fraction of the R_{rms} value? What is the decibel error in the approximation $R_{median} \approx R_{rms}$?

Solution

The median of the distribution satisfies $\mathbf{P}\left[R < r\right] = 0.5$. Consequently, from Eq. (11.38) we have that the median *r* satisfies

$$1 - \exp\left\{\frac{-r^2}{R_{rms}^2}\right\} = \frac{1}{2}$$

Solving for *r* we obtain

$$-\frac{r^2}{R_{rms}^2} = \ln \frac{1}{2}$$
$$r = R_{rms}\sqrt{\ln \frac{1}{2}}$$
$$= 0.83 R_{rms}$$

Consequently, there is a $20\log_{10}(0.83)$ =1.62 dB error when using the rms value of the amplitude instead of the median value.

Problem 11.12 Compute the noise spectral density in watts per hertz of:
(a) an ideal resistor at nominal temperature of 290°K;
(b) an amplifier with an equivalent noise temperature of 22,000°K.

Solution

(a) From Eq. (11.19), the noise power spectral density is

$$\begin{aligned} N_0 &= kT_e \\ &= 1.38 \times 10^{-23} \times 290 \\ &= 4.0 \times 10^{-21} \text{ W/Hz} \end{aligned}$$

(b) From Eq. (11.19), the noise power spectral density is

$$\begin{aligned} N_0 &= kT_e \\ &= 1.38 \times 10^{-23} \times 22000 \\ &= 3.04 \times 10^{-19} \text{ W/Hz} \end{aligned}$$

Problem 11.13 For the two cases of Problem 11.12, compute the pre-detection SNR when the received signal power is:
(a) -60 dBm and the receive bandwidth is 1 MHz;
(b) -90 dBm and the receive bandwidth is 30 kHz.
Express the answers in both absolute terms and decibels.

Solution

(a) The signal power is obtained by converting -60 dBm to watts

$$S = 10^{(-60/10)} = 10^{-6}\,\text{mW} = 10^{-9}\,\text{W}$$

The noise power from the ideal resistor is from Eq. (11.13)

$$\begin{aligned} N &= kT_e B_N \\ &= 4.0\times10^{-21}\times(10^{6}) \\ &= 4.0\times10^{-15}\,\text{W} \end{aligned}$$

The SNR is the ratio of the two

$$SNR = \frac{S}{N} = \frac{10^{-9}}{4.0\times10^{-15}} = 2.5\times10^{5} \sim 54\text{ dB}$$

A similar calculation for the amplifier of the previous problem results in

$$SNR = \frac{S}{N} = \frac{10^{-9}}{3.04\times10^{-19}\times10^{6}} = 2.94\times10^{3} \sim 34.7\text{ dB}$$

(b) The signal power is obtained by converting -90 dBm to watts

$$S = 10^{(-90/10)} = 10^{-9}\,\text{mW} = 10^{-12}\,\text{W}$$

The noise power from the ideal resistor is from Eq. (11.13)

Continued on next slide

Problem 11.13 continued

$$
\begin{aligned}
N &= kT_e B_N \\
&= 4.0\times10^{-21}\times(30\times10^{3}) \\
&= 1.2\times10^{-16}\,\text{W}
\end{aligned}
$$

The SNR is the ratio of the two

$$SNR = \frac{S}{N} = \frac{10^{-12}}{1.2\times10^{-16}} = 8.3\times10^{3} \sim 39.2\ \text{dB}$$

A similar calculation for the amplifier of the previous problem results in

$$SNR = \frac{S}{N} = \frac{10^{-12}}{3.04\times10^{-19}\times\left(30\times10^{3}\right)} = 1.1\times10^{2} \sim 20.4\ \text{dB}$$

Problem 11.14 A wireless local area network transmits a signal that has a noise bandwidth of approximately 6 MHz. If the signal strength at the receiver input terminals is –90 dBm and the receiver noise figure is 8 dB, what is the pre-detection signal-to-noise ratio?

