CHAPTER 19

PROBLEM 19.1

Determine the maximum velocity and maximum acceleration of a particle which moves in simple harmonic motion with an amplitude of 3 mm and a frequency of 20 Hz.

SOLUTION

Frequency: $f = 20 \text{ Hz}$

$$\omega_n = 2\pi f = (2\pi)(20) = 125.66 \text{ rad/s}$$

Amplitude: $x_m = 3 \text{ mm}$

Simple harmonic motion: $x = x_m \sin(\omega_n t + \phi)$

$$v = \dot{x} = \omega_n x_m \cos(\omega_n t + \phi)$$

$$a = \dot{v} = \ddot{x} = -\omega_n^2 x_m \sin(\omega_n t + \phi)$$

Maximum velocity: $v_m = \omega_n x_m = (125.66 \text{ rad/s})(3 \text{ mm})$

$$= 377 \text{ mm/s}$$

$$v_m = 0.377 \text{ m/s} \blacktriangleleft$$

Maximum acceleration: $a_m = \omega_n^2 x_m = (125.66 \text{ rad/s})^2 (3 \text{ mm})$

$$= 47.3 \times 10^3 \text{ mm/s}^2$$

$$a_m = 47.3 \text{ m/s}^2 \blacktriangleleft$$

PROBLEM 19.2

A particle moves in simple harmonic motion. Knowing that the amplitude is 15 in. and the maximum acceleration is 15 ft/s^2, determine the maximum velocity of the particle and the frequency of its motion.

SOLUTION

Simple harmonic motion.

$$x = x_m \sin(\omega_n t + \phi)$$

$$x_m = 15 \text{ in.} = 1.25 \text{ ft}$$

$$\dot{x} = v = x_m \omega_n \cos(\omega_n t + \phi)$$

$$\ddot{x} = a = -x_m \omega_n^2 \sin(\omega_n t + \phi)$$

$$a_m = -x_m \omega_n^2$$

$$|a_m| = 15 \text{ ft/s}^2 = (1.25 \text{ ft})\omega_n^2$$

Natural frequency

$$\omega_n = 3.4641 \text{ rad/s}$$

$$f_n = \frac{\omega_n}{2\pi} = 0.55133 \text{ Hz}$$

$$f_n = 0.551 \text{ Hz} \blacktriangleleft$$

Maximum velocity

$$v_m = x_m \omega_n = (1.25 \text{ ft})(3.4641 \text{ rad/s})$$

$$= 4.3301 \text{ ft/s}$$

$$v_m = 4.33 \text{ ft/s} \blacktriangleleft$$

PROBLEM 19.3

Determine the amplitude and maximum velocity of a particle which moves in simple harmonic motion with a maximum acceleration of 15 ft/s^2 and a frequency of 8 Hz.

SOLUTION

Simple harmonic motion $$x = x_m \sin(\omega_n t + \phi)$$

$$\omega_n = 2\pi f_n = 2\pi(8 \text{ Hz}) = 16\pi \text{ rad/s}$$

$$\dot{x} = v = x_m \omega_n \cos(\omega_n t + \phi)$$

$$v_m = x_m \omega_n$$

$$\ddot{x} = a = -x_m \omega_n^2 \sin(\omega_n t + \phi)$$

$$a_m = x_m \omega_n^2$$

$$15 \text{ ft/s}^2 = x_m (16\pi \text{ rad/s})^2$$

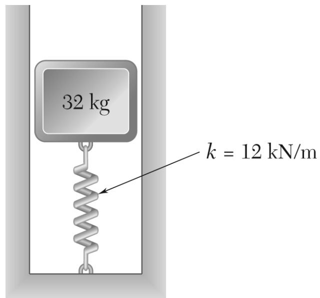
Maximum displacement. $$x_m = 0.005937 \text{ ft} = 0.0712 \text{ in.}$$

$$x_m = 0.0712 \text{ in.} \blacktriangleleft$$

Maximum velocity. $$v_m = x_m \omega_n = (0.005937 \text{ ft})(16\pi \text{ rad/s})$$

$$= 0.2984 \text{ ft/s} = 3.58 \text{ in./s}$$

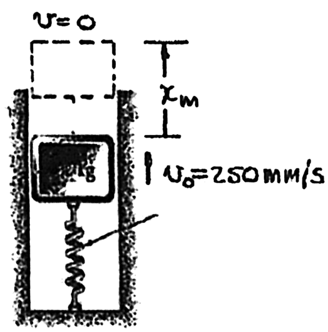
$$v_m = 3.58 \text{ in./s} \blacktriangleleft$$



PROBLEM 19.4

A 32-kg block is attached to a spring and can move without friction in a slot as shown. The block is in its equilibrium position when it is struck by a hammer, which imparts to the block an initial velocity of 250 mm/s. Determine (*a*) the period and frequency of the resulting motion, (*b*) the amplitude of the motion and the maximum acceleration of the block.

SOLUTION



(*a*)

$$x = x_m \sin(\omega_n t + \phi)$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{12\times 10^3 \text{ N/m}}{32 \text{ kg}}}$$

$$\omega_n = 19.365 \text{ rad/s}$$

$$\tau_n = \frac{2\pi}{\omega_n}$$

$$\tau_n = \frac{2\pi}{19.365}$$

$$\tau_n = 0.324 \text{ s} \blacktriangleleft$$

$$f_n = \frac{1}{\tau_n} = \frac{1}{0.324} = 3.08 \text{ Hz} \blacktriangleleft$$

(*b*) At $t = 0$, $x_0 = 0$, $\dot{x}_0 = v_0 = 250 \text{ mm/s}$

Thus, $x_0 = 0 = x_m \sin(\omega_n(0) + \phi)$

and $\phi = 0$

$$\dot{x}_0 = v_0 = x_m \omega_n \cos(\omega_n(0) + 0) = x_m \omega_n$$

$$v_0 = 0.250 \text{ m/s} = x_m (19.365 \text{ rad/s})$$

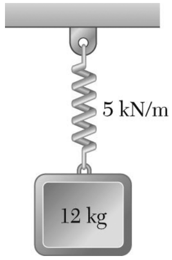
$$x_m = \frac{(0.250 \text{ m/s})}{(19.365 \text{ rad/s})}$$

$$x_m = 12.91\times 10^{-3} \text{ m}$$

$$x_m = 12.91 \text{ mm} \blacktriangleleft$$

$$a_m = x_m \omega_n^2 = (12.91\times 10^{-3} \text{ m})(19.365 \text{ rad/s})^2$$

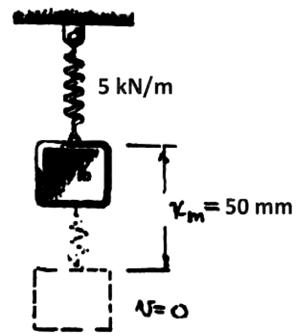
$$a_m = 4.84 \text{ m/s}^2 \blacktriangleleft$$



PROBLEM 19.5

A 12-kg block is supported by the spring shown. If the block is moved vertically downward from its equilibrium position and released, determine (*a*) the period and frequency of the resulting motion, (*b*) the maximum velocity and acceleration of the block if the amplitude of its motion is 50 mm.

SOLUTION



(*a*) Simple harmonic motion. $x = x_m \sin(\omega_n t + \phi)$

Natural frequency. $\omega_n = \sqrt{\dfrac{k}{m}}$ $\quad k = 5\text{ kN/m} = 5000\text{ N/m}$

$$\omega_n = \sqrt{\frac{(5000\text{ N/m})}{12\text{ kg}}}$$

$$\omega_n = 20.412\text{ rad/s}$$

$$\tau_n = \frac{2\pi}{\omega_n}$$

$$\tau_n = \frac{2\pi}{20.412} = 0.30781\text{ s}$$

$\tau_n = 0.308\text{ s}$ ◀

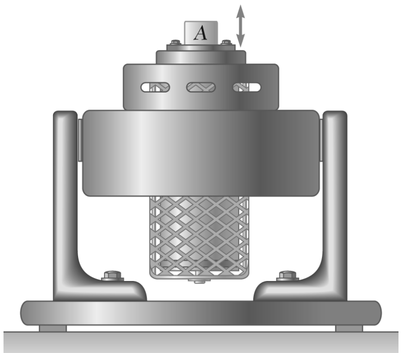
$f_n = \dfrac{1}{\tau_n} = \dfrac{1}{0.30781} = 3.25\text{ Hz}$ ◀

(*b*) $x_m = 50\text{ mm} = 0.05\text{ m}$

$$x = 0.05\sin(20.412t + \phi)$$

Maximum velocity. $v_m = x_m\omega_n = (0.05\text{ m})(20.412\text{ rad/s})$ $\quad v_m = 1.021\text{ m/s}$ ◀

Maximum acceleration. $a_m = x_m\omega_n^2 = (0.05\text{ m})(20.412\text{ rad/s})^2$ $\quad a_m = 20.8\text{ m/s}^2$ ◀



PROBLEM 19.6

An instrument package A is bolted to a shaker table as shown. The table moves vertically in simple harmonic motion at the same frequency as the variable-speed motor which drives it. The package is to be tested at a peak acceleration of 150 ft/s^2. Knowing that the amplitude of the shaker table is 2.3 in., determine (*a*) the required speed of the motor in rpm, (*b*) the maximum velocity of the table.

SOLUTION

In simple harmonic motion,

$$a_{max} = x_{max}\omega_n^2$$

$$150 \text{ ft/s}^2 = \left(\frac{2.3}{12}\text{ ft}\right)\omega_n^2$$

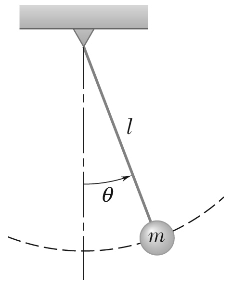
$$\omega_n^2 = (782.6 \text{ rad/s})^2$$

$$\omega_n = 27.98 \text{ rad/s}$$

$$f_n = \frac{\omega_n}{2\pi} = \frac{27.98}{2\pi} = 4.452 \text{ Hz (cycles per second)}$$

(*a*) Motor speed. $(4.452 \text{ rev/s})(60 \text{ s/min})$ speed = 267 rpm ◀

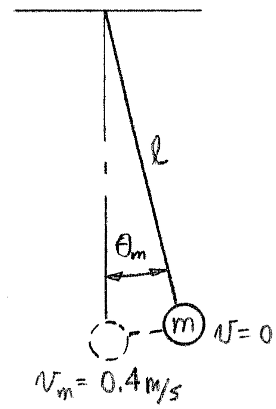
(*b*) Maximum velocity. $v_{max} = x_{max}\omega_n = \left(\frac{2.3}{12}\text{ft}\right)(27.98 \text{ rad/s})$ $v_{max} = 5.36$ ft/s ◀



PROBLEM 19.7

A simple pendulum consisting of a bob attached to a cord oscillates in a vertical plane with a period of 1.3 s. Assuming simple harmonic motion and knowing that the maximum velocity of the bob is 0.4 m/s, determine (*a*) the amplitude of the motion in degrees, (*b*) the maximum tangential acceleration of the bob.

SOLUTION



(*a*) Simple harmonic motion $\theta = \theta_m \sin(\omega_n t + \phi)$

$$\omega_n = \frac{2\pi}{\tau_n} = \frac{2\pi}{(1.3\text{ s})}$$

$$= 4.8332 \text{ rad/s}$$

$$\dot{\theta} = \theta_m \omega_n \cos(\omega_n t + \phi)$$

$$\dot{\theta}_m = \theta_m \omega_n$$

$$v_m = l\dot{\theta}_m = l\theta_m \omega_n$$

Thus, $$\theta_m = \frac{v_m}{l\omega_n} \quad (1)$$

For a simple pendulum, $$\omega_n = \sqrt{\frac{g}{l}}$$

Thus, $$l = \frac{g}{\omega_n^2} = \frac{9.81 \text{ m/s}^2}{(4.8332 \text{ rad/s})^2}$$

$$= 0.41995 \text{ m}$$

Amplitude from (1), $$\theta_m = \frac{v_m}{l\omega_n} = \frac{0.4 \text{ m/s}}{(0.42 \text{ m})(4.833 \text{ rad/s})}$$

$$= 0.19707 \text{ rad}$$

$$= 11.291°$$

$\theta_m = 11.29°$ ◀

(*b*) Maximum tangential acceleration $a_t = l\ddot{\theta}$

The maximum tangential acceleration occurs when $\ddot{\theta}$ is maximum.

$$\ddot{\theta} = -\theta_m \omega_n^2 \sin(\omega_n t + \phi)$$

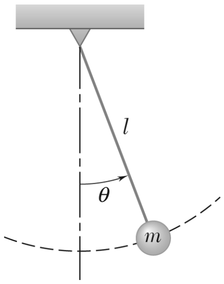
$$\ddot{\theta}_m = \theta_m \omega_n^2$$

$$(a_t)_m = l\theta_m \omega_n^2$$

$$(a_t)_m = (0.41995 \text{ m})(0.19707 \text{ rad})(4.8332 \text{ rad/s})^2$$

$$= 1.933 \text{ m/s}^2$$

$(a_t)_m = 1.933 \text{ m/s}^2$ ◀



PROBLEM 19.8

A simple pendulum consisting of a bob attached to a cord of length $l = 800$ mm oscillates in a vertical plane. Assuming simple harmonic motion and knowing that the bob is released from rest when $\theta = 6°$, determine (*a*) the frequency of oscillation, (*b*) the maximum velocity of the bob.

SOLUTION

(*a*) Frequency.

$$\omega_n = \sqrt{\frac{g}{l}} = \sqrt{\frac{(9.81\ \text{m/s}^2)}{(0.8\ \text{m})}}$$

$$\omega_n = 3.502\ \text{rad/s}$$

$$f_n = \frac{\omega_n}{2\pi} = \frac{(3.502\ \text{rad/s})}{2\pi}$$

$f_n = 0.557$ Hz ◀

(*b*) Simple harmonic motion.

$$\theta = \theta_m \sin(\omega_n t + \phi)$$

where

$$\theta_m = 6° = 0.10472\ \text{rad}$$

Maximum velocity.

$$\dot{\theta} = \theta_m \omega_n \cos(\omega_n t + \phi)$$

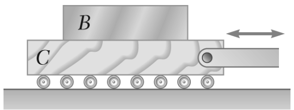
$$\dot{\theta}_m = \theta_m \omega_n$$

$$v_m = l\dot{\theta}_m = l\theta_m\omega_n = (0.8\ \text{m})(0.10472)(3.502)$$

$$v_m = 293.4 \times 10^{-3}\ \text{m/s}$$

$v_m = 293$ mm/s ◀

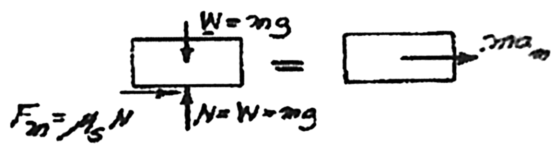
PROBLEM 19.9



An instrument package B is placed on the shaking table C as shown. The table is made to move horizontally in simple harmonic motion with a frequency of 3 Hz. Knowing that the coefficient of static friction is $\mu_s = 0.40$ between the package and the table, determine the largest allowable amplitude of the motion if the package is not to slip on the table. Given the answers in both SI and U.S. customary units.

SOLUTION

Maximum allowable acceleration of B.



$$\mu_s = 0.40$$

$\xrightarrow{+} \Sigma F = ma$:

$$F_m = ma_m$$

$$\mu_s mg = ma_m$$

$$a_m = \mu_s g \qquad a_m = 0.40g$$

Simple harmonic motion.

$$f_n = 3\text{ Hz} = \frac{\omega_n}{2\pi}$$

$$\omega_n = 6\pi \text{ rad/s}$$

$$a_m = x_m \omega_n^2$$

$$0.40g = x_m (6\pi \text{ rad/s})^2$$

$$x_m = 1.1258 \times 10^{-3} g$$

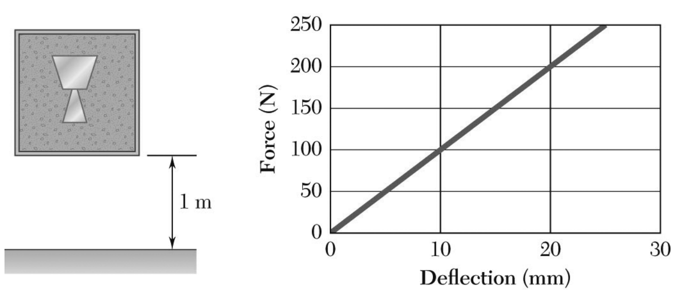
Largest allowable amplitude.

SI: $x_m = 1.1258 \times 10^{-3}(9.81) = 11.044 \times 10^{-3}$ m $\qquad x_m = 11.04$ mm ◀

U.S.: $x_m = 1.1258 \times 10^{-3}(32.2) = 0.03625$ ft $\qquad x_m = 0.435$ in. ◀

PROBLEM 19.10

A 5-kg fragile glass vase is surrounded by packing material in a cardboard box of negligible weight. The packing material has negligible damping and a force-deflection relationship as shown. Knowing that the box is dropped from a height of 1 m and the impact with the ground is perfectly plastic, determine (*a*) the amplitude of vibration for the vase, (*b*) the maximum acceleration the vase experiences in g's.

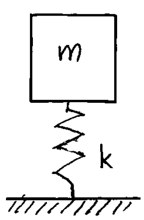


SOLUTION

Velocity at end of free fall: $v = \sqrt{2gh}$

$$v = \sqrt{(2)(9.81 \text{ m/s}^2)(1 \text{ m})} = 4.4294 \text{ m/s}$$

Assume that the spring is unstretched during the free fall. Use a simple spring-mass model for the motion of the vase and the packing material.



$$m = 5 \text{ kg}$$

$$k = \frac{100 \text{ N}}{10 \text{ mm}} \quad \text{(slope from graph)}$$

$$k = 10 \text{ N/m} = 10000 \text{ N/m}$$

Natural frequency: $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{10000 \text{ N/m}}{5 \text{ kg}}} = 44.721 \text{ rad/s}$

Simple harmonic motion: $x = x_m \sin(\omega_n t + \phi)$

$$v = \dot{x} = \omega_n x_m \cos(\omega_n t + \phi)$$

Let $t = 0$ at the instant when the box bottom hits the ground.

Then, at $t = 0$, $x = 0$ and $v = 4.4294 \text{ m/s}$

from which $\phi = 0$

and $\omega_n x_m = 4.4294 \text{ m/s}$

PROBLEM 19.10 (Continued)

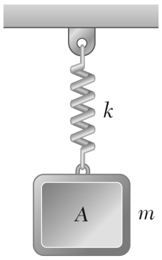
(*a*) Amplitude: $x_m = \dfrac{4.4294 \text{ m/s}}{44.721 \text{ rad/s}} = 0.099045 \text{ m}$

$x_m = 99.0 \text{ mm}$ ◀

(*b*) Maximum acceleration:

$$a_m = \omega_n^2 x_m = (44.721 \text{ rad/s})^2 (0.099045 \text{ m})$$
$$= 198.087 \text{ m/s}^2 = (20.192)(9.81 \text{ m/s}^2)$$

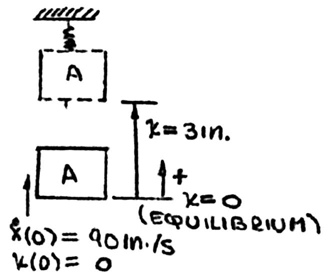
$a_m = 20.2\, g$ ◀



PROBLEM 19.11

A 3-lb block is supported as shown by a spring of constant $k = 2$ lb/in. which can act in tension or compression. The block is in its equilibrium position when it is struck from below by a hammer which imparts to the block an upward velocity of 90 in./s. Determine (*a*) the time required for the block to move 3 in. upward, (*b*) the corresponding velocity and acceleration of the block.

SOLUTION



Simple harmonic motion. $x = x_m \sin(\omega_n t + \phi)$

Natural frequency. $\omega_n = \sqrt{\dfrac{k}{m}}, \quad k = 2 \text{ lb/in.} = 24 \text{ lb/ft}$

$$\omega_n = \sqrt{\dfrac{24 \text{ lb/ft}}{\left(\frac{3 \text{ lb}}{32.2 \text{ ft/s}^2}\right)}}$$

$$\omega_n = 16.05 \text{ rad/s}$$

$$x(0) = 0 = x_m \sin(0 + \phi)$$

$$\phi = 0$$

$$\dot{x}(0) = x_m \omega_n \cos(0 + 0)$$

$$\dot{x}(0) = \frac{90}{12} = 7.5 \text{ ft/s}$$

$$7.5 = x_m (16.05) \qquad x_m = 0.4673 \text{ ft}$$

$$x = (0.4673)\sin(16.05t)(\text{ft/s}) \tag{1}$$

(*a*) Time at $x = 3$ in. ($x = 0.25$ ft)

$$0.25 = 0.4673 \sin(16.05t)$$

$$t = \frac{\sin^{-1}\left(\frac{0.25}{0.4673}\right)}{16.05}$$

$t = 0.0352$ s ◀

(*b*) Velocity and acceleration.

$$\dot{x} = x_m \omega_n \cos(\omega_n t)$$

$$\ddot{x} = -x_m \omega_n^2 \sin \omega_n t$$

$$t = 0.0352$$

$$\dot{x} = (0.4673)(16.05)\cos[(16.05)(0.0352)]$$

$$\dot{x} = 6.34 \text{ ft/s}$$

$\mathbf{v} = 6.34$ ft/s ↑ ◀

$$\ddot{x} = -(0.4673)(16.05)^2 \sin[(16.05)(0.0352)]$$

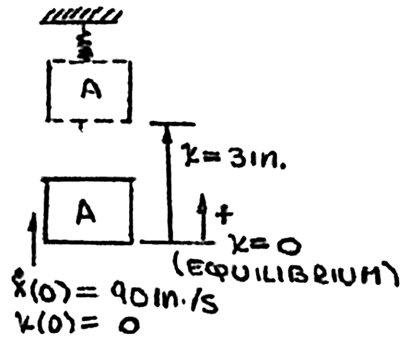
$$= -64.4 \text{ ft/s}^2$$

$\mathbf{a} = 64.4$ ft/s^2 ↓ ◀

PROBLEM 19.12

In Problem 19.11, determine the position, velocity, and acceleration of the block 0.90 s after it has been struck by the hammer.

SOLUTION



Simple harmonic motion. $x = x_m \sin(\omega_n t + \phi)$

Natural frequency. $\omega_n = \sqrt{\dfrac{k}{m}}, \quad k = 2 \text{ lb/in.} = 24 \text{ lb/ft}$

$$\omega_n = \sqrt{\dfrac{24 \text{ lb/ft}}{\left(\frac{3 \text{ lb}}{32.2 \text{ ft/s}^2}\right)}}$$

$$\omega_n = 16.05 \text{ rad/s}$$

$$x(0) = 0 = x_m \sin(0 + \phi)$$

$$\phi = 0$$

$$\dot{x}(0) = x_m \omega_n \cos(0 + 0) \qquad \dot{x}(0) = \frac{90}{12} = 7.5 \text{ ft/s}$$

$$7.5 = x_m (16.05) \qquad x_m = 0.4673 \text{ ft}$$

$$x = (0.4673)\sin(16.05t)(\text{ft/s})$$

Simple harmonic motion. $x = x_m \sin(\omega_n t + \phi)$

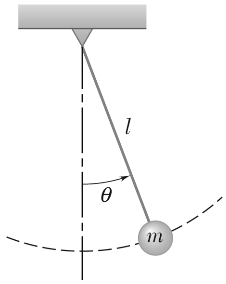
$$\dot{x} = x_m \omega_n \cos(\omega_n t + \phi)$$

$$\ddot{x} = -x_m \omega_n^2 \sin(\omega_n t + \phi)$$

At 0.90 s: $x = (0.4673)\sin[(16.05)(0.90)] = 0.445 \text{ ft}$ $\qquad \mathbf{x} = 0.445 \text{ ft} \uparrow$ ◀

$\dot{x} = (0.4673)(16.05)\cos[(16.05)(0.90)] = -2.27 \text{ ft/s}$ $\qquad \mathbf{v} = 2.27 \text{ ft/s} \downarrow$ ◀

$\ddot{x} = -(0.4673)(16.05)^2 \sin[(16.05)(0.90)] = -114.7 \text{ ft/s}^2$ $\qquad \mathbf{a} = 114.7 \text{ ft/s}^2 \downarrow$ ◀



PROBLEM 19.13

The bob of a simple pendulum of length $l = 40$ in. is released from rest when $\theta = +5°$. Assuming simple harmonic motion, determine 1.6 s after release (*a*) the angle θ, (*b*) the magnitudes of the velocity and acceleration of the bob.

SOLUTION

For simple harmonic motion and $l = 40\text{ in.} = 3.333\text{ ft}$:

$$\omega_n = \sqrt{\frac{g}{l}} = \sqrt{\frac{32.2\text{ ft/s}^2}{3.333\text{ ft}}} = 3.1082\text{ rad/s}$$

Angular displacement: $\theta = \theta_m \sin(\omega_n t + \phi)$

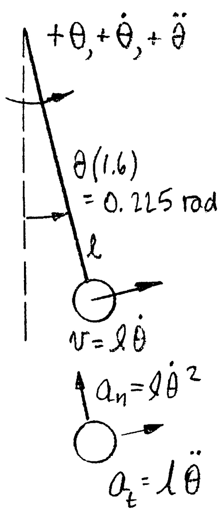
Initial conditions: $\theta(0) = 5° = 0.08727\text{ rad}, \quad \text{and} \quad \dot{\theta}(0) = 0:$

$$\dot{\theta}(0) = 0 = \theta_m \omega_n \cos(0 + \phi) \qquad \phi = \frac{\pi}{2}$$

$$\theta_m = \theta(0) = \frac{5\pi}{180} = 0.08727\text{ rad}$$

$$\theta = \frac{5\pi}{180}\sin\left[(3.1082\text{ rad/s})t + \frac{\pi}{2}\right]$$

$$= (0.08727\text{ rad})\sin\left[(3.1082\text{ rad/s})t + \frac{\pi}{2}\right]$$



(*a*) At $t = 1.6$ s.

$$\theta = \frac{5\pi}{180}\sin\left[(3.1082\text{ rad/s})(1.6\text{ s}) + \frac{\pi}{2}\right]$$

$$= 0.022496\text{ rad} = 1.288°$$

$\theta = 1.288°$ ◀

(*b*) Velocity:

$$\dot{\theta} = \theta_m \omega_n \cos(\omega_n t + \phi)$$

$$= \frac{5\pi}{180}(3.1082\text{ rad/s})\cos\left[(3.1082\text{ rad/s})(1.6\text{ s}) + \frac{\pi}{2}\right]$$

$$= 0.262074\text{ rad/s}$$

$$v = l\dot{\theta} = (3.3333\text{ ft})(0.262074\text{ rad/s}) = 0.874\text{ ft/s}$$

$v = 0.874\text{ ft/s}$ ◀

PROBLEM 19.13 (Continued)

Angular acceleration:

$$\ddot{\theta} = -\theta_m \omega_n^2 \sin(\omega_n t + \phi) = -\frac{5\pi}{180}(3.1082 \text{ rad/s})^2 \cos\left[(3.1082 \text{ rad/s})(1.6 \text{ s}) + \frac{\pi}{2}\right]$$

$$= -0.21733 \text{ rad/s}^2$$

Acceleration:

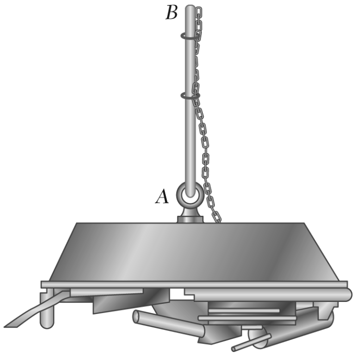
$$a = \sqrt{(a_n)^2 + (a_t)^2}$$

$$a_n = \frac{v^2}{l} = l\dot{\theta}^2 = (3.3333 \text{ ft})(0.26207 \text{ rad/s})^2 = 0.22894 \text{ ft/s}^2$$

$$a_t = l\ddot{\theta} = (3.333 \text{ ft})(-0.21733 \text{ rad/s}^2)^2 = -0.72443 \text{ m/s}^2$$

$$a = 0.75974 \text{ ft/s}^2$$

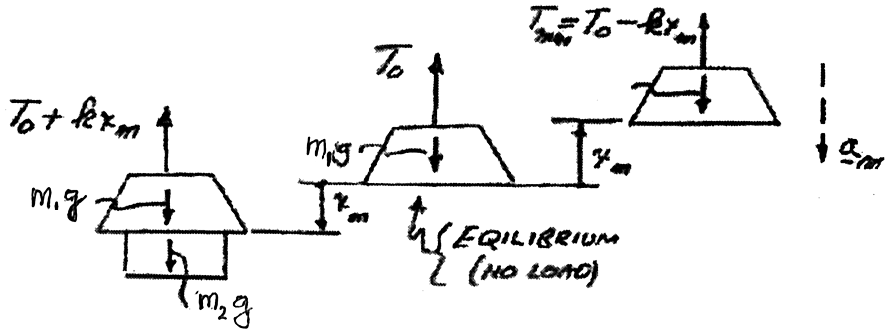
$a = 0.760 \text{ ft/s}^2$ ◀



PROBLEM 19.14

A 150-kg electromagnet is at rest and is holding 100 kg of scrap steel when the current is turned off and the steel is dropped. Knowing that the cable and the supporting crane have a total stiffness equivalent to a spring of constant 200 kN/m, determine (*a*) the frequency, the amplitude, and the maximum velocity of the resulting motion, (*b*) the minimum tension which will occur in the cable during the motion, (*c*) the velocity of the magnet 0.03 s after the current is turned off.

SOLUTION



Data: $m_1 = 150 \text{ kg} \quad m_2 = 100 \text{ kg} \quad k = 200 \times 10^3 \text{ N/m}$

From the first two sketches, $T_0 + kx_m = (m_1 + m_2)g$ (1)

$$T_0 = m_1 g \qquad (2)$$

Subtracting Eq. (2) from Eq. (1), $kx_m = m_2 g$

$$x_m = \frac{m_2 g}{k} = \frac{(100)(9.81)}{200 \times 10^3} = 4.905 \times 10^{-3} \text{ m} = 4.91 \text{ mm}$$

Natural circular frequency: $\omega_n = \sqrt{\frac{k}{m_1}} = \sqrt{\frac{200 \times 10^3}{150}} = 36.515 \text{ rad/s}$

Natural frequency: $f_n = \frac{\omega_n}{2\pi} = \frac{36.515}{2\pi} \qquad f_n = 5.81 \text{ Hz}$

Maximum velocity: $v_m = \omega_n x_m = (36.515)(4.905 \times 10^{-3}) = 0.1791 \text{ m/s}$

(*a*) Resulting motion: amplitude $x_m = 4.91 \text{ mm}$ ◀

frequency $f_n = 5.81 \text{ Hz}$ ◀

maximum velocity $v_m = 0.1791 \text{ m/s}$ ◀

PROBLEM 19.14 (Continued)

(*b*) Minimum value of tension occurs when $x = -x_m$.

$$T_{\min} = T_0 - kx_m$$
$$= m_1 g - m_2 g$$
$$= (m_1 - m_2)g$$
$$= (50)(9.81)$$

$T_{\min} = 491\text{ N}$ ◀

The motion is given by $x = x_m \sin(\omega_n t + \varphi)$

$$\dot{x} = \omega_n x_m \cos(\omega_n t + \varphi)$$

Initally, $x_0 = -x_m$ or $\sin\varphi = -1$

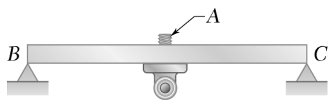
$$\dot{x}_0 = 0 \quad \text{or} \quad \cos\varphi = 0$$
$$\varphi = -\frac{\pi}{2}$$
$$\dot{x} = \omega_n x_m \cos\left(\omega_n t - \frac{\pi}{2}\right)$$

(*c*) Velocity at $t = 0.03$ s.

$$\omega_n t = (36.515)(0.03) = 1.09545\text{ rad}$$
$$\omega_n t - \varphi = -0.47535\text{ rad}$$
$$\cos(\omega_n t - \varphi) = 0.88913$$
$$\dot{x} = (36.515)(4.905\times 10^{-3})(0.88913)$$

$\dot{\mathbf{x}} = 0.1592\text{ m/s}\uparrow$ ◀

PROBLEM 19.15



A variable-speed motor is rigidly attached to beam *BC*. The rotor is slightly unbalanced and causes the beam to vibrate with a frequency equal to the motor speed. When the speed of the motor is less than 600 rpm or more than 1200 rpm, a small object placed at *A* is observed to remain in contact with the beam. For speeds between 600 rpm and 1200 rpm, the object is observed to "dance" and actually to lose contact with the beam. Determine the amplitude of the motion of *A* when the speed of the motor is (*a*) 600 rpm, (*b*) 1200 rpm. Give answers in both SI and U.S. customary units.

SOLUTION

At both 600 rpm and 1200 rpm, the maximum acceleration is just equal to g.

(*a*) $\omega = 600 \text{ rpm} = 62.832 \text{ rad/s}$

Eq. (19.15): $a_m = x_m\omega^2 \qquad x_m = \dfrac{g}{(62.832)^2}$

SI: $x_m = \dfrac{9.81}{(62.832)^2} = 2.4849 \times 10^{-3} \text{ m}$ $\qquad x_m = 2.48 \text{ mm}$ ◀

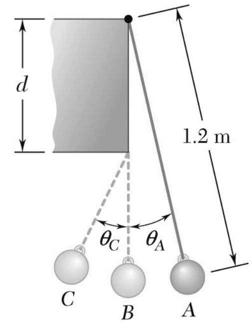
US: $x_m = \dfrac{32.2}{(62.832)^2} = 0.008156 \text{ ft}$ $\qquad x_m = 0.0979 \text{ in.}$ ◀

(*b*) $\omega = 1200 \text{ rpm} = 125.664 \text{ rad/s}$

Eq. (19.15): $a_m = x_m\omega^2 \qquad x_m = \dfrac{g}{(125.664)^2}$

SI: $x_m = \dfrac{9.81}{(125.664)^2} = 621.2 \times 10^{-6} \text{ m}$ $\qquad x_m = 0.621 \text{ mm}$ ◀

US: $x_m = \dfrac{32.2}{(125.664)^2} = 0.002039 \text{ ft}$ $\qquad x_m = 0.0245 \text{ in.}$ ◀



PROBLEM 19.16

A small bob is attached to a cord of length 1.2 m and is released from rest when $\theta_A = 5°$. Knowing that $d = 0.6$ m, determine (*a*) the time required for the bob to return to Point *A*, (*b*) the amplitude θ_C.

SOLUTION

As the pendulum moves between Points *A* and *B*, the length of the pendulum is $l = l_{AB} = 1.2$ m.

$$\omega_n = \omega_{n1} = \sqrt{\frac{g}{l_{AB}}} = \sqrt{\frac{9.81 \text{ m/s}^2}{1.2 \text{ m}}} = 2.8592 \text{ rad/s}$$

$$\tau_1 = \frac{2\pi}{\omega_{n1}} = \frac{2\pi}{2.8592 \text{ rad/s}} = 2.1975 \text{ s}$$

The falling from *A* to *B* is one quarter period.

$$\tau_{AB} = \frac{1}{4}\tau_1 = 0.54938 \text{ s.}$$

As the pendulum moves between Points *B* and *C*, the length of the pendulum is $l = l_{BC} = 1.2 \text{ m} - 0.6 \text{ m} = 0.6 \text{ m}$.

$$\omega_n = \omega_{n2} = \sqrt{\frac{g}{l_{BC}}} = \sqrt{\frac{9.81 \text{ m/s}^2}{0.6 \text{ m}}} = 4.0435 \text{ rad/s}$$

$$\tau_2 = \frac{2\pi}{\omega_{n2}} = \frac{2\pi}{4.0435 \text{ rad/s}} = 1.55389 \text{ s}$$

The motion from *B* to *C* and back to *B* is one half period

$$\tau_{BCB} = \frac{1}{2}\tau_2 = 0.77695 \text{ s}$$

As the pendulum moves from *B* to *A*, the length is again 1.2 meters.

$$\tau_{BA} = \frac{1}{4}\tau_1 = 0.54938 \text{ s}$$

(*a*) Time required to return to *A*.

$$\tau = \tau_{AB} + \tau_{BCB} + \tau_{BA}$$

$$\tau = 1.87571 \text{ s}$$

$\tau = 1.876$ s ◀

For falling from A to B, $\theta_m = \theta_A$

At B, $\dot{\theta}_B = \dot{\theta}_m = \omega_{n1}\theta_A$

$$v_B = l_{AB}\dot{\theta}_B = l_{AB}\omega_{n1}\theta_A$$

For rising from B to C,

$$\dot{\theta}_B = \frac{v_B}{l_{BC}} = \frac{l_{AB}}{l_{BC}}\omega_{n1}\theta_A = \dot{\theta}_{\max}$$

$$\theta_C = \theta_{\max} = \frac{\dot{\theta}_{\max}}{\omega_{n2}} = \frac{l_{AB}\omega_{n1}}{l_{BC}\omega_{n2}}\theta_A$$

$$\theta_C = \frac{(1.2 \text{ m})(2.8592 \text{ rad/s})}{(0.6 \text{ m})(4.0435 \text{ rad/s})}\theta_A = 1.4142\,\theta_A$$

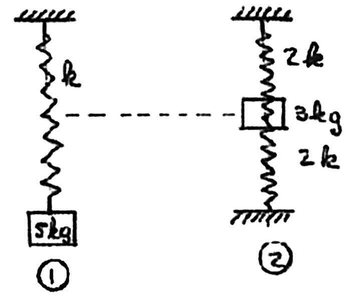
(*b*) Amplitude θ_C:

With $\theta_A = 5°$, $\theta_C = 7.07°$ ◀

PROBLEM 19.17

A 5-kg block, attached to the lower end of a spring whose upper end is fixed, vibrates with a period of 6.8 s. Knowing that the constant k of a spring is inversely proportional to its length, determine the period of a 3-kg block which is attached to the center of the same spring if the upper and lower ends of the spring are fixed.

SOLUTION



Equivalent spring constant.

$$k' = 2k + 2k = 4k \qquad \text{(Deflection of each spring is the same.)}$$

For case ①,

$$\tau_{n1} = 6.8 \text{ s}$$

$$\omega_{n1} = \frac{2\pi}{\tau_{n1}} = \frac{2\pi}{6.8} = 0.924 \text{ rad/s}$$

$$\omega_n^2 = \frac{k}{m_1}$$

$$k = m_1\omega_{n1}^2 = (5)(0.924)^2 = 4.2689 \text{ N/m}$$

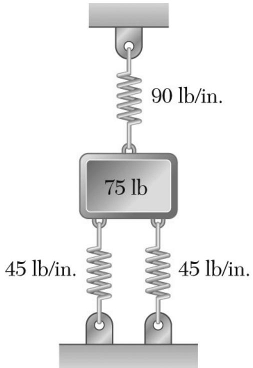
For case ②,

$$\omega_{n2}^2 = \frac{4k}{m_2} = \frac{(4)(4.2689)}{3} = 5.6918 \text{ (rad/s)}^2$$

$$\omega_{n2} = 2.3857 \text{ rad/s}$$

$$\tau_n = \frac{2\pi}{\omega_{n2}} = \frac{2\pi}{2.3857}$$

$\tau_{n2} = 2.63$ s ◀

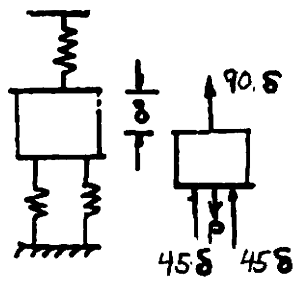


PROBLEM 19.18

A 75-lb block is supported by the spring arrangement shown. The block is moved vertically downward from its equilibrium position and released. Knowing that the amplitude of the resulting motion is 2 in., determine (*a*) the period and frequency of the motion, (*b*) the maximum velocity and maximum acceleration of the block.

SOLUTION

(*a*) Determine the constant k of a single spring equivalent to the three springs



$$P = k\delta$$

$$k\delta = 90\delta + 45\delta + 45\delta$$

$$k = 180 \text{ lb/in.} = 2160 \text{ lb/ft}$$

Natural frequency.

$$\omega_n = \sqrt{\frac{k}{m}}$$

$$= \sqrt{\frac{2160 \text{ lb/ft}}{\frac{75 \text{ lb}}{32.2 \text{ ft/s}^2}}}$$

$$\omega_n = 30.453 \text{ rad/s}$$

$$\tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{30.453} = 0.20633 \text{ s} \qquad \tau_n = 0.206 \text{ s} \blacktriangleleft$$

$$f_n = \frac{1}{\tau_n} \qquad f_n = 4.85 \text{ Hz} \blacktriangleleft$$

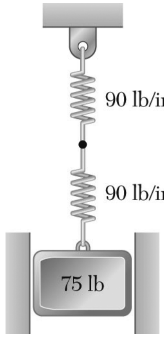
(*b*) $$x = x_m \sin(\omega_n t + \phi) \qquad x_0 = 2 \text{ in.} = 0.16667 \text{ ft} = x_m$$

$$\omega_n = 30.453 \text{ rad/s}$$

$$x = 0.16667 \sin(30.453t + \phi)$$

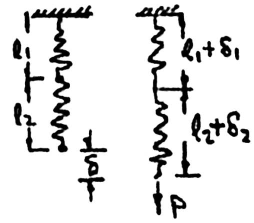
$$\dot{x} = (0.16667)(30.453)\cos(30.453t + \phi) \qquad v_{max} = 5.08 \text{ ft/s} \blacktriangleleft$$

$$\ddot{x} = -(0.16667)(30.453)^2 \sin(30.453t + \phi) \qquad a_{max} = 154.6 \text{ ft/s}^2 \blacktriangleleft$$

	**PROBLEM 19.19** A 75-lb block is supported by the spring arrangement shown. The block is moved vertically downward from its equilibrium position and released. Knowing that the amplitude of the resulting motion is 2 in., determine (*a*) the period and frequency of the motion, (*b*) the maximum velocity and maximum acceleration of the block.

