11.5 Area & length in Polar Coordinates

• Area between the Origin and the Curve $r = f(\theta)$

$$\text{Area} = A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta \qquad \alpha \leq \theta \leq \beta$$

• If you have two curves :-

origin الأبعد عن

$$\text{Area} = \int_{\alpha}^{\beta} \frac{1}{2}\left(r_2^{\,2} - r_1^{\,2}\right) d\theta$$

الأقرب إلى الـorigin

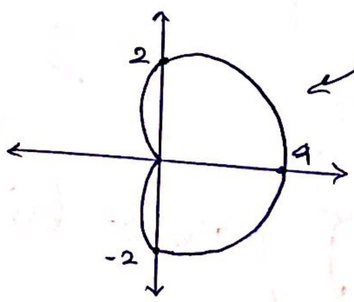
How to find the Area in Polar Coordinates :-

After You Draw the Curve

1- • find the interval of integration $[\alpha, \beta]$

Example :- 1 page 636

The cardioid $r = 2(1 + \cos\theta)$



explained in 11.4

The interval is :- $[0, 2\pi]$

2- write the integration :-

$$A = \int_{0}^{2\pi} \frac{1}{2} r^2 \, d\theta = \int_{0}^{2\pi} \frac{1}{2}\left(2(1+\cos\theta)\right)^2 d\theta$$

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3- • solve the integration

$$= 2\int_0^{2\pi} 1+2\cos\theta+\cos^2\theta \, d\theta$$

$$= 2\left([\theta]_0^{2\pi} + 2\sin\theta\Big]_0^{2\pi} + \int_0^{2\pi}\left(\frac{1+\cos 2\theta}{2}\right)d\theta\right)$$

$$= 2\left(2\pi + 0 + \left(\frac{1}{2}\theta + \frac{\sin 2\theta}{4}\right)\right]_0^{2\pi}$$

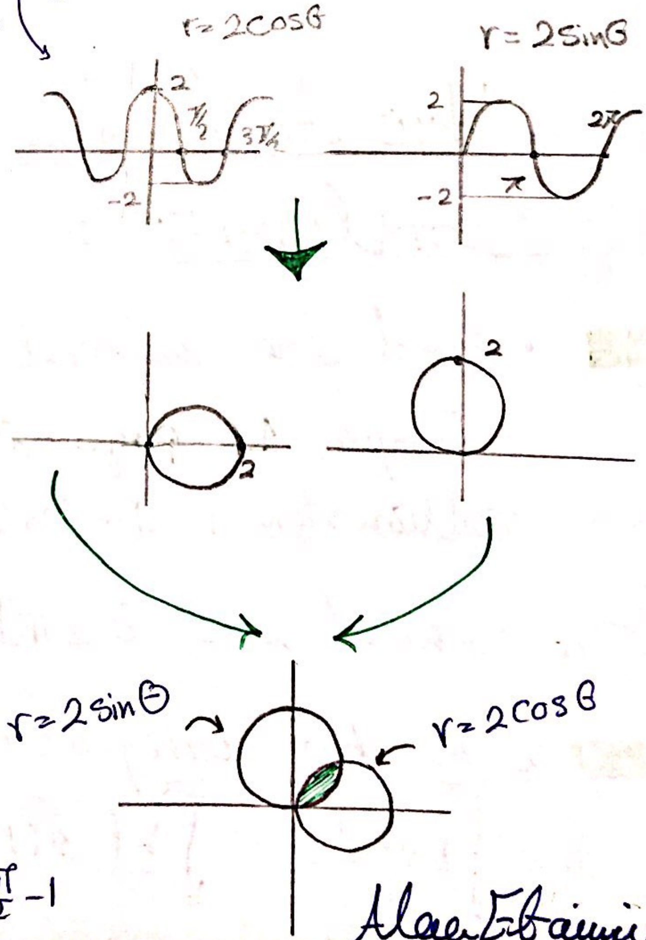
$$= 2\left(2\pi + \frac{1}{2}(2\pi) + 0\right)$$

$$= 2(3\pi) = 6\pi$$

How to find Area between two Curves in Polar Coordinates?