Solution

The signal power is obtained by converting -90 dBm to watts

$$S = 10^{(-90/10)} = 10^{-9}\,\text{mW} = 10^{-12}\,\text{W}$$

The noise power with an 8 dB noise figure F is from Eqs. (11.15) and (11.16)

$$\begin{aligned} N &= kT_0FB \\ &= 1.38\times10^{-23}\times(290)\times10^{8/10}\times(6\times10^{6}) \\ &= 1.52\times10^{-13}\,\text{W} \end{aligned}$$

The pre-detection SNR is the ratio of the two

$$SNR = \frac{S}{N} = \frac{10^{-12}}{1.52\times10^{-13}} = 6.6 \sim 8.2\text{ dB}$$

Problem 11.15 A communications receiver includes a whip antenna whose noise temperature is approximately that of the Earth, that is, 290°K. The receiver pre-amplifier has a noise figure of 4 dB and a gain of 25 dB. What is the equivalent noise temperature of the antenna and the pre-amplifier? What is the combined noise figure?

Solution

(a) Following Example 11.4, the combined noise temperature of the antenna and pre-amplifier is, from Eq. (11.17)

$$\begin{aligned} T_{sys} &= T_{ant} + T_{amp} \\ &= 290 + 290(F-1) \\ &= 728\text{K} \end{aligned}$$

(b) From Eq. (11.16), the combined noise figure is

$$F_{comb} = \frac{T + T_e}{T} = \frac{290 + 728}{290} = 3.51 \sim 5.45\text{dB}$$

Problem 11.16 A parabolic antenna with a diameter of 0.75 meters is used to receive a 12 GHz satellite signal. What is the gain in decibels of this antenna? Assume the antenna efficiency is 60%.

Solution

From Eq.(11.25), the antenna gain is

$$G_R = \frac{4\pi A_{eff}}{\lambda^2} \qquad (1)$$

The signal wavelength is $\lambda = c/f = 3\times10^8 / 12\times10^9 = 0.025\text{m}$, and the effective area is

$$A_{eff} = \eta \frac{\pi d^2}{4} = 0.60 \frac{\pi(.75)^2}{4}$$

Substituting these two results into Eq. (1), the antenna gain is

$$G_R = \frac{4\pi}{(0.025)^2} \times (0.6) \frac{\pi(0.75)^2}{4} = 5.33\times10^3 \sim 37.3\text{dB}$$

Problem 11.17 If the system noise temperature of a satellite receiver is 300°K, what is the required received signal strength to produce a C/N_0 of 80 dB?

Solution

(There is a typo in problem statement, the units should be "dB-Hz".)

From Eq. (11.19), the noise power spectral density is

$$\begin{aligned} N_0 &= kT_s \\ &= 1.38\times10^{-23}\times(300) \\ &= 4.14\times10^{-21}\,\text{W/Hz} \\ &\sim -203.8\,\text{dBW/Hz} \end{aligned}$$

In decibels, the carrier to noise density is given by

$$\begin{aligned} \left(C/N_0\right)_{dB-Hz} &= (C)_{dBW} - \left(N_0\right)_{dBW-Hz} \\ 80 &= \left(C\right)_{dBW} - (-203.8) \end{aligned}$$

Solving for C, we obtain C = -123.8 dBW = -93.8 dBm.

Problem 11.18 If a satellite is 40,000 km from the antenna of Problem 11.16, what satellite EIRP will produce a signal strength of –110 dBm at the antenna terminals? Assume the transmission frequency is 12 GHz.

Solution

The received power is given by Eq. (11.29)

$$P_R = EIRP + G_R - L_P \quad (1)$$

where all quantities are in decibels. From Problem 11.16, the antenna gain is G_R = 37.3 dB. The free-space path loss is given by Eq. (11.32)

$$L_P = 20\log_{10}\left(\frac{4\pi r}{\lambda}\right)$$

From Problem 11.16, the wavelength is λ = 0.025m at 12 GHz. So, at a distance r = 40,000 km, the path loss is

$$L_P = 20\log_{10}\left(\frac{4\pi(40000\times10^3}{0.025}\right) = 206.1\,\text{dB}$$

Substituting these in Eq.(1) with a received power of -110 dBm, we obtain

$$-110\text{ dBm} = EIRP + 37.3\text{ dB} - 206.1\text{dB}$$

Solving this equation, we find the require EIRP is 58.8 dBm.

Problem 11.19 Antennas are placed on two 35-meter office towers that are separated by ten kilometers. What is the minimum height of a building between the two towers that would disturb the assumption of free-space propagation?