SOLUTION

(*a*) Determine the constant *k* of a single spring equivalent to the two springs shown.



$$\delta = \delta_1 + \delta_2 = \frac{P}{90\text{ lb/in.}} + \frac{P}{90\text{ lb/in.}} = \frac{P}{k}$$

$$\frac{1}{k} = \frac{1}{90} + \frac{1}{90} \qquad k = 45\text{ lb/in.} = 540\text{ lb/ft}$$

<u>Period of the motion</u>.

$$\tau_n = \frac{2\pi}{\sqrt{\frac{k}{m}}} = \frac{2\pi}{\sqrt{\frac{540}{75/32.2}}} = 0.41265\text{ s} \qquad \tau_n = 0.413\text{ s} \blacktriangleleft$$

$$f_n = \frac{1}{\tau_n} = \frac{1}{0.41265} = 2.42\text{ Hz} \blacktriangleleft$$

(*b*)

$$x = x_m \sin(\omega_n t + \phi)\ x_0 = 2\text{ in.} = 0.16667\text{ ft} = x_m$$

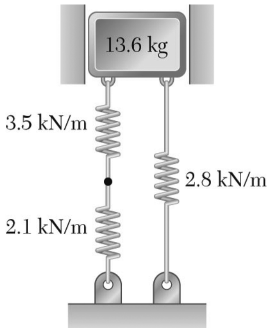
$$\omega_n = 2\pi f_n = 2\pi(2.4233) = 15.226\text{ rad/s}$$

$$x = 0.16667\sin(15.226t + \phi)$$

$$\dot{x} = (0.16667)(15.226)\cos(15.226t + \phi)$$

$$v_{max} = 2.54\text{ ft/s} \blacktriangleleft$$

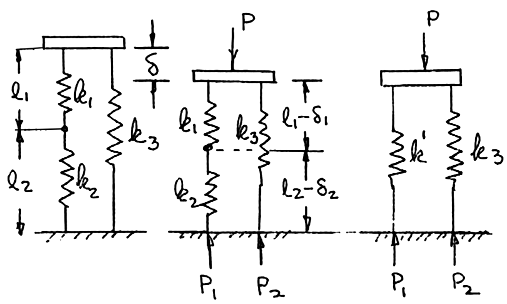
$$\ddot{x} = -(0.16667)(15.226)^2 \sin(15.226t + \phi) \qquad a_{max} = 38.6\text{ ft/s}^2 \blacktriangleleft$$



PROBLEM 19.20

A 13.6-kg block is supported by the spring arrangement shown. If the block is moved from its equilibrium position 44 mm vertically downward and released, determine (*a*) the period and frequency of the resulting motion, (*b*) the maximum velocity and acceleration of the block.

SOLUTION



Determine the constant k of a single spring equivalent to the three springs shown.

Springs 1 and 2: $\delta = \delta_1 + \delta_2$, and $\dfrac{P_1}{k'} = \dfrac{P_1}{k_1} + \dfrac{P_1}{k_2}$

Hence, $k' = \dfrac{k_1 k_2}{k_1 + k_2}$

where k' is the spring constant of a single spring equivalent of springs 1 and 2.
Springs k' and 3: (Deflection in each spring is the same).

So $P = P_1 + P_2$, and $P = k\delta$, $P_1 = k'\delta$, $P_2 = k_3\delta$

Now $k\delta = k'\delta + k_3\delta$

$$k = k' + k_3 = \frac{k_1 k_2}{k_1 + k_2} + k_3$$

or
$$k = \frac{(3.5\text{ kN/m})(2.1\text{ kN/m})}{(3.5\text{ kN/m}) + (2.1\text{ kN/m})} + 2.8\text{ kN/m} = 4.11\text{ kN/m} = 4.11 \times 10^3\text{ N/m}$$

(*a*) Period and frequency: $\tau_n = \dfrac{2\pi}{\sqrt{\frac{k}{m}}} = \dfrac{2\pi}{\sqrt{\frac{4.11\times 10^3\text{ N/m}}{13.6\text{ kg}}}}$ $\qquad t_n = 0.361\text{ s}$ ◀

$f_n = \dfrac{1}{\tau_n} = \dfrac{1}{0.3614\text{ s}}$ $\qquad f_n = 2.77\text{ Hz}$ ◀

PROBLEM 19.20 (Continued)

(*b*) Displacement:

$$x = x_m \sin(\omega_n t + \phi)$$

$$x_m = 44 \text{ mm} = 0.044 \text{ m}$$

$$\omega_n = 2\pi f_n = (2\pi)(2.77 \text{ Hz}) = 17.384 \text{ rad/s}$$

$$x = (0.044 \text{ m})\sin[(17.384 \text{ rad/s})t + \phi]$$
$$\dot{x} = (0.044 \text{ m})(17.384 \text{ rad/s})\cos[(17.384 \text{ rad/s})t + \phi]$$

Velocity: $$v_{\max} = (0.044 \text{ m})(17.384 \text{ rad/s}) = 0.765 \text{ m/s}$$

$$\ddot{x} = -(0.044 \text{ m})(17.384 \text{ rad/s})^2 \sin[(17.384 \text{ rad/s})t + \phi]$$

Acceleration: $$a_{\max} = (0.044 \text{ m})(17.384 \text{ rad/s})^2 = 13.30 \text{ m/s}^2$$

$v_{\max} = 0.765 \text{ m/s}$ ◀

$a_{\max} = 13.30 \text{ m/s}^2$ ◀

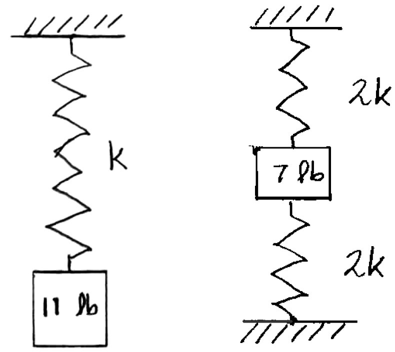
PROBLEM 19.21

A 11-lb block, attached to the lower end of a spring whose upper end is fixed, vibrates with a period of 7.2 s. Knowing that the constant k of a spring is inversely proportional to its length (e.g., if you cut a 10 lb/in. spring in half, the remaining two springs each have a spring constant of 20 lb/in.), determine the period of a 7-lb block which is attached to the center of the same spring if the upper and lower ends of the spring are fixed.

SOLUTION

<u>Equivalent spring constant</u>.

$$k' = 2k + 2k = 4k \qquad \text{(Deflection of each spring is the same.)}$$

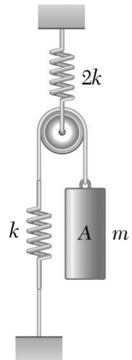


For case ①,
$$\tau_{n1} = 7.2 \text{ s}$$
$$\omega_{n1} = \frac{2\pi}{\tau_{n1}} = \frac{2\pi}{7.2} = 0.87266 \text{ rad/s}$$
$$\omega_n^2 = \frac{k}{m_1}$$
$$k = m_1\omega_{n1}^2 = \left(\frac{11}{32.2}\right)(0.87266)^2 = 0.26015 \text{ lb/ft}$$

For case ②,
$$\omega_{n2}^2 = \frac{4k}{m_2} = \frac{(4)(0.26015)}{\frac{7}{32.2}} = 4.7868 \text{ (rad/s)}^2$$
$$\omega_{n2} = 2.1879 \text{ rad/s}$$
$$\tau_n = \frac{2\pi}{\omega_{n2}} = \frac{2\pi}{2.1879}$$

$\tau_{n2} = 2.87 \text{ s}$ ◀

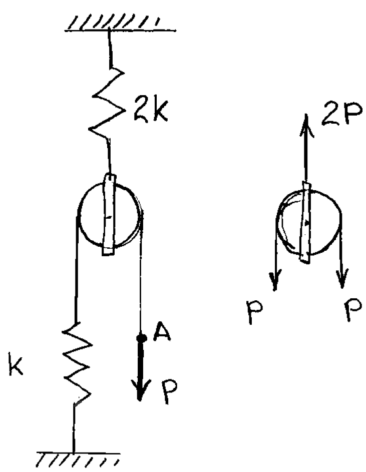
PROBLEM 19.22



Block A of mass m is supported by the spring arrangement as shown. Knowing that the mass of the pulley is negligible and that the block is moved vertically downward from its equilibrium position and released, determine the frequency of the motion.

SOLUTION

We first determine the constant k_{eq} of a single spring equivalent to the spring and pulley system supporting the block by finding the total displacement δ_A of the end of the cable under a given static load **P**. Owing to the force $2P$ in the upper spring the pulley moves down a distance



$$\delta_1 = \frac{2P}{2k}$$

Owing to the force P in the lower spring, Point A moves down an additional distance

$$\delta_2 = \frac{P}{k}$$

The total displacement is

$$\delta_A = \delta_1 + \delta_2 = \frac{2P}{2k} + \frac{P}{k} = \frac{2P}{k}$$

But $\delta_A = \dfrac{P}{k_{eq}}$ so that

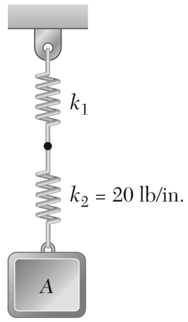
$$k_{eq} = \frac{k}{2}$$

Natural frequency:

$$\omega_n = \sqrt{\frac{k_{eq}}{m}}$$

$$\omega_n = \sqrt{\frac{k}{2\,m}} \blacktriangleleft$$

PROBLEM 19.23



The period of vibration of the system shown is observed to be 0.2 s. After the spring of constant $k_2 = 20$ lb/in. is removed and block A is connected to the spring of constant k_1, the period is observed to be 0.12 s. Determine (*a*) the constant k_1 of the remaining spring, (*b*) the weight of block A.

SOLUTION

Equivalent spring constant for springs in series.

$$k_e = \frac{k_1 k_2}{(k_1 + k_2)}$$

For k_1 and k_2,

$$\tau = \frac{2\pi}{\sqrt{\frac{k_e}{m_A}}} = \frac{2\pi}{\sqrt{\frac{(k_1 k_2)}{(m_A)(k_1+k_2)}}}$$

For k_1 alone,

$$\tau' = \frac{2\pi}{\sqrt{\frac{k_1}{m_A}}}$$

(*a*)

$$\frac{\tau}{\tau'} = \sqrt{\frac{(k_1 + k_2)(k_1)}{(k_1 k_2)}} = \sqrt{\frac{k_1 + k_2}{k_2}}$$

$$k_2 \left(\frac{\tau}{\tau'}\right)^2 = k_1 + k_2$$

$$\frac{\tau}{\tau'} = \frac{0.2 \text{ s}}{0.12 \text{ s}} = 1.6667$$

$$k_2 = 20 \text{ lb/in.}$$

$$(20 \text{ lb/in.})(1.6667)^2 = k_1 + 20 \text{ lb/in.}$$

$k_1 = 35.6$ lb/in. ◀

(*b*)

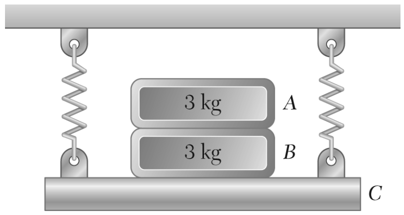
$$\tau' = \frac{2\pi}{\sqrt{\frac{k_1}{m_A}}} \qquad m_A = \frac{W_A}{g}$$

$$m_A = \frac{(\tau')^2 k_1}{(2\pi)^2}$$

$$k_1 = 35.6 \text{ lb/in.} = 426.7 \text{ lb/ft}$$

$$W_A = \frac{(32.2 \text{ ft/s}^2)(0.12 \text{ s})^2 (426.7 \text{ lb/ft})}{(2\pi)^2}$$

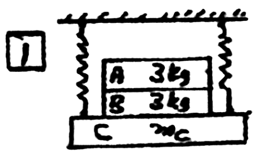
$W_A = 5.01$ lb ◀



PROBLEM 19.24

The period of vibration of the system shown is observed to be 0.8 s. If block A is removed, the period is observed to be 0.7 s. Determine (a) the mass of block C, (b) the period of vibration when both blocks A and B have been removed.

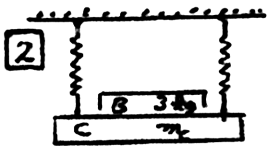
SOLUTION



$$m_1 = m_C + 6 \text{ kg} \qquad \tau_1 = 0.8 \text{ s}$$

$$\omega_1 = \frac{2\pi}{\tau_1} = \frac{2\pi}{0.8 \text{ s}} = \frac{2\pi}{0.8} \text{ rad/s}$$

$$\omega_1^2 = \frac{k}{m_1}; \quad k = m_1\omega_1^2 = (m_C + 6)\left(\frac{2\pi}{0.8}\right)^2 \tag{1}$$



$$m_2 = m_C + 3 \text{ kg} \qquad \tau_2 = 0.7 \text{ s}$$

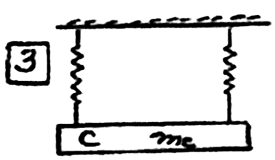
$$\omega_2 = \frac{2\pi}{\tau_2} = \frac{2\pi}{0.7 \text{ s}} = \frac{2\pi}{0.7} \text{ rad/s}$$

$$\omega_2^2 = \frac{k}{m_2}; \quad k = m_2\omega_2^2 = (m_C + 3)\left(\frac{2\pi}{0.7}\right)^2 \tag{2}$$

Equating the expressions found for k in Eqs. (1) and (2):

$$(m_C + 6)\left(\frac{2\pi}{0.8}\right)^2 = (m_C + 3)\left(\frac{2\pi}{0.7}\right)^2$$

$$\frac{m_C + 6}{m_C + 3} = \left(\frac{0.8}{0.7}\right)^2; \quad \text{Solve for } m_C: \qquad m_C = 6.80 \text{ kg} \blacktriangleleft$$



$$\omega_3 = \frac{2\pi}{\tau_3}$$

$$\omega_3^2 = \frac{k}{m_C}; \quad k = m_C\omega_3^2 = m_C\left(\frac{2\pi}{\tau_3}\right)^2 \tag{3}$$

Equating expressions for k from Eqs. (2) and (3),

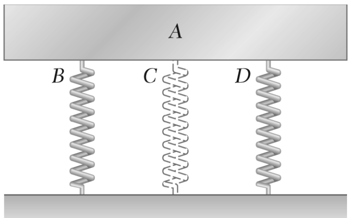
$$(m_C + 3)\left(\frac{2\pi}{0.7}\right)^2 = m_C\left(\frac{2\pi}{\tau_3}\right)^2$$

Recall $m_C = 6.8$ kg:

$$(6.8 + 3)\left(\frac{2\pi}{0.7}\right)^2 = 6.8\left(\frac{2\pi}{\tau_3}\right)^2$$

$$\left(\frac{\tau_3}{0.7}\right)^2 = \frac{6.8}{9.8}; \quad \frac{\tau_3}{0.7} = 0.833 \qquad \tau_3 = 0.583 \text{ s} \blacktriangleleft$$

PROBLEM 19.25



The 100-lb platform A is attached to springs B and D, each of which has a constant $k = 120$ lb/ft. Knowing that the frequency of vibration of the platform is to remain unchanged when an 80-lb block is placed on it and a third spring C is added between springs B and D, determine the required constant of spring C.

SOLUTION

<u>Frequency of the original system</u>.

Springs B and D are in parallel.

$$k_e = k_B + k_D = 2(120 \text{ lb/ft}) = 240 \text{ lb/ft}$$

$$\omega_n^2 = \frac{k_e}{m_A} = \frac{240 \text{ lb/ft}}{\left(\frac{100 \text{ lb}}{32.2 \text{ ft/s}^2}\right)}$$

$$\omega_n^2 = 77.28\,(\text{rad/s})^2$$

<u>Frequency of new system</u>.

Springs A, B, and C are in parallel.

$$k_e' = k_B + k_D + k_C = (2)(120) + k_e$$

$$(\omega_n')^2 = \frac{k_e'}{m_A + m_B} = \frac{(240 + k_C)(32.2 \text{ ft/s}^2)}{(100 \text{ lb} + 80 \text{ lb})}$$

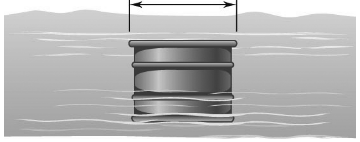
$$(\omega_n')^2 = (0.1789)(240 + k_C)$$

$$\omega_n^2 = (\omega_n')^2$$

$$77.28 = (0.1789)(240 + k_C)$$

$$k_C = 191.97 \text{ lb/ft}$$

$k_C = 192.0$ lb/ft ◀



PROBLEM 19.26

The period of vibration for a barrel floating in salt water is found to be 0.58 s when the barrel is empty and 1.8 s when it is filled with 55 gallons of crude oil. Knowing that the density of the oil is 900 kg/m^3, determine (*a*) the mass of the empty barrel, (*b*) the density of the salt water, ρ_{sw}. [*Hint*: the force of the water on the bottom of the barrel can be modeled as a spring with constant $k = \rho_{sw} gA$.]

SOLUTION

Area of bottom of barrel:
$$A = \frac{\pi D^2}{4} = \frac{\pi(0.572\text{ m})^2}{4} = 0.2570\text{ m}^2$$

Mass of oil:
$$m_{oil} = (55\text{ gal})\left(\frac{1\text{ m}^3}{264.172\text{ gal}}\right)(900\text{ kg/m}^3) = 187.378\text{ kg}$$

Barrel empty:
$$\tau_1 = 0.58\text{ s}$$
$$\omega_{n1} = \frac{2\pi}{\tau_1} = \frac{2\pi}{0.58\text{ s}} = 10.833\text{ rad/s}$$
$$\omega_{n1} = \sqrt{\frac{k}{m_b}} \qquad (1)$$

Barrel full:
$$\tau_2 = 1.8\text{ s}$$
$$\omega_{n2} = \frac{2\pi}{\tau_2} = \frac{2\pi}{1.8\text{ s}} = 3.4907\text{ rad/s}$$
$$\omega_{n2} = \sqrt{\frac{k}{m}} = \sqrt{\frac{k}{m_{oil} + m_b}} \qquad (2)$$

(*a*) Mass m_b of empty barrel.

Divide Eq. (1) by Eq. (2) and square both sides.
$$\frac{\omega_{n1}^2}{\omega_{n2}^2} = \frac{(10.833)^2}{(3.4907)^2} = 9.6310 = \frac{m_{oil} + m_b}{m_b}$$
$$9.6310\, m_b = m_{oil} + m_b$$
$$m_b = \frac{m_{oil}}{9.6310 - 1} = \frac{187.378\text{ kg}}{8.6310} = 21.710\text{ kg}$$

$m_b = 21.7\text{ kg}$ ◀

Spring constant:
$$k = m_b\omega_{n1}^2 = (21.710)(10.833)^2 = 2.5477 \times 10^3\text{ N/m}$$

PROBLEM 19.26 (Continued)

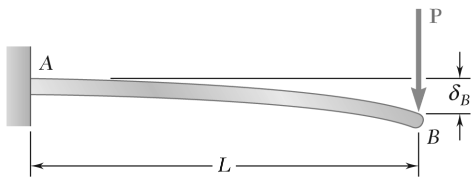
(*b*) Density of the salt water.

$$k = \rho_{sw} gA$$

$$\rho_{sw} = \frac{k}{gA} = \frac{2.5477 \times 10^3 \text{ N/m}}{(9.81 \text{ m/s}^2)(0.2570 \text{ m}^2)}$$

$$\rho_{sw} = 1011 \text{ kg/m}^3 \blacktriangleleft$$

PROBLEM 19.27



From mechanics of materials it is known that for a cantilever beam of constant cross section, a static load **P** applied at end B will cause a deflection $\delta_B = PL^3/3EI$, where L is the length of the beam, E is the modulus of elasticity, and I is the moment of inertia of the cross-sectional area of the beam. Knowing that $L = 10$ ft, $E = 29\times10^6$ lb/in.^{2}, and $I = 12.4$ in.^{4}, determine (*a*) the equivalent spring constant of the beam, (*b*) the frequency of vibration of a 520-lb block attached to end B of the same beam.

SOLUTION

(*a*) Equivalent spring constant.

$$k_e = \frac{P}{\delta_B}$$

$$P = k_e\delta_B$$

$$\delta_B = \frac{PL^3}{3EI}$$

$$P = \left(\frac{3EI}{L^3}\right)\delta_B$$

$$k_e = \frac{3EI}{L^3}$$

$$= \frac{(3)(29\times10^6 \text{ lb/in.}^2)(12.4 \text{ in.}^4)}{(10\times12 \text{ in.})^3}$$

$$k_e = 624.3 \text{ lb/in.}$$

$k_e = 624.3$ lb/in. ◀

(*b*) Natural frequency.

$$f_n = \frac{\sqrt{\frac{k_e}{m}}}{2\pi}$$

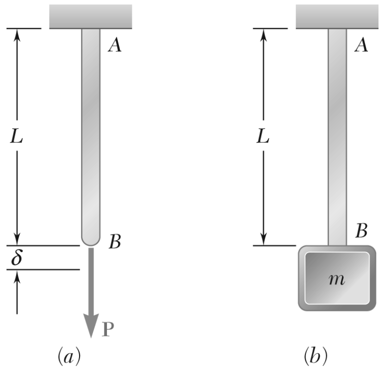
$$k_e = 624.3 \text{ lb/in.}$$

$$= 7.492\times10^3 \text{ lb/ft}$$

$$f_n = \frac{\sqrt{\frac{(7.492\times10^3 \text{ lb/ft})}{\left(\frac{(520 \text{ lb})}{(32.2 \text{ ft/s}^2)}\right)}}}{2\pi}$$

$$f_n = 3.428 \text{ Hz}$$

$f_n = 3.43$ Hz ◀



PROBLEM 19.28

From mechanics of materials it is known that when a static load **P** is applied at the end B of a uniform metal rod fixed at end A, the length of the rod will increase by an amount $\delta = PL/AE$, where L is the length of the undeformed rod. A is its cross-sectional area, and E is the modulus of elasticity of the metal. Knowing that $L = 450$ mm and $E = 200$ GPa and that the diameter of the rod is 8 mm, and neglecting the mass of the rod, determine (*a*) the equivalent spring constant of the rod, (*b*) the frequency of the vertical vibrations of a block of mass $m = 8$ kg attached to end B of the same rod.

SOLUTION

(*a*)

$$P = k_e \delta$$

$$\delta = \frac{PL}{AE}$$

$$P = \left(\frac{AE}{L}\right)\delta$$

$$k_e = \frac{AE}{L}$$

$$A = \frac{\pi d^2}{4} = \frac{\pi(8\times10^{-3}\text{ m})^2}{4}$$

$$A = 5.027\times10^{-5}\text{ m}^2$$

$$L = 0.450\text{ m}$$

$$E = 200\times10^9\text{ N/m}^2$$

$$k_e = \frac{(5.027\times10^{-5}\text{ m}^2)(200\times10^9\text{ N/m}^2)}{(0.450\text{ m})}$$

$$k_e = 22.34\times10^6\text{ N/m}$$

$k_e = 22.3$ MN/m ◀

(*b*)

$$f_n = \frac{\sqrt{\frac{k_e}{m}}}{2\pi}$$

$$= \frac{\sqrt{\frac{22.3\times10^6}{8}}}{2\pi}$$

$$= 265.96\text{ Hz}$$

$f_n = 266$ Hz ◀

PROBLEM 19.29

Denoting by δ_{st} the static deflection of a beam under a given load, show that the frequency of vibration of the load is

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{\delta_{st}}}$$

Neglect the mass of the beam, and assume that the load remains in contact with the beam.

SOLUTION

$$k = \frac{W}{\delta_{st}}$$

$$m = \frac{W}{g}$$

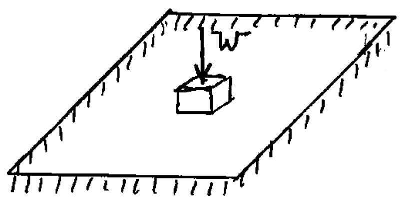
$$\omega_n^2 = \frac{k}{m} = \frac{\frac{W}{\delta_{st}}}{\frac{W}{g}} = \frac{g}{\delta_{st}}$$

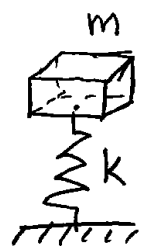
$$f_n = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{g}{\delta_{st}}} \blacktriangleleft$$

PROBLEM 19.30

A 40-mm deflection of the second floor of a building is measured directly under a newly installed 3500-kg piece of rotating machinery, which has a slightly unbalanced rotor. Assuming that the deflection of the floor is proportional to the load it supports, determine (*a*) the equivalent spring constant of the floor system, (*b*) the speed in rpm of the rotating machinery that should be avoided if it is not to coincide with the natural frequency of the floor-machinery system.

SOLUTION





(*a*) Equivalent spring constant.

$$W = k_e \delta_s$$

$$k_e = \frac{mg}{\delta}$$

$$= \frac{3500(9.81)\text{ N}}{40\text{ mm}}$$

$k_e = 858$ N/mm ◀

(*b*) Natural frequency.

$$f_n = \frac{\sqrt{\frac{k_e}{m}}}{2\pi}$$

$$= \frac{\sqrt{\frac{(858.38 \times 1000\text{ N/m})}{(3500\text{ kg})}}}{2\pi}$$

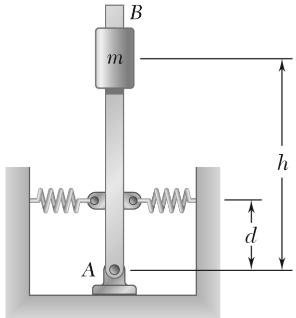
$$f_n = 2.4924\text{ Hz}$$

$$1\text{ Hz} = 1\text{ cycle/s}$$

$$= 60\text{ rpm}$$

$$\text{Speed} = (2.424\text{ Hz})\frac{(60\text{ rpm})}{\text{Hz}}$$

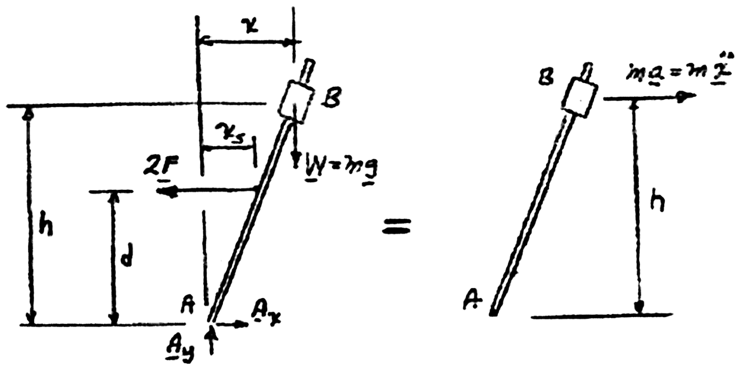
Speed = 149.5 rpm ◀



PROBLEM 19.31

If $h = 700$ mm and $d = 500$ mm and each spring has a constant $k = 600$ N/m, determine the mass m for which the period of small oscillations is (*a*) 0.50 s, (*b*) infinite. Neglect the mass of the rod and assume that each spring can act in either tension or compression.

SOLUTION



$$x_s = x\frac{d}{h}$$

$$2F = 2kx_s = 2k\frac{d}{h}x$$

$$+\circlearrowleft \Sigma M_A = \Sigma(M_A)_{\text{eff}}: \quad 2Fd - mgx = -(m\ddot{x})h$$

$$2k\left(\frac{d}{h}x\right)d - mgx = -m\ddot{x}h$$

$$\ddot{x} + \left[\frac{2kd^2}{mh^2} - \frac{g}{h}\right]x = 0$$

$$\omega_n^2 = \left[\frac{2kd^2}{mh^2} - \frac{g}{h}\right]$$

$$\omega_n^2 = \frac{2k}{m}\left(\frac{d}{h}\right)^2 - \frac{g}{h} \qquad (1)$$

<u>Data</u>:

$$d = 0.5 \text{ m}$$

$$h = 0.7 \text{ m}$$

$$k = 600 \text{ N/m}$$

PROBLEM 19.31 (Continued)

(*a*) For $\tau = 0.5$ s: $\tau = \dfrac{2\pi}{\omega_n}; \quad 0.5 = \dfrac{2\pi}{\omega_n} \qquad \omega_n = 4\pi$

Eq. (1): $$(4\pi)^2 = \frac{2(600)}{m}\left(\frac{0.5}{0.7}\right)^2 - \frac{9.81}{0.7}$$

$m = 3.561$ kg $\qquad m = 3.56$ kg ◀

(*b*) For $\tau =$ infinite: $\tau = \dfrac{2\pi}{\omega_n} \quad \omega_n = 0$

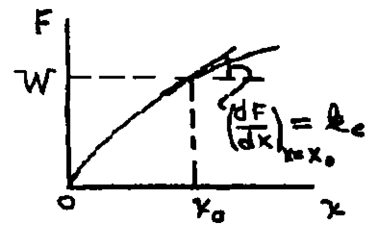
Eq. (1): $$0 = \frac{2(600)}{m}\left(\frac{0.5}{0.7}\right)^2 - \frac{9.81}{0.7}$$

$m = 43.69$ kg $\qquad m = 43.7$ kg ◀

PROBLEM 19.32

The force-deflection equation for a nonlinear spring fixed at one end is $F = 1.5x^{1/2}$ where F is the force, expressed in newtons, applied at the other end, and x is the deflection expressed in meters. (a) Determine the deflection x_0 if a 4-oz block is suspended from the spring and is at rest. (b) Assuming that the slope of the force-deflection curve at the point corresponding to this loading can be used as an equivalent spring constant, determine the frequency of vibration of the block if it is given a very small downward displacement from its equilibrium position and released.

SOLUTION



(a) Deflection x_0.

$$W = 4 \text{ oz} = 0.25 \text{ lb}$$

$$F = W = 1.5x_0^{1/2}$$

$$x_0 = \left(\frac{0.25}{1.5}\right)^2 = 0.027778 \text{ ft}$$

$x_0 = 0.333$ in. ◀

Equivalent spring constant.

At x_0,

$$\left(\frac{dF}{dx}\right)_{x_0} = \frac{1.5}{2}(x_0)^{-1/2} = \frac{1.5}{2}(0.027778)^{-1/2}$$

$$\left(\frac{dF}{dx}\right)_{x_0} = 4.5 \text{ lb/ft}$$

$$k_e = 4.5 \text{ lb/ft}$$

(b) Natural frequency.

$$f_n = \frac{\sqrt{\frac{k_e}{m}}}{2\pi} = \frac{\sqrt{\frac{(4.5 \text{ lb/ft})}{(0.25/32.2)}}}{2\pi}$$

$$f_n = 3.8316 \text{ Hz}$$

$f_n = 3.83$ Hz ◀

PROBLEM 19.33*

Expanding the integrand in Equation (19.19) of Section 19.4 into a series of even powers of $\sin\phi$ and integrating, show that the period of a simple pendulum of length l may be approximated by the formula

$$\tau = 2\pi\sqrt{\frac{l}{g}}\left(1+\frac{1}{4}\sin^2\frac{\theta_m}{2}\right)$$

where θ_m is the amplitude of the oscillations.

SOLUTION

Using the Binomial Theorem, we write

$$\frac{1}{\sqrt{1-\sin^2\left(\frac{\theta_m}{2}\right)\sin^2\phi}} = \left[1-\sin^2\left(\frac{\theta_m}{2}\right)\sin\phi\right]^{-1/2}$$

$$= 1+\frac{1}{2}\sin^2\frac{\theta_m}{2}\sin^2\phi+\cdots$$

Neglecting terms of order higher than 2 and setting $\sin^2\phi = \frac{1}{2}(1-\cos 2\phi)$, we have

$$\tau_n = 4\sqrt{\frac{l}{g}}\int_0^{\pi/2}\left\{1+\frac{1}{2}\sin^2\frac{\theta_m}{2}\left[\frac{1}{2}(1-\cos 2\phi)\right]\right\}d\phi$$

$$= 4\sqrt{\frac{l}{g}}\int_0^{\pi/2}\left\{1+\frac{1}{4}\sin^2\frac{\theta_m}{2}-\frac{1}{4}\sin^2\frac{\theta_m}{2}\cos 2\phi\right\}d\phi$$

$$= 4\sqrt{\frac{l}{g}}\left[\phi+\frac{1}{4}\left(\sin^2\frac{\theta_m}{2}\right)\phi-\frac{1}{8}\sin^2\frac{\theta_m}{2}\sin 2\phi\right]_0^{\pi/2}$$

$$= 4\sqrt{\frac{l}{g}}\left[\frac{\pi}{2}+\frac{1}{4}\left(\sin^2\frac{\theta_m}{2}\right)\frac{\pi}{2}+0\right] \qquad \tau_n = 2\pi\sqrt{\frac{l}{g}}\left(1+\frac{1}{4}\sin^2\frac{\theta_m}{2}\right) \blacktriangleleft$$

PROBLEM 19.34*

Using the formula given in Problem 19.33, determine the amplitude θ_m for which the period of a simple pendulum is $\frac{1}{2}$ percent longer than the period of the same pendulum for small oscillations.

SOLUTION

For small oscillations,

$$(\tau_n)_0 = 2\pi\sqrt{\frac{l}{g}}$$

We want

$$\tau_n = 1.005(\tau_n)_0$$

$$= 1.005\ 2\pi\sqrt{\frac{l}{g}}$$

Using the formula of Problem 19.33, we write

$$\tau_n = (\tau_n)_0\left(1+\frac{1}{4}\sin^2\frac{\theta_m}{2}\right)$$

$$= 1.005(\tau_n)_0$$

$$\sin^2\frac{\theta_m}{2} = 4[1.005-1] = 0.02$$

$$\sin\frac{\theta_m}{2} = \sqrt{0.02}$$

$$\frac{\theta_m}{2} = 8.130°$$

$\theta_m = 16.26°$ ◀

PROBLEM 19.35*

Using the data of Table 19.1, determine the period of a simple pendulum of length $l = 750$ mm (*a*) for small oscillations, (*b*) for oscillations of amplitude $\theta_m = 60°$, (*c*) for oscillations of amplitude $\theta_m = 90°$.

SOLUTION

(*a*) $\tau_n = 2\pi\sqrt{\dfrac{l}{g}}$ (Equation 19.18 for small oscillations):

$$\tau_n = 2\pi\sqrt{\frac{0.750 \text{ m}}{9.81 \text{ m/s}^2}}$$

$$= 1.737 \text{ s}$$

$\tau_n = 1.737$ s ◀

(*b*) For large oscillations (Eq. 19.20),

$$\tau_n = \left(\frac{2K}{\pi}\right)\left(2\pi\sqrt{\frac{l}{g}}\right)$$

$$= \frac{2K}{\pi}(1.737 \text{ s})$$

For $\theta_m = 60°$, $K = 1.686$ (Table 19.1)

$$\tau_n(60°) = \frac{2(1.686)(1.737 \text{ s})}{\pi}$$

$$= 1.864 \text{ s}$$

$\tau_n = 1.864$ s ◀

(*c*) For $\theta_m = 90°$, $K = 1.854$ $\tau_n = \dfrac{2(1.854)(1.737 \text{ s})}{\pi} = 2.05$ s ◀

PROBLEM 19.36*

Using the data of Table 19.1, determine the length in inches of a simple pendulum which oscillates with a period of 2 s and an amplitude of 90°.

SOLUTION

For large oscillations (Eq. 19.20),

$$\tau_n = \left(\frac{2K}{\pi}\right)\left(2\pi\sqrt{\frac{l}{g}}\right)$$

for $$\theta_m = 90°$$

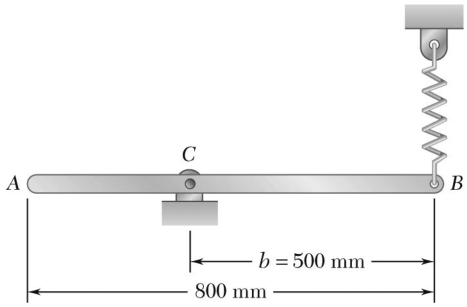
$$K = 1.854 \text{ (Table 19.1)}$$

$$(2\text{ s}) = (2)(1.854)(2)\sqrt{\frac{l}{32.2\text{ ft/s}^2}}$$

$$l = \frac{(2\text{ s})^2(32.2\text{ ft/s}^2)}{[(4)(1.854)]^2}$$

$$= 2.342\text{ ft}$$

$l = 28.1$ in. ◀

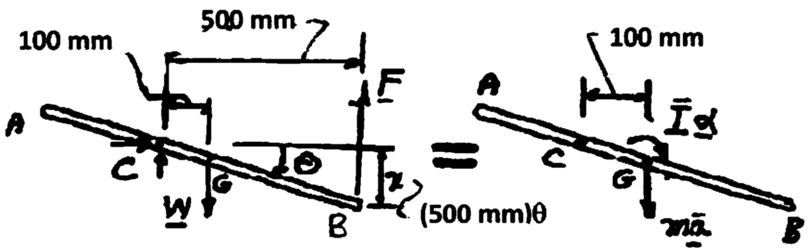


PROBLEM 19.37

The uniform rod shown has mass 6 kg and is attached to a spring of constant $k = 700$ N/m. If end B of the rod is depressed 10 mm and released, determine (*a*) the period of vibration, (*b*) the maximum velocity of end B.

SOLUTION

$$k = 700 \text{ N/m}$$
$$W = mg$$



where
$$F = k(x + \delta_{\text{st}})$$
$$= k(0.5\theta + \delta_{\text{st}})$$
$$m\bar{a} = m\bar{r}\alpha = 6(0.1 \text{ m})\ddot{\theta} = 0.6\ddot{\theta}$$
$$\bar{I}\alpha = \frac{1}{12}(6)(0.8 \text{ m})^2\ddot{\theta}$$
$$= 0.32\ddot{\theta}$$

(*a*) <u>Equation of motion</u>.

$$+\circlearrowright \Sigma M_C = \bar{I}\alpha + m\bar{a}d: \quad W(0.1 \text{ m}) - F(0.5 \text{ m}) = \bar{I}\alpha + m\bar{a}(0.1 \text{ m})$$
$$W(0.1) - k(0.5\theta + \delta_{\text{st}})(0.5 \text{ m}) = 0.32\ddot{\theta} + 0.6\ddot{\theta}(0.1)$$

But in equilibrium, we have $\quad +\circlearrowright W(0.1 \text{ m}) - k\delta_{\text{st}}(0.5 \text{ m}) = 0$

Thus,
$$-k(0.5)^2\theta = [0.32 + 0.06]\ddot{\theta}$$
$$-(700 \text{ N/m})(0.5)^2\theta = 0.38\ddot{\theta}$$
$$\ddot{\theta} + (460.53)\theta = 0$$

Natural frequency and period.

$$\omega_n^2 = 460.53$$

$$\omega_n = 21.46 \text{ rad/s}$$

$$\tau = \frac{2\pi}{\omega_n} = \frac{2\pi}{21.46 \text{ rad/s}}$$

$\tau = 0.293$ s ◀

(*b*) At end *B*.

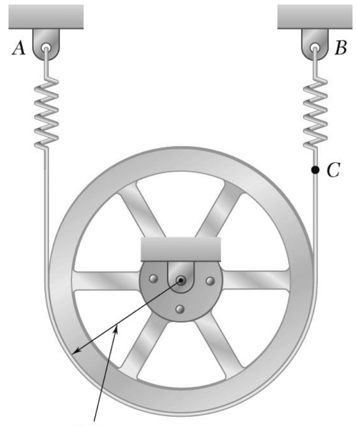
$$x_m = 0.010 \text{ m}$$

$$v_m = x_m \omega_n$$

$$= (10 \text{ mm})(21.46 \text{ rad/s})$$

$$= 214.6 \text{ mm/s}$$

$v_m = 0.215$ m/s ◀

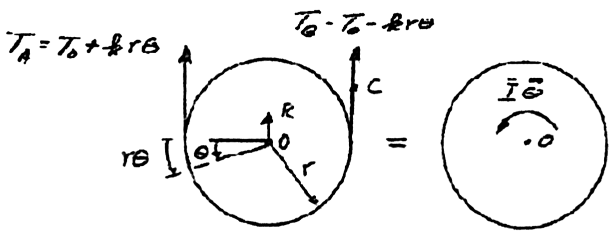


PROBLEM 19.38

A belt is placed around the rim of a 500-lb flywheel and attached as shown to two springs, each of constant $k = 85$ lb/in. If end C of the belt is pulled 1.5 in. down and released, the period of vibration of the flywheel is observed to be 0.5 s. Knowing that the initial tension in the belt is sufficient to prevent slipping, determine (*a*) the maximum angular velocity of the flywheel, (*b*) the centroidal radius of gyration of the flywheel.

SOLUTION

Denote the initial tension by T_0.



<u>Equation of motion</u>. $\circlearrowleft + \Sigma M_0 = \bar{I}\ddot{\theta}: \quad -T_A r + T_B r = \bar{I}\ddot{\theta}$

$$-(T_0 + kr\theta)r + (T_0 - kr\theta)\theta = \bar{I}\ddot{\theta}$$

$$\ddot{\theta} + \frac{2kr^2}{\bar{I}}\theta = 0$$

$$\omega_n^2 = \frac{2kr^2}{\bar{I}} \qquad (1)$$

<u>Data</u>:

$$m = \frac{W}{g} = \frac{500 \text{ lb}}{32.2} \qquad k = 85 \text{ lb/in.} = 1020 \text{ lb/ft}$$

$$\tau = 0.5 \text{ s} \qquad r = 18 \text{ in.} = 1.5 \text{ ft}$$

$$\tau = \frac{2\pi}{\omega_n}; \qquad \omega_n = \frac{2\pi}{\tau} = \frac{2\pi}{0.5} = 4\pi \text{ rad/s}$$

(*a*) <u>Maximum angular velocity</u>. If Point C is pulled down 1.5 in. and released,

$$\theta_m = \theta_{\max} = \left(\frac{1.5 \text{ in.}}{18 \text{ in.}}\right) = 83.333\times 10^{-3} \text{ rad}$$

$$\dot{\theta}_m = \theta_m \omega_n = (83.333\times 10^{-3} \text{ rad})(4\pi \text{ rad/s}) \qquad \dot{\theta}_m = 1.047 \text{ rad/s} \blacktriangleleft$$

PROBLEM 19.38 (Continued)

(*b*) Centroidal radius of gyration.

$$\omega_n^2 = \frac{2kr^2}{I}$$

$$(4\pi \text{ rad/s})^2 = \frac{2(1020 \text{ lb/ft})(1.5 \text{ ft})^2}{\bar{I}}$$

$$\bar{I} = 29.067 \text{ slug} \cdot \text{ft}^2$$

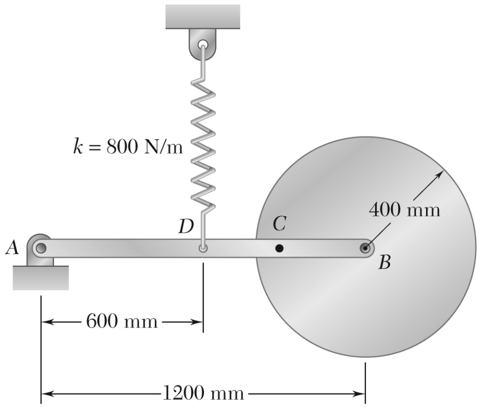
or since

$$\bar{I} = m\bar{k}^2$$

$$\left(\frac{500 \text{ lb}}{32.2 \text{ ft/s}^2}\right)\bar{k}^2 = 29.067 \text{ slug} \cdot \text{ft}^2$$

$$\bar{k} = 1.3682 \text{ ft}$$

$\bar{k} = 16.42$ in. ◀

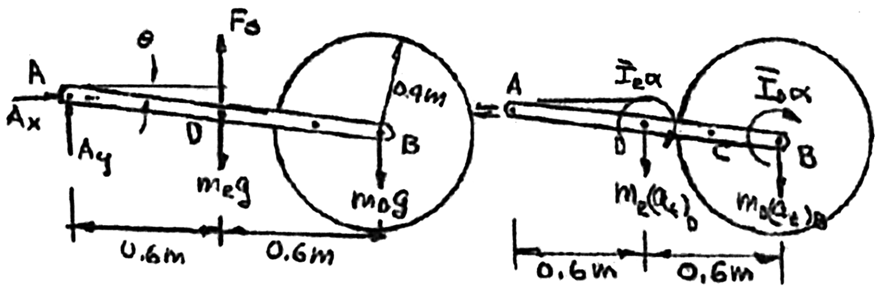


PROBLEM 19.39

An 8-kg uniform rod *AB* is hinged to a fixed support at *A* and is attached by means of pins *B* and *C* to a 12-kg disk of radius 400 mm. A spring attached at *D* holds the rod at rest in the position shown. If Point *B* is moved down 25 mm and released, determine (*a*) the period of vibration, (*b*) the maximum velocity of Point *B*.

