2.1 The Determinante of a Matrix

* Let A be 2×2 matrix given by $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$.

The determinante of A denoted by

$$|A| = \det(A) = a_{11}a_{22} - a_{12}a_{21}$$

* If A is 1×1 matrix, $A = (a)$, then $|A| = a$

---

* The matrix A is non singular iff $|A| \neq 0$

Proof (By Th* in 1.5) A is non singular iff A is row equivalent to I.

(Assume $a_{11} \neq 0$) • Multiply the 2^{nd} row of A by a_{11}: $\begin{bmatrix} a_{11} & a_{12} \\ a_{11}a_{21} & a_{11}a_{22} \end{bmatrix}$

• $R_2 - a_{21}R_1 \Rightarrow \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{11}a_{22} - a_{21}a_{12} \end{bmatrix}$

Since $a_{11} \neq 0$, the resulting matrix will be row equivalent to I iff $a_{11}a_{22} - a_{12}a_{21} \neq 0$

(If $a_{11} = 0$) • we switch the two rows of $A \Rightarrow \begin{bmatrix} a_{21} & a_{22} \\ 0 & a_{12} \end{bmatrix}$ is row equivalent to I iff $a_{12}a_{21} \neq 0$.

Thus, A is non singular iff $\det(A) \neq 0$

Exp • The matrix $\begin{bmatrix} -1 & 3 \\ -2 & 6 \end{bmatrix}$ is singular since $\begin{vmatrix} -1 & 3 \\ -2 & 6 \end{vmatrix} = -6 + 6 = 0$

• The matrix $\begin{bmatrix} -2 & 3 \\ 4 & -1 \end{bmatrix}$ is non singular since $\begin{vmatrix} -2 & 3 \\ 4 & -1 \end{vmatrix} = 2 - 12 = -10 \neq 0$

Def. Let $A = (a_{ij})$ be $n \times n$ matrix.

- Let M_{ij} be $(n-1)\times(n-1)$ matrix obtained from A by deleting the i^{th} row and the j^{th} column.
- The minor of a_{ij} is $|M_{ij}|$
- The cofactor of a_{ij} is $A_{ij} = (-1)^{i+j} |M_{ij}|$

Th. The determinante of A is $|A| = \sum_{k=1}^{n} a_{ik} A_{ik}$ " i^{th} row "

for any $i = 1, \dots, n$, $j = 1, \dots, n$

$$= \sum_{k=1}^{n} a_{kj} A_{kj} \quad \text{" } j^{th} \text{ column"}$$

---

* Note that
- if $n=1$, then $|A| = a_{11}$
- if $n>1$, then $|A|$ is as given in Th above.
- the cofactors can be associated with the entries in the i^{th} row or in the j^{th} column.
- Hence, we choose the row or the column that has maximum number of zeros.

Exp Find $|A|$ if $A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 0 & 1 \\ 2 & 1 & -1 \end{bmatrix}$

- The cofactor expansion of $|A|$ along the 2^{nd} row is

$$|A| = a_{21} A_{21} + a_{22} A_{22} + a_{23} A_{23}$$

$$= (-1)(-1)^{2+1} |M_{21}| + (0)(-1)^{2+2} |M_{22}| + (1)(-1)^{2+3} |M_{23}|$$

$$= \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} + 0 - \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}$$

$$= (1-1) - (1+2) = 0 - 3 = -3$$

- The cofactor expansion of $|A|$ along the 2^{nd} column is

$$|A| = a_{12} A_{12} + a_{22} A_{22} + a_{32} A_{32}$$

$$= (-1)(-1)^{1+2} |M_{12}| + (0)(-1)^{2+2} |M_{22}| + (1)(-1)^{3+2} |M_{32}|$$

$$= \begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix} + 0 - \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix}$$

$$= (1-2) - (1+1) = -1 - 2 = -3$$

Exp Find $|A|$ if $A = \begin{bmatrix} 0 & 2 & 3 & 0 \\ 0 & 4 & 5 & 0 \\ 0 & 1 & 0 & 3 \\ 2 & 0 & 1 & 3 \end{bmatrix}$