Solution

From Eq. (11.35), the radius of the first Fresnel zone is

$$h = \sqrt{\frac{\lambda d_1 d_2}{d_1 + d_2}}$$

This radius is maximized midway between the two towers and must be kept clear to approximate free-space propagation. With $d_1 = d_2$ = 5km, the radius in meters is

$$h = \sqrt{2500\lambda} = 50\sqrt{\lambda}$$

The maximum building height (in meters) is

$$\begin{aligned} b &= 35 - h \\ &= 35 - 50\sqrt{\lambda} \end{aligned}$$

For example, at a transmission frequency of 4 GHz, the maximum height is b = 21.3 m.

Problem 11.20 If a receiver has a sensitivity of –90 dBm and a 12 dB noise figure what is minimum pre-detection signal-to-noise ratio of an 8 kHz signal?

Solution

The noise in an 8 kHz bandwidth for a receiver with an 8 dB noise figure is, from Eqs. (11.15) and (11.16),

$$\begin{aligned} N &= kT_0FB \\ &= 1.38\times10^{-23}\times(290)\times\left(10^{12/10}\right)\times(8\times10^3) \\ &= 5.07\times10^{-16}\,\mathrm{W} \end{aligned}$$

The receiver sensitivity is defined as the minimum received signal power that will provide a demodulated signal with acceptable performance, thus the minimum signal power is S = -90 dBm ~ 10^{-12} W. The minimum pre-detection SNR is the ratio of the two

$$SNR = \frac{S}{N} = \frac{10^{-12}}{5.07\times10^{-16}} = 1.97\times10^{3} \sim 32.9\text{ dB}$$

Problem 11.21 A satellite antenna is installed on the tail of an aircraft and has a noise temperature of 100°K. The antenna is connected by a coaxial cable to a low-noise amplifier in the equipment bay at the front of the aircraft. The cable causes 2 dB attenuation of the signal. The low-noise amplifier has a gain of 60 dB and a noise temperature of 120°K. What is the system noise temperature? Where would a better place for the low-noise amplifier be?

Solution

Following Example 11.4, the system noise temperature is

$$\begin{aligned} T_s &= T_{ant} + \frac{T_{cable}}{G_{ant}} + \frac{T_{amp}}{G_{cable}} \\ &= 100 + \frac{290}{1} + \frac{120}{.631} \\ &= 580\text{K} \end{aligned}$$

where we have used the facts that the antenna does not provide any electrical gain, thus $G_{ant} = 1$; and the fact the fact that cable causes a 2 dB loss so $G_{cable} = 10^{-2/10} = 0.631$. Locating the low-noise amplifier in the tail of the aircraft, close to the antenna would be a better system design. With the amplifier in the antenna tail, the system noise temperature would be approximately 220 K.

Problem 11.22 A wireless local area network transmitter radiates 200 milliwatts. Experimentation indicates that the path loss may be accurately described by

$$L_p = 31 + 33\log_{10}(r)$$

where the path loss is in decibels and r is the range in meters. If the minimum receiver sensitivity is –85 dBm, what is the range of the transmitter?

Solution

Since the problem says nothing about the transmit and receive antennas, we shall assume they are omni-directional with a gain of 0 dB. In this case, the Friis equation (in decibels) for the received signal strength reduces to

$$\begin{aligned} P_R &= P_T - L_P \\ &= P_T - \left(31 + 33\log_{10} r\right) \end{aligned} \quad (1)$$

With a transmit power of 200 mW, equivalent to 23 dBm, and a minimum signal strength of -85 dBm, Eq. (1) becomes

$$-85 = 23 - \left(31 + 33\log_{10} r\right)$$

Solving this equation for the maximum range, we find r is 215.4 meters.

Problem 11.23 A mobile radio transmits 30 watts and the median path loss may be approximated by

$$L_p = 69 + 31 \log_{10}(r)$$

where the path loss is in decibels and r is the range in kilometers. If the receiver sensitivity is -110 dBm and 12 dB of margin must be included to compensate for variations about the median path loss, what is the range of the transmitter?