SOLUTION

(*a*)



Equation of motion. $\Sigma M_A = (\Sigma M_A)_{\text{eff}}: \quad F_S = k(0.6\theta + \delta_{\text{st}})$

$$+\circlearrowright\; 0.6(m_R g - F_S) + 1.2 m_D g = (\bar{I}_R + \bar{I}_D)\alpha + 0.6(m_R)(a_t)_D + 1.2(m_D)(a_t)_B \qquad (1)$$

At equilibrium $(\theta = 0)$, $\quad F_S = k\delta_{\text{st}}$

$$\Sigma M_A = 0 = 0.6(m_R g - k(\delta_{\text{st}})) + 1.2 m_D g \qquad (2)$$

Substituting Eq. (2) into Eq. (1), $\quad (\bar{I}_R + \bar{I}_D)\alpha + 0.6\, m_R (a_t)_D + 1.2 m_D (a_t)_B + (0.6)^2 k\theta = 0$

$$\alpha = \ddot{\theta}$$

$$(a_t)_B = 0.6\ddot{\theta}$$

$$(a_t)_D = 1.2\ddot{\theta}$$

$$\bar{I}_R = \frac{1}{12} m_R l^2 = \frac{1}{12}(8)(1.2)^2$$

$$= 0.960 \text{ kg} \cdot \text{m}$$

$$\bar{I}_D = \frac{1}{2} m_D R^2 = \frac{1}{2}(12)(0.4)^2 = 0.960 \text{ kg} \cdot \text{m}$$

$$[0.960 + 0.960 + (0.6)^2(8) + (1.2)^2(12)]\ddot{\theta} + (0.6)^2(800)\theta = 0$$

$$\ddot{\theta} + \frac{288 \text{ N} \cdot \text{m}}{(22.08 \text{ kg} \cdot \text{m}^2)}\theta = 0$$

PROBLEM 19.39 (Continued)

(*a*) Natural frequency and period.

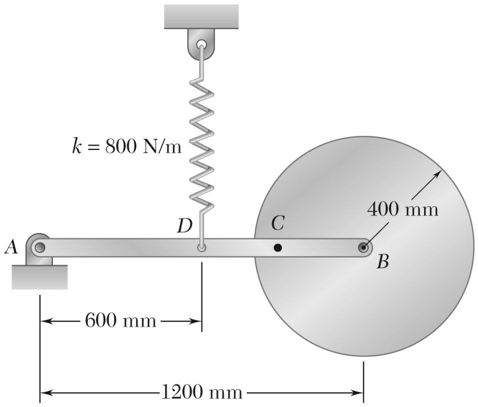
$$\omega_n = \sqrt{\frac{288}{22.08}}$$
$$= 3.6116 \text{ rad/s}$$
$$\tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{3.6116}$$

$\tau_n = 1.740$ s ◀

(*b*) Maximum velocity at *B*.

$$(v_B)_{\max} = (1.2)(\dot{\theta}_{\max})$$
$$\theta_m = \frac{y_B}{1.2}$$
$$\theta_m = \frac{0.025}{1.2} = 0.02083 \text{ rad}$$
$$\theta = \theta_m \sin(\omega_n t + \phi)$$
$$\dot{\theta} = \theta_m \omega_n \cos(\omega_n t + \phi)$$
$$\dot{\theta}_{\max} = \theta_m \omega_n = (0.02083)(3.612) = 0.07524 \text{ rad/s}$$
$$(v_B)_{\max} = (1.2)(\dot{\theta}_{\max}) = (1.2 \text{ m})(0.07524) \text{ rad/s}$$
$$(v_B)_{\max} = 0.09029 \text{ m/s}$$

$(v_B)_{\max} = 90.3$ mm/s ◀



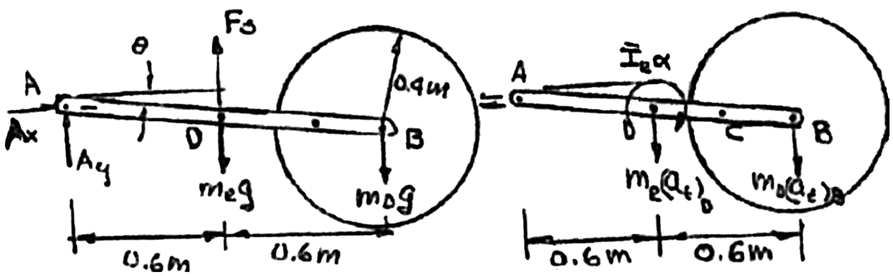
PROBLEM 19.40

Solve Problem 19.39, assuming that pin C is removed and that the disk can rotate freely about pin B.

PROBLEM 19.39 An 8-kg uniform rod AB is hinged to a fixed support at A and is attached by means of pins B and C to a 12-kg disk of radius 400 mm. A spring attached at D holds the rod at rest in the position shown. If Point B is moved down 25 mm and released, determine (*a*) the period of vibration, (*b*) the maximum velocity of Point B.

SOLUTION

(*a*)



Note: This problem is the same as Problem 19.39, *except* that the disk does not rotate, so that the effective moment $I_D\alpha = 0$.

Equation of motion. $\Sigma M_A = (\Sigma M_A)_{\text{eff}}: \quad F_S = k(0.60 + \delta_{\text{st}})$

$$\circlearrowright (0.6)(m_R g - F_S) + 1.2 m_D g = \bar{I}_R\alpha + (0.6)(m_R)(a_t)_D + 1.2(m_D)(a_t)_B \qquad (1)$$

At equilibrium $(\theta = 0)$, $\quad F_S = k\delta_{\text{st}}$

$$\circlearrowright \Sigma M_A = 0 = 0.6(m_R g - \delta_{\text{st}}) + 1.2\, m_D g \qquad (2)$$

Substituting Eq. (2) into Eq. (1), $\quad I_R\alpha + 0.6\, m_R(a_t)_D + 1.2 m_D(a_t)_B + (0.6)^2 k\theta = 0$

$$\alpha = \ddot{\theta}$$

$$(a_t)_B = 0.6\ddot{\theta}$$

$$(a_t)_D = 1.2\ddot{\theta}$$

$$I_R = \frac{1}{12}m_R l^2 = \frac{1}{12}(8)(1.2)^2$$

$$= 0.960 \text{ kg} \cdot \text{m}$$

$$[0.960 + (0.6)^2(8) + (1.2)^2(12)]\ddot{\theta} + (0.6)^2(800)\theta = 0$$

$$\ddot{\theta} + \frac{(288 \text{ N} \cdot \text{m})}{21.12 \text{ kg} \cdot \text{m}^2}\theta = 0$$

PROBLEM 19.40 (Continued)

(*a*) Natural frequency and period.

$$\omega_n = \sqrt{\frac{288}{21.12}}$$
$$= 3.693 \text{ rad/s}$$
$$\tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{3.693}$$

$\tau_n = 1.701$ s ◀

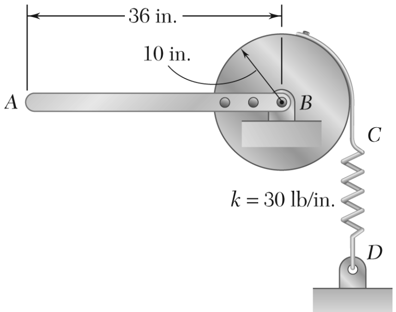
(*b*) Maximum velocity at *B*.

$$(v_B)_{\max} = (1.2)(\dot{\theta})_{\max}$$

$$\theta_m = \frac{y_B}{1.2} = \frac{0.025}{1.20} = 0.02083 \text{ rad}$$
$$\theta = \theta_m \sin(\omega_n t + \phi)$$
$$\ddot{\theta} = \theta_m \omega_n \cos(\omega_n t + \phi)$$
$$\ddot{\theta}_{\max} = \theta_m \omega_n$$
$$= (0.02083)(3.693)$$
$$= 0.07694 \text{ rad/s}$$
$$(v_B)_{\max} = (1.2)(\dot{\theta}_{\max})$$
$$= (1.2)(0.07694)$$
$$= 0.09233 \text{ m/s}$$

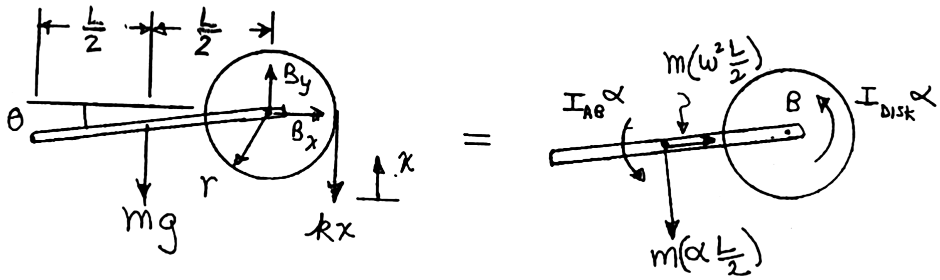
$(v_B)_{\max} = 92.3$ mm/s ◀

PROBLEM 19.41



A 15-lb slender rod *AB* is riveted to a 12-lb uniform disk as shown. A belt is attached to the rim of the disk and to a spring which holds the rod at rest in the position shown. If end *A* of the rod is moved 0.75 in. down and released, determine (*a*) the period of vibration, (*b*) the maximum velocity of end *A*.

SOLUTION



Equation of motion. $+\circlearrowleft \Sigma M_B = \Sigma(M_B)_{\text{eff}}: \quad mg\dfrac{L}{2}\cos\theta - kxr = I_{AB}\alpha + m\left(\alpha\dfrac{L}{2}\right)\left(\dfrac{L}{2}\right) + I_{\text{disk}}\alpha$ (1)

where $x = r\theta + \delta_{\text{st}}$ and from statics, $mg\dfrac{L}{2} = k\delta_{\text{st}}r$

Assuming small angles $(\cos\theta \approx 1)$, Equation (1) becomes

$$\cancel{mg\frac{L}{2}}\theta - kr^2\theta - \cancel{kr\delta_{\text{st}}} = \left(I_{AB} + m\left(\frac{L}{2}\right)^2 + I_{\text{disk}}\right)\alpha$$

$$\left(I_{AB} + \frac{mL^2}{4} + I_{\text{disk}}\right)\ddot{\theta} + kr^2\theta = 0$$

Data:

$$m = \frac{15}{32.2} = 0.46584 \text{ lb}\cdot\text{s}^2/\text{ft}$$

$$m_{\text{disk}} = \frac{12}{32.2} = 0.37267 \text{ lb}\cdot\text{s}^2/\text{ft}$$

$$L = 36 \text{ in.} = 3.0 \text{ ft}$$

$$r = 10 \text{ in.} = 0.83333 \text{ ft}$$

$$k = 30 \text{ lb/in.} = 360 \text{ lb/ft}$$

PROBLEM 19.41 (Continued)

$$I_{AB} = \frac{1}{12}mL^2$$
$$= \frac{1}{12}(0.46584)(3.0)^2$$
$$= 0.34938 \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$
$$I_{\text{disk}} = \frac{1}{2}m_{\text{disk}}r^2$$
$$= \frac{1}{2}(0.37267)(0.83333)^2$$
$$= 0.1294 \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

$$\left[0.34935 + \frac{1}{4}(0.46584)(3.0)^2 + 0.1294\right]\ddot{\theta} + (360)(0.83333)^2\theta = 0$$

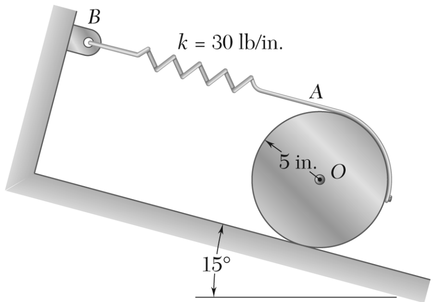
$$1.5269\ddot{\theta} + 250\theta = 0 \quad \text{or} \quad \ddot{\theta} + 163.73\theta = 0$$

(*a*) Natural frequency and period. $\omega_n^2 = 163.73 \text{ (rad/s)}^2$

$$\omega_n = 12.796 \text{ rad/s}$$

$$\tau = \frac{2\pi}{\omega_n} = \frac{2\pi}{12.796}$$ $\tau = 0.491$ s ◀

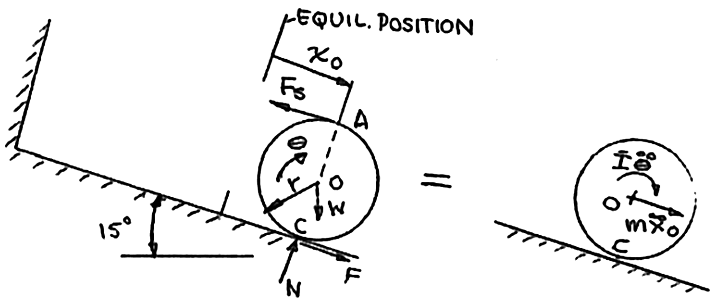
(*b*) Maximum velocity. $v_m = \omega_n x_m = (12.796)(0.75)$ $v_m = 9.60$ in./s ◀



PROBLEM 19.42

A 30-lb uniform cylinder can roll without sliding on a 15°-incline. A belt is attached to the rim of the cylinder, and a spring holds the cylinder at rest in the position shown. If the center of the cylinder is moved 2 in. down the incline and released, determine (*a*) the period of vibration, (*b*) the maximum acceleration of the center of the cylinder.

SOLUTION



Spring deflection.

$$x_A = x_0 + x_{A/0}$$

$$x_{A/0} = r\theta$$

$$\theta = \frac{x_0}{r}$$

$$x_A = 2x_0$$

$$F_S = k(x_A + \delta_{st}) = k(2x_0 + \delta_{st})$$

$$\curvearrowright + \ \Sigma M_C = (\Sigma M)_{\text{eff}}: \quad -2rk(2x_0 + \delta_{st}) + rW\sin 15° = rm\ddot{x}_0 + \bar{I}\ddot{\theta} \qquad (1)$$

But in equilibrium, $x_0 = 0$

$$\Sigma M_C = 0 = -2rk\delta_{st} + rW\sin 15° \qquad (2)$$

Substituting Eq. (2) into Eq. (1) and noting that $\theta = \dfrac{x_0}{r}, \quad \ddot{\theta} = \dfrac{\ddot{x}_0}{r}$

$$rm\ddot{x}_0 + \bar{I}\frac{\ddot{x}_0}{r} + 4rkx_0 = 0$$

$$\bar{I} = \frac{1}{2}mr^2$$

$$\frac{3}{2}mr\ddot{x}_0 + 4rkx_0 = 0$$

$$\ddot{x}_0 + \left(\frac{8}{3}\frac{k}{m}\right)x_0 = 0$$

PROBLEM 19.42 (Continued)

Natural frequency. $\omega_n = \sqrt{\frac{8}{3}\frac{k}{m}} = \sqrt{\frac{(8)(30\times 12 \text{ lb/ft})}{(3)\frac{(30 \text{ lb})}{(32.2 \text{ ft/s}^2)}}} = 32.1 \text{ s}^{-1}$

(*a*) Period. $\tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{32.1} = 0.1957 \text{ s}$ $\qquad \tau_n = 0.1957 \text{ s}$ ◀

(*b*) $x_0 = (x_0)_m \sin(\omega_n t + \phi)$

At $t = 0$, $x_0 = \frac{2}{12}$ ft $\qquad \dot{x}_0 = 0$

$$\dot{x}_0 = (x_0)_m \omega_n \cos(\omega_n t + \phi)$$

$$t = 0$$

$$0 = (x_0)_m \omega_n \cos\phi$$

Thus, $\phi = \frac{\pi}{2}$

$$t = 0$$

$$x_0(0) = \frac{1}{6} \text{ ft} = (x_0)_m \sin\phi = (x_0)_m (1)$$

$$(x_0)_m = \frac{1}{6} \text{ ft}$$

$$\ddot{x}_0 = -(x_0)_m \omega_n^2 \sin(\omega_n t + \phi)$$

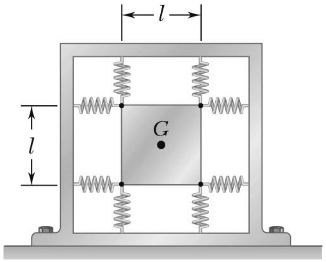
$$(a_0)_{max} = (\ddot{x}_0)_{max}$$

$$= -(x_0)_m \omega_n^2$$

$$= -\left(\frac{1}{6} \text{ ft}\right)(32.1 \text{ s}^{-1})^2$$

$$= 171.7 \text{ ft/s}^2$$

$(a_0)_{max} = 171.7 \text{ ft/s}^2$ ◀



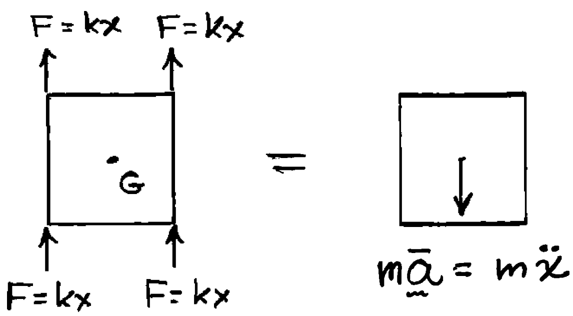
PROBLEM 19.43

A square plate of mass m is held by eight springs, each of constant k. Knowing that each spring can act in either tension or compression, determine the frequency of the resulting vibration (*a*) if the plate is given a small vertical displacement and released, (*b*) if the plate is rotated through a small angle about G and released.

SOLUTION

(*a*) Small vertical displacement.

Let the plate be displaced downward a distance x from the equilibrium position. Each corner moves downward a distance x and the four vertical springs exert additional forces kx for each spring. The horizontal springs exert negligible change.



$$+\downarrow \Sigma F = ma: \quad -4kx = m\ddot{x}$$

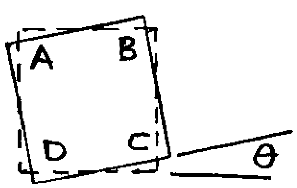
$$\ddot{x} + \frac{4k}{m}x = 0$$

$$\omega_n^2 = \frac{4k}{m}$$

Frequency: $\quad f = \dfrac{\omega_n}{2\pi} = \dfrac{1}{2\pi}\sqrt{\dfrac{4k}{m}}$ $\qquad f = 0.318\sqrt{\dfrac{k}{m}}$ ◀

(*b*) Small rotation about G.

Let the plate be rotated through a small counterclockwise angle θ from the equilibrium position. The corners A, B, C, and D move as indicated below:



A: $\quad (l/2)\theta\downarrow + (l/2)\theta \longleftarrow$

B: $\quad (l/2)\theta \longleftarrow + (l/2)\theta\uparrow$

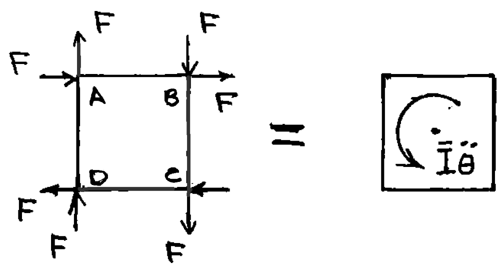
C: $\quad (l/2)\theta\uparrow + (l/2)\theta \longleftarrow$

D: $\quad (l/2)\theta \longrightarrow + (l/2)\theta\downarrow$

The additional force exerted by each of the eight springs is $F = (kl/2)\theta$ and directed as shown on the free body diagram. The eight forces reduce to four clockwise couples, each of magnitude Fl. For a square plate

$$\bar{I} = \frac{1}{6}ml^2$$

PROBLEM 19.43 (Continued)



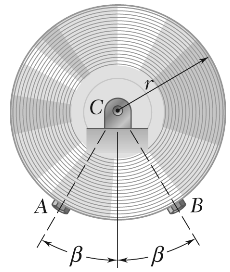
$$+\circlearrowleft M_G = \bar{I}\alpha: \quad 4Fl = \bar{I}\ddot{\theta} - 4(kl^2/2)\theta$$

$$= \frac{1}{6}ml^2\ddot{\theta}$$

$$\ddot{\theta} + \frac{12k}{\text{m}}\theta = 0$$

$$\omega_n^2 = \frac{12k}{\text{m}}$$

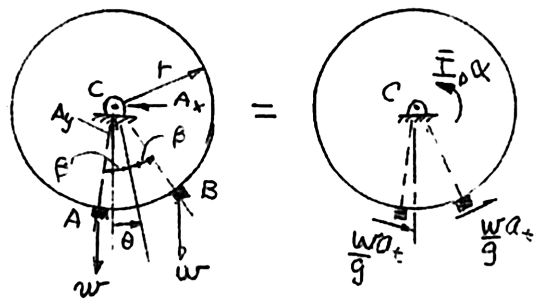
Frequency: $f = \dfrac{\omega_n}{2\pi} = \dfrac{1}{2\pi}\sqrt{\dfrac{12k}{m}}$ $\qquad f = 0.551\sqrt{\dfrac{k}{m}}$ ◀



PROBLEM 19.44

Two small weights w are attached at A and B to the rim of a uniform disk of radius r and weight W. Denoting by τ_0 the period of small oscillations when $\beta = 0$, determine the angle β for which the period of small oscillations is $2\tau_0$.

SOLUTION



$$\alpha = \ddot{\theta}$$

$$a_t = r\alpha = r\ddot{\theta}$$

$$\bar{I}_D = \frac{1}{2}\frac{W}{g}r^2$$

Equation of motion.

$$\Sigma M_C = (\Sigma M_C)_{\text{eff}}: \quad wr\sin(\beta-\theta) - wr\sin(\beta+\theta) = \frac{2w}{g}ra_t + \bar{I}\alpha$$

$$wr[\sin(\beta-\theta) - \sin(\beta+\theta)] = -2wr\sin\theta\cos\beta$$

$$\sin\theta \approx \theta$$

$$\left(\frac{2w}{g}r^2 + \frac{W}{2g}r^2\right)\ddot{\theta} + (2wr\cos\beta)\theta = 0$$

Natural frequency.

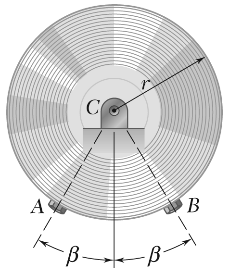
$$\omega_n = \sqrt{\frac{2wg\cos\beta}{\left(2w+\frac{W}{2}\right)r}} = \sqrt{\frac{4g\cos\beta}{(4+\frac{W}{w})r}} \quad (1)$$

$$\beta = 0 \qquad \tau_0 = \frac{2\pi}{\omega_0} = \frac{2\pi}{\sqrt{\frac{4g}{(4+\frac{W}{w})r}}}$$

$$\tau_n = \frac{2\pi}{\sqrt{\frac{\cos\beta}{(4+\frac{W}{w})r}}} = 2\tau_0 = \frac{4\pi}{\sqrt{\frac{4g}{(4+\frac{W}{w})r}}}$$

$$\cos\beta = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

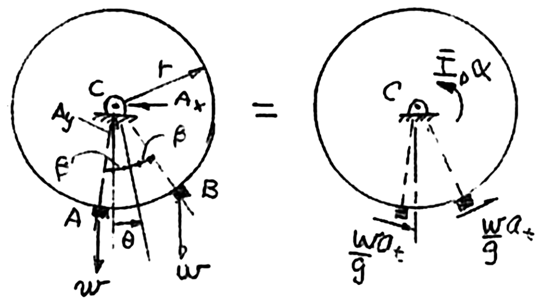
$\beta = 75.5°$ ◀



PROBLEM 19.45

Two 40-g weights are attached at A and B to the rim of a 1.5-kg uniform disk of radius $r = 100$ mm. Determine the frequency of small oscillations when $\beta = 60°$.

SOLUTION



$$\alpha = \ddot{\theta}$$

$$a_t = r\alpha = r\ddot{\theta}$$

$$\bar{I}_D = \frac{1}{2} m_D r^2$$

<u>Equation of motion</u>.

$$\Sigma M_C = \bar{I}\alpha + m\bar{a}d: \quad wr\sin(\beta - \theta) - wr\sin(\beta + \theta) = \frac{2w}{g} r a_t + \bar{I}\alpha$$

$$wr[\sin(\beta - \theta) - \sin(\beta + \theta)] = -2wr\sin\theta\cos\beta$$

$$\sin\theta \approx \theta$$

$$\left(\frac{2w}{g} r^2 + \frac{W}{2g} r^2\right)\ddot{\theta} + (2wr\cos\beta)\theta = 0$$

<u>Natural frequency</u>.

$$\omega_n = \sqrt{\frac{2wg\cos\beta}{\left(2w + \frac{W}{2}\right)r}} = \sqrt{\frac{4g\cos\beta}{(4 + \frac{W}{w})r}} \qquad (1)$$

<u>Data</u>:

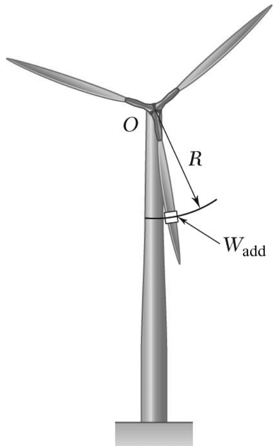
$$w = mg = 0.04g \qquad W = m_D g = 1.5g \qquad \frac{W}{w} = \frac{1.5g}{0.04g} = 37.5$$

$$r = 0.100 \text{ m} \qquad \beta = 60°$$

$$\omega_n = \sqrt{\frac{(4)(9.81)\cos 60°}{(4 + 37.5)(0.10)}} = 2.1743 \text{ rad/s}$$

$$f_n = \frac{\omega_n}{2\pi} = \frac{2.1743}{2\pi}$$

$f_n = 0.346$ Hz ◀

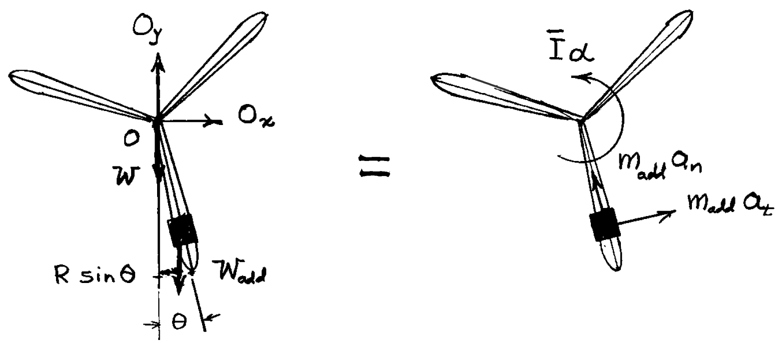


PROBLEM 19.46

A three-bladed wind turbine used for research is supported on a shaft so that it is free to rotate about O. One technique to determine the centroidal mass moment of inertia of an object is to place a known weight at a known distance from the axis of rotation and to measure the frequency of oscillations after releasing it from rest with a small initial angle. In this case, a weight of $W_{add} = 50$ lb is attached to one of the blades at a distance $R = 20$ ft from the axis of rotation. Knowing that when the blade with the added weight is displaced slightly from the vertical axis, the system is found to have a period of 7.6 s, determine the centroidal mass moment of inertia of the 3-bladed rotor.

SOLUTION

Let the turbine rotor be turned counterclockwise through a small angle θ. The moment of the added weight about Point O is



$$M = -W_{add} R \sin\theta$$

$$+\circlearrowleft \Sigma M_0 = \Sigma(M_0)_{eff}: \quad -W_{add} R \sin\theta = \bar{I}\alpha + m_{add} R a_a$$

$$= (\bar{I} + m_{add} R^2)\alpha$$

$$= (\bar{I} + m_{add} R^2)\ddot{\theta}$$

$$\ddot{\theta} + \frac{W_{add} R}{\bar{I} + m_{add} R^2} \sin\theta = 0$$

Using $\sin\theta \approx \theta$ gives

$$\ddot{\theta} + \frac{W_{add} R}{\bar{I} + m_{add} R^2} \theta = 0$$

$$\ddot{\theta} + \omega_n^2 \theta = 0 \qquad \omega_n^2 = \frac{W_{add} R}{\bar{I} + m_{add} R^2}$$

PROBLEM 19.46 (Continued)

Solving for $\bar{I}$,

$$\bar{I} = \frac{W_{add}R}{\omega_n^2} - m_{add}R^2 \qquad (1)$$

Data: $R = 20\text{ ft}, \quad W_{add} = 50\text{ lb}$

$$m_{add} = \frac{W_{add}}{g} = \frac{50\text{ lb}}{32.2\text{ ft/s}^2} = 1.5528\text{ lb}\cdot\text{s}^2/\text{ft}$$

Period and frequency: $\tau = 7.6\text{ s}$

$$f = \frac{1}{\tau} = \frac{1}{7.6}\text{Hz}$$

$$\omega_n = 2\pi f = \frac{2\pi}{7.6} = 0.82673\text{ rad/s}$$

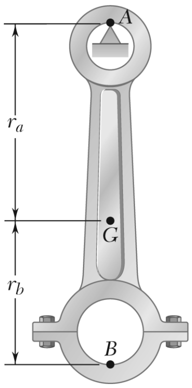
From Eq. (1),

$$\bar{I} = \frac{(50\text{ lb})(20\text{ ft})}{(0.82673\text{ rad/s})^2} - (1.5528\text{ lb}\cdot\text{s}^2/\text{ft})(20\text{ ft})^2$$

$$= 1463.10 - 621.12$$

$$\bar{I} = 842\text{ lb}\cdot\text{s}^2\cdot\text{ft} \blacktriangleleft$$

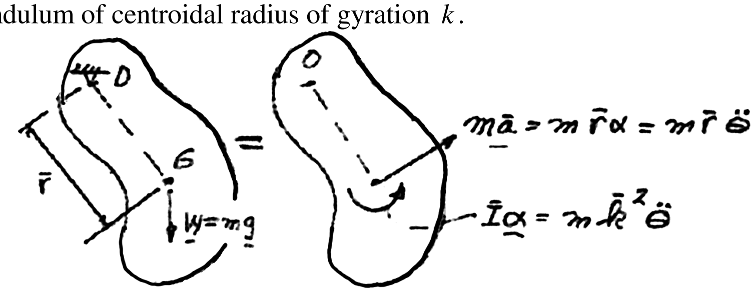
PROBLEM 19.47



A connecting rod is supported by a knife-edge at Point A; the period of its small oscillations is observed to be 0.87 s. The rod is then inverted and supported by a knife-edge at Point B and the period of small oscillations is observed to be 0.78 s. Knowing that $r_a + r_b = 10$ in. determine (*a*) the location of the mass center G, (*b*) the centroidal radius of gyration $\bar{k}$.

SOLUTION

Consider general pendulum of centroidal radius of gyration $\bar{k}$.



<u>Equation of motion</u>. $\quad +\circlearrowleft \Sigma M_0 = \Sigma (M_0)_{\text{eff}}: \quad -mg\bar{r}\sin\theta = (m\bar{r}\ddot{\theta})\bar{r} + m\bar{k}^2\ddot{\theta}$

$$\ddot{\theta} + \left[\frac{g\bar{r}}{\bar{r}^2 + \bar{k}^2}\right]\sin\theta = 0$$

For small oscillations, $\sin\theta \approx \theta$, we have

$$\ddot{\theta} + \left[\frac{g\bar{r}}{\bar{r}^2 + \bar{k}^2}\right]\theta = 0$$

$$\omega_n^2 = \frac{g\bar{r}}{\bar{r}^2 + \bar{k}^2}$$

$$\tau = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{\bar{r}^2 + \bar{k}^2}{g\bar{r}}}$$

For rod suspended at A,

$$\tau_A = 2\pi\sqrt{\frac{r_a^2 + \bar{k}^2}{g r_a}}$$

$$g\tau_A^2 r_a = 4\pi^2\left(r_a^2 + \bar{k}^2\right) \qquad (1)$$

For rod suspended at B,

$$\tau_B = 2\pi\sqrt{\frac{r_b^2 + \bar{k}^2}{g r_b}}$$

$$g\tau_B^2 r_b = 4\pi^2\left(r_b^2 + \bar{k}^2\right) \qquad (2)$$

PROBLEM 19.47 (Continued)

(*a*) Value of r_a.

Subtracting Eq. (2) from Eq. (1), $g\tau_A^2 r_a - g\tau_B^2 r_b = 4\pi^2\left(r_a^2 - r_b^2\right)$

$$g\tau_A^2 r_a - g\tau_B^2 r_b = 4\pi^2(r_a + r_b)(r_a - r_b)$$

Applying the numerical data with $r_a + r_b = 10 \text{ in.} = 0.83333 \text{ ft}$

$$(32.2)(0.87)^2 r_a - (32.2)(0.78)^2 r_b = 4\pi^2(0.83333)(r_a - r_b)$$

$$24.372 r_a - 19.590 r_b = 32.899(r_a - r_b)$$

$$13.309 r_b = 8.527 r_a \qquad r_b = 0.6407 r_a$$

$$0.83333 = r_a + 0.6407 r_a \qquad r_a = 0.5079 \text{ ft} \qquad r_a = 6.09 \text{ in.} \blacktriangleleft$$

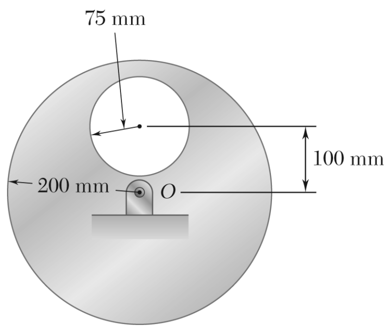
$$r_b = 0.83333 - 0.5079 \qquad r_b = 0.32543 \text{ ft} \qquad r_b = 3.91 \text{ in.} \blacktriangleleft$$

(*b*) Centroidal radius of gyration.

From Eq. (1), $4\pi^2 \bar{k}^2 = g\,\tau_A^2\, r_a - 4\pi^2 r_a^2$

$$= (32.2)(0.87)^2(0.5079) - 4\pi^2(0.5079)^2 = 2.1947 \text{ ft}^2$$

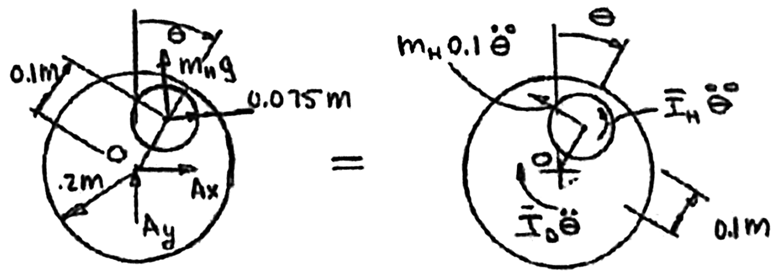
$$\bar{k} = 0.2398 \text{ ft} \qquad \bar{k} = 2.83 \text{ in.} \blacktriangleleft$$



PROBLEM 19.48

A 75-mm-radius hole is cut in a 200-mm-radius uniform disk, which is attached to a frictionless pin at its geometric center *O*. Determine (*a*) the period of small oscillations of the disk, (*b*) the length of a simple pendulum which has the same period.

SOLUTION



Equation of motion.

$$\Sigma M_0 = (\Sigma M_0)_{\text{eff}}: \quad \circlearrowleft + \; -m_H g(0.1)\sin\theta = \bar{I}_D\theta - I_H\ddot{\theta} - (0.1)^2 m_H\ddot{\theta}$$

$$\begin{aligned} m_D &= \rho t\pi R^2 \\ &= (\rho t\pi)(0.2)^2 \\ &= (0.04)\pi\rho t \\ m_H &= \rho t\pi r^2 \\ &= (\rho t\pi)(0.075)^2 \\ &= (0.005625)\pi\rho t \\ I_D &= \frac{1}{2}m_D R^2 = \frac{1}{2}(0.04\pi\rho t)(.2)^2 \\ &= 800\times 10^{-6}\pi\rho t \\ I_H &= \frac{1}{2}m_H r^2 \\ &= \frac{1}{2}(0.005625\pi\rho t)(0.75)^2 \\ &= 15.82\times 10^{-6}\pi\rho t \end{aligned}$$

PROBLEM 19.48 (Continued)

Small angles. $\sin\theta \approx \theta$

$$[800\times10^{-6}\pi\rho t - 15.82\times10^{-6}\pi\rho t - (0.1)(0.005625)\pi\rho t]\ddot{\theta} + (0.005625\ \pi\rho t)\ (9.81)(0.1)\theta = 0$$

$$727.9\times10^{-6}\ddot{\theta} + 5.518\times10^{-3}\theta = 0$$

(*a*) Natural frequency and period.

$$\omega_n^2 = \frac{5.518\times10^{-3}}{727.9\times10^{-6}} = 7.581$$

$$\omega_n = 2.753 \text{ rad/s}$$

$$\tau_n = \frac{2\pi}{\omega_n}$$

$$\tau_n = \frac{2\pi}{2.753} = 2.28 \text{ s} \blacktriangleleft$$

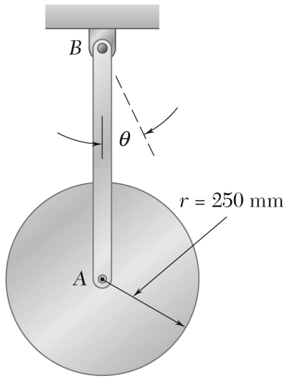
(*b*) Length and period of a simple pendulum.

$$\tau_n = 2\pi\sqrt{\frac{l}{g}}$$

$$l = \left(\frac{\tau_n}{2\pi}\right)^2 g$$

$$l = \left[\frac{(2.753)}{2\pi}\right]^2 (9.81 \text{ m/s}^2)$$

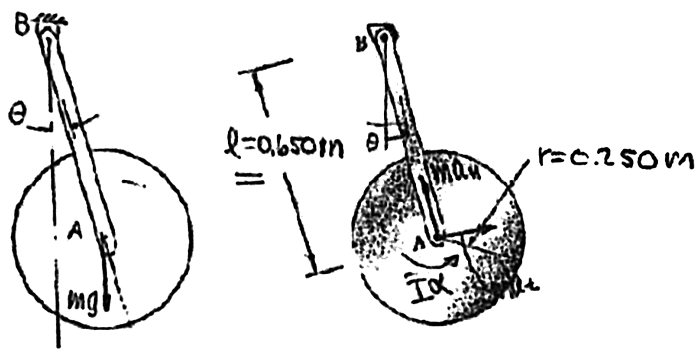
$$l = 1.294 \text{ m} \blacktriangleleft$$



PROBLEM 19.49

A uniform disk of radius $r = 250$ mm is attached at A to a 650-mm rod AB of negligible mass, which can rotate freely in a vertical plane about B. Determine the period of small oscillations (a) if the disk is free to rotate in a bearing at A, (b) if the rod is riveted to the disk at A.

SOLUTION



$$\bar{I} = \frac{1}{2}mr^2$$

$$= \frac{1}{2}(0.250)^2 m = \frac{m}{32}$$

$$a_t = l\alpha = 0.650\alpha$$

$$\alpha = \ddot{\theta}$$

(a) The disk is free to rotate and is in curvilinear translation.

Thus, $\bar{I}\alpha = 0$

$$\Sigma M_B = \Sigma(M_B)_{\text{eff}}: \quad \circlearrowleft + -mgl\sin\theta = lma_t \qquad \sin\theta \approx \theta$$

$$ml^2\ddot{\theta} - mgl\theta = 0$$

$$\omega_n^2 = \frac{g}{l} = \frac{9.81 \text{ m/s}^2}{0.650 \text{ m}}$$

$$= 15.092$$

$$\omega_n = 3.885 \text{ rad/s}$$

$$\tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{3.885}$$

$\tau_n = 1.617$ s ◀

PROBLEM 19.49 (Continued)

(*b*) When the disk is riveted at A, it <u>rotates</u> at an angular acceleration α.

$$\Sigma M_B = \Sigma(M_B)_{\text{eff}}: \quad \circlearrowleft + -mgl\sin\theta = \bar{I}\alpha + lma_t \quad I = \frac{1}{2}mr^2$$

$$\left(\frac{1}{2}mr^2 + ml^2\right)\ddot{\theta} + mgl\theta = 0$$

$$\omega_n^2 = \frac{gl}{\left(\frac{r^2}{2} + l^2\right)}$$

$$= \frac{(9.81 \text{ m/s}^2)(0.650 \text{ m})}{\left[\left(\frac{0.250^2}{2}\right) + (0.650)^2\right]}$$

$$= 14.053$$

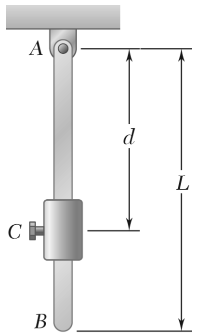
$$\omega_n = 3.749 \text{ rad/s}$$

$$\tau_n = \frac{2\pi}{\omega_n}$$

$$= \frac{2\pi}{3.749}$$

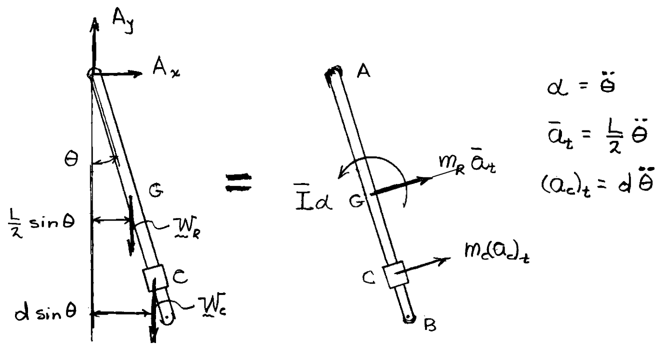
$\tau_n = 1.676$ s ◀

PROBLEM 19.50



A small collar of mass 1 kg is rigidly attached to a 3-kg uniform rod of length $L = 750$ mm. Determine (*a*) the distance d to maximize the frequency of oscillation when the rod is given a small initial displacement, (*b*) the corresponding period of oscillation.