• outline (Question 9 page 638) :- shared by CirCles $r=2\cos\theta$ and $r=2\sin\theta$

Draw them :-



• To find the interval

$r_1 = r_2$

$2\sin\theta = 2\cos\theta$

$\theta = \pi/4$

Using symmetry :-

$$A = 2\left(\frac{1}{2}\right)\int_0^{\pi/4} (2\sin\theta)^2 \, d\theta$$

$$= \int_0^{\pi/4} 4\sin^2\theta \, d\theta$$

$$= \cancel{4}^{2}\int_0^{\pi/4} \frac{1-\cos 2\theta}{\cancel{2}} \, d\theta$$

$$= 2\int_0^{\pi/4} 1-\cos 2\theta \, d\theta = 2\theta - \frac{\cancel{2}\sin 2\theta}{\cancel{2}}\Big]_0^{\pi/4} = \frac{\pi}{2} - 1$$

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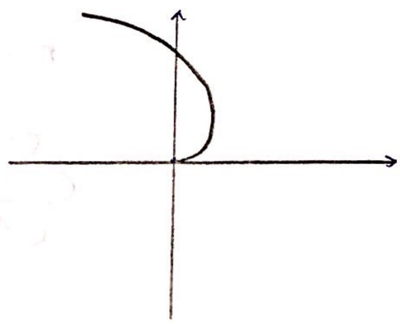
- Sometimes you can't be sure of symmetry so for example in the previous Question You can write integration as :-

$$A = \int_0^{\pi/4} \frac{1}{2}(2\sin\theta)^2\,d\theta + \int_{\pi/4}^{\pi/2} \frac{1}{2}(2\cos\theta)^2\,d\theta$$

- <u>How to find length of the Curve</u>

- outline :- The spiral $r = \theta^2$ $\quad \underline{0 \leq \theta \leq \sqrt{5}}$ interval



- The General Rule is :-

$$\boxed{L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta}$$

$r = \theta^2 \Rightarrow r^2 = \theta^4$

$\frac{dr}{d\theta} = 2\theta\,d\theta$

$$L = \int_0^{\sqrt{5}} \sqrt{\theta^4 + 4\theta^2}\,d\theta$$

let $u = \theta^2 + 4$

$du = 2\theta\,d\theta$

$\theta = 0 \rightarrow u = 4$

$\theta = \sqrt{5} \rightarrow u = 9$

$$= \int_0^{\sqrt{5}} \theta\sqrt{\theta^2 + 4}\,d\theta$$

$$= \int_4^9 \theta\sqrt{u}\,\frac{du}{2\theta}$$

$$= \frac{1}{2}\,\frac{(u)^{3/2}}{\frac{3}{2}}\Bigg]_4^9 = \frac{(9)^{3/2} - (4)^{3/2}}{3} = \frac{27 - 8}{3} = \frac{19}{3}$$

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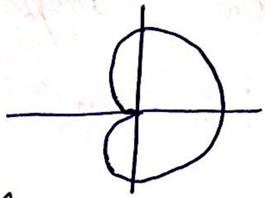
Notes :-

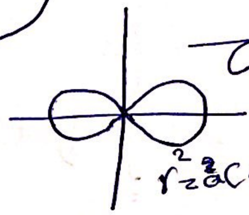
- **Cardiod** : Heart shaped :-



$r = 1 + \cos\theta$

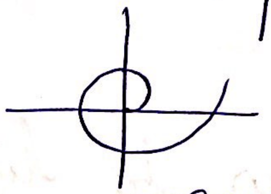
$r^2 = a\sin 2\theta$

- **lemniscate** :- look like infinity



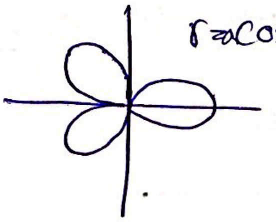
$r^2 = a^2\cos 2\theta$

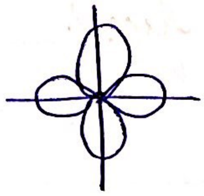
- **Spiral** :- Cochleate حلزوني



- **Roses** :

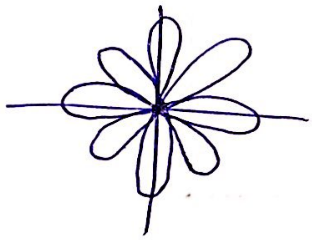
$r = a\cos 3\theta$



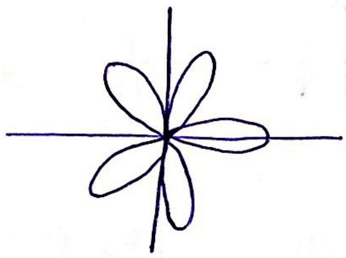


$r = a\cos 2\theta$

$r = a\cos 4\theta$



$r = a\cos 5\theta$



Alaa Etaimi