Solution

Since the problem says nothing about the transmit and receive antennas, we shall assume they are omni-directional with a gain of 0 dB. In this case, the Friis equation for the received signal strength reduces to

$$\begin{aligned} P_R &= P_T - L_P - L_0 \\ &= P_T - \left(69 + 31\log_{10} r\right) - L_0 \end{aligned} \qquad (1)$$

where L_0 represents the required margin. A transmit power of 30 W is equivalent to 14.8 dBW, and a minimum signal strength of -110 dBm is equivalent to -140 dBW. Thus, Eq. (1) becomes

$$-140 = 14.8 - \left(69 + 31\log_{10} r\right) - 12$$

Solving this equation for the maximum range, we find r is 240.2 kilometers. In practice, the range will likely be somewhat less than this due to the curvature of the earth and depending on the height of the base station antenna.

Problem 11.24 A cellular telephone transmits 600 milliwatts of power. If the receiver sensitivity is –90 dBm, what would the range of the telephone be under free space propagation? Assume the transmitting and receiving antennas have unity gain and the transmissions are at 900 MHz. If propagation conditions actually show a path-loss exponent of 3.1 with a fixed loss β = 36 dB, what would the range be in this case?

Solution

(a) The Friis equation for the received power in decibels is

$$P_R = P_T + G_R + G_T - L_P \tag{1}$$

where the antenna gains are $G_R = G_T = 0$ dB. The transmit power of 600 mW is equivalent to or 27.8 dBm. For free-space propagation, the path loss is

$$L_P = 20\log_{10}\left(\frac{4\pi r}{\lambda}\right)$$

At 900 MHz, the wavelength is $\lambda = c/f = 3\times10^8/900\times10^6 = 0.33\,\text{m}$. Making these substitutions, we have

$$-90 = 27.8 + 0 + 0 - 20\log_{10}\left(\frac{4\pi r}{0.33}\right)$$

Solving this equation for the maximum range, we find the *r* is 20.4 kilometers.

(b) In this case, the Friis equation still applies but the path loss is given by Eq. (11.37)

$$\begin{aligned} L_P &= \left(\frac{10^{-36/10}}{r^{3.1}}\right)^{-1} \\ &\sim 36 + 31\log_{10}(r) \end{aligned}$$

Substituting the second line into Eq. (1), we have

$$-90 = 27.8 + 0 + 0 - (36 + 31\log_{10} r)$$

Solving this equation for the maximum range, we find that *r* is 435 meters. Clearly, the propagation conditions can make a huge difference on the range.

Problem 11.25 A line-of-sight 10-kilometer radio link is required to transmit data at a rate of 1 megabit per second at a center frequency of 4 GHz. The transmitter uses an antenna with 10 dB gain and QPSK modulation with a root-raised cosine pulse shape spectrum having a roll-off factor of 0.5. The receiver also has an antenna with 10 dB gain and has a system noise temperature of 900 K. What is the minimum transmit power required to achieve a bit error rate of 10^{-5}?

Solution

From the BER performance of QPSK in Fig. 10.16, we find that a BER of 10^{-5} implies an E_b/N_0 of 9.5 dB is required. From this, we obtain the required C/N_0 using knowledge of the transmission rate R = 1 Mbps.

$$\begin{aligned}\left(\frac{C}{N_0}\right)_{dB-Hz} &= \left(\frac{E_b}{N_0}\right)_{dB} + 10\log_{10} R \\ &= 69.5 \text{ dB-Hz}\end{aligned}$$

The system noise temperature of 900 K implies

$$\begin{aligned}N_0 &= kT_e \\ &= 1.38\times10^{-23}\times900 \\ &= 1.24\times10^{-20} \text{ W/Hz} \\ &\sim -199.1 \text{ dBW/Hz}\end{aligned}$$

Using this information, the received power level may be calculated from

$$\begin{aligned}P_R &= C \\ &= \left(\frac{C}{N_0}\right)_{dB-Hz} + \left(N_0\right)_{dBW-Hz} \\ &= 69.5 + (-199.1) \\ &= -129.6 \text{ dBW}\end{aligned}$$

We now appeal to the decibel form of the Friis equation:

$$P_R = P_T + G_R + G_T - L_P \qquad (1)$$

where the antenna gains are $G_R = G_T$ = 10 dB. Since the problem sight says line-of-sight transmission, we shall assume free-space propagation, and the path loss is

Continued on next slide

Problem 11.25 continued

$$L_P = 20\log_{10}\left(\frac{4\pi r}{\lambda}\right)$$

At 4 GHz, the wavelength is $\lambda = c/f = 3\times10^8 / 4\times10^9 = 0.075\ \mathrm{m}$. Making all these substitutions into Eq. (1) with a range r = 10 km, we obtain

$$-129.6 = P_T + 10 + 10 - 20\log_{10}\left(\frac{4\pi 10\times10^3}{0.075}\right)$$

Solving this equation for the transmitted power, we find that the required P_T is -25.1 dBW or 4.9 dBm.