SOLUTION



Equation of motion.

$$+\Sigma M_A = (\Sigma M_A)_{\text{eff}}: \quad -W_R\frac{L}{2}\sin\theta - W_C d\sin\theta = \bar{I}_R\alpha + m_R\frac{L}{2}(\bar{a}_t)_R + m_C d(a_t)_C$$

$$\sin\theta \approx \theta \qquad \alpha = \ddot{\theta}, \quad (\bar{a}_t)_R = \frac{L}{2}\alpha = \frac{L}{2}\ddot{\theta}, \quad (a_t)_C = d\alpha = d\ddot{\theta}$$

$$\left(\bar{I}_R + m_R\left(\frac{L}{2}\right)^2 + m_C d^2\right)\ddot{\theta} + \left(m_R g\frac{L}{2} + m_C g d\right)\theta = 0$$

$$\bar{I}_R = \frac{1}{12}m_R L^2$$

$$\bar{I}_R + m_R\left(\frac{L}{2}\right)^2 = \frac{m_R L^2}{3}$$

$$\left(\frac{m_R L^2}{3} + m_C d^2\right)\ddot{\theta} + \left(m_R g\frac{L}{2} + m_C g d\right)\theta = 0$$

$$\ddot{\theta} + \frac{\left(\frac{L}{2} + \frac{m_C}{m_R}d\right)g}{\left(\frac{L^2}{3} + \frac{m_C}{m_R}d^2\right)}\theta = 0$$

$$\frac{m_C}{m_R} = \frac{1}{3}$$

PROBLEM 19.50 (Continued)

Natural frequency. $$\omega_n^2 = \frac{\left(\frac{L}{2} + \frac{m_C}{m_R} d\right) g}{\frac{L^2}{3} + \frac{m_C}{m_R} d^2} = \frac{\left(\frac{3L}{2} + d\right) g}{(L^2 + d^2)}$$

(*a*) To maximize the frequency, we need to take the derivative with respect to d and set it equal to zero.

$$\frac{1}{g}\frac{d\left(\omega_n^2\right)}{d(d)} = \frac{(L^2 + d^2)(1) - \left(\frac{3L}{2} + d\right)(2d)}{(L^2 + d^2)^2} = 0$$

$$d^2 + L^2 - 3Ld - 2d^2 = 0$$

$$d^2 + 3Ld - L^2 = 0$$

Solve for d knowing that $L = 0.75$ m $\quad d = 0.22708 \quad$ or $\quad -2.4771$

$$d = 0.22708 \text{ m}$$

$d = 227$ mm ◀

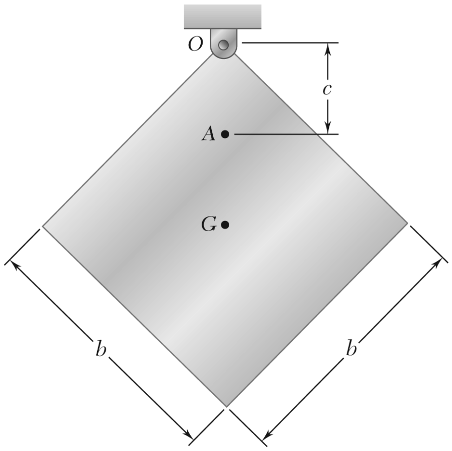
$$\omega_n^2 = \frac{\left(\frac{3(0.75)}{2} + 0.22708\right) 9.81}{(0.75^2 + 0.22708^2)}$$

$$= 21.6$$

$\omega_n = 4.6476$ rad/s

(*b*) Period of oscillation. $$\tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{4.6476}$$

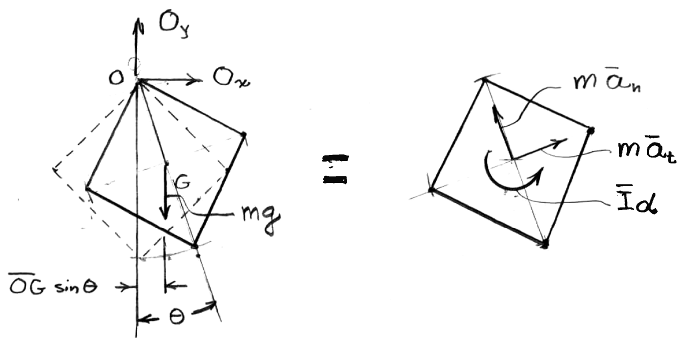
$\tau_n = 1.352$ s ◀



PROBLEM 19.51

For the uniform square plate of side $b = 12$ in, determine (*a*) the period of small oscillations if the plate is suspended as shown, (*b*) the distance *c* from *O* to a Point *A* from which the plate should be suspended for the period to be a minimum.

SOLUTION



(*a*) <u>Equation of motion</u>.

$$\Sigma M_0 = \bar{I}\alpha + m\bar{a}d: \quad \alpha = \ddot{\theta}$$

$$\bar{I} = \frac{1}{6}mb^2$$

$$a_t = (OG)(\alpha)$$

$$OG = b\frac{\sqrt{2}}{2}$$

$$a_t = \left(b\frac{\sqrt{2}}{2}\right)\ddot{\theta}$$

$$\circlearrowright + \; (OG)(\sin\theta)(mg) = -(OG)ma_t - \bar{I}\alpha \qquad \sin\theta \approx \theta$$

$$\left(b\frac{\sqrt{2}}{2}\right)m\left(b\frac{\sqrt{2}}{2}\right)\ddot{\theta} + \frac{1}{6}mb^2\ddot{\theta} + \left(b\frac{\sqrt{2}}{2}\right)mg\,\theta = 0$$

$$(b)\left(\frac{1}{2} + \frac{1}{6}\right)m\ddot{\theta} + \left(\frac{\sqrt{2}}{2}\right)mg\,\theta = 0$$

$$\ddot{\theta} + \frac{\left(\frac{\sqrt{2}}{2}\right)g}{\left(\frac{2}{3}\right)b}\theta = 0 \quad \text{or} \quad \ddot{\theta} + \frac{3\sqrt{2}}{4}\frac{g}{b} = 0$$

Natural frequency and period.

$$\omega_{n0}^2 = \frac{3\sqrt{2}}{4}\frac{g}{b} = \frac{3\sqrt{2}(32.2)}{(4)(1)} = 34.153$$

$$\omega_{n0} = 5.8441 \text{ rad/s}$$

$$\tau_{n0} = \frac{2\pi}{\omega_{n0}} \qquad \tau_{n0} = 1.075 \text{ s} \blacktriangleleft$$

(*b*) Suspended about *A*. Let $e = (OG - c)$

$$a_t = e\alpha$$

Equation of motion.

$$+\circlearrowleft \Sigma M_A = \bar{I}\alpha + m\bar{a}d: \quad mge\sin\theta = -ema_t - \bar{I}\alpha = -(me^2 + \bar{I})\alpha$$

$$m\left(e^2 + \frac{1}{6}b^2\right)\ddot{\theta} + mge\theta = 0$$

Frequency and period.

$$\omega_n^2 = \frac{eg}{e^2 + \frac{1}{6}b^2}$$

$$\tau_n^2 = \frac{4\pi^2}{\omega_n^2} = \frac{4\pi^2(e^2 + \frac{1}{6}b^2)}{eg}$$

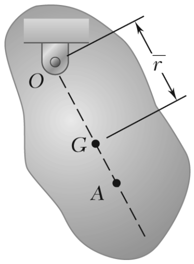
$$\tau_n^2 = \frac{4\pi^2}{g}\left(e + \frac{b^2}{6e}\right)$$

For τ_n to be minimum, $\frac{d}{de}\left(e + \frac{b^2}{6e}\right) = 0$

$$1 - \frac{b^2}{6e^2} = 0 \qquad \frac{b^2}{e^2} = 6 \qquad e = \frac{b}{\sqrt{6}}$$

$$c = OG - e = \frac{\sqrt{2}}{2}b - \frac{b}{\sqrt{6}} = 0.29886b$$

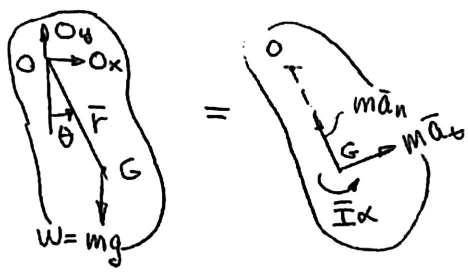
$$c = (0.29886)(12 \text{ in.}) \qquad c = 3.59 \text{ in.} \blacktriangleleft$$



PROBLEM 19.52

A *compound pendulum* is defined as a rigid slab which oscillates about a fixed Point O, called the center of suspension. Show that the period of oscillation of a compound pendulum is equal to the period of a simple pendulum of length OA, where the distance from A to the mass center G is $GA = \bar{k}^2/\bar{r}$. Point A is defined as the center of oscillation and coincides with the center of percussion defined in Problem 17.66.

SOLUTION

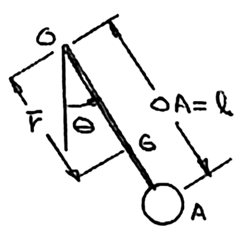


$$+\circlearrowleft \Sigma M_0 = \Sigma(M_0)_{\text{eff}}: \quad -W\bar{r}\sin\theta = \bar{I}\alpha + m\bar{a}_t\bar{r}$$

$$-mg\bar{r}\sin\theta = m\bar{k}^2\ddot{\theta} + m\bar{r}^2\ddot{\theta}$$

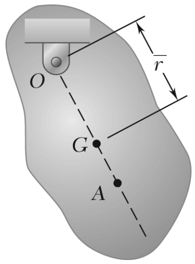
$$\ddot{\theta} + \frac{g\bar{r}}{\bar{r}^2 + \bar{k}^2}\sin\theta = 0 \qquad (1)$$

For a simple pendulum of length $OA = l$,



$$\ddot{\theta} + \frac{g}{l}\theta = 0 \qquad (2)$$

Comparing Equations (1) and (2), $\quad l = \dfrac{\bar{r}^2 + \bar{k}^2}{\bar{r}} \qquad GA = l - \bar{r} = \dfrac{\bar{k}^2}{\bar{r}}$ Q.E.D. ◀



PROBLEM 19.53

A rigid slab oscillates about a fixed Point O. Show that the smallest period of oscillation occurs when the distance $\bar{r}$ from Point O to the mass center G is equal to $\bar{k}$.

SOLUTION

See Solution to Problem 19.52 for derivation of

$$\ddot{\theta} + \frac{g\bar{r}}{\bar{r}^2 + \bar{k}^2}\sin\theta = 0$$

For small oscillations, $\sin\theta \approx \theta$ and

$$\tau_n = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{\bar{r}^2 + \bar{k}^2}{g\bar{r}}} = \frac{2\pi}{\sqrt{g}}\sqrt{\bar{r} + \frac{\bar{k}^2}{\bar{r}}}$$

For smallest τ_n, we must have $\bar{r} + \frac{\bar{k}^2}{\bar{r}}$ as a minimum:

$$\frac{d\left(\bar{r} + \frac{\bar{k}^2}{\bar{r}}\right)}{d\bar{r}} = 1 - \frac{\bar{k}^2}{\bar{r}^2} = 0$$

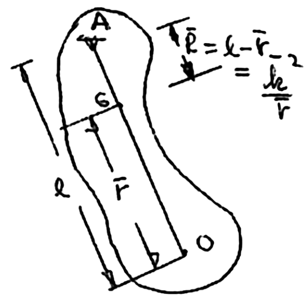
$$\bar{r}^2 = \bar{k}^2$$

$\bar{r} = \bar{k}$ Q.E.D. ◀

PROBLEM 19.54

Show that if the compound pendulum of Problem 19.52 is suspended from *A* instead of *O*, the period of oscillation is the same as before and the new center of oscillation is located at *O*.

SOLUTION



Same derivation as in Problem 19.52 with $\bar{r}$ replaced by $\bar{R}$.

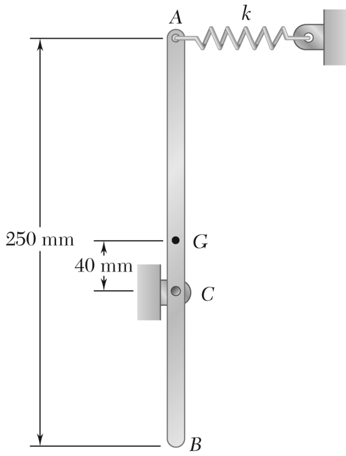
Thus, $$\ddot{\theta} + \frac{g\bar{R}}{\bar{R}^2 + \bar{k}}\theta = 0$$

Length of the equivalent simple pendulum is

$$L = \frac{\bar{R}^2 + \bar{k}^2}{\bar{R}} = \bar{R} + \frac{\bar{k}^2}{\bar{R}}$$

$$L = (l - \bar{r}) + \frac{\bar{k}^2}{\frac{\bar{k}^2}{\bar{r}}} = l$$

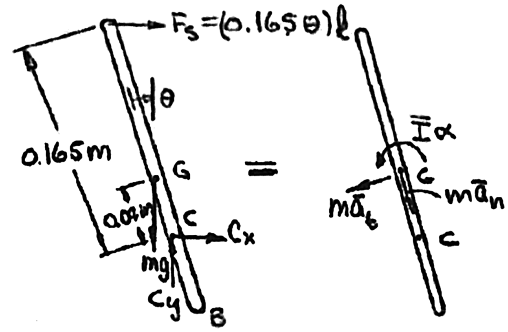
Thus, the length of the equivalent simple pendulum is the same as in Problem 19.52. It follows that the period is the same and that the new center of oscillation is at *O*. Q.E.D.



PROBLEM 19.55

The 8-kg uniform bar AB is hinged at C and is attached at A to a spring of constant $k = 500$ N/m. If end A is given a small displacement and released, determine (*a*) the frequency of small oscillations, (*b*) the smallest value of the spring constant k for which oscillations will occur.

SOLUTION



$$\bar{I} = \frac{1}{12}ml^2 = \left(\frac{1}{12}\right)(8)(0.250)^2$$

$$\bar{I} = 0.04167 \text{ kg}\cdot\text{m}^2$$

$$\alpha = \ddot{\theta}$$

$$a_t = 0.04\alpha = 0.04\ddot{\theta}$$

$$\sin\theta \approx \theta$$

<u>Equation of motion.</u>

$$+\circlearrowleft \Sigma M_C = \Sigma(M_C)_{\text{eff}}: \quad -(0.165)^2 k\theta + 0.04mg\theta = \bar{I}\ddot{\theta} + (0.04)^2 m\ddot{\theta}$$

$$(0.04167 + 0.01280)\ddot{\theta} + (0.02722k - 0.32g)\theta = 0 \qquad (1)$$

(*a*) <u>Frequency if $k = 500$ N $\cdot$ m.</u>

$$0.05447\ddot{\theta} + (10.47)\theta = 0$$

$$f_n = \frac{\omega_n}{2\pi} = \frac{\left(\sqrt{\frac{10.47}{0.05447}}\right)}{2\pi}$$

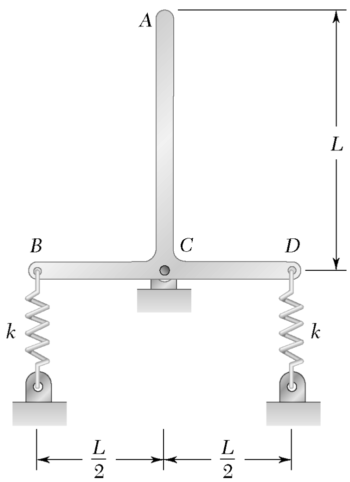
$f_n = 2.21$ Hz ◀

(*b*) For $\tau_n \longrightarrow \infty$ or $\omega_n \longrightarrow 0$, oscillations will not occur.

From Equation (1), $$\omega_n^2 = \frac{0.02722k - 0.32g}{(0.05447)} = 0$$

$$k = \frac{0.32g}{0.02722} = \frac{(0.32)(9.81)}{(0.02722)}$$

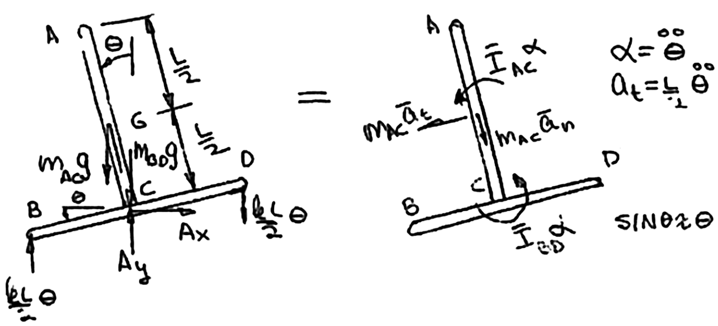
$k = 115.3$ N/m ◀



PROBLEM 19.56

Two uniform rods, each of mass $m = 12$ kg and length L = 800 mm, are welded together to form the assembly shown. Knowing that the constant of each spring is $k = 500$ N/m and that end A is given a small displacement and released, determine the frequency of the resulting motion.

SOLUTION



Equation of motion. $+\circlearrowleft \Sigma M_0 = \Sigma(M_0)_{\text{eff}}$: $\left[m_{AC}g\frac{L}{2} - 2k\left(\frac{L}{2}\right)^2\right]\theta = (\bar{I}_{AC} + \bar{I}_{BD})\alpha + m_{AC}\left(\frac{L}{2}\right)^2\alpha$

$$m_{BD} = m_{AC} = m$$

$$\bar{I}_{BD} = \bar{I}_{AC} = \bar{I} = \frac{1}{2}mL^2$$

$$\left(\frac{1}{6} + \frac{1}{4}\right)mL^2\ddot{\theta} + \left[2k\left(\frac{L}{2}\right)^2 - mg\frac{L}{2}\right]\theta = 0$$

$$\frac{10}{24}mL^2\ddot{\theta} + \left[\frac{kL^2}{2} - \frac{mgL}{2}\right]\theta = 0 \Rightarrow \omega_n^2 = \frac{6(kL - mg)}{5mL}$$

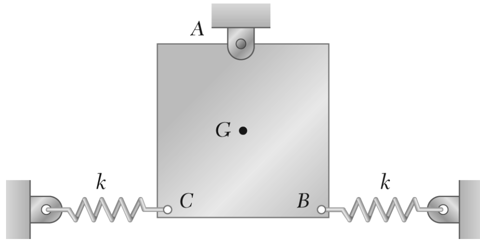
Data: $L = 800 \text{ mm} = 0.8\text{m}, \quad m = 12 \text{ kg}, \quad k = 500 \text{ N/m}$

Frequency. $\omega_n^2 = \dfrac{6[(500)(0.8) - (12)(9.81)]}{(5)(12)(0.8)} = 35.285$

$\omega_n = 5.9401 \text{ rad/s} \quad f_n = \dfrac{\omega_n}{2\pi}$

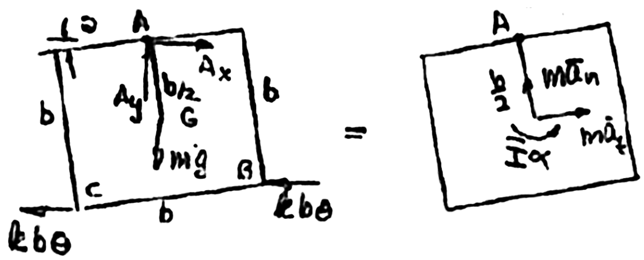
$f_n = 0.945 \text{ Hz}$ ◀

PROBLEM 19.57



A 45-lb uniform square plate is suspended from a pin located at the midpoint A of one of its 1.2-ft edges and is attached to springs, each of constant $k = 8$ lb/in. If corner B is given a small displacement and released, determine the frequency of the resulting vibration. Assume that each spring can act in either tension or compression.

SOLUTION



$$\alpha = \ddot{\theta}$$

$$a_t = \frac{b}{2}\alpha - \frac{b}{2}\ddot{\theta}$$

$$\sin\theta \approx \theta$$

Equation of motion. $\quad (+\Sigma M_0 = \Sigma(M_0)_{\text{eff}}: \quad -mg\frac{b}{2}\theta + 2kb^2\theta = \bar{I}\alpha + \left(\frac{b}{2}\right)^2 m\alpha$

$$\bar{I} + m\left(\frac{b}{2}\right)^2 = \frac{1}{6}mb^2 + m\frac{b^2}{4} = \frac{5}{12}mb^2$$

$$\frac{5}{12}mb^2\ddot{\theta} + \left(mg\frac{b}{2} + 2kb\right)\theta = 0$$

$$\ddot{\theta} + \left(\frac{12}{10}\frac{g}{b} + \frac{24k}{5mb}\right)\theta = 0$$

Data: $\quad b = 1.2$ ft; $\quad m = \dfrac{45}{32.2} = 1.3975$ lb·s²/ft

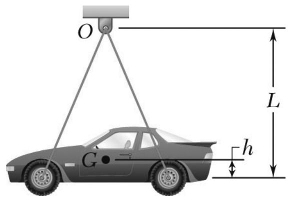
$$k = 8 \text{ lb/in.} = 96 \text{ lb/ft}$$

$$\ddot{\theta} + \left[\frac{(12)(32.2)}{(10)(1.2)} + \frac{(24)(96)}{(5)(1.3975)(1.2)}\right]\theta = 0$$

$$\ddot{\theta} + 306.98\theta = 0$$

$$\omega_n^2 = 306.98 \quad \omega_n = 17.521 \text{ rad/s}$$

Frequency. $\quad f_n = \dfrac{\omega_n}{2\pi} = \dfrac{17.521}{2\pi}$ $\qquad f_n = 2.79$ Hz ◀



PROBLEM 19.58

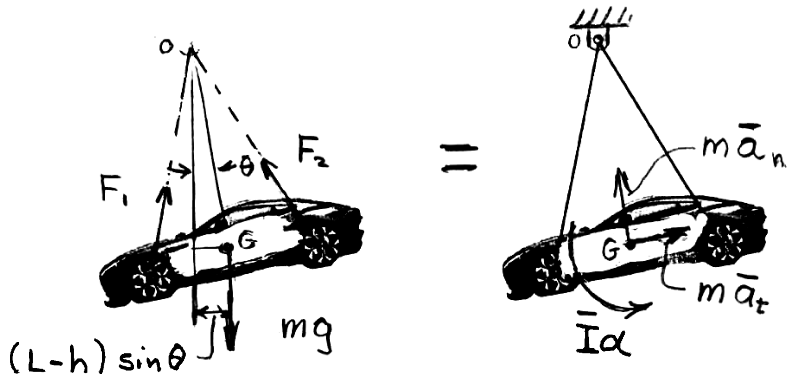
A 1300-kg sports car has a center of gravity G located a distance h above a line connecting the front and rear axles. The car is suspended from cables that are attached to the front and rear axles as shown. Knowing that the periods of oscillation are 4.04 s when $L = 4$ m and 3.54 s when $L = 3$ m, determine h and the centroidal radius of gyration.

SOLUTION

Let the mass center of the car be displaced a small distance x to the right. The mass center is moves on a circular arc of radius $L - h$, so that $x = (L - h)\sin\theta$, where θ is the counterclockwise rotation of the car. From kinematics

$$\alpha = \ddot{\theta} \qquad \bar{a}_t = (L-h)\ddot{\theta}$$

The moment of the weight force about O is



$$M_0 = -mg(L-h)\sin\theta$$

$$+\circlearrowleft M_0 = \bar{I}\alpha + (L-h)m\bar{a}_t$$

$$-mg(L-h)\sin\theta = \bar{I}\ddot{\theta} + m(L-h)^2\ddot{\theta}$$

Dividing by m and transposing terms yields

$$[\bar{k}^2 + (L-h)^2]\ddot{\theta} + g(L-h)\sin\theta = 0$$

For small angle θ, $\quad \sin\theta \approx \theta$

$$\ddot{\theta} + \frac{g(L-h)}{\bar{k}^2 + (L-h)^2}\theta = 0$$

$$\ddot{\theta} + \omega_n^2\theta = 0 \qquad \omega_n^2 = \frac{g(L-h)}{\bar{k}^2 + (L-h)^2}$$

$$\bar{k}^2 + (L-h)^2 = \frac{g}{\omega_n^2}(L-h)$$

Using the two different values (L_1 and L_2) for L,

$$\bar{k}^2 + (L_1 - h)^2 = \frac{g}{\omega_{n1}^2}(L_1 - h) \qquad (1)$$

$$\bar{k}^2 + (L_2 - h)^2 = \frac{g}{\omega_{n2}^2}(L_2 - h) \qquad (2)$$

Subtracting Eq. (2) from Eq. (1) to eliminate $\bar{k}^2$,

$$(L_1 - h)^2 - (L_2 - h)^2 = \frac{gL_1}{\omega_{n1}^2} - \frac{gL_2}{\omega_{n2}^2} - \left(\frac{g}{\omega_{n1}^2} - \frac{g}{\omega_{n2}^2}\right)h$$

$$\left(L_1^2 - L_2^2\right) - 2(L_1 - L_2)h = A - Bh$$

where $$A = \frac{gL_1}{\omega_{n1}^2} - \frac{gL_2}{\omega_{n2}^2}$$

and $$B = \frac{g}{\omega_{n1}^2} - \frac{g}{\omega_{n2}^2}$$

$$h = \frac{L_1^2 - L_2^2 - A}{2(L_1 - L_2) - B}$$

Data: $L_1 = 4\text{ m}, \quad L_2 = 3\text{ m}, \quad g = 9.81\text{ m/s}^2$

$$\omega_{n1} = \frac{2\pi}{\tau_1} = \frac{2\pi}{4.04\text{ s}} = 1.55524\text{ rad/s}$$

$$\omega_{n2} = \frac{2\pi}{\tau_2} = \frac{2\pi}{3.54\text{ s}} = 1.77491\text{ rad/s}$$

$$A = \frac{(9.81)(4)}{(1.55524)^2} - \frac{(9.81)(3)}{(1.77491)^2} = 6.8812\text{ m}^2$$

$$B = \frac{9.81}{(1.55524)^2} - \frac{9.81}{(1.77491)^2} = 0.94190\text{ m}$$

$$h = \frac{(4)^2 - (3)^2 - 6.8812}{2(4-3) - 0.94190} = 0.11228\text{ m}$$

$h = 0.1123\text{ m}$ ◀

$$L_1 - h = 3.88772\text{ m} \qquad L_2 - h = 2.88772\text{ m}$$

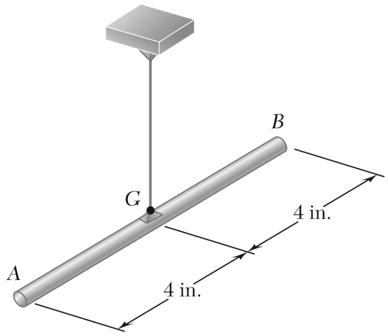
From Eq. (1), $$\bar{k}^2 + (3.88772)^2 = \frac{(9.81)(3.88772)}{(1.55524)^2}$$

$$\bar{k}^2 = 0.65336\text{ m}^2$$

$\bar{k} = 0.808\text{ m}$ ◀

Checking, using Eq. (2), $$\bar{k}^2 + (2.88772)^2 = \frac{(9.81)(2.88772)}{(1.77491)^2}$$

$$\bar{k}^2 = 0.65339\text{ m}^2$$



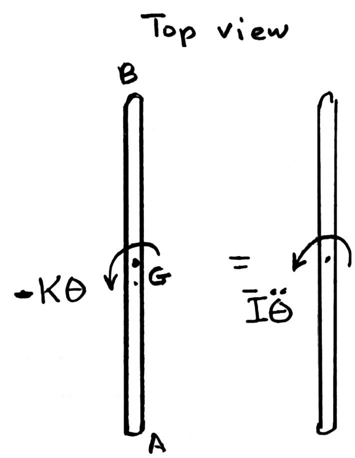
PROBLEM 19.59

A 6-lb slender rod is suspended from a steel wire which is known to have a torsional spring constant $K = 1.5$ ft·lb/rad. If the rod is rotated through 180° about the vertical and released, determine (*a*) the period of oscillation, (*b*) the maximum velocity of end *A* of the rod.

SOLUTION

Equation of motion. $\Sigma M_G = \Sigma(M_G)_{\text{eff}}: \quad -K\theta = \bar{I}\ddot{\theta} \qquad \ddot{\theta} + \frac{K}{\bar{I}}\theta = 0$

$$\ddot{\theta} + \omega_n^2\theta = 0 \qquad \omega_n^2 = \frac{K}{\bar{I}}$$



Data:

$$W = 6 \text{ lb.}$$

$$m = \frac{W}{g} = \frac{6}{32.2} = 0.186335 \text{ lb}\cdot\text{s}^2/\text{ft}$$

$$l = 8 \text{ in.} = \frac{2}{3}\text{ft}$$

$$\bar{I} = \frac{1}{12}ml^2 = \frac{1}{12}(0.186335)\left(\frac{2}{3}\right)^2$$

$$= 0.006901 \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

$$K = 1.5 \text{ lb}\cdot\text{ft/rad}$$

$$\omega_n^2 = \frac{1.5}{0.006901} = 217.35$$

$$\omega_n = 14.743 \text{ rad/s}$$

(*a*) Period of oscillation.

$$\tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{14.743}$$

$\tau_n = 0.426$ s ◀

Simple harmonic motion:

$$\theta = \theta_m \sin(\omega_n t + \varphi)$$

$$\dot{\theta} = \omega_n\theta_m \cos(\omega_n t + \varphi)$$

$$\dot{\theta}_m = \omega_n\theta_m$$

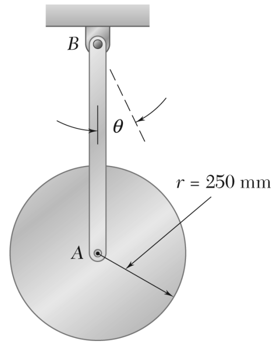
$$(v_A)_m = \frac{l}{2}\dot{\theta}_m = \frac{1}{2}l\omega_A\theta_m$$

$$\theta_m = 180° = \pi \text{ radians}$$

(*b*) Maximum velocity at end *A*.

$$(v_A)_m = \left(\frac{1}{2}\right)\left(\frac{2}{3}\right)(14.743)(\pi)$$

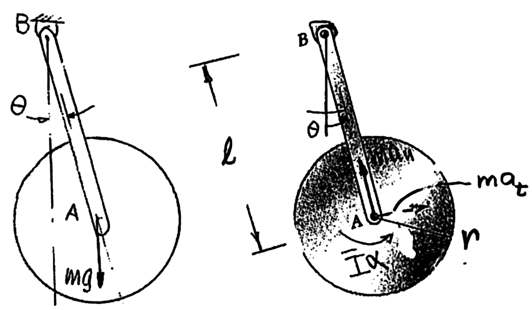
$(v_A)_m = 15.44$ ft/s ◀



PROBLEM 19.60

A uniform disk of radius $r = 250$ mm is attached at A to a 650-mm rod AB of negligible mass which can rotate freely in a vertical plane about B. If the rod is displaced 2° from the position shown and released, determine the magnitude of the maximum velocity of Point A, assuming that the disk (a) is free to rotate in a bearing at A, (b) is riveted to the rod at A.

SOLUTION



$$\bar{I} = \frac{1}{2}mr^2$$

Kinematics: $\alpha = \ddot{\theta}$

$$a_t = l\alpha = l\ddot{\theta}$$

(a) The disk is free to rotate and is in curvilinear translation.

Thus, $\bar{I}\alpha = 0$

Equation of motion. $\Sigma M_B = (\Sigma M_B)_{\text{eff}}: \quad -mgl\sin\theta = lma_t, \quad \sin\theta \approx \theta$

$$ml^2\ddot{\theta} + mgl\theta = 0$$

Frequency. $\omega_n^2 = \dfrac{g}{l}$

$$= \frac{9.81}{0.650}$$

$$= 15.092$$

$$\omega_n = 3.8849 \text{ rad/s}$$

$$\theta_m = 2° = 0.034906 \text{ rad}$$

$$\dot{\theta}_m = \omega_n\theta_m = (3.8849)(0.034906) = 0.13561 \text{ rad/s}$$

$$(v_A)_m = l\dot{\theta}_m = (0.650)(0.13561)$$

$(v_A)_m = 0.0881$ m/s ◀

(*b*) When the disk is riveted at *A*, it rotates at an angular acceleration α.

Equation of motion. $\Sigma M_B = (\Sigma M_B)_{\text{eff}}: \quad -mgl\sin\theta = \bar{I}\alpha + lma_t, \quad \bar{I} = \frac{1}{2}mr^2, \quad \sin\theta \approx \theta$

$$\left(\frac{1}{2}mr^2 + ml^2\right)\ddot{\theta} + mgl\theta = 0$$

Frequency.

$$\omega_n^2 = \frac{gl}{\left(\frac{r^2}{2} + l^2\right)}$$

$$= \frac{(9.81)(0.650)}{\frac{1}{2}(0.250)^2 + (0.650)^2}$$

$$= 14.053$$

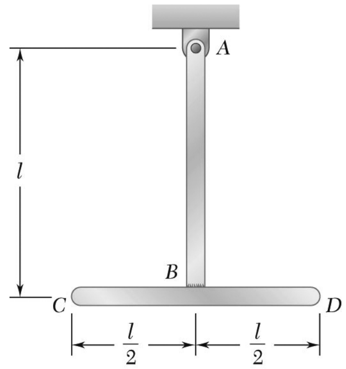
$$\omega_n = 3.7487 \text{ rad/s}$$

$$\theta_m = 2° = 0.034906 \text{ rad}$$

$$\dot{\theta}_m = \omega_n \theta_m = (3.7487)(0.034906) = 0.13085 \text{ rad/s}$$

$$(v_A)_m = l\dot{\theta}_m = (0.650)(0.13085)$$

$(v_A)_m = 0.0851$ m/s ◀

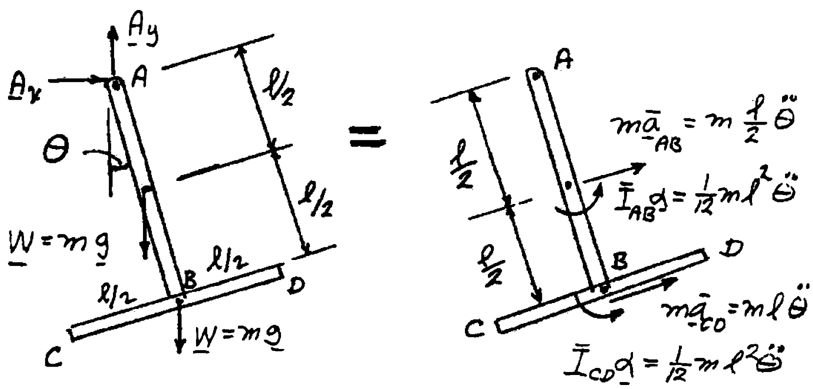


PROBLEM 19.61

Two uniform rods, each of mass m and length l, are welded together to form the T-shaped assembly shown. Determine the frequency of small oscillations of the assembly.

SOLUTION

Let the assembly be rotated counterclockwise through the small angle θ about the fixed Point A.



Equation of motion. $+\circlearrowleft \Sigma M_A = \Sigma(M_A)_{\text{eff}}$: $-mg\frac{l}{2}\sin\theta - mgl\sin\theta = \bar{I}_{AB}\alpha + m\bar{a}_{AB}\frac{l}{2} + \bar{I}_{CD}\alpha + m\bar{a}_{CD}l$

$$-\frac{3}{2}mgl\sin\theta = \frac{1}{12}ml^2\ddot{\theta} + m\left(\frac{l}{2}\right)^2\ddot{\theta} + \frac{1}{12}ml^2\ddot{\theta} + ml^2\ddot{\theta}$$

$$-\frac{3}{2}mgl\sin\theta = \frac{17}{12}ml^2\ddot{\theta}$$

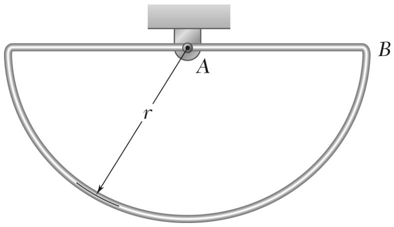
$$\ddot{\theta} + \frac{18}{17}\frac{g}{l}\sin\theta = 0$$

For small oscillations, $\sin\theta \approx \theta$

$$\ddot{\theta} + \frac{18}{17}\frac{g}{l}\theta = 0$$

$$\omega_n^2 = \frac{18}{17}\frac{g}{l} \qquad \omega_n = \sqrt{\frac{18g}{17l}}$$

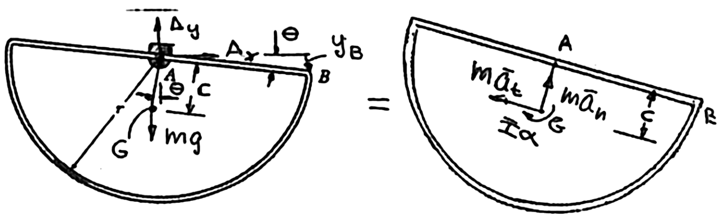
Frequency. $f = \frac{\omega_n}{2\pi} \qquad f = \frac{1}{2\pi}\sqrt{\frac{18g}{17l}}$ $\qquad f = 0.1638\sqrt{\frac{g}{l}}$ ◀



PROBLEM 19.62

A homogeneous wire bent to form the figure shown is attached to a pin support at A. Knowing that $r = 220$ mm and that Point B is pushed down 20 mm and released, determine the magnitude of the velocity of B, 8 s later.

SOLUTION



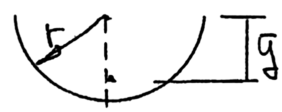
Determine location of the centroid G.

Let $\rho =$ mass per unit length

Then total mass $m = \rho(2r + \pi r) = \rho r(2+\pi)$

About C: $mgc = 0 + \pi r \rho \left(\dfrac{2r}{\pi}\right) g = 2r^2 \rho g$

$$\bar{y} = \frac{2r}{\pi} \quad \text{for a semicircle}$$



$$\rho r(2+\pi)c = 2r^2\rho, \quad c = \frac{2r}{(2+\pi)}$$

<u>Equation of motion</u>. $\circlearrowleft + \; \Sigma M_0 = \Sigma (M_0)_{\text{eff}}: \quad \alpha = \ddot{\theta} \quad a_t = c\alpha = c\ddot{\theta}$

$$-mgc\sin\theta = \bar{I}\alpha + mca_n \qquad \sin\theta \approx \theta$$

$$(\bar{I} + mc^2)\ddot{\theta} + mgc\theta = 0 \qquad I_0\ddot{\theta} + mgc\theta = 0$$

<u>Moment of inertia</u>. $\bar{I} + mc^2 = I_0$

$$I_0 = (I_0)_{\text{semicirc.}} + (I_0)_{\text{line}} = m_{\text{semicirc.}} r^2 + m_{\text{line}} \frac{(2r)^2}{12}$$

$$m_{\text{semicirc.}} = \rho\pi r \qquad m_{\text{line}} = 2\rho r \qquad \rho = \frac{m}{(2+\pi)r}$$

$$I_0 = \rho\left[\pi r^2 \cdot r + 2r \cdot \frac{r^2}{3}\right] = \frac{mr^2}{(2+\pi)}\left[\pi + \frac{2}{3}\right]$$

$$\frac{mr^2}{(2+\pi)}\left[\pi + \frac{2}{3}\right]\ddot{\theta} + mg\frac{2r}{(2+\pi)}\theta = 0$$

PROBLEM 19.62 (Continued)

Frequency.
$$\omega_n^2 = \frac{2g}{\left(\pi + \frac{2}{3}\right)r} = \frac{(2)(9.81)}{\left(\pi + \frac{2}{3}\right)(0.220)}$$

$$\omega_n^2 = 23.418 \text{ s}^{-2} \qquad \omega_n = 4.8392 \text{ rad/s}$$

$$\theta = \theta_m \sin(\omega_n t + \phi) \qquad y_B = r\theta$$

At $t = 0$,
$$y_B = 20 \text{ mm}, \quad \dot{y}_B = 0$$

$$\dot{y}_B = 0 = (y_B)_m \omega_n \cos(0 + \phi), \quad \phi = \frac{\pi}{2}$$

$$y_B = 20 \text{ mm} = (y_B)_m \sin\left(0 + \frac{\pi}{2}\right), \quad (y_B)_m = 20 \text{ mm}$$

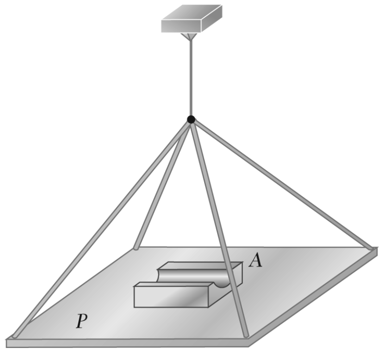
$$y_B = (20 \text{ mm}) \sin\left(\omega_n t + \frac{\pi}{2}\right) \qquad \omega_n = 4.8392 \text{ rad/s}$$

$$\dot{y}_B = 20\omega \cos\left(\omega_n t + \frac{\pi}{2}\right) = -(20 \text{ mm})\omega_n \sin \omega_n t$$

At $t = 8$ s,
$$\dot{y}_B = -(20)(4.8392)\sin[(4.8392)(8)] = -(96.78)(0.8492)$$

$$= -82.2 \text{ mm/s}$$

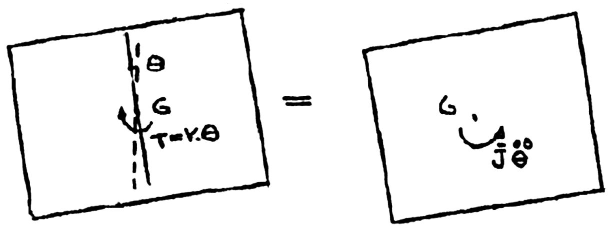
$v_B = 82.2$ mm/s ◀



PROBLEM 19.63

A horizontal platform P is held by several rigid bars which are connected to a vertical wire. The period of oscillation of the platform is found to be 2.2 s when the platform is empty and 3.8 s when an object A of unknown moment of inertia is placed on the platform with its mass center directly above the center of the plate. Knowing that the wire has a torsional constant $K = 27\ \text{N}\cdot\text{m/rad}$, determine the centroidal moment of inertia of object A.