- We expan over the first column. Hence, the cofactor expansion of $|A|$ along the 1^{st} column is

$$|A| = a_{11} A_{11} + a_{21} A_{22} + a_{31} A_{31} + a_{41} A_{41}$$

$$= 0 + 0 + 0 + (2)(-1)^{4+1} |M_{41}|$$

$$= -2 \underbrace{\begin{vmatrix} 2 & 3 & 0 \\ 4 & 5 & 0 \\ 1 & 0 & 3 \end{vmatrix}}_{B}$$

$$= -2 \left[b_{13} B_{13} + b_{23} B_{23} + b_{33} B_{33} \right]$$

$$= -2 \left[0 + 0 + (3)(-1)^{3+3} \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} \right]$$

$$= (-2)(3)(10-12) = (-6)(-2) = 12 .$$

Th If A is $n\times n$ matrix, then $|A| = |A^T|$

Proof "By Induction":

- The result holds for $n=1$ since $A=(a)$ is symmetric.
- * Assume that the result holds for $k\times k$ matrix
- We need to show the results holds for $(k+1)\times(k+1)$ matrix

$\Rightarrow$ Expanding $|A|$ along the 1^{st} row

$$|A| = a_{11} A_{11} + a_{12} A_{12} + \cdots + a_{1,k+1} A_{1,k+1}$$

$$= a_{11}(-1)^2 |M_{11}| + a_{12}(-1)^3 |M_{12}| + \cdots + a_{1,k+1}(-1)^{k+2} |M_{1,k+1}|$$

$$= a_{11}|M_{11}| - a_{12}|M_{12}| + \cdots + a_{1,k+1}|M_{1,k+1}|$$

$\boxed{M_{ij} \text{ are } k\times k \text{ by } *}$

$$= a_{11}|M_{11}^T| - a_{12}|M_{12}^T| + \cdots + a_{1,k+1}|M_{1,k+1}^T|$$

$$= |A^T|$$

Q8

Th If A is $n\times n$ triangular matrix, then $|A| = \prod_{i=1}^{n} a_{ii}$

Proof: Expand $|A|$ over 1^{st} column:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

$$|A| = a_{11} A_{11} + a_{21} A_{21} + \cdots + a_{n1} A_{n1}$$

$$= a_{11} A_{11} \quad \text{expand } |M_{11}| \text{ over } 1^{st} \text{ column}$$

$$= a_{11} |M_{11}|$$

$$A_{11} = \begin{bmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

$$= a_{11}\left[a_{22}|M_{22}| + 0|M_{32}| + \cdots + 0|M_{n2}|\right]$$

$$= a_{11} a_{22} |M_{22}| \quad \text{expand } |M_{22}| \text{ over } 1^{st} \text{ column}$$

$$\vdots$$

$$= a_{11} a_{22} a_{33} \cdots a_{nn} = \prod_{i=1}^{n} a_{ii}$$

Th Let A be $n \times n$ matrix.

1. If A has a zero row or zero column, then $|A| = 0$
2. If A has two identical rows or two identical columns, then, $|A| = 0$
3. If A has two rows (or columns) one is a multiple of the other, then $|A| = 0$

Q9

Proof (1) • Assume that the K^{th} row "colume" of A has zero entries

• Expand $|A|$ over this K^{th} row

$$|A| = a_{K1} A_{K1} + a_{K2} A_{K2} + \cdots + a_{Kn} A_{Kn}$$

$$= 0 + 0 + \cdots + 0 = 0$$

(2) By Induction "Q10" (3) Next section

Exp

• $\begin{vmatrix} 4 & 0 & 2 & 1 \\ 5 & 0 & 4 & 2 \\ 2 & 0 & 3 & 4 \\ 1 & 0 & 2 & 3 \end{vmatrix} = 0$

• $\begin{vmatrix} 3 & -1 & 3 & -2 \\ 0 & 2 & 0 & 3 \\ 5 & 6 & 5 & 1 \\ 4 & 3 & 4 & 0 \end{vmatrix} = 0$

• $\begin{vmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 0 & 2 \\ 4 & -1 & 5 & 2 \\ 0 & 3 & 0 & 6 \end{vmatrix} = 0$