Problem 11.26 A land-mobile radio transmits 128 kbps at a frequency of 700 MHz. The transmitter uses an omni-directional antenna and 16-QAM modulation with a root-raised cosine pulse spectrum having a roll-off of 0.4. The receiver has an antenna with 3 dB gain and a noise figure of 6 dB. If the path loss between the transmitter and receiver is given by

$$L_P(r) = 30 + 28\log_{10}(r) \text{ dB}$$

where r is in meters, what is the maximum range at which the bit error rate of 10^{-4} may be achieved?

Solution

From the BER performance of 16-QAM in Fig. 10.16, we find that a BER of 10^{-4} implies an E_b/N_0 of 13 dB. From this, we obtain the C/N_0 by using knowledge of the transmission rate R = 128 kbps.

$$\begin{aligned}\left(\frac{C}{N_0}\right)_{dB-Hz} &= \left(\frac{E_b}{N_0}\right)_{dB} + 10\log_{10} R \\ &= 64.1 \text{ dB-Hz}\end{aligned}$$

The noise figure of 6 dB implies

$$\begin{aligned}N_0 &= kFT_0 \\ &= 1.38\times10^{-23}\times\left(10^{6/10}\right)\times\left(290\right) \\ &= 1.59\times10^{-20} \text{ W/Hz} \\ &\sim -198.0 \text{ dBW/Hz}\end{aligned}$$

and the received power level is

$$\begin{aligned}P_R = C &= \left(\frac{C}{N_0}\right)_{dB-Hz} + \left(N_0\right)_{dBW-Hz} \\ &= 64.1 + \left(-198.0\right) \\ &= -133.9 \text{ dBW}\end{aligned}$$

We now appeal to the decibel form of the Friis equation:

$$P_R = P_T + G_R + G_T - L_P \qquad (1)$$

Continued on next slide

Problem 11.26 continued

where the antenna gains are $G_R = G_T = 0$ dB. The path loss is

$$L_P = 30 + 28\log_{10}(r)$$

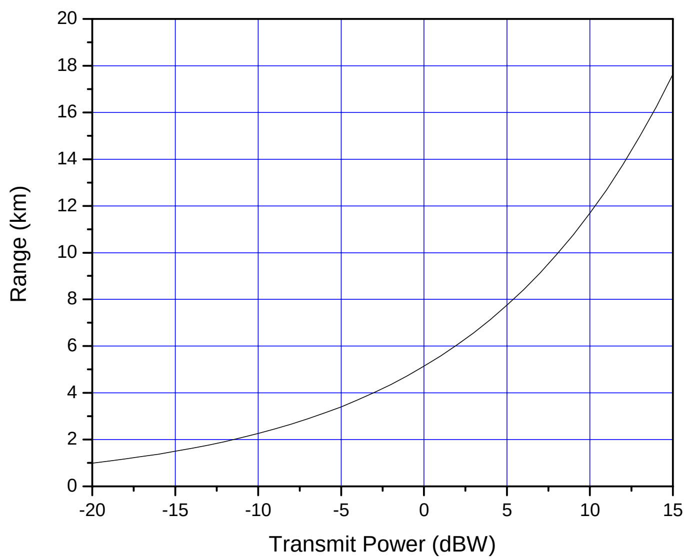
Making all these substitutions into Eq. (1), we obtain

$$-133.9 = P_T + 0 + 0 - (30 + 28\log_{10} r)$$

or

$$r = 10^{(P_T + 103.9)/28}$$

In the following figure, we plot the range in kilometres versus the transmit power in dBW.



For example, with a transmit power of 10 W or 10 dBW, we find that range is 11.7 km.