SOLUTION



Equation of motion. $\Sigma M_G = \bar{I}\alpha: \quad -K\theta = \bar{I}\ddot{\theta}$

$$\ddot{\theta} + \frac{K}{\bar{I}}\theta = 0 \qquad \omega_n^2 = \frac{K}{\bar{I}}$$

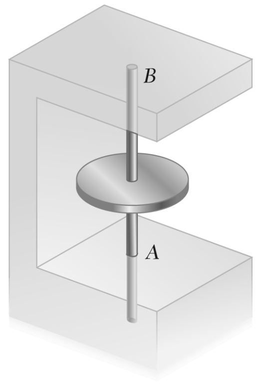
Case 1. The platform is empty. $\omega_{n1} = \frac{2\pi}{\tau_1} = \frac{2\pi}{2.2} = 2.856\ \text{rad/s}$

$$\bar{I}_1 = \frac{K}{\omega_{n1}^2} = \frac{27}{(2.856)^2} = 3.31\ \text{kg}\cdot\text{m}^2$$

Case 2. Object A is on the platform. $\omega_{n2} = \frac{2\pi}{\tau_2} = \frac{2\pi}{3.8} = 1.653\ \text{rad/s}$

$$\bar{I}_2 = \frac{K}{\omega_{n2}^2} = \frac{27}{(1.653)^2} = 9.8814\ \text{kg}\cdot\text{m}^2$$

Moment of inertia of object A. $\bar{I}_A = \bar{I}_2 - \bar{I}_1$ $\qquad \bar{I}_A = 6.57\ \text{kg}\cdot\text{m}^2$ ◀



PROBLEM 19.64

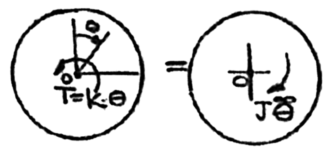
A uniform disk of radius $r = 120$ mm is welded at its center to two elastic rods of equal length with fixed ends at A and B. Knowing that the disk rotates through an $8°$ angle when a $500\text{-mN}\cdot\text{m}$ couple is applied to the disk and that it oscillates with a period of 1.3 s when the couple is removed, determine (a) the mass of the disk, (b) the period of vibration if one of the rods is removed.

SOLUTION

Torsional spring constant.

$$k = \frac{T}{\theta} = \frac{0.5\ \text{N}\cdot\text{m}}{(8)\left(\frac{\pi}{180}\right)}$$

$$k = 3.581\ \text{N}\cdot\text{m/rad}$$



Equation of motion.

$$\Sigma M_0 = \Sigma(M_0)_{\text{eff}}: \quad -K\theta = I\ddot{\theta} \quad \ddot{\theta} + \frac{K}{I}\theta = 0$$

Natural frequency and period. $\omega_n^2 = \frac{K}{I}$

Period. $\tau_n = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{I}{K}}$

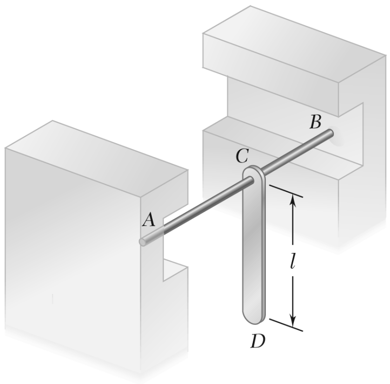
Mass moment of inertia. $I = \frac{\tau_n^2 K}{(2\pi)^2} = \frac{(1.35)^2(3581\ \text{N}\cdot\text{m/r})}{(2\pi)^2}$

$$I = 0.1533\ \text{N}\cdot\text{m}\cdot\text{s}^2 = \frac{1}{2}mr^2 = \frac{1}{2}m(0.120\ \text{m})^2$$

(a) Mass of the disk. $m = \frac{(0.1533\ \text{N}\cdot\text{m}\cdot\text{s}^2)(2)}{(0.120\ \text{m})^2}$ $\qquad m = 21.3$ kg ◀

(b) With one rod removed: $K' = \frac{K}{2} = \frac{3.581}{2} = 1.791\ \text{N}\cdot\text{m/rad}$

Period. $\tau = 2\pi\sqrt{\frac{I}{K'}} = 2\pi\sqrt{\frac{(0.1533\ \text{N}\cdot\text{m}\cdot\text{s}^2)}{1.791\ \text{N}\cdot\text{m/rad}}}$ $\qquad \tau = 1.838$ s ◀

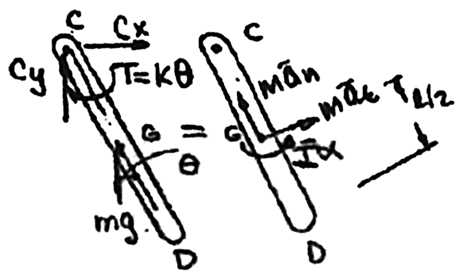


PROBLEM 19.65

A 5-kg uniform rod CD of length $l = 0.7$ m is welded at C to two elastic rods, which have fixed ends at A and B and are known to have a combined torsional spring constant $K = 24$ N·m/rad. Determine the period of small oscillation, if the equilibrium position of CD is (*a*) vertical as shown, (*b*) horizontal.

SOLUTION

(*a*) Equation of motion.



$$\alpha = \ddot{\theta} \qquad a_t = \frac{l}{2}\alpha = \frac{l}{2}\ddot{\theta}$$

$$+\circlearrowleft \Sigma M_C = \bar{I}\alpha + m\bar{a}d: \quad -K\theta - (mg)\frac{l}{2}\sin\theta = \bar{I}\alpha + \frac{l}{2}(ma_t)$$

$$-K\theta - \frac{1}{2}mgl\sin\theta = \bar{I}\ddot{\theta} + \frac{1}{4}ml^2\ddot{\theta}$$

$$\left(\bar{I} + \frac{1}{4}ml^2\right)\ddot{\theta} + \frac{1}{2}mgl\sin\theta + K\theta = 0$$

$$\left(\frac{1}{12}ml^2 + \frac{1}{4}ml^2\right)\ddot{\theta} + K\theta + \frac{1}{2}mgl\theta = 0$$

$$\frac{1}{3}ml^2\ddot{\theta} + \left(K + \frac{1}{2}mgl\right)\theta = 0$$

$$\ddot{\theta} + \left(\frac{3K}{ml^2} + \frac{3g}{2l}\right)\theta = 0$$

Data: $K = 24\ \text{N}\cdot\text{m/rad}, \quad m = 5\ \text{kg}, \quad l = 0.7\ \text{m}$

$$\ddot{\theta} + \left[\frac{(3)(24)}{(5)(0.7)^2} + \frac{(3)(9.81)}{(2)(0.7)}\right]\theta = 0$$

$$\ddot{\theta} + 50.409\theta = 0$$

Frequency. $\omega_n^2 = 50.409 \qquad \omega_n = 7.1\ \text{rad/s}$

Period. $\tau = \dfrac{2\pi}{\omega_n} = \dfrac{2\pi}{7.1}$ $\qquad \tau_n = 0.885\ \text{s}$ ◀

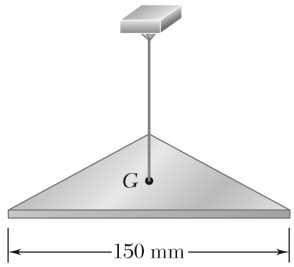
(*b*) If the rod is horizontal, the gravity term is not present and the equation of motion is

$$\ddot{\theta} + \frac{3K}{ml^2}\theta = 0$$

$$\omega_n^2 = \frac{3K}{ml^2} = \frac{(3)(24)}{(5)(0.7)^2} = 29.388$$

$$\omega_n = 5.4210\ \text{rad/s} \qquad \tau = \frac{2\pi}{\omega_n} = \frac{2\pi}{5.4210}$$

$\tau_n = 1.159\ \text{s}$ ◀

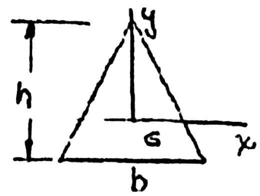


PROBLEM 19.66

A 1.8-kg uniform plate in the shape of an equilateral triangle is suspended at its center of gravity from a steel wire which is known to have a torsional constant $K = 35\text{ mN}\cdot\text{m/rad}$. If the plate is rotated 360° about the vertical and then released, determine (*a*) the period of oscillation, (*b*) the maximum velocity of one of the vertices of the triangle.

SOLUTION

Mass moment of inertia of plate about a vertical axis:



$$h = \frac{\sqrt{3}}{2}b \qquad A = \frac{1}{2}bh = \frac{\sqrt{3}}{4}b^2$$

For area,

$$\bar{I}_x = \bar{I}_y = \frac{1}{36}bh^3 = \frac{\sqrt{3}b^4}{96}$$

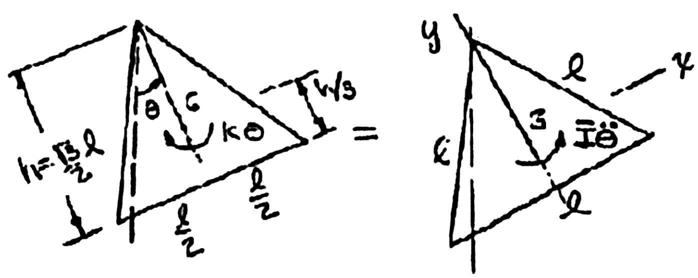
$$\bar{I}_z = \bar{I}_x + \bar{I}_y = \frac{\sqrt{3}}{48}b^4$$

For mass,

$$\bar{I} = \frac{m}{A}(\bar{I}_z)_{\text{area}}$$

$$= \left(\frac{4\,m}{\sqrt{3}b}\right)\left(\frac{\sqrt{3}}{48}b^4\right) = \frac{1}{12}mb^2$$

$$\bar{I} = \frac{1}{12}(1.8)(0.150)^2 = 3.375\times10^{-3}\text{ kg}\cdot\text{m}^2$$



Equation of motion. $\circlearrowleft + \Sigma M_G = \Sigma(M_G)_{\text{eff}}:\quad -K\theta = \bar{I}\ddot{\theta}$

$$\ddot{\theta} + \frac{K}{\bar{I}}\theta = 0$$

Frequency.

$$\omega_n^2 = \frac{K}{\bar{I}} = \frac{35\times10^{-3}}{3.375\times10^{-3}} = 10.37$$

$$\omega_n = 3.2203\text{ rad/s}$$

(*a*) Period. $\tau = \dfrac{2\pi}{\omega_n} = \dfrac{2\pi}{3.2203}$ $\tau = 1.951 \text{ s}$ ◀

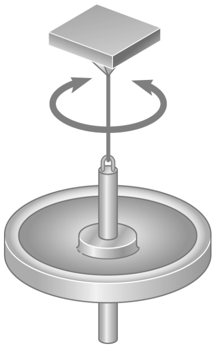
Maximum rotation. $\theta_m = 360° = 2\pi \text{ rad}$

Maximum angular velocity. $\dot{\theta}_m = \omega_n \theta_m = (3.2203)(2\pi) = 20.234 \text{ rad/s}$

(*b*) Maximum velocity at a vertex.

$$v_m = r\dot{\theta}_m = \frac{2}{3}h\dot{\theta}_m = \frac{2}{3}\frac{\sqrt{3}}{2}b = \left(\frac{2}{3}\right)\left(\frac{\sqrt{3}}{2}\right)(0.150)(20.234)$$

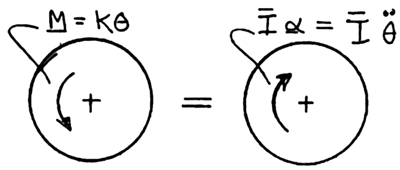
$v_m = 1.752 \text{ m/s}$ ◀



PROBLEM 19.67

A period of 6.00 s is observed for the angular oscillations of a 4-oz gyroscope rotor suspended from a wire as shown. Knowing that a period of 3.80 s is obtained when a 1.25-in.-diameter steel sphere is suspended in the same fashion, determine the centroidal radius of gyration of the rotor. (Specific weight of steel = 490 lb/ft^3.)

SOLUTION



$$\circlearrowleft \Sigma M = \Sigma(M)_{\text{eff}}: \quad -K\theta = \bar{I}\ddot{\theta}$$

$$\ddot{\theta} + \frac{K}{\bar{I}}\theta = 0$$

$$\omega_n^2 = \frac{K}{\bar{I}}$$

$$\tau = 2\pi\sqrt{\frac{\bar{I}}{K}} \qquad (1)$$

$$K = \frac{4\pi^2\bar{I}}{\tau^2} \qquad (2)$$

$$\bar{I} = \frac{K\tau^2}{4\pi^2} \qquad (3)$$

For the sphere, $r = \frac{d}{2} = 0.625 \text{ in.} = 52.083\times10^{-3} \text{ ft}$

Volume: $V_s = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(52.083\times10^{-3})^3$

$= 591.81\times10^{-6} \text{ ft}^3$

Weight: $W_s = \gamma V_s$

$= (490 \text{ lb/ft}^3)(591.81\times10^{-6} \text{ ft}^3)$

$= 0.28999 \text{ lb}$

Mass: $m_s = \frac{W_s}{g} = \frac{0.28999}{32.2}$

$= 9.0059\times10^{-3} \text{ lb}\cdot\text{s}^2/\text{ft}$

Moment of inertia: $\bar{I} = \frac{2}{5}m_s r^2 = \frac{2}{5}(9.0059\times10^{-3})(52.083\times10^{-3})^2$

$= 9.7719\times10^{-6} \text{ lb}\cdot\text{s}^2\cdot\text{ft}$

Period: $\tau_s = 3.80 \text{ s}$

PROBLEM 19.67 (Continued)

From Eq. (2):

$$K = \frac{4\pi^2(9.7719\times10^{-6})}{(3.80)^2}$$

$$= 26.716\times10^{-6}\ \text{lb}\cdot\text{ft/rad}$$

For the rotor,

$$m = \frac{W}{g} = \left(\frac{4}{16}\right)\left(\frac{1}{32.2}\right)$$

$$= 7.764\times10^{-3}\ \text{lb}\cdot\text{s}^2/\text{ft}$$

$$\tau = 6.00\ \text{s}$$

From Eq. (3):

$$\bar{I} = (26.716\times10^{-6})\frac{(6.00)^2}{4\pi^2}$$

$$= 24.362\times10^{-6}\ \text{lb}\cdot\text{s}^2\cdot\text{ft}$$

Radius of gyration.

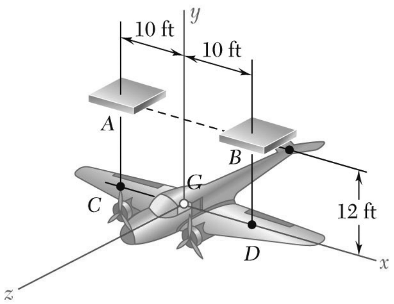
$$\bar{I} = m\bar{k}^2$$

$$\bar{k} = \sqrt{\frac{\bar{I}}{m}}$$

$$= \sqrt{\frac{24.362\times10^{-6}}{7.764\times10^{-3}}}$$

$$= 0.056016\ \text{ft}$$

$\bar{k} = 0.672$ in. ◀



PROBLEM 19.68

The centroidal radius of gyration $\bar{k}_y$ of an airplane is determined by suspending the airplane by two 12-ft-long cables as shown. The airplane is rotated through a small angle about the vertical through G and then released. Knowing that the observed period of oscillation is 3.3 s, determine the centroidal radius of gyration $\bar{k}_y$.

SOLUTION

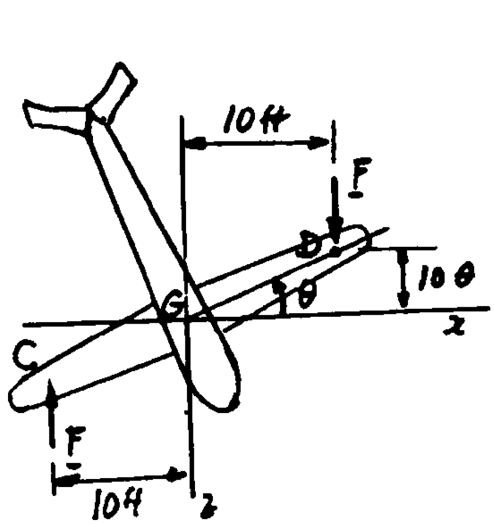
Let the airplane rotate through the small angle θ about a vertical axis. Suspension Points C and D on the airplane each move horizontally a distance (10 ft) sin $\theta \approx$ (10 ft) θ. Let φ be the angle between a cable and the vertical direction. Then, $\sin\varphi = (10\text{ ft})\theta/(12\text{ ft}) = \frac{5}{6}\theta.$

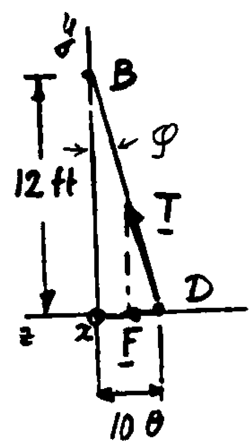
$$+\uparrow \Sigma F = 0: \quad 2T\cos\varphi - W = 0$$

$$T = \frac{W}{2\cos\varphi} \approx \frac{W}{2}$$

Let F be the horizontal component of T.

$$F = T\sin\varphi \approx \frac{W}{2}\cdot\frac{5}{6}\theta = \frac{5}{12}W\theta$$





The two forces F form a couple of moment

$$M = -(20\text{ ft})F = -(20)\left(\frac{5}{12}\right)W\theta$$

Equation of motion: $$+\circlearrowleft \Sigma M_y = \bar{I}_y\alpha: \quad -20\left(\frac{5W}{12}\theta\right) = \frac{W}{g}\bar{k}_y^2\ddot{\theta}$$

$$\ddot{\theta} + \frac{(20)(5)g}{12\bar{k}_y^2}\theta = 0$$

Natural frequency:

$$\omega_n^2 = \frac{(20)(5)g}{12\,\bar{k}_y^2} = \frac{(20)(5)(32.2)}{12\,\bar{k}_y^2} = \frac{268.33}{\bar{k}_y^2}$$

$$\bar{k}_y^2 = \frac{268.33}{\omega_n^2}$$

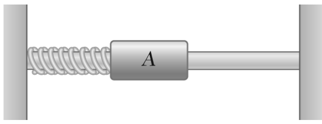
$$\bar{k}_y = \frac{16.381}{\omega_n} = \frac{16.381}{2\pi f}$$

$$= \frac{2.607}{f} = 2.607\tau$$

With $\tau = 3.3$ s,

$$\bar{k}_y = (2.607)(3.3)$$

$\bar{k}_y = 8.60$ ft ◀



PROBLEM 19.69

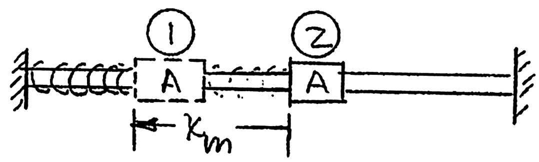
A 1.8-kg collar A is attached to a spring of constant 800 N/m and can slide without friction on a horizontal rod. If the collar is moved 70 mm to the left from its equilibrium position and released, determine the maximum velocity and maximum acceleration of the collar during the resulting motion.

SOLUTION

Datum at ①:

Position ①

$$T_1 = 0 \qquad V_1 = \frac{1}{2}kx_m^2$$



Position ②

$$T_2 = \frac{1}{2}mv_2^2 \qquad V_2 = 0 \qquad v_2 = \dot{x}_m$$

$$\dot{x}_m = \omega_n x_m$$

$$T_1 + V_1 = T_2 + V_2 \qquad 0 + \frac{1}{2}kx_m^2 = \frac{1}{2}m\dot{x}_m^2 + 0$$

$$\frac{1}{2}kx_m^2 = \frac{1}{2}m\omega_n^2 x_m^2 \qquad \omega_n^2 = \frac{k}{m} = \frac{800 \text{ N/m}}{1.8 \text{ kg}}$$

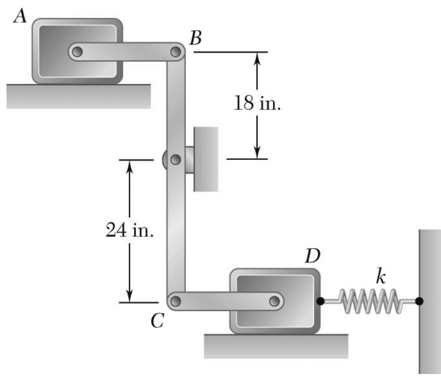
$$\omega_n^2 = 444.4 \text{ s}^{-2} \qquad \omega_n = 21.08 \text{ rad/s}$$

$$\dot{x}_m = \omega_n x_m = (21.08 \text{ s}^{-1})(0.070 \text{ m})$$

$\dot{x}_m = 1.476$ m/s ◀

$$\ddot{x}_m = \omega_n^2 x_m = (21.08 \text{ s}^{-2})(0.070 \text{ m})$$

$\dot{x}_m = 31.1$ m/s^2 ◀



PROBLEM 19.70

Two blocks, each of weight 3 lb, are attached to links which are pin-connected to bar *BC* as shown. The weights of the links and bar are negligible, and the blocks can slide without friction. Block *D* is attached to a spring of constant $k = 4$ lb/in. Knowing that block *A* is moved 0.5 in. from its equilibrium position and released, determine the magnitude of the maximum velocity of block *D* during the resulting motion.

SOLUTION

$$T = \frac{1}{2}m(b^2 + c^2)\dot{\theta}^2$$

$$V = \frac{1}{2}kc^2\theta^2$$

$$\omega_n^2 = \frac{kc^2}{m(b^2 + c^2)}$$

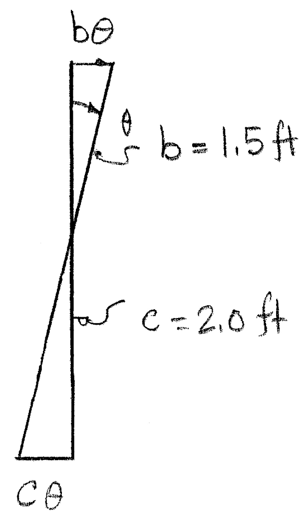
$$k = 48 \text{ lb/ft}$$

$$m = \frac{3 \text{ lb}}{32.2 \text{ ft/s}^2} = 0.093167 \text{ lb} \cdot \text{s}^2/\text{ft}$$

$$\omega_n^2 = \frac{(48)(2)^2}{(0.093167)(1.5^2 + 2^2)} = 329.73$$

$$\omega_n = 18.158 \text{ rad/s}$$

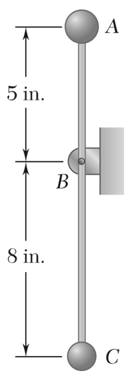
$$\theta_0 = \frac{\frac{0.5 \text{ in.}}{12 \text{ in./ft}}}{1.5 \text{ ft}} = 0.0277 \text{ rad}$$



$$|v_D|_m = c\omega_n\theta_0$$

$$= (2)(18.158)(0.02778) = 1.009 \text{ ft}$$

$|v_D|_m = 12.11$ in./s ◀

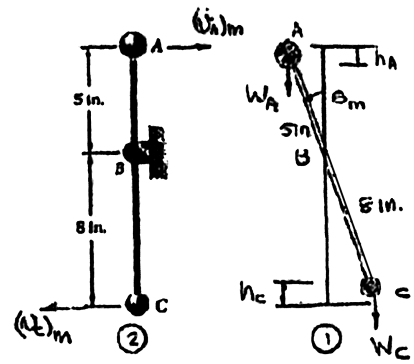


PROBLEM 19.71

A 14-oz sphere *A* and a 10-oz sphere *C* are attached to the ends of a rod *AC* of negligible weight which can rotate in a vertical plane about an axis at *B*. Determine the period of small oscillations of the rod.

SOLUTION

Datum at ①:



Position ①

$$T_1 = 0$$
$$V_1 = W_C h_C - W_A h_A$$
$$h_C = BC(1-\cos\theta_m)$$
$$h_A = BA(1-\cos\theta_m)$$

Small angles.

$$1-\cos\theta_m \approx \frac{\theta_m^2}{2}$$

$$V_1 = [(W_C)(BC)-(W_A)(BA)]\frac{\theta_m^2}{2}$$

$$V_1 = \left[\left(\frac{10}{16}\text{lb}\right)\left(\frac{8}{12}\text{ft}\right)-\left(\frac{14}{16}\text{lb}\right)\left(\frac{5}{12}\text{ft}\right)\right]\frac{\theta_m^2}{2}$$

$$V_1 = (0.4167-0.3646)\frac{\theta_m^2}{2} = 0.05208\frac{\theta_m^2}{2}$$

Position ②

$$V_2 = 0$$

$$(v_C)_m = \frac{8}{12}\dot{\theta}_m \qquad (v_A)_m = \frac{5}{12}\dot{\theta}_m$$

$$m_C = \frac{W_C}{g} = \frac{10}{(16)(32.2)} = 0.019410 \text{ lb}\cdot\text{s}^2/\text{ft}$$

$$m_A = \frac{W_A}{g} = \frac{14}{(16)(32.2)} = 0.027174 \text{ lb}\cdot\text{s}^2/\text{ft}$$

PROBLEM 19.71 (Continued)

$$T_2 = \frac{1}{2}m_C(v_C)_m^2 + \frac{1}{2}m_A(v_A)_m^2$$

$$= \frac{1}{2}\left[(0.019410)\left(\frac{8}{12}\right)^2 + \frac{1}{2}(0.027174)\left(\frac{5}{12}\right)^2\right]\dot{\theta}_m^2$$

$$= 0.013344\frac{\dot{\theta}_m^2}{2}$$

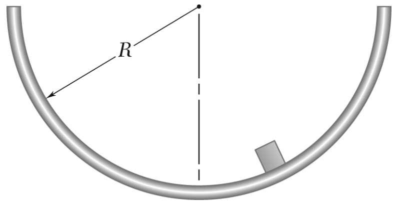
$$= 0.013344\frac{\omega_n^2\theta_m^2}{2}$$

Conservation of energy. $T_1 + V_1 = T_2 + V_2: \quad 0 + 0.05208\frac{\theta_m^2}{2} = 0.013344\frac{\omega_n^2\theta_m^2}{2} + 0$

$$\omega_n^2 = \frac{0.05208}{0.013344} = 3.902$$

$$\omega_n = 1.9755 \text{ rad/s}$$

Period of oscillations. $\tau_n = \dfrac{2\pi}{\omega_n}$ $\tau_n = 3.18 \text{ s}$ ◀

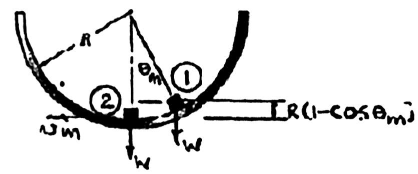


PROBLEM 19.72

Determine the period of small oscillations of a small particle which moves without friction inside a cylindrical surface of radius R.

SOLUTION

Datum at ②:



Position ①

$$T_1 = 0$$

$$V_1 = WR(1-\cos\theta_m)$$

Small oscillations:

$$(1-\cos\theta_m) = 2\sin^2\frac{\theta_m}{2} \approx \frac{\theta_m^2}{2}$$

$$V_1 = \frac{WR\theta_m^2}{2}$$

Position ②

$$v_m = R\dot{\theta}_m \quad T_2 = \frac{1}{2}mv_m^2 = \frac{1}{2}mR^2\dot{\theta}_m^2$$

$$V_2 = 0$$

Conservation of energy. $T_1 + Y_1 = T_2 + V_2$

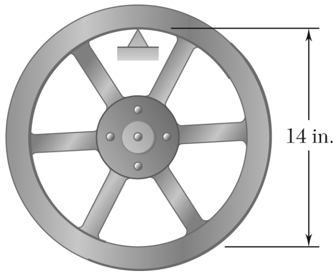
$$0 + WR\frac{\theta_m^2}{2} = \frac{1}{2}mR^2\dot{\theta}_m^2 + 0 \qquad \dot{\theta}_m = \omega_n\theta_m$$

$$W = mg$$

$$mgR\frac{\theta_m^2}{2} = \frac{1}{2}mR^2\omega_n^2\theta_m^2$$

$$\omega_n = \sqrt{\frac{g}{R}}$$

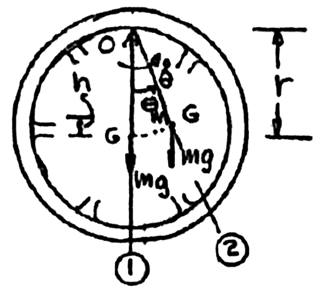
Period of oscillations. $\tau_n = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{R}{g}}$ ◀



PROBLEM 19.73

The inner rim of an 85-lb flywheel is placed on a knife edge, and the period of its small oscillations is found to be 1.26 s. Determine the centroidal moment of inertia of the flywheel.

SOLUTION



Datum at ①:

Position ①

$$T_1 = \frac{1}{2} I_0 \dot{\theta}_m^2 \quad V_1 = 0$$

Position ②

$$T_2 = 0 \quad V_2 = mgh$$

$$h = r(1 - \cos\theta_m) = r\left(2\sin^2\frac{\theta_m}{2}\right)$$

$$\approx r\frac{\theta_m^2}{2}$$

$$V_2 = mgr\frac{\theta_m^2}{2}$$

Conservation of energy.

$$T_1 + V_1 = T_2 + V_2: \quad \frac{1}{2} I_0 \dot{\theta}_m^2 + 0 = 0 + mgr\frac{\theta_m^2}{2}$$

For simple harmonic motion, $\dot{\theta}_m = \omega_n \theta_m$

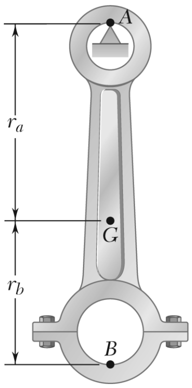
$$I_0 \omega_n^2 \theta_m^2 = mgr\,\theta_m^2 \qquad \omega_n^2 = \frac{mgr}{I_0} \qquad \tau_n^2 = \frac{4\pi^2}{\omega_n^2} = \frac{(4\pi^2)I_0}{mgr}$$

Moment of inertia. $I_0 = \bar{I} + mr^2 \quad \bar{I} + mr^2 = \dfrac{(\tau_n^2)(mgr)}{4\pi^2}$

$$\bar{I} = \frac{(\tau_n^2)(mgr)}{4\pi^2} - mr^2 = \frac{(1.26\text{ s})^2(85\text{ lb})}{4\pi^2}\left(\frac{7}{12}\text{ ft}\right) - \frac{(85\text{ lb})}{(32.2\text{ ft/s}^2)}\left(\frac{7}{12}\text{ ft}\right)^2$$

$$\bar{I} = 1.994 - 0.8983$$

$\bar{I} = 1.096\text{ lb}\cdot\text{ft}\cdot\text{s}^2$ ◀



PROBLEM 19.74

A connecting rod is supported by a knife edge at Point A; the period of its small oscillations is observed to be 1.03 s. Knowing that the distance r_a is 6 in. determine the centroidal radius of gyration of the connecting rod.

SOLUTION

Position ① Displacement is maximum.

$$T_1 = 0, \quad V_1 = mgr_a(1-\cos\theta_m) \approx \frac{1}{2}mgr_a\theta_m^2$$

Position ② Velocity is maximum.

$$(v_G)_m = r_a\dot{\theta}_m$$

$$T_2 = \frac{1}{2}mv_G^2 + \frac{1}{2}\bar{I}\dot{\theta}_m^2 = \frac{1}{2}mr_a^2\dot{\theta}_m^2 + \frac{1}{2}m\bar{k}^2\dot{\theta}_m^2$$

$$= \frac{1}{2}m\left(r_a^2 + \bar{k}^2\right)\dot{\theta}_m^2$$

$$V_2 = 0$$

For simple harmonic motion, $\dot{\theta}_m = \omega_n\theta_m$

Conservation of energy. $T_1 + V_1 = T_2 + V_2$

$$0 + \frac{1}{2}mgr_a\theta_m^2 = \frac{1}{2}m\left(r_a^2 + \bar{k}^2\right)\omega_n^2\theta_m^2 + 0$$

$$\omega_n^2 = \frac{gr_a}{r_a^2 + \bar{k}^2} \quad \text{or} \quad \bar{k}^2 = \frac{gr_a}{\omega_n^2} - r_a^2$$

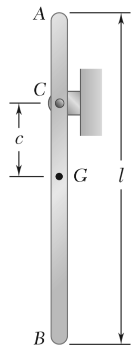
Data: $\tau_n = 1.03 \text{ s} \qquad \omega_n = \frac{2\pi}{\tau_n} = \frac{2\pi}{1.03} = 6.1002 \text{ rad/s}$

$$r_a = 6 \text{ in.} = 0.5 \text{ ft} \qquad g = 32.2 \text{ ft/s}^2$$

$$\bar{k}^2 = \frac{(32.2)(0.5)}{(6.1002)^2} - (0.5)^2 = 0.43265 - 0.25 = 0.18265 \text{ ft}^2$$

$$\bar{k} = 0.42738 \text{ ft}$$

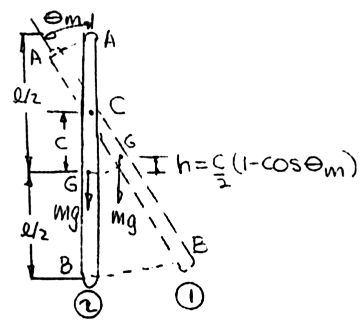
$\bar{k} = 5.13 \text{ in.}$ ◀



PROBLEM 19.75

A uniform rod AB can rotate in a vertical plane about a horizontal axis at C located at a distance c above the mass center G of the rod. For small oscillations, determine the value of c for which the frequency of the motion will be maximum.

SOLUTION



Find ω_n as a function of c.

Datum at ②:

Position ①

$$T_1 = 0 \qquad V_1 = mgh$$

$$V_1 = mgc(1-\cos\theta_m)$$

$$1-\cos\theta_m = 2\sin^2\frac{\theta_m}{2} \approx \frac{\theta_m^2}{2}$$

$$V_1 = mgc\frac{\theta_m^2}{2}$$

Position ②

$$T_2 = \frac{1}{2}I_C\dot{\theta}_m^2$$

$$I_C = \overline{I} + mc^2 = \frac{1}{12}ml^2 + mc^2$$

$$T_2 = \frac{1}{2}m\left(\frac{l^2}{12}+c^2\right)\dot{\theta}_m^2 \qquad V_2 = 0$$

$$T_1 + V_1 = T_2 + V_2 \qquad 0 + mgc\frac{\theta_m^2}{2} = m\left(\frac{l^2}{12}+c^2\right)\frac{\dot{\theta}_m^2}{2} + 0$$

$$\dot{\theta}_m = \omega_n\theta_m$$

$$gc = m\left(\frac{l^2}{12}+c^2\right)\omega_n^2$$

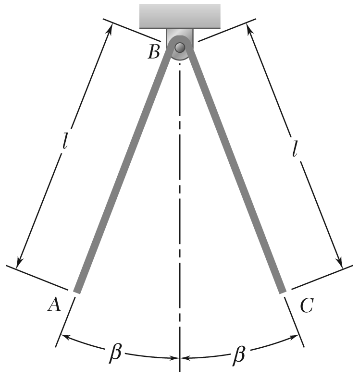
$$\omega_n^2 = \frac{gc}{\left(\frac{l^2}{12}+c^2\right)}$$

Maximum c, when

$$\frac{d\omega_n^2}{dc} = 0 = \frac{g\left(\frac{l^2}{12}+c^2\right) - 2c^2g}{\left(\frac{l^2}{12}+c^2\right)} = 0$$

$$\frac{l^2}{12} - c^2 = 0$$

$$c = \frac{l}{\sqrt{12}} \blacktriangleleft$$

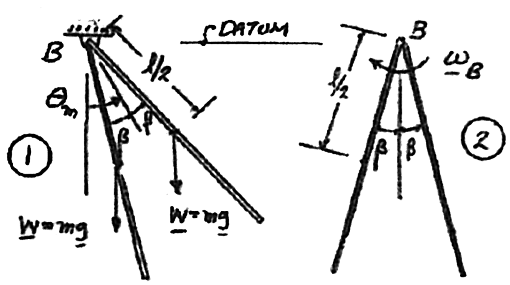


PROBLEM 19.76

A homogeneous wire of length $2l$ is bent as shown and allowed to oscillate about a frictionless pin at B. Denoting by τ_0 the period of small oscillations when $\beta = 0$, determine the angle β for which the period of small oscillations is $2\tau_0$.

SOLUTION

We denote by m the mass of half the wire.



Position ① Maximum deflections:

$$T_1 = 0, \quad V_1 = -mg\frac{l}{2}\cos(\theta_m - \beta) - mg\frac{l}{2}\cos(\theta_m + \beta)$$

$$= -mg\frac{l}{2}(\cos\theta_m\cos\beta + \sin\theta_m\sin\beta + \cos\theta_m\cos\beta - \sin\theta_m\sin\beta)$$

$$V_1 = -mgl\cos\beta\cos\theta_m$$

For small oscillations, $\quad \cos\theta_m \approx 1 - \frac{1}{2}\theta_m^2$

$$V_1 = -mgl\cos\beta + \frac{1}{2}mgl\cos\beta\,\theta_m^2$$

Position ② Maximum velocity:

$$T_2 = \frac{1}{2}I_B\dot{\theta}_m^2 \quad \text{but} \quad I_B = 2\left(\frac{1}{3}ml^2\right)$$

Thus, $\quad T_2 = \frac{1}{2}\left(\frac{2}{3}ml^2\right)\dot{\theta}_m^2$

$$V_2 = -2mg\left(\frac{l}{2}\cos\beta\right) = -mgl\cos\beta$$

PROBLEM 19.76 (Continued)

Conservation of energy. $T_1 + V_1 = T_2 + V_2$

$$-mgl\cos\beta + \frac{1}{2}mgl\cos\beta\,\theta_m^2 = \frac{1}{3}ml^2\dot{\theta}_m^2 - mgl\cos\beta$$

Setting $\dot{\theta}_m = \theta_m\omega_n$, $\frac{1}{2}mgl\cos\beta\,\theta_m^2 = \frac{1}{3}ml^2\,\theta_m^2\omega_n^2$

$$\omega_n^2 = \frac{3}{2}\frac{g}{l}\cos\beta \qquad \tau = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{2l}{3g\cos\beta}} \qquad (1)$$

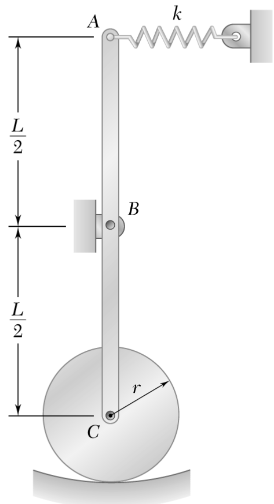
But for $\beta = 0$, $\tau_0 = 2\pi\sqrt{\frac{2l}{3g}}$

For $\tau = 2\tau_0$, $2\pi\sqrt{\frac{2l}{3g\cos\beta}} = 2\left(2\pi\sqrt{\frac{2l}{3g}}\right)$

Squaring and reducing,

$$\frac{1}{\cos\beta} = 4 \qquad \cos\beta = \frac{1}{4}$$

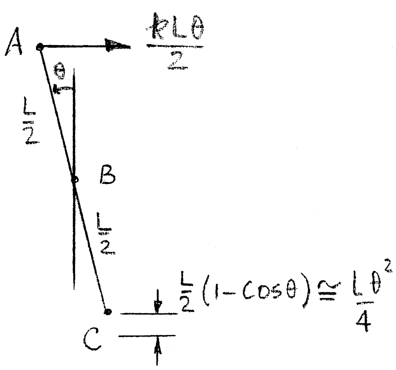
$\beta = 75.5°$ ◀



PROBLEM 19.77

A uniform disk of radius r and mass m can roll without slipping on a cylindrical surface and is attached to bar ABC of length L and negligible mass. The bar is attached to a spring of constant k and can rotate freely in the vertical plane about Point B. Knowing that end A is given a small displacement and released, determine the frequency of the resulting oscillations in terms of m, L, k, and g.

SOLUTION



$$V = \frac{1}{2}k\left(\frac{L\theta}{2}\right)^2 + mg\frac{L\theta^2}{4}$$

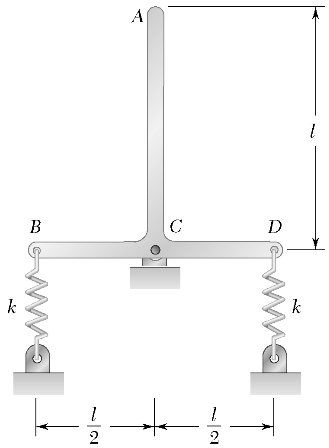
$$T = \frac{1}{2}m\left(\frac{L^2\dot{\theta}^2}{4}\right) + \frac{1}{2}\frac{mr^2}{2}\left(\frac{L^2\dot{\theta}^2}{4r^2}\right)$$

$$= \frac{3mL^2\dot{\theta}^2}{16}$$

$$\omega_n^2 = \frac{\frac{kL^2}{8} + \frac{mgL}{4}}{\frac{3mL^2}{16}}$$

$$= \frac{2}{3}\left(\frac{k}{m} + \frac{2g}{L}\right)$$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{2k}{3m} + \frac{4g}{3L}} \blacktriangleleft$$



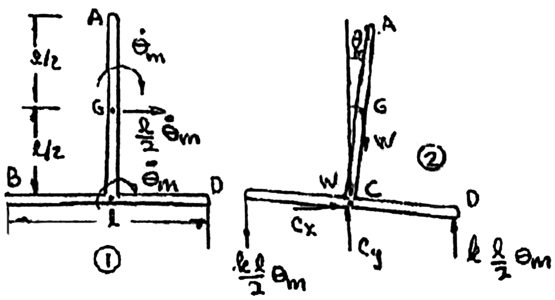
PROBLEM 19.78

Two uniform rods, each of weight $W = 1.2$ lb and length $l = 8$ in., are welded together to form the assembly shown. Knowing that the constant of each spring is $k = 0.6$ lb/in. and that end A is given a small displacement and released, determine the frequency of the resulting motion.

SOLUTION

Mass and moment of inertia of one rod. $\quad m = \frac{W}{g} = \frac{1.2}{32.2} = 0.037267 \text{ lb}\cdot\text{s}^2/\text{ft}$

$$\bar{I} = \frac{1}{12}ml^2 = \frac{1}{12}(0.037267)\left(\frac{8}{12}\right)^2 = 1.38026\times10^{-3} \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$



Approximation.

$$\sin\theta_m \approx \tan\theta_m \approx \theta_m$$

$$1-\cos\theta_m = 2\sin^2\frac{\theta_m}{2} \approx \frac{1}{2}\theta_m^2$$

Spring constant: $\quad k = 0.6 \text{ lb/in.} = 7.2 \text{ lb/ft}$

Position ①

$$T_1 = 2\left(\frac{1}{2}\bar{I}\dot{\theta}_m^2\right) + \frac{1}{2}m\left(\frac{l}{2}\dot{\theta}_m\right)^2$$

$$= (2)\left(\frac{1}{2}\right)(1.38026\times10^{-3})\dot{\theta}_m^2 + \left(\frac{1}{2}\right)(0.037267)\left(\frac{4}{12}\dot{\theta}_m\right)^2$$

$$= 3.4506\times10^{-3}\dot{\theta}_m^2$$

$$V_1 = 0$$

PROBLEM 19.78 (Continued)

<u>*Position* ②</u> $T_2 = 0$

$$V_2 = -\frac{Wl}{2}(1-\cos\theta_m) + 2\left(\frac{1}{2}\right)k\left(\frac{l}{2}\theta_m\right)^2$$

$$\approx -\frac{(1.2)(0.66667)}{2}\left(\frac{1}{2}\theta_m^2\right) + (2)\left(\frac{1}{2}\right)(7.2)\left(\frac{0.66667}{2}\theta_m\right)^2$$

$$\approx 0.6\theta_m^2$$

<u>Conservation of energy</u>. $T_1 + V_1 = T_2 + V_2$

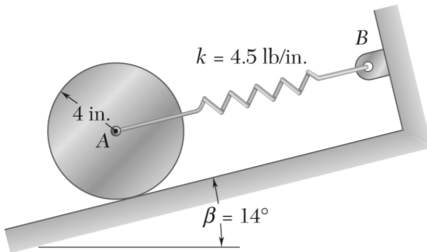
$$3.4506\times10^{-3}\dot{\theta}_m^2 + 0 = 0 + 0.6\theta_m^2$$

$$\dot{\theta}_m = 13.186\theta_m$$

<u>Simple harmonic motion</u>. $\dot{\theta}_m = \omega_n\theta_m$

$$\omega_n = 13.186 \text{ rad/s}$$

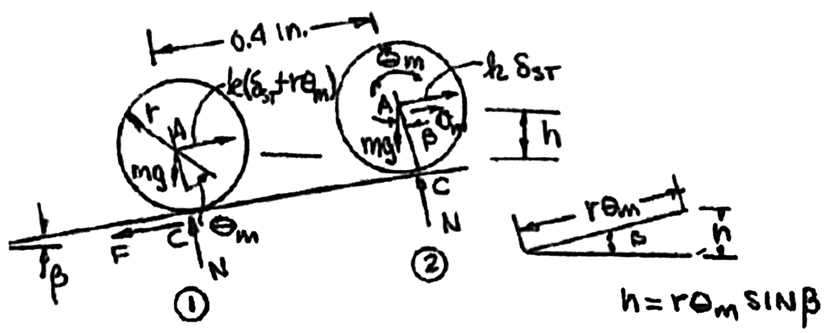
<u>Frequency</u>. $f_n = \dfrac{\omega_n}{2\pi}$ $f_n = 2.10$ Hz ◀



PROBLEM 19.79

A 15-lb uniform cylinder can roll without sliding on an incline and is attached to a spring *AB* as shown. If the center of the cylinder is moved 0.4 in. down the incline and released, determine (*a*) the period of vibration, (*b*) the maximum velocity of the center of the cylinder.

SOLUTION



(*a*) *Position ①*

$$T_1 = 0 \quad V_1 = \frac{1}{2}k(\delta_{st} + r\theta_m)^2$$

Position ②

$$T_2 = \frac{1}{2}\bar{I}\dot{\theta}_m^2 + \frac{1}{2}m\bar{v}_m^2$$

$$V_2 = mgh + \frac{1}{2}k(\delta_{st})^2$$

Conservation of energy.

$$T_1 + V_1 = T_2 + V_2: \quad 0 + \frac{1}{2}k(\delta_{st} + r\theta_m)^2 = \frac{1}{2}\bar{I}\dot{\theta}_m^2 + \frac{1}{2}m\bar{v}_m^2 + mgh + \frac{1}{2}k(\delta_{st})^2$$

$$k\delta_{st}^2 + 2k\delta_{st}r\theta_m + kr^2\theta_m^2 = \bar{I}\dot{\theta}_m^2 + mv_m^2 + 2mgh + k\delta_{st}^2 \qquad (1)$$

When the disk is in equilibrium,

$$\circlearrowleft + \Sigma M_c = 0 = mg\sin\beta r - k\delta_{st}r$$

Also,

$$h = r\sin\beta\,\theta_m$$

Thus,

$$mgh - k\delta_{st}r = 0 \qquad (2)$$

PROBLEM 19.79 (Continued)

Substituting Eq. (2) into Eq. (1)

$$kr^2\theta_m^2 = \bar{I}\dot{\theta}_m^2 + m\bar{v}_m^2$$

$$\dot{\theta}_m = \omega_n\theta_m \qquad v_m = r\dot{\theta}_m = r\omega_n\theta_m$$

$$kr^2\theta_m^2 = (\bar{I} + mr^2)\theta_m^2\omega_n^2$$

$$\omega_n^2 = \frac{kr^2}{\bar{J} + mr^2} \qquad \bar{I} = \frac{1}{2}mr^2$$

$$\omega_n^2 = \frac{kr^2}{\frac{1}{2}mr^2 + mr^2} = \frac{2}{3}\frac{k}{m}$$

$$\tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{\sqrt{\frac{2}{3}\frac{(4.5\times 12 \text{ lb/ft})}{\left(\frac{15 \text{ lb}}{32.2}\text{ ft/s}^2\right)}}}$$

$\tau_n = 0.715 \text{ s}$ ◀

(*b*) Maximum velocity.

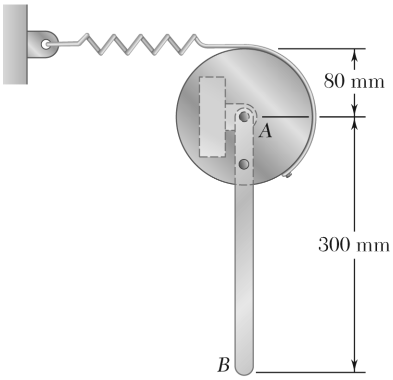
$$v_m = r\dot{\theta}_m$$

$$\dot{\theta}_m = \theta_m\omega_n$$

$$v_m = r\theta_m\omega_n \qquad r\theta_m = \frac{0.4}{12}\text{ ft}$$

$$v_m = \left(\frac{0.4}{12}\text{ ft}\right)\left(\frac{2\pi}{0.715 \text{ s}}\right)$$

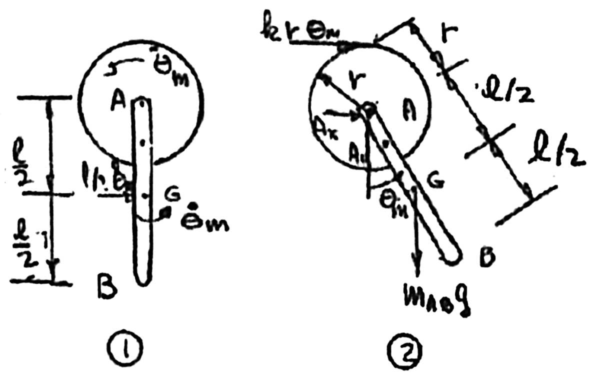
$v_m = 0.293 \text{ ft/s}$ ◀



PROBLEM 19.80

A 3-kg slender rod *AB* is bolted to a 5-kg uniform disk. A spring of constant 280 N/m is attached to the disk and is unstretched in the position shown. If end *B* of the rod is given a small displacement and released, determine the period of vibration of the system.

SOLUTION



$$r = 0.08 \text{ m}$$
$$l = 0.3 \text{ m}$$

Position ①

$$T_1 = \frac{1}{2}\overline{I}_{\text{disk}}\dot{\theta}_m^2 + \frac{1}{2}(I_A)_{\text{rod}}\dot{\theta}_m^2$$
$$V_1 = 0$$
$$I_{\text{disk}} = \frac{1}{2}m_0 r^2$$
$$I_{A\ \text{rod}} = \frac{1}{3}m_{AB}l^2$$

Position ②

$$T_2 = 0$$
$$V_2 = \frac{1}{2}k(r\theta_m)^2 + m_{AB}g\frac{l}{2}(1-\cos\theta_m)$$
$$1-\cos\theta_m = 2\sin^2\frac{\theta_m}{2} \approx \frac{\theta_m^2}{2}$$
$$V_2 = \frac{1}{2}kr^2\theta_m^2 + \frac{m_{AB}g\left(\frac{l}{2}\right)}{2}\theta_m^2$$

Conservation of energy.

$$T_1 + V_1 = T_2 + V_2: \quad \frac{1}{2}\left(\frac{1}{2}m_0 r^2 + \frac{1}{3}m_{AB} l^2\right)\dot{\theta}_m^2 + 0 = 0 + \frac{1}{2}kr^2\theta_m^2 + \frac{1}{2}m_{AB} g \frac{l}{2}\theta_m^2$$

For simple harmonic motion,

$$\dot{\theta}_m = \omega_n \theta_m$$

$$\left(\frac{1}{2}m_0 t^2 + \frac{1}{3}m_{AB} l^2\right)\omega_n^2 \theta_m^2 = \left(kr^2 + m_{AB} g \frac{l}{2}\right)\theta_m^2$$

$$\omega_n^2 = \frac{kr^2 + m_{AB} g l}{\frac{1}{2}m_0 r^2 + \frac{1}{3}m_{AB} l^2}$$

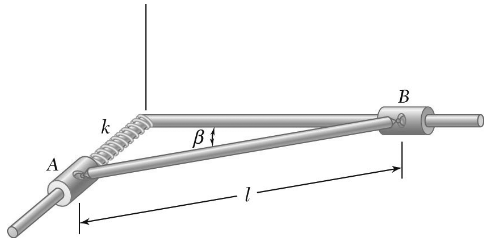
$$\omega_n^2 = \frac{(280\text{ N/m})(0.08\text{ m})^2 + (3\text{ kg})(9.81\text{ m/s}^2)\left(\frac{0.3}{2}\text{ m}\right)}{\frac{1}{2}(5\text{ kg})(0.08\text{ m})^2 + \frac{1}{3}(3\text{ kg})(0.300\text{ m})^2}$$

$$\omega_n^2 = \frac{6.207}{0.106} = 58.55$$

Period of vibration.

$$\tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{\sqrt{58.55}}$$

$\tau_n = 0.821\text{ s}$ ◀



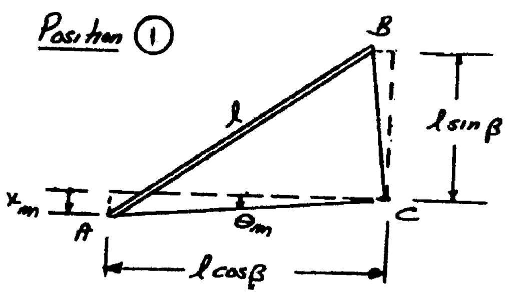
PROBLEM 19.81

A slender rod AB of mass m and length l is connected to two collars of negligible mass in a horizontal plane as shown. Collar A is attached to a spring of constant k. Knowing that the collars can slide freely on their respective rods and the system is in equilibrium in the position shown, determine the period of vibration if collar A is given a small displacement and released.

SOLUTION

Moment of inertia: $$\bar{I} = \frac{1}{12}ml^2$$

Position ① Maximum deflection: Let collar A be moved a small distance x_m as shown. Since the movement is horizontal, there is no change in gravitational potential energy.



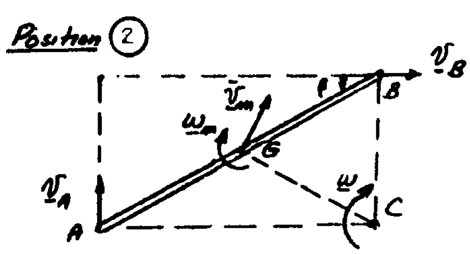
$$x_m = (l\cos\beta)\theta_m$$

$$V_1 = \frac{1}{2}kx_m^2 = \frac{1}{2}k(l\cos\beta\theta_m)^2$$

$$V_1 = \frac{1}{2}kl^2\cos^2\beta\theta_m^2$$

$$T_1 = 0$$

Position ② Maximum velocity: The instantaneous center of rotation lies at Point C, the intersection of lines perpendicular, respectively, to $\mathbf{v}_A$ and $\mathbf{v}_B$.



$$\bar{v}_m = (GC)\omega_m = \frac{1}{2}l\omega_m$$

$$T_2 = \frac{1}{2}m\bar{v}_m^2 + \frac{1}{2}\bar{I}\omega_m^2$$

$$= \frac{1}{2}m\left(\frac{1}{2}l\omega_m\right)^2 + \frac{1}{2}\left(\frac{1}{12}ml^2\right)\omega_m^2$$

$$T_2 = \frac{1}{6}ml^2\omega_m^2$$

But, $$\omega_m = -\dot{\theta}_m$$

so that $$T_2 = \frac{1}{6}ml^2\dot{\theta}_m^2$$

$$V_2 = 0$$

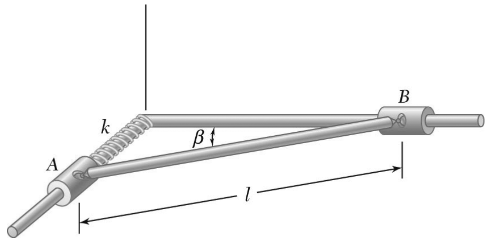
PROBLEM 19.81 (Continued)

For simple harmonic motion, $\dot{\theta}_m = \omega_n \theta_m$

Conservation of energy: $T_1 + V_1 = T_2 + V_2$: $0 + \frac{1}{2} k l^2 \cos^2 \beta \theta_m^2 = \frac{1}{6} m l^2 \omega_n^2 \theta_m^2 + 0$

Natural frequency: $\omega_n^2 = \frac{3k}{m} \cos^2 \beta$

Period of vibration: $\tau = \frac{2\pi}{\omega_n}$ $\tau = 2\pi \sqrt{m/3k \cos^2 \beta}$ ◀



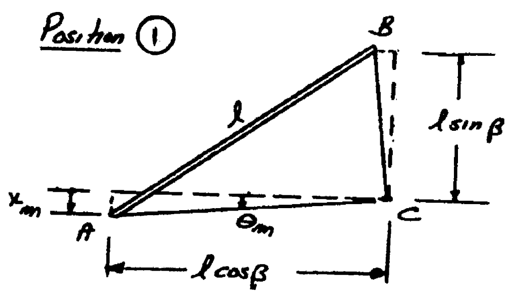
PROBLEM 19.82

A slender rod AB of mass m and length l is connected to two collars of mass m_C in a horizontal plane as shown. Collar A is attached to a spring of constant k. Knowing that the collars can slide freely on their respective rods and the system is in equilibrium in the position shown, determine the period of vibration if collar A is given a small displacement and released.

SOLUTION

Moment of inertia of rod: $\bar{I} = \frac{1}{12}ml^2$

Position ① Maximum deflection: Let collar A be moved a small distance x_m as shown. Since the movement is horizontal, there is no change in gravitational potential energy.



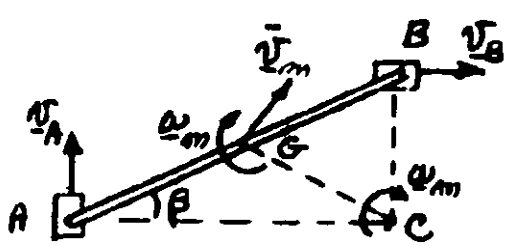
$$x_m = (l\cos\beta)\theta_m$$

$$V_1 = \frac{1}{2}kx_m^2 = \frac{1}{2}k(l\cos\beta\theta_m)^2$$

$$V_1 = \frac{1}{2}kl^2\cos^2\beta\theta_m^2$$

$$T_1 = 0$$

Position ② Maximum velocity: The instantaneous center of rotation lies at Point C, the intersection of lines perpendicular, respectively, to $\mathbf{v}_A$ and $\mathbf{v}_B$.



$$v_A = (AC)\omega_m = l\cos\beta\omega_m$$

$$v_B = (BC)\omega_m = l\sin\beta\omega_m$$

$$\bar{v}_m = (CG)\omega_m = \frac{1}{2}l\omega_m$$

$$T_2 = \frac{1}{2}m\bar{v}^2 + \frac{1}{2}\bar{I}\omega_m^2 + \frac{1}{2}m_C v_A^2 + \frac{1}{2}m_C v_B^2$$

$$= \frac{1}{2}m\left(\frac{1}{2}lv_m\right)^2 + \frac{1}{2}\left(\frac{1}{12}ml^2\right)\omega_m^2$$

$$+\frac{1}{2}m_C(l\sin\beta\omega_m)^2 + \frac{1}{2}m_C(l\cos\beta\omega_m)^2$$

$$= \frac{1}{2}\frac{1}{3}ml^2\omega_m^2 + \frac{1}{2}m_C l^2(\sin^2\beta + \cos^2\beta)\omega_m^2$$

$$T_2 = \frac{1}{2}\left(\frac{1}{3}m + m_C\right)l^2\omega_m^2$$

PROBLEM 19.82 (Continued)

But, $\omega_m = -\dot{\theta}_m$

so that

$$T_2 = \frac{1}{2}\left(\frac{1}{3}m + m_C\right)l^2\dot{\theta}_m^2$$

$$V_2 = 0$$

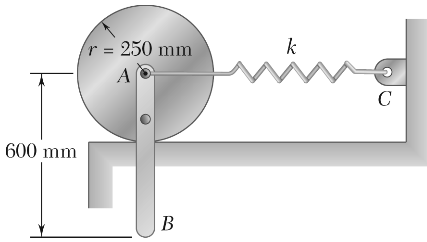
Conservation of energy: $T_1 + V_1 = T_2 + V_2$: $\quad 0 + \frac{1}{2}kl^2\cos^2\beta\theta_m^2 = \frac{1}{2}\left(\frac{1}{3}m + m_C\right)l^2\omega_n^2\theta_m^2$

Natural frequency:

$$\omega_n^2 = \frac{k\cos^2\beta}{\frac{m}{3} + m_C}$$

Period of vibration: $\tau = \dfrac{2\pi}{\omega_n}$

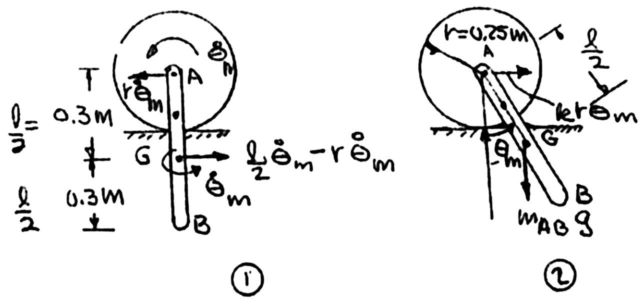
$$\tau = 2\pi\sqrt{\left(\frac{m}{3} + m_C\right)/k\cos^2\beta} \quad \blacktriangleleft$$



PROBLEM 19.83

An 800-g rod AB is bolted to a 1.2-kg disk. A spring of constant $k = 12$ N/m is attached to the center of the disk at A and to the wall at C. Knowing that the disk rolls without sliding, determine the period of small oscillations of the system.

SOLUTION



Position ①

$$T_1 = \frac{1}{2}(\bar{I}_G)_{AB}\dot{\theta}_m^2 + \frac{1}{2}m_{AB}\left(\frac{l}{2} - r\right)^2\dot{\theta}_m + \frac{1}{2}(\bar{I}_G)_{\text{disk}}\dot{\theta}_m^2 + \frac{1}{2}m_{\text{disk}}r^2\dot{\theta}_m^2$$

$$(\bar{I}_G)_{AB} = \frac{1}{12}ml^2 = \frac{1}{12}(0.8)(0.6)^2 = 0.024 \text{ kg}\cdot\text{m}^2$$

$$m_{AB}\left(\frac{l}{2} - r\right)^2 = (0.8)(0.3 - 0.25)^2 = 0.002 \text{ kg}\cdot\text{m}^2$$

$$(I_G)_{\text{disk}} = \frac{1}{2}m_{\text{disk}}r^2 = \frac{1}{2}(1.2)(0.25)^2 = 0.0375 \text{ kg}\cdot\text{m}^2$$

$$m_{\text{disk}}r^2 = 1.2(0.25)^2 = 0.0750 \text{ kg}\cdot\text{m}^2$$

$$T_1 = \frac{1}{2}[0.024 + 0.002 + 0.0375 + 0.0750]\dot{\theta}_m^2$$

$$T_1 = \frac{1}{2}[0.1385]\dot{\theta}_m^2$$

$$V_1 = 0$$

Position ②

$$T_2 = 0$$

$$V_2 = \frac{1}{2}k(r\theta_m)^2 + m_{AB}g\frac{l}{2}(1 - \cos\theta_m)$$

$$1 - \cos\theta_m = 2\sin^2\frac{\theta_m}{2} \approx \frac{\theta_m^2}{2} \text{ (small angles)}$$

$$V_2 = \frac{1}{2}(12 \text{ N/m})(0.25 \text{ m})^2\theta_m^2 + (0.8 \text{ kg})(9.81 \text{ m/s}^2)\left(\frac{0.6 \text{ m}}{2}\right)\frac{\theta_m^2}{2}$$

PROBLEM 19.83 (Continued)

$$V_2 = \frac{1}{2}[0.750 + 2.354]\theta_m^2$$

$$= \frac{1}{2}(3.104)\ \theta_m^2\ \text{N}\cdot\text{m}$$

$$T_1 + V_1 = T_2 + V_2$$

$$\dot{\theta}_m^2 = \omega_n^2 \theta_m^2$$

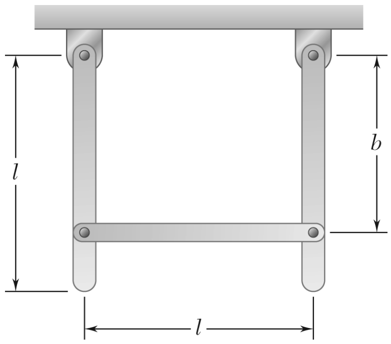
$$\frac{1}{2}(0.1385)\theta_m^2 \omega_n^2 + 0 = 0 + \frac{1}{2}(3.104)\theta_m^2$$

$$\omega_n^2 = \frac{(3.104\ \text{N}\cdot\text{m})}{(0.1385\ \text{kg}\cdot\text{m}^2)}$$

$$= 22.41\ \text{s}^{-2}$$

$$\tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{\sqrt{22.41}}$$

$\tau_n = 1.327\ \text{s}$ ◀



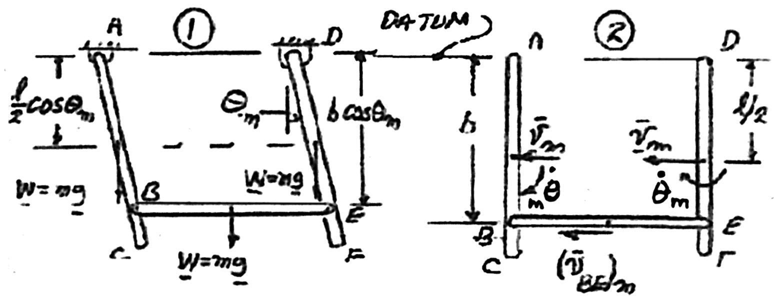
PROBLEM 19.84

Three identical rods are connected as shown. If $b = \frac{3}{4}l$, determine the frequency of small oscillations of the system.

SOLUTION

$l =$ length of each rod

$m =$ mass of each rod



Kinematics:

$$\bar{v}_m = \frac{l}{2}\dot{\theta}_m$$

$$(\bar{v}_{BE})_m = b\dot{\theta}_m$$

Position ①

$$T_1 = 0$$

$$V_1 = -2mg\frac{l}{2}\cos\theta_m - mgb\cos\theta_m$$

$$V_1 = -mg(l+b)\cos\theta_m$$

Position ②

$$V_2 = -2mg\frac{l}{2} - mgb$$

$$= -mg(l+b)$$

$$T_2 = 2\left[\frac{1}{2}\bar{I}\dot{\theta}_m^2 + \frac{1}{2}m\bar{v}_m^2\right] + \frac{1}{2}m(\bar{v}_{BE})_m^2$$

$$= \frac{1}{12}ml^2\dot{\theta}_m^2 + m\left(\frac{l}{2}\dot{\theta}_m\right)^2 + \frac{1}{2}m(b\dot{\theta}_m)^2$$

$$T_2 = \left(\frac{1}{3}l^2 + \frac{1}{2}b^2\right)m\dot{\theta}_m^2$$

Conservation of energy. $T_1 + V_1 = T_2 + V_2$: $0 - mg(l+b)\cos\theta_m = \left(\frac{1}{3}l^2 + \frac{1}{2}b^2\right)m\dot{\theta}_m^2 - mg(l+b)$

$$mg(l+b)(1-\cos\theta_m) = \left(\frac{1}{3}l^2 + \frac{1}{2}b^2\right)m\dot{\theta}_m^2$$

For small oscillations, $(1-\cos\theta_m) = \frac{1}{2}\theta_m^2$

$$\frac{1}{2}mg(l+b)\theta_m^2 = \left(\frac{1}{3}l^2 + \frac{1}{2}b^2\right)m\dot{\theta}_m^2$$

But for simple harmonic motion, $\dot{\theta}_m = \omega_n\theta_m$: $\frac{1}{2}mg(l+b)\theta_m^2 = \left(\frac{1}{3}l^2 + \frac{1}{2}b^2\right)m(\omega_n\theta_m)^2$

$$\omega_n^2 = \frac{1}{2}g\frac{l+b}{\frac{1}{3}l^2 + \frac{1}{2}b^2}$$

or

$$\omega_n^2 = 3g\frac{l+b}{2l^2 + 3b^2} \quad (1)$$

For $b = \frac{3}{4}l$, we have

$$\omega_n^2 = 3g\frac{l + \frac{3}{4}l}{2l^2 + 3\left(\frac{3}{4}l\right)^2}$$

$$= 3g\frac{\frac{7}{4}l}{\frac{59}{16}l^2}$$

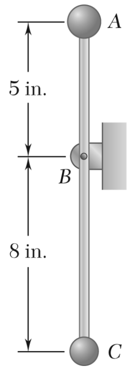
$$= 1.4237\frac{g}{l}$$

$$\omega_n = 1.1932\sqrt{\frac{g}{l}}$$

$$f_n = \frac{\omega_n}{2\pi}$$

$$= \frac{1.1932}{2\pi}\sqrt{\frac{g}{l}}$$

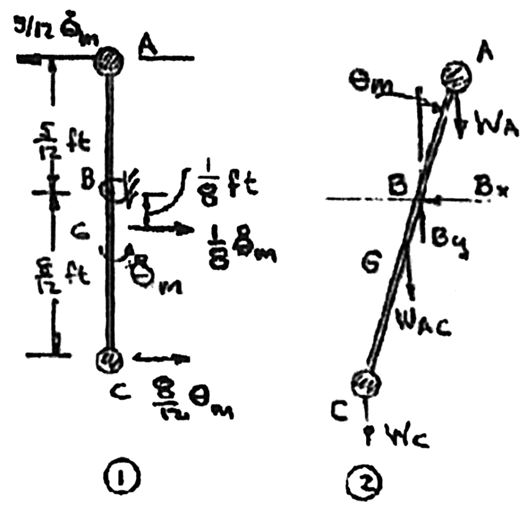
$f_n = 0.1899\sqrt{\frac{g}{l}}$ ◀



PROBLEM 19.85

A 14-oz sphere A and a 10-oz sphere C are attached to the ends of a 20-oz rod AC which can rotate in a vertical plane about an axis at B. Determine the period of small oscillations of the rod.

SOLUTION



Position ①

$$T_1 = \frac{1}{2}\frac{W_A}{g}\left(\frac{5}{12}\dot{\theta}_m\right)^2 + \frac{1}{2}\frac{W_C}{g}\left(\frac{8}{12}\dot{\theta}_m\right)^2 + \frac{1}{2}\frac{W_{AC}}{g}\left(\frac{1}{8}\dot{\theta}_m\right)^2 + \frac{1}{2}\bar{I}_{AC}\dot{\theta}_m^2$$

$$\bar{I}_{AC} = \frac{1}{12}\frac{W_{AC}}{g}\left(\frac{13}{12}\right)^2$$

$$T_1 = \frac{1}{2g}\left[\frac{14}{16}\left(\frac{5}{12}\right)^2 + \frac{10}{16}\left(\frac{8}{12}\right)^2 + \frac{20}{16}\left(\frac{1}{8}\right)^2 + \frac{1}{12}\left(\frac{20}{16}\right)\left(\frac{13}{12}\right)^2\right]\dot{\theta}_m^2$$

$$T_1 = \frac{1}{2(32.2\ \text{ft/s}^2)}[0.1519 + 0.2778 + 0.01953 + 0.1223]\dot{\theta}_m^2$$

$$T_1 = \frac{1}{2}\left(\frac{0.5715\ \text{lb}\cdot\text{ft}^2}{32.2\ \text{ft/s}^2}\right)\dot{\theta}_m^2 = \frac{1}{2}(0.01775)\dot{\theta}_m^2\,(\text{lb}\cdot\text{ft})$$

$$V_1 = 0$$

PROBLEM 19.85 (Continued)

<u>*Position* ②</u>

$$T_2 = 0$$

$$V_2 = -W_A \frac{5}{12}(1-\cos\theta_m) + W_C \frac{8}{12}(1-\cos\theta_m) + W_{AC}\frac{1}{8}(1-\cos\theta_m)$$

$$1-\cos\theta_m = 2\sin^2\frac{\theta_m}{2} \approx \frac{\theta_m^2}{2}$$

$$V_2 = \left[-\left(\frac{14}{16}\right)\left(\frac{5}{12}\right)+\left(\frac{10}{16}\right)\left(\frac{8}{12}\right)+\left(\frac{20}{16}\right)\left(\frac{1}{8}\right)\right]\frac{\theta_m^2}{2}\ (\text{lb}\cdot\text{ft})$$

$$V_2 = [-0.3646 + 0.4167 + 0.1563]\frac{\theta_m^2}{2}$$

$$V_2 = \frac{0.2084\theta_m^2}{2}$$

<u>Conservation of energy</u>. $T_1 + V_1 = T_2 + V_2$: $\frac{1}{2}(0.01775)\dot{\theta}_m^2 + 0 = 0 + \frac{0.2084}{2}\theta_m^2$

<u>Simple harmonic motion</u>.

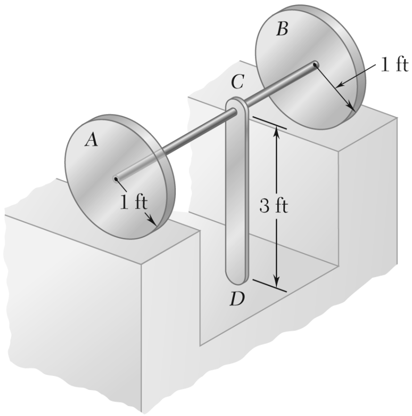
$$\dot{\theta}_m = \omega_n \theta_m$$

$$\omega_n^2 = \frac{0.2084}{0.01775} = 11.738$$

$$\tau_n = \frac{2\pi}{\sqrt{\omega_n}} = \frac{2\pi}{\sqrt{11.738}}$$

$\tau_n = 1.834$ s ◀

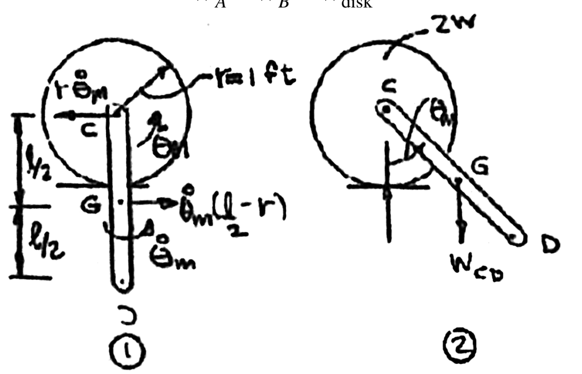
PROBLEM 19.86



A 10-lb uniform rod *CD* is welded at *C* to a shaft of negligible mass which is welded to the centers of two 20-lb uniform disks *A* and *B*. Knowing that the disks roll without sliding, determine the period of small oscillations of the system.

SOLUTION

$$W_A = W_B = W_{\text{disk}}$$



<u>*Position ①*</u>

$$T_1 = \frac{1}{2}2(\bar{I}_A)_{\text{disk}}\,\dot{\theta}_m^2 + \frac{1}{2}\left(\frac{2W_{\text{disk}}}{g}\right)(r\dot{\theta}_m)^2$$

$$+\frac{1}{2}\bar{I}_{CD}\dot{\theta}_m^2 + \frac{1}{2}\frac{W_{CD}}{g}\left(\frac{l}{2}-r\right)^2\dot{\theta}_m^2$$

$$(\bar{I}_A)_{\text{disk}} = \frac{1}{2}\frac{W_{\text{disk}}}{g}r^2 = \frac{1}{2}\frac{(20)}{g}(1)^2 = \frac{10}{g}$$

$$\bar{I}_{CD} = \frac{1}{12}\frac{W_{CD}}{g}l^2 = \frac{1}{12}\frac{(10)}{g}(3)^2 = \frac{15}{2g}$$

$$T_1 = \frac{1}{2g}\left[20+40+\frac{15}{2}+\frac{5}{2}\right]\dot{\theta}_m^2$$

$$T_1 = \frac{1}{2}g(70)\dot{\theta}_m^2 \qquad V_1 = 0$$

<u>*Position ②*</u>

$$T_2 = 0$$

$$V_2 = W_{CD}\frac{l}{2}(1-\cos\theta_m)$$

PROBLEM 19.86 (Continued)

Small angles: $$1-\cos\theta_m = 2\sin^2\frac{\theta_m}{2} \approx \frac{\theta_m^2}{2}$$

$$V_2 = \frac{1}{2}W_{CD}l\frac{\theta_m^2}{2}$$

$$= \frac{1}{2}(10)(1.5)\theta_m^2$$

$$= \frac{1}{2}(15)\theta_m^2$$

Conservation of energy and simple harmonic motion.

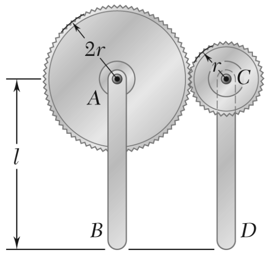
$$T_1 + V_1 = T_2 + V_2$$

$$\dot{\theta}_m = \omega_n\theta_m$$

$$\frac{1}{2g}(70)\omega_n^2\theta_m^2 + 0 = 0 + \frac{1}{2}(15)\theta_m^2$$

$$\omega_n^2 = \frac{15g}{70}$$

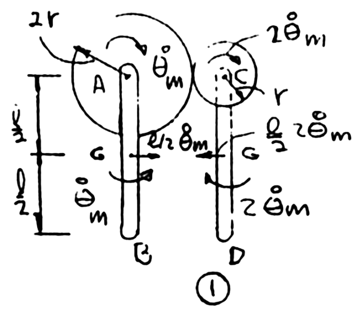
Period of oscillations. $$\tau_n = \frac{2\pi}{\sqrt{\omega_n}} = 2\pi\sqrt{\frac{70}{(15)(32.2)}}$$ $\tau_n = 2.39\text{ s}$ ◀

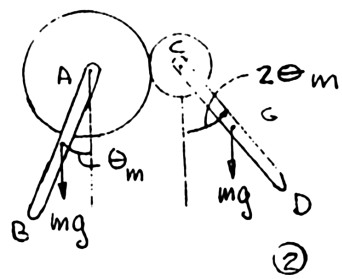


PROBLEM 19.87

Two uniform rods *AB* and *CD*, each of length *l* and mass *m*, are attached to gears as shown. Knowing that the mass of gear *C* is *m* and that the mass of gear *A* is 4*m*, determine the period of small oscillations of the system.

SOLUTION





Kinematics:

$$2r\theta_A = r\theta_C$$
$$2\theta_A = \theta_C$$
$$2\dot{\theta}_A = \dot{\theta}_C$$

Let

$$\theta_A = \theta_m$$
$$2\theta_m = (\theta_C)_m$$
$$2\dot{\theta}_m = (\dot{\theta}_C)_m$$

<u>*Position* ①</u>

$$T_1 = \frac{1}{2}\bar{I}_A\dot{\theta}_m^2 + \frac{1}{2}\bar{I}_C(2\dot{\theta}_m)^2 + \frac{1}{2}\bar{I}_{AB}\dot{\theta}_m^2 + \frac{1}{2}\bar{I}_{CD}(2\dot{\theta}_m)^2 + \frac{1}{2}m_{AB}\left(\frac{l}{2}\dot{\theta}_m\right)^2 + \frac{1}{2}m_{CD}\left(\frac{l}{2}2\dot{\theta}_m\right)^2$$

$$\bar{I}_A = \frac{1}{2}(4m)(2r)^2 = 8mr^2$$

$$\bar{I}_C = \frac{1}{2}(m)(r)^2 = \frac{1}{2}mr^2$$

$$\bar{I}_{AB} = \frac{1}{12}ml^2 \qquad \bar{I}_{CD} = \frac{1}{12}ml^2$$

$$T_1 = \frac{1}{2}m\left[8r^2 + \left(\frac{r^2}{2}\right)4 + \frac{l^2}{12} + \frac{l^2}{3} + \frac{l^2}{4} + l^2\right]$$

$$T_1 = \frac{1}{2}m\left[10r^2 + \frac{5}{3}l^2\right]\dot{\theta}_m^2$$

$$V_1 = 0$$

Position ②

$$T_1 = 0$$

$$V_1 = mg\frac{l}{2}(1-\cos\theta_m) + \frac{mgl}{2}(1-\cos 2\theta_m)$$

For small angles,

$$1-\cos\theta_m = 2\sin^2\frac{\theta_m}{2} \approx \frac{\theta_m^2}{2}$$

$$1-\cos 2\theta_m = 2\sin^2\theta_m \approx 2\theta_m^2$$

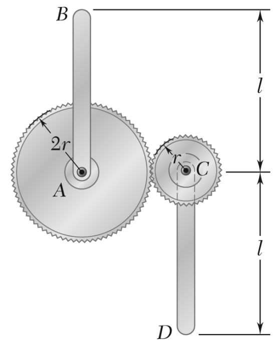
$$V_1 = \frac{1}{2}mgl\left(\frac{\theta_m^2}{2} + 2\theta_m^2\right) = \frac{1}{2}mgl\frac{5\theta_m^2}{2}$$

$$T_1 + V_1 = T_2 + V_2 \qquad \dot{\theta}_m^2 = \omega_n^2\theta_m^2$$

$$\frac{1}{2}m\left[10r^2 + \frac{5}{3}l^2\right]\omega_n^2\theta_m^2 + 0 = 0 + \frac{1}{2}mgl\frac{5\theta_m^2}{2}$$

$$\omega_n^2 = \frac{\frac{5}{2}gl}{10r^2 + \frac{5}{3}l^2}$$

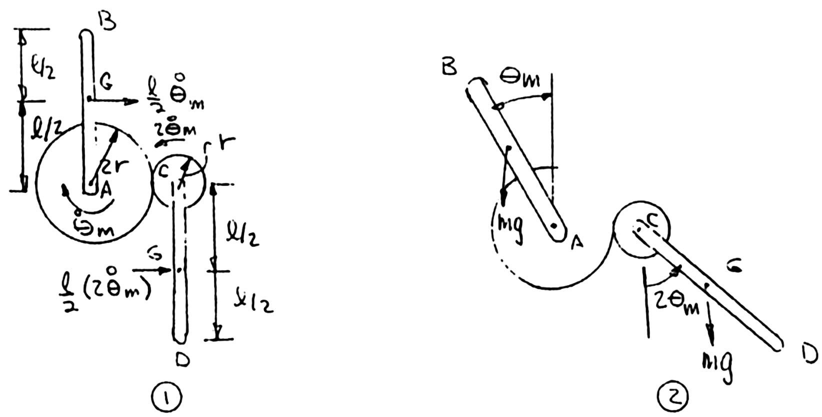
$$= \frac{3gl}{12r^2 + 2l^2} \qquad \tau_n = \frac{2\pi}{\sqrt{\omega_n}} = 2\pi\sqrt{\frac{12r^2 + 2l^2}{3gl}} \blacktriangleleft$$



PROBLEM 19.88

Two uniform rods AB and CD, each of length l and mass m, are attached to gears as shown. Knowing that the mass of gear C is m and that the mass of gear A is $4m$, determine the period of small oscillations of the system.

SOLUTION



Kinematics: $2r\theta_A = r\theta_C \qquad 2\theta_A = \theta_C$

$$2\dot{\theta}_A = \dot{\theta}_C$$

Let $\theta_A = \theta_m \qquad 2\theta_m = (\theta_C)_m$

$$2\dot{\theta}_m = (\dot{\theta}_C)_m$$

<u>*Position* ①</u>

$$T_1 = \frac{1}{2}\overline{I}_A\dot{\theta}_m^2 + \frac{1}{2}\overline{I}_C(2\dot{\theta}_m)^2 + \frac{1}{2}\overline{I}_{AB}\dot{\theta}_m^2 + \frac{1}{2}\overline{I}_{CD}(2\dot{\theta}_m)^2 + \frac{1}{2}m_{AB}\left(\frac{l}{2}\dot{\theta}_m\right)^2 + \frac{1}{2}m_{CD}\left(\frac{l}{2}2\dot{\theta}_m\right)^2$$

$$\overline{I}_A = \frac{1}{2}(4m)(2r)^2 = 8mr^2$$

$$\overline{I}_C = \frac{1}{2}(m)(r^2) = \frac{1}{2}mr^2$$

$$\overline{I}_{AB} = \frac{1}{12}ml^2 \qquad \overline{I}_{CD} = \frac{1}{12}ml^2$$

$$T_1 = \frac{1}{2}m\left[8r^2 + \left(\frac{r^2}{2}\right)4 + \frac{l^2}{12} + \frac{l^2}{3} + \frac{l^2}{4} + l^2\right]\dot{\theta}_m^2$$

$$T_1 = \frac{1}{2}m\left[10r^2 + \frac{5}{3}l^2\right]\dot{\theta}_m^2 \qquad V_1 = 0$$

Position ②

$$T_2 = 0$$

$$V_2 = -mg\frac{l}{2}(1-\cos\theta_m) + \frac{mgl}{2}(1-\cos 2\theta_m)$$

For small angles,

$$1-\cos\theta_m = 2\sin^2\frac{\theta_m}{2} \approx \frac{\theta_m^2}{2}$$

$$1-\cos 2\theta_m = 2\sin^2\theta_m \approx 2\theta_m^2$$

$$V_2 = -mg\frac{l}{2}\frac{\theta_m^2}{2} + \frac{mgl}{2}2\theta_m^2$$

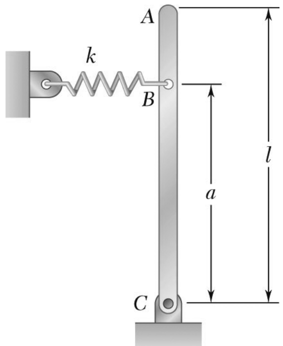
$$= \frac{1}{2}mgl\frac{3}{2}\theta_m^2$$

$$T_1 + V_1 = T_2 + V_2 \qquad \dot{\theta}_m = \omega_n\theta_m$$

$$\frac{1}{2}m\left[10r^2 + \frac{5}{3}l^2\right]\theta_m^2\omega_n^2 + 0 = 0 + \frac{1}{2}mgl\frac{3}{2}\theta_m^2$$

$$\omega_n^2 = \frac{\frac{3}{2}gl}{10r^2 + \frac{5}{3}l^2}$$

$$= \frac{9gl}{60r^2 + 10l^2} \qquad \tau_n = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{60r^2+10l^2}{9gl}} \blacktriangleleft$$



PROBLEM 19.89

An inverted pendulum consisting of a rigid bar *ABC* of length l and mass m is supported by a pin and bracket at *C*. A spring of constant k is attached to the bar at *B* and is undeformed when the bar is in the vertical position shown. Determine (*a*) the frequency of small oscillations, (*b*) the smallest value of a for which these oscillations will occur.

SOLUTION

Moment of inertia: $\bar{I} = \frac{1}{12} ml^2$

Position ① Maximum deflection. Let rod *AC* rotate through angle θ_m. The spring stretches an amount

$$x_m = a \sin \theta_m$$

and the center of gravity moves down an amount

$$-y_m = \frac{l}{2}(1 - \cos \theta_m)$$

$$V_1 = \frac{1}{2} k x_m^2 + mg y_m$$

$$= \frac{1}{2} k (a \sin \theta_m)^2 - mg \frac{l}{2}(1 - \cos \theta_m)$$

$$\approx \frac{1}{2} k a^2 \theta_m^2 - mg \left(\frac{l}{2}\right)\left(\frac{1}{2}\theta_m^2\right)$$

$$= \frac{1}{2}\left(ka^2 - \frac{1}{2} mgl\right)\theta_m^2$$

$$T_1 = 0$$

Position ② Maximum velocity:

For simple harmonic motion, $\dot{\theta} = -\omega_n \theta_m$

Velocity of the mass center of the rod: $\bar{v} = \frac{l}{2}\dot{\theta}$

PROBLEM 19.89 (Continued)

Kinetic energy: $T_2 = \frac{1}{2}m\bar{v}^2 + \frac{1}{2}\bar{I}\dot{\theta}^2$

$$= \frac{1}{2}\left[m\left(\frac{l\dot{\theta}}{2}\right)^2 + \frac{1}{12}ml^2\dot{\theta}^2\right]$$

$$= \frac{1}{2}\left[\frac{1}{3}ml^2\dot{\theta}^2\right]$$

$$= \frac{1}{2}\left(\frac{1}{3}ml^2\omega_n^2\theta_m^2\right)$$

$$V_2 = 0$$

Conservation of energy: $T_1 + V_1 = T_2 + V_2$

$$0 + \frac{1}{2}\left(ka^2 - \frac{1}{2}mgl\right)\theta_m^2 = \frac{1}{2}\left(\frac{1}{3}ml^2\omega_n^2\theta_m^2\right)$$

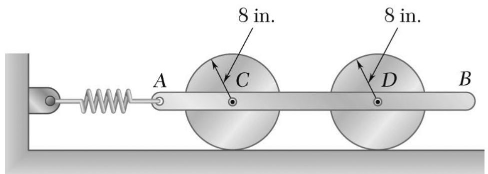
$$\omega_n^2 = \frac{6\,ka^2 - 3mgl}{2ml^2}$$

(*a*) Frequency: $f = 2\pi\omega_n$

$$f = 2\pi\sqrt{(6ka^2 - 3mgl)/2ml^2} \blacktriangleleft$$

(*b*) Smallest value of *a* for oscillations. *f* is real for $6\,ka^2 > 3mgl$

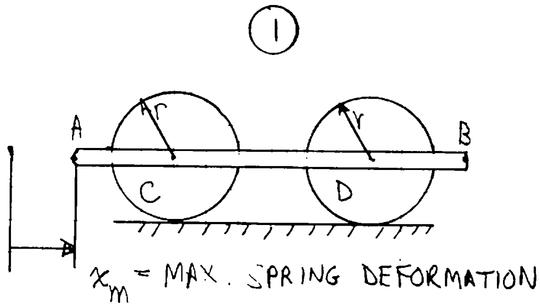
$$a > \sqrt{\frac{mgl}{2k}} \qquad a_{\min} = \sqrt{\frac{mgl}{2k}} \blacktriangleleft$$

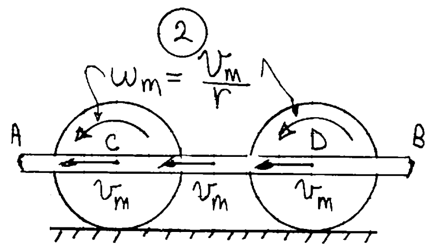


PROBLEM 19.90

Two 12-lb uniform disks are attached to the 20-lb rod *AB* as shown. Knowing that the constant of the spring is 30 lb/in. and that the disks roll without sliding, determine the frequency of vibration of the system.

SOLUTION





Position 1: $\quad T_1 = 0, \quad V_1 = \frac{1}{2}kx_m^2$

Position 2: $\quad V_2 = 0, \quad T_2 = \frac{1}{2}m_{AB}v_m^2 + 2\left(\frac{1}{2}\bar{I}\omega_m^2 + \frac{1}{2}m_{\text{disk}}\bar{v}_m^2\right)$

$$T_2 = \frac{1}{2}m_{AB}v_m^2 + \left(\frac{1}{2}m_{\text{disk}}r^2\right)\left(\frac{v_m}{r}\right)^2 + m_{\text{disk}}v_m^2$$

$$T_2 = \frac{1}{2}(m_{AB} + 3m_{\text{disk}})v_m^2$$

Conservation of energy

$$T_1 + V_1 = T_2 + V_2: \quad 0 + \frac{1}{2}kx_m^2 = \frac{1}{2}(m_{AB} + 3m_{\text{disk}})v_m^2$$

But for simple harmonic motion, $\quad v_m = \omega_n x_m$:

$$\frac{1}{2}kx_m^2 = \frac{1}{2}(m_{AB} + 3m_{\text{disk}})(\omega_n x_m)^2$$

$$\omega_n^2 = \frac{k}{m_{AB} + 3m_{\text{disk}}}$$

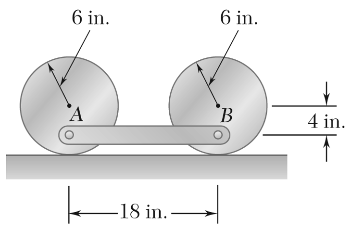
Note: Result is independent of r

Data: $\quad k = 30$ lb/in. $\quad W_{AB} = 20$ lb $\quad W_{\text{disk}} = 12$ lb $\quad m_{AB} = \frac{W_{AB}}{g} \quad m_{\text{disk}} = \frac{W_{\text{disk}}}{g}$

$$\omega_n^2 = \frac{30(12)\text{ lb/ft}}{20\text{ lb}/32.2 + 3(12/32.2)} = 207 \qquad \omega_n = 14.387\text{ rad/s}$$

$$f = \frac{\omega_n}{2\pi} = \frac{14.387\text{ rad/s}}{2\pi}$$

$f = 2.29$ Hz ◀



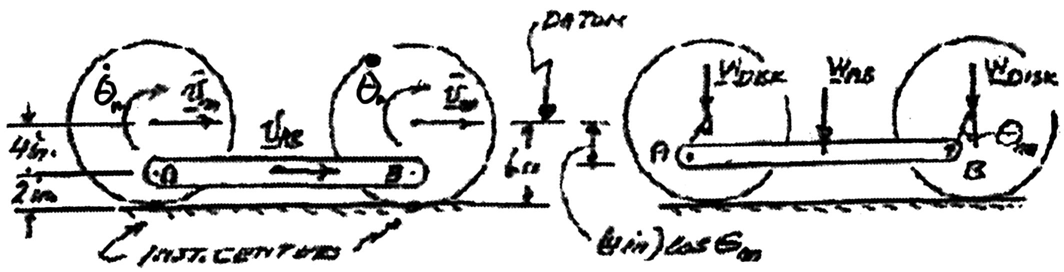
PROBLEM 19.91

The 20-lb rod AB is attached to two 8-lb disks as shown. Knowing that the disks roll without sliding, determine the frequency of small oscillations of the system.

SOLUTION

Position ② $\qquad\qquad$ *Position ①*

$r = 6$ in.



Masses and moments of inertia.

$$m_A = m_B = \frac{8}{32.2} = 0.24845 \text{ lb}\cdot\text{s}^2/\text{ft}$$

$$\bar{I}_A = \bar{I}_B = \frac{1}{2}m_A r_A^2 = \frac{1}{2}(0.24845)\left(\frac{6}{12}\right)^2$$

$$= 0.031056 \text{ lb}\cdot\text{s}\cdot\text{ft}$$

$$m_{AB} = \frac{20}{32.2} = 0.62112 \text{ lb}\cdot\text{s}^2/\text{ft}$$

Kinematics:

$$\bar{v}_m = r_A\dot{\theta}_m = \frac{6}{12}\dot{\theta}_m = 0.5\dot{\theta}_m$$

$$v_{AB} = \left(\frac{2}{12}\right)\dot{\theta}_m = \frac{1}{6}\dot{\theta}_m$$

Position ① (Maximum displacement) $\quad T_1 = 0$

$$V_1 = -W_{AB}\left(\frac{4}{12}\cos\theta_m\right) = -\frac{80}{12}\cos\theta_m$$

Position ② (Maximum speed)

$$T_2 = \frac{1}{2}m_A v_m^2 + \frac{1}{2}\bar{I}_A\dot{\theta}_m^2 + \frac{1}{2}m_B v_m^2 + \frac{1}{2}\bar{I}_B\dot{\theta}_m^2 + \frac{1}{2}m_{AB}v_{AB}^2$$

$$= 2\left[\frac{1}{2}(0.24845)(0.5\dot{\theta}_m)^2 + \frac{1}{2}(0.031056)\dot{\theta}_m^2\right] + \frac{1}{2}(0.62112)\left(\frac{1}{6}\dot{\theta}_m\right)^2$$

$$= 0.101795\dot{\theta}_m^2$$

$$V_2 = -W_{AB}\left(\frac{4}{12}\right) = -\frac{80}{12}$$

PROBLEM 19.91 (Continued)

Conservation of energy. $T_1 + V_1 = T_2 + V_2$

$$0 - \frac{80}{12}\cos\theta_m = 0.101795\dot{\theta}_m^2 - \frac{80}{12}$$

$$\dot{\theta}_m^2 = 65.491(1 - \cos\theta_m)$$

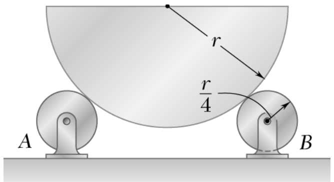
$$\approx 65.491\left(\frac{1}{2}\theta_m^2\right)$$

$$= 32.745\theta_m^2$$

$$\theta_m = 5.7224\,\theta_m$$

Simple harmonic motion. $\dot{\theta}_m = \omega_n \theta_m$ $\qquad \omega_n = 5.7224 \text{ rad/s}$

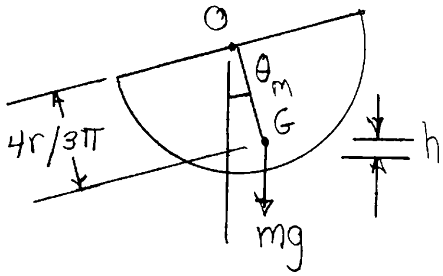
Frequency. $f_n = \frac{\omega_n}{2\pi} = \frac{5.7224}{2\pi}$ $\qquad f_n = 0.911 \text{ Hz}$ ◀



PROBLEM 19.92

A half section of a uniform cylinder of radius r and mass m rests on two casters A and B, each of which is a uniform cylinder of radius $r/4$ and mass $m/8$. Knowing that the half cylinder is rotated through a small angle and released and that no slipping occurs, determine the frequency of small oscillations.

SOLUTION



$$V_1 = mgh = mg\left(\frac{4r}{3\pi}\right)(1-\cos\theta)$$

$$1-\cos\theta \approx \frac{\theta_m^2}{2}$$

$$V_1 = 2mgr\frac{\theta_m^2}{3\pi}$$

$$T_2 = \frac{1}{2}I_A\omega_A^2 + \frac{1}{2}I_B\omega_B^2 + \frac{1}{2}\left(\frac{1}{2}mr^2\right)\omega^2$$

Where $$I_A = I_B = \frac{1}{2}\left(\frac{m}{8}\right)\left(\frac{r}{4}\right)^2 = \frac{mr^2}{256}$$

and $$\omega_A = \omega_B = 4\omega$$

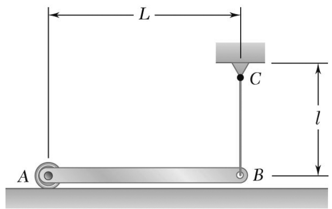
$$\therefore T_2 = \left(\frac{mr^2}{16} + \frac{mr^2}{4}\right)\omega^2 = \frac{5mr^2\omega^2}{16}$$

$$V_1 = T_2, \quad \frac{2\cancel{m}gr\cancel{\theta_m^2}}{3\pi} = \frac{5\cancel{m}r^2\omega_n^2\cancel{\theta_m^2}}{16}$$

$$\omega_n^2 = \frac{32g}{15\pi r},$$

$$f_n = \left(\frac{1}{2\pi}\right)\sqrt{\frac{32g}{15\pi r}}$$

$$f_n = 0.1312\sqrt{\frac{g}{r}}$$ ◄



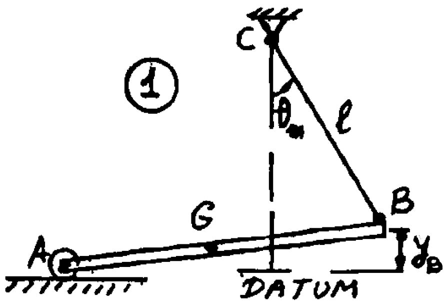
PROBLEM 19.93

The motion of the uniform rod *AB* is guided by the cord *BC* and by the small roller at *A*. Determine the frequency of oscillation when the end *B* of the rod is given a small horizontal displacement and released.

SOLUTION

Position ①. (Maximum deflection):

Let θ_m be the small angle between the cord *CB* and the vertical. As the rod is moved from the equilibrium position the center of gravity *G* moves up an amount $\bar{y}_m$.



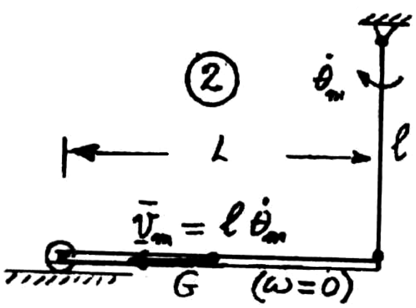
$$y_B = l(1-\cos\theta_m) \approx l\left(1-1+\frac{1}{2}\theta_m^2\right) = \frac{1}{2}l\theta_m^2$$

$$\bar{y}_m = y_G = \frac{1}{2}y_B = \frac{1}{4}l\theta_m^2$$

$$V_1 = mg\bar{y}_m = \frac{1}{4}mgl\theta_m^2$$

$$T_1 = 0,$$

Position ②. (Maximum velocity): At the equilibrium position the motion of the rod is a translation.



$$\bar{v}_m = l\omega = l\dot{\theta}_m$$

$$T_2 = \frac{1}{2}m\bar{v}_m^2 = \frac{1}{2}ml^2\dot{\theta}_m^2$$

$$V_2 = 0$$

Conservation of energy: $T_1 + V_1 = T_2 + V_2$: $\frac{1}{4}mgl\theta_m^2 = \frac{1}{2}ml^2\dot{\theta}_m^2$

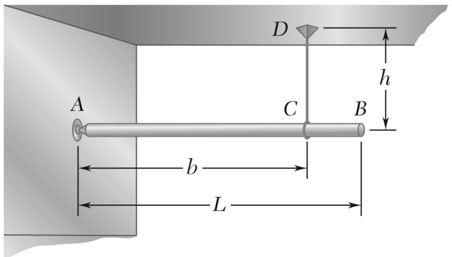
For simple harmonic motion; $\dot{\theta}_m = \omega_n\theta_m$ so that

$$\frac{1}{4}mgl\theta_m^2 = \frac{1}{2}ml^2\omega_n^2\theta_m^2$$

Natural frequency: $\omega_n^2 = \frac{g}{2l}$

$$f = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{g}{2l}}$$

$$f = 0.1125\sqrt{\frac{g}{l}} \blacktriangleleft$$

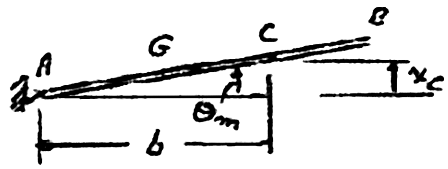


PROBLEM 19.94

A uniform rod of length L is supported by a ball-and-socket joint at A and by a vertical wire CD. Derive an expression for the period of oscillation of the rod if end B is given a small horizontal displacement and then released.

SOLUTION

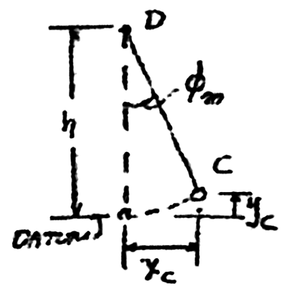
Position ① (Maximum deflection)



Looking from above:

Horizontal displacement of C: $x_C = b\theta_m$

Looking from right: $\phi_m = \dfrac{x_C}{h} = \dfrac{b}{h}\theta_m$



$$y_C = h(1-\cos\phi_m) \approx \frac{1}{2}h\phi_m^2$$

$$y_C = \frac{1}{2}h\left(\frac{b}{h}\theta_m\right)^2 = \frac{1}{2}\frac{b^2}{h}\theta_m^2$$

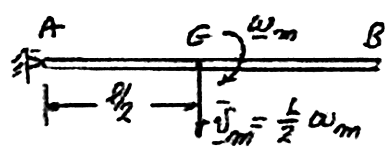
$$\overline{y}_m = y_G = \frac{AG}{AC} - y_C = \frac{\frac{1}{2}L}{b}\left(\frac{1}{2}\frac{b^2}{h}\theta_m^2\right)$$

$$\overline{y}_m = \frac{1}{4}\cdot\frac{bL}{h}\theta_m^2$$

We have $T_1 = 0$

$$V_1 = mgy_m = \frac{1}{4}\frac{mgbL}{h}\theta_m^2$$

Position ② (Maximum velocity)



Looking from above: $T_2 = \dfrac{1}{2}\overline{I}\dot{\theta}_m^2 + \dfrac{1}{2}m\overline{v}_m^2$

$$= \frac{1}{2}\left(\frac{1}{12}mL^2\right)\dot{\theta}_m^2 + \frac{1}{2}m\left(\frac{L}{2}\dot{\theta}\right)^2$$

$$T_2 = \frac{1}{6}mL^2\dot{\theta}_m^2$$

$$V_2 = 0$$

Conservation of energy. $T_1 + V_1 = T_2 + V_2$: $\quad 0 + \dfrac{1}{4}\dfrac{mgbL}{h}\theta_m^2 = \dfrac{1}{6}mL^2\dot{\theta}_m^2$

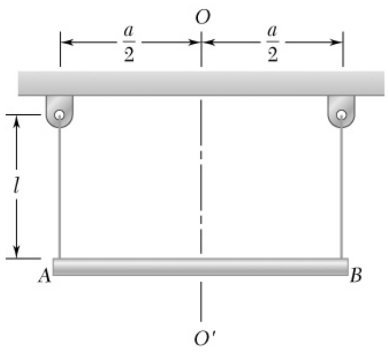
PROBLEM 19.94 (Continued)

But for simple harmonic motion, $\dot{\theta}_m = \omega_n \theta_m$

$$\frac{1}{4}\frac{mgbL}{h}\theta_m^2 = \frac{1}{6}mL^2(\omega_n\theta_m)^2$$

$$\omega_n^2 = \frac{3}{2}\frac{bg}{hL}$$

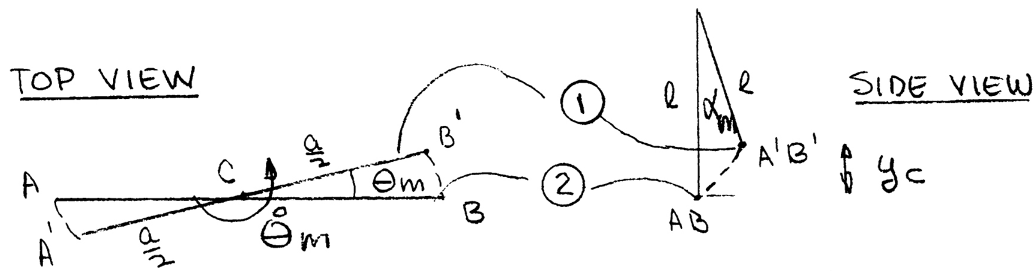
Period of vibration. $\tau_n = \frac{2\pi}{\omega_n}$ $\tau_n = 2\pi\sqrt{\frac{2hL}{3bg}}$ ◄



PROBLEM 19.95

A section of uniform pipe is suspended from two vertical cables attached at A and B. Determine the frequency of oscillation when the pipe is given a small rotation about the centroidal axis OO' and released.

SOLUTION



$$AA' = BB' = \frac{a}{2}\theta_m = l\alpha_m \qquad \alpha_m = \frac{a}{2l}\theta_m$$

Position ①

$$T_1 = 0 \qquad V_1 = mgy_c = mgl(1-\cos\alpha)$$

For small angles

$$1-\cos\alpha_m = 2\sin\frac{\alpha_m}{2} \approx \frac{\alpha_m^2}{2} = \frac{a^2}{8l^2}\theta_m^2$$

$$V_1 = mgl\left(\frac{a^2}{8l^2}\right)\theta_m^2$$

Position ②

$$T_2 = \frac{1}{2}\bar{I}\dot{\theta}_m^2 = \frac{1}{2}\left(\frac{1}{12}ma^2\right)\dot{\theta}_m^2 \qquad V_2 = 0$$

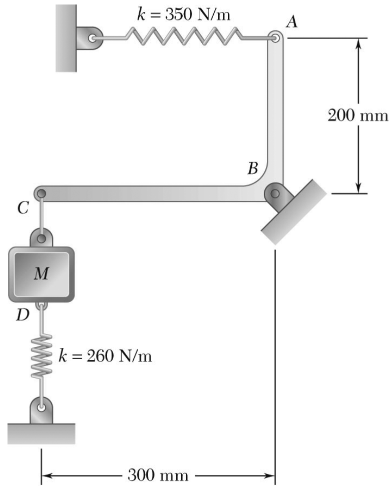
$$\dot{\theta}_m = \omega_n\theta_m$$

$$T_1 + V_1 = T_2 + V_2$$

$$mgl\left(\frac{a^2}{8l^2}\right) + 0 + \frac{1}{24}ma^2\omega_n^2\theta_m^2$$

$$\omega_n^2 = \frac{3g}{l}$$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{3g}{l}} \blacktriangleleft$$



PROBLEM 19.96

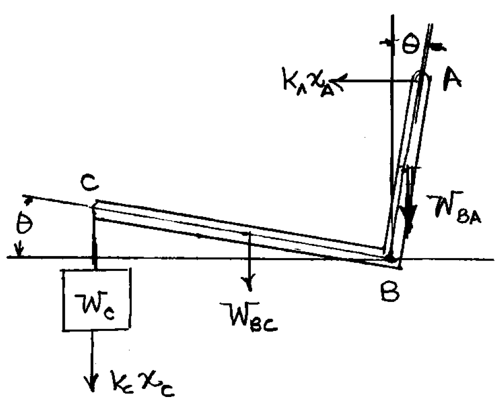
A 0.6-kg uniform arm ABC is supported by a pin at B and is attached to a spring at A. It is connected at C to a 1.4-kg mass M which is attached to a spring. Knowing that each spring can act in tension or compression, determine the frequency of small oscillations of the system when the weight is given a small vertical displacement and released.

SOLUTION

Data:

$$k_A = 260 \text{ N/m} \qquad k_C = 350 \text{ N/m}$$
$$l_{AB} = 0.200 \text{ m} \qquad l_{BC} = 0.300 \text{ m}$$
$$l_{ABC} = l_{BA} + l_{BA} = 0.500 \text{ m}$$
$$m_{ABC} = 0.6 \text{ kg} \qquad m_C = 1.4 \text{ kg}$$
$$m_{BA} = \frac{0.200}{0.500} m_{ABC} = \frac{2}{5}(0.6 \text{ kg}) = 0.24 \text{ kg}$$
$$m_{BC} = \frac{0.300}{0.500} m_{ABC} = \frac{3}{5}(0.6 \text{ kg}) = 0.36 \text{ kg}$$
$$W_{BA} = m_{BA} g = (0.24 \text{ kg})(9.81 \text{ m/s}^2) = 2.3544 \text{ N}$$
$$W_{BC} = m_{BC} g = (0.36 \text{ kg})(9.81 \text{ m/s}^2) = 3.5316 \text{ N}$$
$$W_C = m_C g = (1.4 \text{ kg})(9.81 \text{ m/s}^2) = 13.734 \text{ N}$$

Let x_A and x_C be the amounts of stretch from their zero force lengths of the springs at locations A and C, respectively. Let θ be the small clockwise rotation of arm ABC about the fixed Point B, measured from the equilibrium position. Let $\bar{y}_{BA}$ and $\bar{y}_{BC}$ be the upward movement of the mass centers of portions BA and BC of the arm ABC.



Potential energy:

$$V = W_C y_C + W_{BC}\bar{y}_{BC} + W_{BA}\bar{y}_{BA} + \frac{1}{2}k_A x_A^2 + \frac{1}{2}k_B x_B^2$$

$$= W_C l_{BC}\sin\theta + W_{BC}\left(\frac{1}{2}l_{BC}\sin\theta\right) - W_{BA}\left[\frac{1}{2}l_{BA}(1-\cos\theta)\right]$$

$$+\frac{1}{2}k_A(l_{BA}\sin\theta + \delta_A)^2 + \frac{1}{2}k_C(l_{BC}\sin\theta + \delta_C)^2$$

$$= W_C l_{BC}\sin\theta + \frac{1}{2}W_{BC}l_{BC}\sin\theta - \frac{1}{2}W_{BA}l_{BA}(1-\cos\theta)$$

$$+\frac{1}{2}k_A l_{BA}^2\sin^2\theta + k_A l_{BA}\delta_A\sin\theta + \frac{1}{2}k_A\delta_A^2$$

$$+\frac{1}{2}k_C l_{BC}^2\sin^2\theta + k_C l_{BC}\delta_C\sin\theta + \frac{1}{2}k_C\delta_C^2 \qquad (1)$$

where δ_A and δ_C are the spring elongations at the equilibrium position.

In the static equilibrium position,

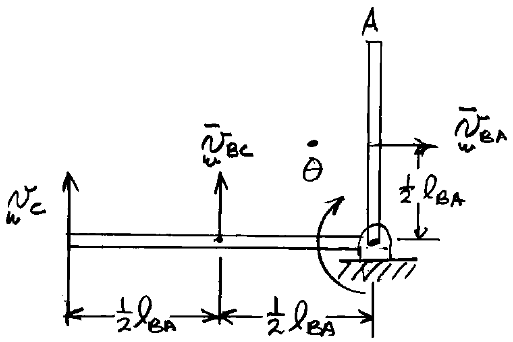
$$+\circlearrowleft \Sigma M_B = 0:\quad W_C l_{BC} + \frac{1}{2}W_{BC}l_{BC} + (k_A\delta_A)l_{BA} + (k_C\delta_C)l_{BC} = 0 \qquad (2)$$

Substiting Eq. (2) into Eq. (1) gives

$$V = -\frac{1}{2}W_{BA}l_{BA}(1-\cos\theta) + \frac{1}{2}k_A l_{BA}^2\sin^2\theta$$

$$+\frac{1}{2}k_C l_{BC}^2\sin^2\theta + \frac{1}{2}k_A\delta_A^2 + \frac{1}{2}k_C\delta_C^2 \qquad (3)$$

Kinematics for position with $\theta = 0$.



$$v_C = l_{BC}\dot{\theta}$$

$$\bar{v}_{BC} = \frac{1}{2}l_{BC}\dot{\theta}$$

$$\bar{v}_{BA} = \frac{1}{2}l_{BA}\dot{\theta}$$

Kinetic energy:

$$T = \frac{1}{2}m_C v_C^2 + \frac{1}{2}m_{BC}\bar{v}_{BC}^2 + \frac{1}{2}\bar{I}_{BC}\dot{\theta}^2 + \frac{1}{2}m_{BA}\bar{v}_{BA}^2 + \frac{1}{2}\bar{I}_{BA}\dot{\theta}^2$$

$$= \frac{1}{2}m_C l_{BC}^2\dot{\theta}^2 + \frac{1}{2}m_{BC}\left(\frac{1}{2}l_{BC}\dot{\theta}\right)^2 + \frac{1}{2}\left(\frac{1}{12}m_{BC}l_{BC}^2\right)\dot{\theta}^2$$

$$+\frac{1}{2}m_{BA}\left(\frac{1}{2}l_{BA}\dot{\theta}\right)^2 + \frac{1}{2}\left(\frac{1}{12}m_{BA}l_{BA}^2\right)\dot{\theta}$$

$$= \frac{1}{2}\left(m_C l_{BC}^2 + \frac{1}{3} m_{BC} l_{BC}^2 + \frac{1}{3} m_B l_{BA}^2 \right)\dot{\theta}^2 \qquad (4)$$

Conservation of energy: $T_1 + V_1 = T_2 + V_2$ (5)

Position ①. (Maximum deflection) $\theta = \theta_m$

$$T_1 = 0 \qquad (6)$$

$$V_1 = -\frac{1}{2} W_{AB} l_{AB} (1 - \cos\theta_m) + \frac{1}{2} k_A l_{BA}^2 \sin^2\theta_m + \frac{1}{2} k_C l_{BC}^2 \sin^2\theta_m + \frac{1}{2} k_A \delta_A^2 + \frac{1}{2} k_C \delta_C^2$$

For small angle θ_m, $\sin\theta_m \approx \theta_m$

$$1 - \cos\theta_m = 2\sin^2\frac{\theta_m}{2} \approx \frac{1}{2}\theta_m^2$$

$$V_1 \approx \frac{1}{2}\left(-\frac{1}{2} W_{BA} l_{BA} + k_A l_{BA}^2 + k_C l_{BC}^2 \right)\theta_m^2 + \frac{1}{2} k_A \delta_A^2 + \frac{1}{2} k_C \delta_C^2 \qquad (7)$$

Position ②: Maximum velocity. $\theta = 0$

For simple harmonic motion $\dot{\theta} = \omega_n \theta_m$ (8)

$$T_2 = \frac{1}{2}\left(m_{BC} l_{BC}^2 + \frac{1}{3} m_{BC} l_{BC}^2 + \frac{1}{3} m_{BA} l_{BA}^2 \right)\omega_n^2 \theta_m^2 \qquad (9)$$

Substituting Eqs. (6), (7), (8), and (9) into Eq. (5) and noting that the terms containing δ_A and δ_C cancel,

$$0 + \frac{1}{2}\left(-\frac{1}{2} W_{BA} l_{BC} + k_A l_{BA}^2 + k_C l_{BC}^2 \right)\theta_m^2 = \frac{1}{2}\left(m_C l_{BC}^2 + \frac{1}{3} m_{BC} l_{BC}^2 + \frac{1}{3} m_{BA} l_{BA}^2 \right)\omega_n^2 \theta_m^2 + 0$$

Applying the numerical data: $-\frac{1}{2} W_{BA} l_{BA} + k_A l_{BA}^2 + k_C l_{BC}^2$

$$= -\frac{1}{2}(2.3544)(0.2) + (350)(0.2)^2 + (260)(0.3)^2$$

$$= -0.23544 + 14.0 + 23.4 = 37.165 \text{ N} \cdot \text{m}$$

$$m_C l_{BC}^2 + \frac{1}{3} m_{BC} l_{BC}^2 + \frac{1}{3} m_{BA} l_{BA}^2$$

$$= (1.4)(0.3)^2 + \frac{1}{3}(0.36)(0.3)^2 + \frac{1}{3}(0.24)(0.2)^2$$

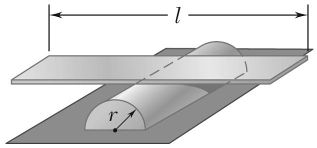
$$= 0.126 + 0.0108 + 0.0032 = 0.1400 \text{ kg} \cdot \text{m}^2$$

Then, $$\frac{1}{2}(37.165)\theta_m^2 = \frac{1}{2}(0.1400)\omega_n^2 \theta_m^2$$

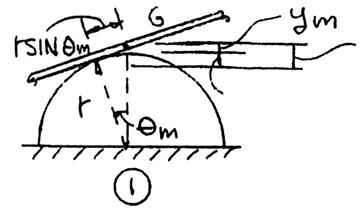
Natural frequency: $$\omega_n^2 = \frac{37.165}{0.1400} = 265.46$$

$$\omega_n = 16.293 \text{ rad/s}$$

$$f = \frac{\omega_n}{2\pi}$$ $f = 2.59$ Hz ◀

	**PROBLEM 19.97*** A thin plate of length l rests on a half cylinder of radius r. Derive an expression for the period of small oscillations of the plate.

SOLUTION



$$(r\sin\theta_m)\sin\theta_m \approx r\theta_m^2$$

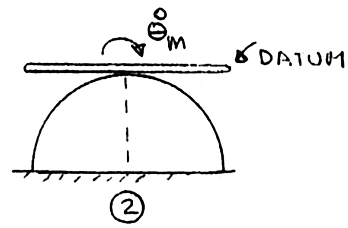
$$r(1-\cos\theta_m) \approx r\frac{\theta_m^2}{2}$$

<u>*Position*</u> ① (Maximum deflection)

$$T_1 = 0$$

$$V_1 = Wy_m$$

$$= mgr\frac{\theta_m^2}{2}$$



<u>*Position*</u> ② $(\theta = 0)$:

$$T_2 = \frac{1}{2}\bar{I}\dot{\theta}_m^2$$

$$= \frac{1}{2}\left(\frac{1}{12}\right)ml^2\dot{\theta}_m^2$$

$$\dot{\theta}_m = \omega_n\theta_m$$

$$T_2 = \frac{1}{2}\left(\frac{1}{12}\right)ml^2\omega_n^2\theta_m^2$$

$$T_1 + V_1 = T_2 + V_2$$

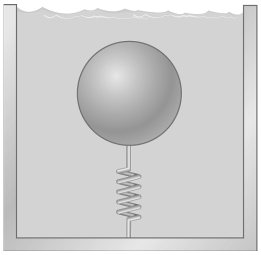
$$0 + \frac{1}{2}mgr\theta_m^2 = \frac{1}{2}\left(\frac{1}{12}\right)ml^2\omega_n^2\theta_m^2$$

$$\omega_n^2 = \frac{12gr}{l^2}$$

$$\tau_n = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{l^2}{12gr}}$$

$$\tau_n = \frac{\pi l}{\sqrt{3gr}} \blacktriangleleft$$

PROBLEM 19.98*



As a submerged body moves through a fluid, the particles of the fluid flow around the body and thus acquire kinetic energy. In the case of a sphere moving in an ideal fluid, the total kinetic energy acquired by the fluid is $\frac{1}{4}\rho V v^2$, where ρ is the mass density of the fluid, V is the volume of the sphere, and v is the velocity of the sphere. Consider a 500-g hollow spherical shell of radius 80 mm, which is held submerged in a tank of water by a spring of constant 500 N/m. (*a*) Neglecting fluid friction, determine the period of vibration of the shell when it is displaced vertically and then released. (*b*) Solve Part *a*, assuming that the tank is accelerated upward at the constant rate of 8 m/s^2.

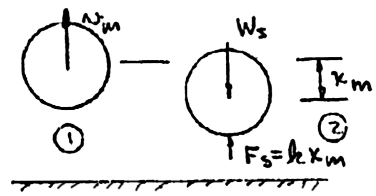
SOLUTION

This is not a damped vibration. However, the kinetic energy of the fluid must be included.

(*a*) *Position ②*

$$T_2 = 0$$

$$V_2 = \frac{1}{2}kx_m^2$$



Position ①

$$T_1 = T_{\text{spere}} + T_{\text{fluid}} = \frac{1}{2}m_s v_m^2 + \frac{1}{4}\rho V v_m^2$$

$$V_1 = 0$$

Conservation of energy and simple harmonic motion.

$$T_1 + V_1 = T_2 + V_2: \quad \frac{1}{2}m_s v_m^2 + \frac{1}{4}\rho V v_m^2 + 0 = 0 + \frac{1}{2}kx_m^2$$

$$v_m = \dot{x}_m = x_m \omega_n$$

$$\frac{1}{2}\left(m_s + \frac{1}{2}\rho V\right)x_m^2 \omega_n^2 = \frac{1}{2}kx_m^2$$

$$\omega_n^2 = \frac{k}{m_s + \frac{1}{2}\rho V}$$

$$\omega_n^2 = \frac{500 \text{ N/m}}{(0.5 \text{ kg}) + \left(\frac{1}{2}\rho V\right)}$$

$$\frac{1}{2}\rho V = \frac{1}{2}(1000 \text{ kg/m}^3)\left(\frac{4}{3}\pi(0.08 \text{ m})^3\right)$$

$$\frac{1}{2}\rho V = 1.0723 \text{ kg}$$

$$\omega_n^2 = \frac{500 \text{ N/m}}{(0.5 \text{ kg}) + (1.0723 \text{ kg})} = 318 \text{ s}^{-2}$$

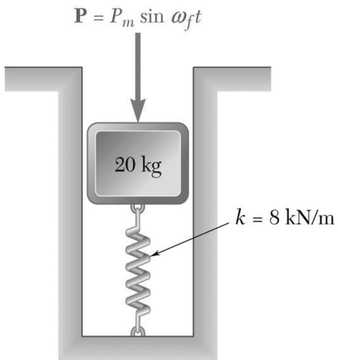
Period of vibration.

$$\tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{\sqrt{318}}$$

$\tau_n = 0.352$ s ◀

(*b*) Acceleration does not change mass.

$\tau_n = 0.352$ s ◀



PROBLEM 19.99

A 20-kg block is attached to a spring of constant $k = 8$ N/m and can move without friction in a vertical slot as shown. The block is acted upon by a periodic force of magnitude $P = P_m \sin \omega_f t$, where $P_m = 100$ N. Determine the amplitude of the motion of the block if (*a*) $\omega_f = 10$ rad/s, (*b*) $\omega_f = 19$ rad/s, (*c*) $\omega_f = 30$ rad/s.

SOLUTION

Equation of motion: $m\ddot{x} + kx = P_m \sin \omega_f t$

The steady state response is

$$x_m = \frac{P_m/k}{1 - \left(\frac{\omega_f}{\omega_n}\right)^2}$$

where

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{8000 \text{ N/m}}{20 \text{ kg}}} = 20 \text{ rad/s}$$

and

$$P_m/k = \frac{100 \text{ N}}{8000 \text{ N/m}} = 0.0125 \text{ m}$$

(*a*) $\omega_f = 10$ rad/s:

$$\frac{\omega_f}{\omega_n} = \frac{10}{20} = 0.5$$

$$x_m = \frac{0.0125}{1-(0.5)^2} = 0.01667 \text{ m}$$

$x_m = 166.7$ mm ◀
(in-phase)

(*b*) $\omega_f = 19$ rad/s:

$$\frac{\omega_f}{\omega_n} = \frac{19}{20} = 0.95$$

$$x_m = \frac{0.0125}{1-(0.95)^2} = 0.1282 \text{ m}$$

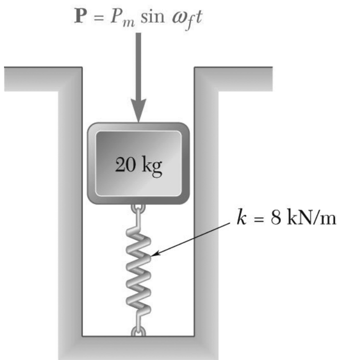
$x_m = 128.2$ mm ◀
(in-phase)

(*c*) $\omega_f = 30$ rad/s:

$$\frac{\omega_f}{\omega_n} = \frac{30}{20} = 1.5$$

$$x_m = \frac{0.0125}{1-(1.5)^2} = -0.0100 \text{ m}$$

$x_m = 10.00$ mm ◀
(out-of-phase)



PROBLEM 19.100

A 20-kg block is attached to a spring of constant $k = 8$ kN/m and can move without friction in a vertical slot as shown. The block is acted upon by a periodic force of magnitude $P = P_m \sin \omega_f t$, where $P_m = 10$ N. Knowing that the amplitude of the motion is 3 mm, determine the value of ω_f.

SOLUTION

Equation of motion: $m\ddot{x} + kx = P_m \sin \omega_f t$

The steady state response is
$$x_m = \frac{P_m/k}{1 - \left(\frac{\omega_f}{\omega_n}\right)^2}$$

where
$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{8000 \text{ N/m}}{20 \text{ kg}}} = 20 \text{ rad/s}$$

and
$$P_m/k = \frac{10 \text{ N}}{8000 \text{ N/m}} = 0.00125 \text{ m}$$

Solve for ω_f/ω_n:
$$\frac{\omega_f}{\omega_n} = \left(1 - \frac{P_m}{kx_m}\right)^{1/2}$$

The amplitude is 3 mm so that $x_m = \pm 0.003$ m.

so that
$$\frac{P_m}{kx_m} = \frac{0.00125 \text{ m}}{\pm 0.003} = \pm 0.41667$$

For the in-phase motion,
$$\frac{\omega_f}{\omega_n} = (1 - 0.41667)^{1/2} = 0.76376$$

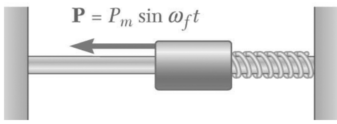
$$\omega_f = (0.76376)(20 \text{ rad/s})$$

$\omega_f = 15.28$ rad/s ◀

For the out-of-phase motion,
$$\frac{\omega_f}{\omega_n} = (1 + 0.41667)^{1/2} = 1.19024$$

$$\omega_f = (1.19024)(20 \text{ rad/s})$$

$\omega_f = 23.8$ rad/s ◀



PROBLEM 19.101

A 9-lb collar can slide on a frictionless horizontal rod and is attached to a spring of constant k. It is acted upon by a periodic force of magnitude $P = P_m \sin \omega_f t$, where $P_m = 2$ lb and $\omega_f = 5$ rad/s. Determine the value of the spring constant k knowing that the motion of the collar has an amplitude of 6 in. and is (*a*) in phase with the applied force, (*b*) out of phase with the applied force.

SOLUTION

Eq. (19.33):

$$x_m = \frac{\frac{P_m}{k}}{1-\left(\frac{\omega_f}{\omega_n}\right)^2} \qquad \omega_n^2 = \frac{k}{m}$$

$$x_m = \frac{P_m}{k - m\omega_f^2}$$

$$k = \frac{P_m}{x_m} + m\omega_f^2$$

Data:

$$P_m = 2 \text{ lb}, \qquad m = \frac{W}{g} = \frac{9}{32.2} = 0.2795 \text{ lb} \cdot \text{s}^2/\text{ft}$$

$$\omega_f = 5 \text{ rad/s}$$

$$k = \frac{P_m}{x_m} + (0.2795)(5)^2$$

$$= \frac{P_m}{x_m} + 6.9876$$

(*a*) (In phase)

$$x_m = 6 \text{ in.} = 0.5 \text{ ft}$$

$$k = \frac{2}{0.5} + 6.9876$$

$k = 10.99$ lb/ft ◀

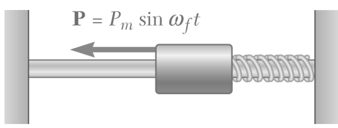
(*b*) (Out of phase)

$$x_m = -6 \text{ in.} = -0.5 \text{ ft}$$

$$k = \frac{2}{-0.5} + 6.9876$$

$k = 2.99$ lb/ft ◀

PROBLEM 19.102



A collar of mass m which slides on a frictionless horizontal rod is attached to a spring of constant k and is acted upon by a periodic force of magnitude $P = P_m \sin \omega_f t$. Determine the range of values of ω_f for which the amplitude of the vibration exceeds two times the static deflection caused by a constant force of magnitude P_m.

SOLUTION

Circular natural frequency. $\omega_n = \sqrt{\frac{k}{m}}$

For forced vibration, the equation of motion is

$$m\ddot{x} + kx = P_m \sin(\omega_f t + \varphi)$$

The amplitude of vibration is

$$x_m = \frac{\frac{P_m}{k}}{\left|1 - \left(\frac{\omega_f}{\omega_n}\right)^2\right|} = \frac{\delta_{st}}{\left|\left(1 - \frac{\omega_f}{\omega_n}\right)^2\right|}$$

For $\omega_f < \omega_n$ and $x_m = 2\,\delta_{st}$, we have

$$2\,\delta_{st} = \frac{\delta_{st}}{1 - \left(\frac{\omega_f}{\omega_n}\right)^2} \quad \text{or} \quad 1 - \left(\frac{\omega_f}{\omega_n}\right)^2 = \frac{1}{2}$$

$$\omega_f^2 = \frac{1}{2}\omega_n^2 = \frac{1}{2}\frac{k}{m} \qquad \omega_f = \sqrt{\frac{k}{2m}} \qquad (1)$$

For $\sqrt{\frac{k}{2m}} < \omega_f \le \omega_n,$ $|x_m|$ exceeds $2\delta_{st}$

For $\omega_f > \omega_n$ and $x_m = 2\delta_{st}$, we have

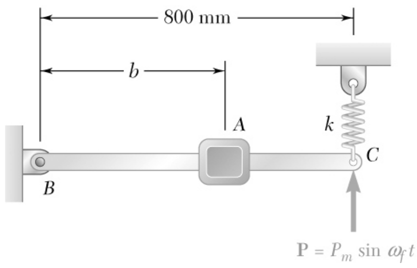
$$2\delta_{st} = \frac{\delta_{st}}{(\omega_f - \omega_n)^2 - 1} \quad \text{or} \quad \frac{\omega_f^2}{\omega_n^2} - 1 = \frac{1}{2}$$

$$\omega_f^2 = \frac{3}{2}\omega_n^2 = \frac{3}{2}\frac{k}{m} \qquad \omega_f = \sqrt{\frac{3k}{2m}} \qquad (2)$$

For $\omega_n \le \omega_f \le \sqrt{\frac{3k}{2m}}$ $|x_m|$ exceeds $2\delta_{st}$

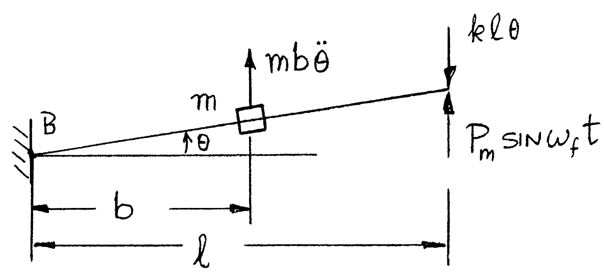
From Eqs. (1) and (2), Range: $\sqrt{\frac{k}{2m}} < \omega_f < \sqrt{\frac{3k}{2m}}$ ◀

PROBLEM 19.103



A small 20-kg block A is attached to the rod BC of negligible mass which is supported at B by a pin and bracket and at C by a spring of constant $k = 2$ kN/m. The system can move in a vertical plane and is in equilibrium when the rod is horizontal. The rod is acted upon at C by a periodic force **P** of magnitude $P = P_m \sin \omega_f t$, where $P_m = 6$ N. Knowing that $b = 200$ mm, determine the range of values of ω_f for which the amplitude of vibration of block A exceeds 3.5 mm.

SOLUTION



$$+\circlearrowleft \Sigma M_B = mb^2\ddot{\theta} = -kl^2\theta + P_m l \sin \omega_f t$$

$$mb^2\ddot{\theta} + kl^2\theta = P_m l \sin \omega_f t$$

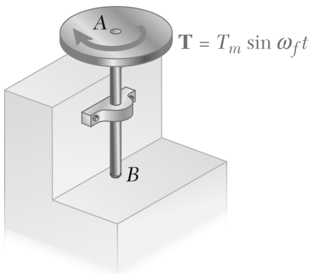
$$\omega_n = \sqrt{\frac{kl^2}{mb^2}} = 40 \text{ rad/s}, \ \theta = \theta_m \sin \omega_f t$$

$$\theta_m = \frac{\pm 3.5 \text{ mm}}{b} = \pm 0.0175 \text{ rad} = \frac{\frac{P_m l}{mb^2}}{\omega_n^2 - \omega_f^2} = \frac{6}{1600 - \omega_f^2}$$

Lower frequency: $6 = 0.0175\left(1600 - \omega_f^2\right), \quad \omega_f = 35.5 \text{ rad/s}$

Upper frequency: $6 = -0.0175\left(1600 - \omega_f^2\right), \quad \omega_f = 44.1 \text{ rad/s}$

$35.5 \text{ rad/s} < \omega_f < 44.1 \text{ rad/s}$ ◀



PROBLEM 19.104

An 8-kg uniform disk of radius 200 mm is welded to a vertical shaft with a fixed end at B. The disk rotates through an angle of $3°$ when a static couple of magnitude $50\ \text{N}\cdot\text{m}$ is applied to it. If the disk is acted upon by a periodic torsional couple of magnitude $T = T_m \sin \omega_f t$, where $T_m = 60\ \text{N}\cdot\text{m}$, determine the range of values of ω_f for which the amplitude of the vibration is less than the angle of rotation caused by a static couple of magnitude T_m.

SOLUTION

Mass moment of inertia: $\bar{I} = \frac{1}{2}mr^2 = \frac{1}{2}(8)(0.200)^2 = 0.16\ \text{kg}\cdot\text{m}^2$

Torsional spring constant:

$$K = \frac{T}{\theta}$$

$$T = 50\ \text{N}\cdot\text{m}$$

$$\theta = 3° = 0.05236\ \text{rad}$$

$$K = \frac{50}{0.05236} = 954.93\ \text{N}\cdot\text{m/rad}$$

Natural circular frequency: $\omega_n = \sqrt{\frac{K}{I}} = \sqrt{\frac{954.93}{0.16}} = 77.254\ \text{rad/s}$

For forced vibration,

$$\theta_m = \frac{\frac{T_m}{K}}{1 - \left(\frac{\omega_f}{\omega_n}\right)^2} = \frac{\theta_{\text{st}}}{1 - \left(\frac{\omega_f}{\omega_n}\right)^2}$$

For the amplitude $|\theta_m|$ to be less than θ_{st}, we must have $\omega_f > \omega_n$.

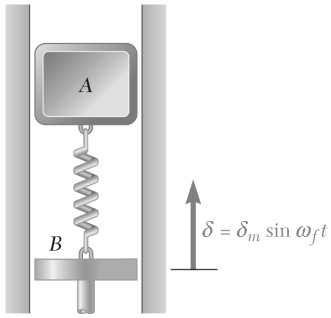
Then

$$|\theta_m| = \frac{\theta_{\text{st}}}{\left(\frac{\omega_f}{\omega_n}\right)^2 - 1} < \theta_{\text{st}}$$

$$\left(\frac{\omega_f}{\omega_n}\right)^2 - 1 > 1$$

$$\left(\frac{\omega_f}{\omega_n}\right)^2 > 2 \qquad \omega_f > \sqrt{2}\omega_n = (\sqrt{2})(77.254)$$

$$\omega_f > 109.3\ \text{rad/s} \blacktriangleleft$$



PROBLEM 19.105

An 18-lb block A slides in a vertical frictionless slot and is connected to a moving support B by means of a spring AB of constant $k = 10$ lb/in. Knowing that the displacement of the support is $\delta = \delta_m \sin \omega_f t$, where $\delta_m = 6$ in., determine the range of values of ω_f for which the amplitude of the fluctuating force exerted by the spring on the block is less than 30 lb.

SOLUTION

Natural circular frequency:
$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{10 \times 12}{\frac{18 \text{ lb}}{32.2}}} = 14.652 \text{ rad/s}$$

Eq. (19.33′):
$$x_m = \frac{\delta_m}{1 - \left(\frac{\omega_f}{\omega_n}\right)^2}$$

Spring force:
$$F_m = -k(x_m - \delta_m) = -k\delta_m \left[1 - \frac{1}{1 - \left(\frac{\omega_f}{\omega_n}\right)^2}\right]$$

$$= k\delta_m \frac{\left(\frac{\omega_f}{\omega_n}\right)^2}{1 - \left(\frac{\omega_f}{\omega_n}\right)^2}$$

$$= (120)(0.50)\frac{\left(\frac{\omega_f}{\omega_n}\right)^2}{1 - \left(\frac{\omega_f}{\omega_n}\right)^2} = 60\frac{\left(\frac{\omega_f}{\omega_n}\right)^2}{1 - \left(\frac{\omega_f}{\omega_n}\right)^2}$$

Limit on spring force:
$$|F_m| < 30 \text{ lb}$$

$$60\left|\frac{\left(\frac{\omega_f}{\omega_n}\right)^2}{1 - \left(\frac{\omega_f}{\omega_n}\right)^2}\right| < 30 \quad \text{or} \quad \left|\frac{\left(\frac{\omega_f}{\omega_n}\right)^2}{1 - \left(\frac{\omega_f}{\omega_n}\right)^2}\right| < \frac{1}{2}$$

In phase motion.

$$\frac{\left(\frac{\omega_f}{\omega_n}\right)^2}{1-\left(\frac{\omega_f}{\omega_n}\right)^2} < \frac{1}{2}$$

$$\left(\frac{\omega_f}{\omega_n}\right)^2 < \frac{1}{2} - \frac{1}{2}\left(\frac{\omega_f}{\omega_n}\right)^2$$

$$\frac{3}{2}\left(\frac{\omega_f}{\omega_n}\right)^2 < \frac{1}{2} \qquad \frac{\omega_f}{\omega_n} > \frac{1}{3}$$

$$\omega_f < \frac{1}{\sqrt{3}}\omega_n$$

$\omega_f < 8.46$ rad/s ◀

Out of phase motion.

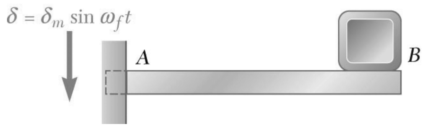
$$\frac{\left(\frac{\omega_f}{\omega_n}\right)^2}{\left(\frac{\omega_f}{\omega_n}\right)^2 - 1} < \frac{1}{2}$$

$$\left(\frac{\omega_f}{\omega_n}\right)^2 < \frac{1}{2}\left(\frac{\omega_f}{\omega_n}\right)^2 - \frac{1}{2}$$

$$\left(\frac{\omega_f}{\omega_n}\right)^2 < -\frac{1}{2}$$

No solution for ω_f.

PROBLEM 19.106



A cantilever beam AB supports a block which causes a static deflection of 8 mm at B. Assuming that the support at A undergoes a vertical periodic displacement $\delta = \delta_m \sin \omega_f t$, where $\delta_m = 2$ mm, determine the range of values of ω_f for which the amplitude of the motion of the block will be less than 4 mm. Neglect the weight of the beam and assume that the block does not leave the beam.

SOLUTION

For the static condition. $\quad mg = k\delta_{st}$

Natural circular frequency. $\quad \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{\delta_{st}}}$

$$g = 9.81 \text{ m/s}, \quad \delta_{st} = 8 \text{ mm} = 0.008 \text{ m}$$

$$\omega_n = \sqrt{\frac{9.81}{0.008}} = 35.018 \text{ rad/s}$$

From Eqs. (19.31 and 19.33′): $\quad (x_m)_B = \dfrac{\delta_m}{1-\left(\frac{\omega_f}{\omega_n}\right)^2}$

Conditions: $\quad |x_m|_B < 4 \text{ mm} \qquad \delta_m = 2 \text{ mm}$

<u>In phase motion</u>.

$$\frac{\delta_m}{1-\left(\frac{\omega_f}{\omega_n}\right)^2} < x_m$$

$$\frac{1-\left(\frac{\omega_f}{\omega_n}\right)^2}{\delta_m} > \frac{1}{x_m}$$

$$1-\left(\frac{\omega_f}{\omega_n}\right)^2 > \frac{\delta_m}{x_m}$$

$$1-\frac{\delta_m}{x_m} > \left(\frac{\omega_f}{\omega_n}\right)^2$$

$$\omega_f < \left(\sqrt{1-\frac{\delta_m}{x_m}}\right)\omega_n$$

$$\omega_f < \left(\sqrt{1-\frac{2}{4}}\right)(35.018)$$

$\omega_f < 24.8$ rad/s ◀

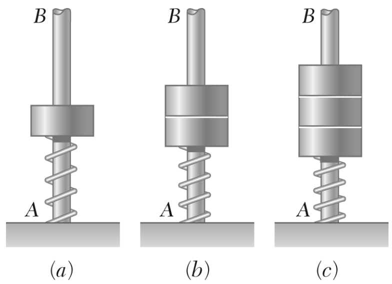
PROBLEM 19.106 (Continued)

Out of phase motion.

$$\frac{\delta_m}{\left(\frac{\omega_f}{\omega_n}\right)^2 - 1} < x_m$$

$$\frac{\left(\frac{\omega_f}{\omega_n}\right)^2 - 1}{\delta_m} > \frac{1}{x_m} \qquad \left(\frac{\omega_f}{\omega_n}\right)^2 - 1 > \frac{\delta_m}{x_m}$$

$$\left(\frac{\omega_f}{\omega_n}\right)^2 > 1.5 \qquad \omega_f > \sqrt{1.5}\omega_n \qquad \omega_f > 42.9 \text{ rad/s} \blacktriangleleft$$



PROBLEM 19.107

Rod AB is rigidly attached to the frame of a motor running at a constant speed. When a collar of mass m is placed on the spring, it is observed to vibrate with an amplitude of 15 mm. When two collars, each of mass m, are placed on the spring, the amplitude is observed to be 18 mm. What amplitude of vibration should be expected when three collars, each of mass m, are placed on the spring? (Obtain two answers.)

SOLUTION

(*a*) *One collar*: $(x_m)_1 = 15 \text{ mm} \qquad (\omega_n)_1^2 = \dfrac{k}{m}$

(*b*) *Two collars*: $(x_m)_2 = 18 \text{ mm} \qquad (\omega_n)_2^2 = \dfrac{k}{2m} = \dfrac{1}{2}(\omega_n)_1^2$

$$\left(\frac{\omega}{\omega_n}\right)_2 = \sqrt{2}\left(\frac{\omega}{\omega_n}\right)_1$$

(*c*) *Three collars*:

$$(x_m)_3 = \text{unknown}, \quad (\omega_n)_3^2 = \frac{k}{3m} = \frac{1}{3}(\omega_n)_1^2, \qquad \left(\frac{\omega}{\omega_n}\right)_3 = \sqrt{3}\left(\frac{\omega}{\omega_n}\right)_1$$

We also note that the amplitude δ_m of the displacement of the base remains constant.

Referring to Section 19.7, Figure 19.9, we note that, since $(x_m)_2 > (x_m)_1$ and $\frac{\omega}{(\omega_n)_2} > \frac{\omega}{(\omega_n)_1}$, we must have $\frac{\omega}{(\omega_n)_1} < 1$ and $(x_m)_1 > 0$. However, $\frac{\omega}{(\omega_n)_2}$ may be either <1 or >1, with $(x_m)_2$ being correspondingly either >0 or <0.

1. *Assuming* $(x_m)_2 > 0$:

For one collar,

$$(x_m)_1 = \frac{\delta_m}{1-\left(\frac{\omega}{\omega_n}\right)_1^2} \qquad +15 \text{ mm} = \frac{\delta_m}{1-\left(\frac{\omega}{\omega_n}\right)_1^2} \tag{1}$$

For two collars,

$$(x_m)_2 = \frac{\delta_m}{1-\left(\frac{\omega}{\omega_n}\right)_2^2} \qquad +18 \text{ mm} = \frac{\delta_m}{1-2\left(\frac{\omega}{\omega_n}\right)_1^2} \tag{2}$$

Dividing Eq. (2) by Eq. (1), member by member:

$$1.2 = \frac{1 - \left(\frac{\omega}{\omega_n}\right)_1^2}{1 - 2\left(\frac{\omega}{\omega_n}\right)_1^2}; \quad \text{we find } \left(\frac{\omega}{\omega_n}\right)_1^2 = \frac{1}{7}$$

Substituting into Eq. (1), $\delta_m = (15 \text{ mm})\left(1 - \frac{1}{7}\right) = \frac{90}{7} \text{ mm}$

For three collars,

$$(x_m)_3 = \frac{\delta_m}{1 - 3\left(\frac{\omega}{\omega_n}\right)_1^2} = \frac{\left(\frac{90}{7}\right) \text{ mm}}{1 - 3\left(\frac{1}{7}\right)} = \frac{90}{4} \text{ mm}, \qquad (x_m)_3 = 22.5 \text{ mm} ◀$$

2. *Assuming* $(x_m)_2 < 0$:

For two collars, we have $-18 \text{ mm} = \dfrac{\delta_m}{1 - 2\left(\frac{\omega}{\omega_n}\right)_1^2}$ (3)

Dividing Eq. (3) by Eq. (1), member by member:

$$-1.2 = \frac{1 - \left(\frac{\omega}{\omega_n}\right)_1^2}{1 - 2\left(\frac{\omega}{\omega_n}\right)_1^2}$$

$$-1.2 + 2.4\left(\frac{\omega}{\omega_n}\right)_1^2 = 1 - \left(\frac{\omega}{\omega_n}\right)_1^2$$

$$\left(\frac{\omega}{\omega_n}\right)_1^2 = \frac{2.2}{3.4} = \frac{1.1}{1.7}$$

Substitute into Eq. (1), $\delta_m = (15 \text{ mm})\left(1 - \frac{1.1}{1.7}\right) = \frac{9}{1.7} \text{ mm}$

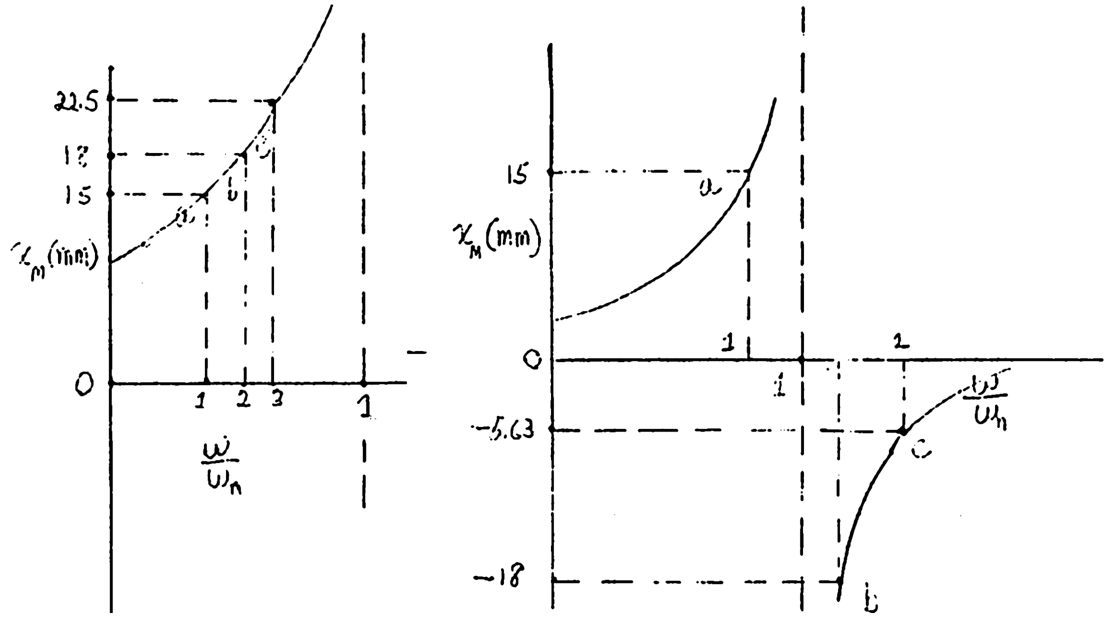
For three collars,

$$(x_m)_3 = \frac{\delta_m}{1 - 3\left(\frac{\omega}{\omega_n}\right)_1^2} = \frac{\left(\frac{9}{1.7}\right)}{1 - 3\left(\frac{1.1}{1.7}\right)} = \frac{9 \text{ mm}}{-1.6}, \qquad (x_m)_3 = -5.63 \text{ mm} ◀$$

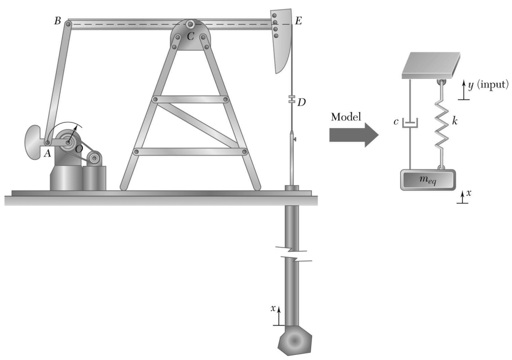
(out of phase)

PROBLEM 19.107 (Continued)

Points corresponding to the two solutions are indicated below:



PROBLEM 19.108



The crude-oil-pumping rig shown in the accompanying figure is driven at 20 rpm. The inside diameter of the well pipe is 2 in., and the diameter of the pump rod is 0.75 in. The length of the pump rod and the length of the column of oil lifted during the stroke are essentially the same, and equal to 6000 ft. During the downward stroke, a valve at the lower end of the pump rod opens to let a quantity of oil into the well pipe, and the column of oil is then lifted to obtain a discharge into the connecting pipeline. Thus, the amount of oil pumped in a given time depends upon the stroke of the lower end of the pump rod. Knowing that the upper end of the rod at *D* is essentially sinusoidal with a stroke of 45 in. and the specific weight of crude oil is 56.2 lb/ft^3, determine (*a*) the output of the well in ft^3/min if the shaft is rigid, (*b*) the output of the well in ft^3/min if the stiffness of the rod is 2210 N/m, the equivalent mass of the oil and shaft is 290 kg and damping is negligible.

SOLUTION

Forcing frequency: $\omega_f = 20 \text{ rpm} = 2.0944 \text{ rad/s}$

Cross sectional area of the flow chamber

$$A_{\text{oil}} = \frac{\pi}{4}\left[(2 \text{ in.})^2 - (0.75 \text{ in.})^2\right] = 2.6998 \text{ in}^2 = 0.018749 \text{ ft}^2$$

Let s be the stroke at the lower end of the pump in feet. Stroke is twice the amplitude. $s = 2x_m$

Volume of oil pumped per revolution:

$$V_{\text{oil}} = A_{\text{oil}} s = 0.018749s$$

Amplitude of motion at top of shaft:

$$\delta_m = \frac{1}{2}(45 \text{ in.}) = 22.5 \text{ in.} = 1.875 \text{ ft}$$

Amplitude of motion at bottom of shaft:

$$x_m = \frac{\delta_m}{1 - \left(\frac{\omega_f}{\omega_n}\right)^2}$$

(*a*) Rigid shaft:

$$\omega_n = \infty$$

$$x_m = \delta_m = 1.875 \text{ ft}$$

$$s = (2)(1.875) = 3.75 \text{ ft}$$

$$V_{\text{oil}} = (0.018749 \text{ ft}^2)(3.75 \text{ ft}) = 0.070309 \text{ ft}^3/\text{rev}$$

PROBLEM 19.108 (Continued)

output rate: $(0.070309 \text{ ft}^3/\text{rev})(20 \text{ rev/min})$ $1.406 \text{ ft}^3/\text{min}$ ◀

(*b*) Flexible shaft.

$$k = 2210 \text{ N/m} \qquad m_{\text{eq}} = 290 \text{ kg}$$

$$\omega_n = \sqrt{\frac{k}{m_{\text{eq}}}} = \sqrt{\frac{2210 \text{ N/m}}{290 \text{ kg}}} = 2.7606 \text{ rad/s}$$

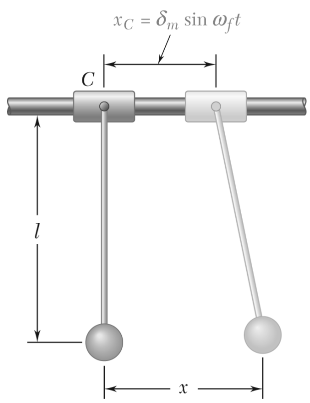
$$\frac{\omega_f}{\omega_n} = \frac{2.0944}{2.7606} = 0.75869$$

$$x_m = \frac{1.875}{1-(0.75869)^2} = 4.4178 \text{ ft}$$

$$s = (2)(4.4178) = 8.8358 \text{ ft}$$

$$V_{\text{oil}} = (0.018749 \text{ ft}^2)(8.8358 \text{ ft}) = 0.16566 \text{ ft}^3/\text{rev}$$

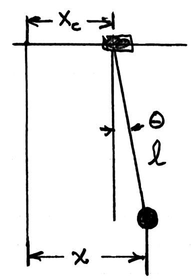
output rate: $(0.16566 \text{ ft}^3/\text{rev})(20 \text{ rev/min})$ $3.31 \text{ ft}^3/\text{min}$ ◀



PROBLEM 19.109

A simple pendulum of length l is suspended from a collar C which is forced to move horizontally according to the relation $x_C = \delta_m \sin \omega_f t$. Determine the range of values of ω_f for which the amplitude of the motion of the bob is less than δ_m. (assume that δ_m is small compared with the length l of the pendulum).

SOLUTION



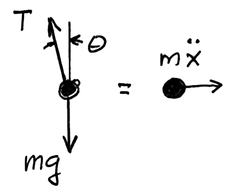
<u>Geometry</u>.

$$x = x_C + l \sin\theta$$

$$\sin\theta = \frac{x - x_C}{l}$$

$$+\uparrow \Sigma F_y = ma_y \approx 0: \quad T\cos\theta - mg = 0 \qquad T \approx mg$$

$$\xrightarrow{+} \Sigma F_x = ma_x: \qquad -T\sin\theta = m\ddot{x}$$



$$m\ddot{x} + \frac{mg(x - x_C)}{l} = 0$$

$$\ddot{x} + \frac{g}{l}x = \frac{g}{l}x_C$$

Using the given motion of x_C,

$$\ddot{x} + \frac{g}{l}x = \frac{g}{l}\delta_m \sin\omega_f t$$

<u>Circular natural frequency</u>.

$$\omega_n = \sqrt{\frac{g}{l}}$$

$$\ddot{x} + \omega_n^2 x = \omega_n^2 \delta_m \sin\omega_f t$$

The steady state response is

$$x_m = \frac{\delta_m}{1 - \left(\frac{\omega_f}{\omega_n}\right)^2}$$

$$x_m^2 = \frac{\delta_m^2}{\left[1 - \left(\frac{\omega_f}{\omega_n}\right)^2\right]^2} \le \delta_m^2$$

Consider $\quad x_m^2 = \delta_m^2.$

PROBLEM 19.109 (Continued)

Then $$\left[1-\left(\frac{\omega_f}{\omega_n}\right)^2\right]^2 = 1-2\left(\frac{\omega_f}{\omega_n}\right)^2+\left(\frac{\omega_f}{\omega_n}\right)^4 = 1$$

$$\left(\frac{\omega_f}{\omega_n}\right)^2 = 0 \quad \text{and} \quad \left(\frac{\omega_f}{\omega_n}\right)^2 = 2$$

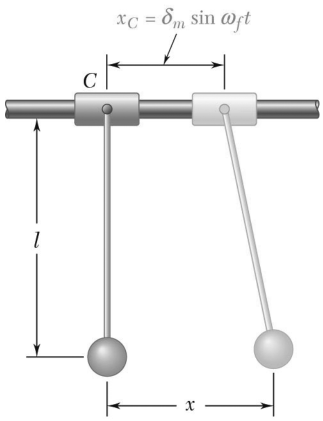
$$\left(\frac{\omega_f}{\omega_n}\right) = 0 \quad \text{and} \quad \frac{\omega_f}{\omega_n} = \sqrt{2}$$

For $$0 < \frac{\omega_f}{\omega_n} < \sqrt{2}, \quad |x_m| > \delta_m$$

For $$\frac{\omega_f}{\omega_n} > \sqrt{2}, \quad |x_m| < \delta_n$$

Then $$\omega_f > \sqrt{2}\,\omega_n = \sqrt{\frac{2g}{l}}$$

$$\omega_f > \sqrt{\frac{2g}{l}} \blacktriangleleft$$

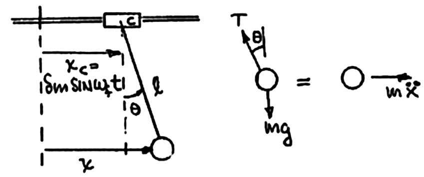


PROBLEM 19.110

The 2.75-lb bob of a simple pendulum of length $l = 24$ in. is suspended from a 3-lb collar C. The collar is forced to move according to the relation $x_C = \delta_m \sin \omega_f t$, with an amplitude $\delta_m = 0.4$ in. and a frequency $f_f = 0.5$ Hz. Determine (*a*) the amplitude of the motion of the bob, (*b*) the force that must be applied to collar C to maintain the motion.

SOLUTION

(*a*)



$$\Sigma F_x = ma_x$$
$$-T \sin\theta = m\ddot{x}$$
$$\Sigma F_y = T\cos\theta - mg = 0$$

For small angles $\cos\theta \approx 1$. Acceleration in the y direction is second order and is neglected.

$$T = mg$$
$$m\ddot{x} = -mg\sin\theta$$
$$\sin\theta = \frac{x - x_c}{l}$$
$$m\ddot{x} + \frac{mg}{l}x = \frac{g}{l}x_c = \frac{mg}{l}\delta_m \sin\omega_f t$$
$$\omega_n^2 = \frac{g}{l}$$
$$\ddot{x} + \omega_n^2 x = \omega_n^2 \delta_m \sin\omega_f t$$

From Equation (19.33′):

$$x_m = \frac{\delta_m}{1 - \frac{\omega_f^2}{\omega_n^2}}$$

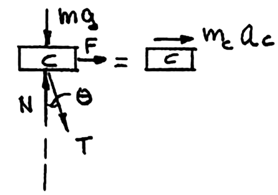
So $$\omega_f^2 = (2\pi f_f)^2 = 4\pi^2(0.5)^2 = \pi^2 \ s^{-2}$$

$$\omega_n^2 = \frac{g}{l} = \frac{32.2 \text{ ft/s}^2}{2 \text{ ft}} = 16.1 \text{ s}^{-2}$$

$$x_m = \frac{\frac{0.4}{12}\text{ ft}}{1 - \frac{\pi^2}{16.1}} = 0.086137 \text{ ft}$$

$x_m = 1.034$ in. ◀

(*b*)



$$a_c = \ddot{x}_c = -\delta_m \omega_f^2 \sin \omega_f t$$

$$\xrightarrow{+} \Sigma F_x = m_c a_c$$

$$F - T \sin\theta = m_c a_c$$

From Part (*a*): $T = mg, \quad \sin\theta = \dfrac{x - x_c}{l}$

Thus, $F = -mg\left[\dfrac{x - x_c}{l}\right] + m_c \ddot{x}_c$

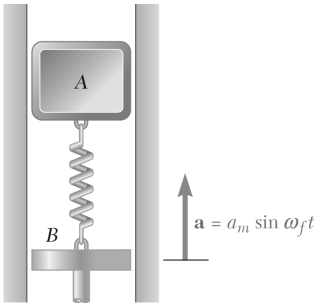
$$= -m\omega_n^2 x + m\omega_n^2 x_c + m_c \ddot{x}_c$$

$$= -m\omega_n^2 x_m \sin \omega_f t + m\omega_n^2 \delta_m \sin \omega_f t - m_c \omega_f^2 \delta_m \sin \omega_f t$$

$$= \left[-\left(\frac{2.75 \text{ lb}}{32.2}\right)(16.1)(0.086137) + \left(\frac{2.75 \text{ lb}}{32.2}\right)(16.1)\left(\tfrac{0.4}{12}\right) - \left(\frac{3 \text{ lb}}{32.2}\right)\pi^2\left(\tfrac{0.4}{12}\right)\right]\sin \pi t$$

$$= -0.10326 \sin \pi t$$

$F = -0.1033 \sin \pi t$ (lb) ◀



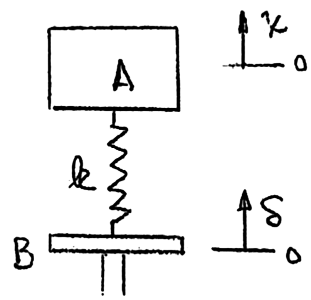
PROBLEM 19.111

An 18-lb block A slides in a vertical frictionless slot and is connected to a moving support B by means of a spring AB of constant $k = 8$ lb/ft. Knowing that the acceleration of the support is $a = a_m \sin \omega_f t$, where $a_m = 5$ ft/s^2 and $\omega_f = 6$ rad/s, determine (a) the maximum displacement of block A, (b) the amplitude of the fluctuating force exerted by the spring on the block.

SOLUTION

(a) Support motion.

$$a = \ddot{\delta} = a_m \sin \omega_f t$$



$$\delta = -\left(\frac{a_m}{\omega_f^2}\right)\sin \omega_f t$$

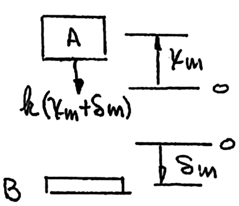
$$\delta_m = \frac{-a_m}{\omega_f^2} = -\frac{5 \text{ ft/s}^2}{(6 \text{ rad/s})^2} = -0.13889 \text{ ft}$$

From Equations (19.31 and 19.33′):

$$x_m = \frac{\delta_m}{1 - \frac{\omega_f^2}{\omega_n^2}} \qquad \omega_n^2 = \frac{k}{m} = \frac{8 \text{ lb/ft}}{\frac{18}{32.2}} = 14.311 \text{ (rad/s)}^2$$

$$x_m = \frac{-0.13889}{1 - \left(\frac{36}{14.311}\right)} = 0.091643 \text{ ft}$$

$x_m = 1.100$ in. ◀



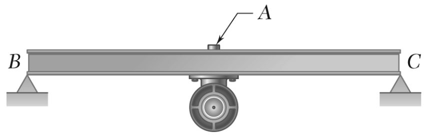
(b) x is out of phase with δ for $\omega_f = 6$ rad/s.

Thus,

$$F_m = k(x_m + \delta_m) = 8 \text{ lb/ft}(0.091643 \text{ ft} + 0.13889 \text{ ft})$$

$$= 1.8443 \text{ lb}$$

$F_m = 1.844$ lb ◀



PROBLEM 19.112

A variable-speed motor is rigidly attached to a beam *BC*. When the speed of the motor is less than 600 rpm or more than 1200 rpm, a small object placed at *A* is observed to remain in contact with the beam. For speeds between 600 and 1200 rpm the object is observed to “dance” and actually to lose contact with the beam. Determine the speed at which resonance will occur.

SOLUTION

Let m be the unbalanced mass and $\bar{r}$ the eccentricity of the unbalanced mass. The vertical force exerted on the beam due to the rotating unbalanced mass is

$$P = m\bar{r}\omega_f^2 \sin\omega_f t = P_m \sin\omega_f t$$

Then from Eq. 19.33, $$x_m = \frac{\frac{P_m}{k}}{1-\left(\frac{\omega_f}{\omega_n}\right)^2} = \frac{\frac{m\bar{r}\omega_f^2}{k}}{1-\left(\frac{\omega_f}{\omega_n}\right)^2}$$

For simple harmonic motion, the acceleration is

$$a_m = -\omega_f^2 x_m = \frac{\frac{m\bar{r}\omega_f^4}{k}}{1-\left(\frac{\omega_f}{\omega_n}\right)^2}$$

When the object loses contact with the beam, the acceleration $|a_m|$ is greater than g.
Let $\omega_1 = 600 \text{ rpm} = 62.832 \text{ rad/s}$.

$$|a_m|_1 = \frac{\frac{m\bar{r}\omega_1^4}{k}}{1-\left(\frac{\omega_1}{\omega_n}\right)^2} = \frac{\frac{mr\omega_n^4 U^4}{k}}{1-U^2} \qquad (1)$$

where $$U = \frac{\omega_1}{\omega_n}.$$

Let $$\omega_2 = 1200 \text{ rpm} = 125.664 \text{ rad/s} = 2\omega_1$$

$$|a_m|_2 = \frac{\frac{m\bar{r}\omega_2^4}{k}}{\left(\frac{\omega_2}{\omega_n}\right)^2 - 1} = \frac{\frac{mr\omega_n^4 (2U)^4}{k}}{4U^2 - 1} \qquad (2)$$

PROBLEM 19.112 (Continued)

Dividing Eq. (1) by Eq. (2),

$$1 = \frac{4U^2 - 1}{16(1 - U^2)} \quad \text{or} \quad 16 - 16U^2 = 4U^2 - 1$$

$$20U^2 = 17 \qquad U = \sqrt{\frac{17}{20}}$$

$$\frac{\omega_1}{\omega_n} = \sqrt{\frac{17}{20}} \qquad \omega_n = \sqrt{\frac{20}{17}}\omega_1 = 1.08465\,\omega_1$$

$$\omega_n = (1.08465)(600 \text{ rpm}) \qquad \omega_n = 651 \text{ rpm} \blacktriangleleft$$

PROBLEM 19.113

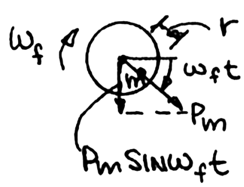
A motor of mass M is supported by springs with an equivalent spring constant k. The unbalance of its rotor is equivalent to a mass m located at a distance r from the axis of rotation. Show that when the angular velocity of the motor is ω_f, the amplitude x_m of the motion of the motor is

$$x_m = \frac{r\left(\frac{m}{M}\right)\left(\frac{\omega_f}{\omega_n}\right)^2}{1-\left(\frac{\omega_f}{\omega_n}\right)^2}$$

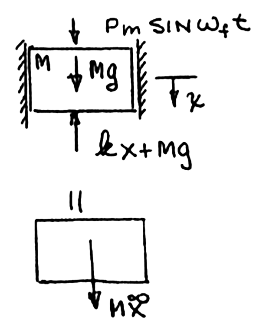
where $\omega_n = \sqrt{\frac{k}{M}}$.

SOLUTION

Rotor



Motor



$$+\downarrow \Sigma F = ma \quad P_m \sin \omega_f t - kx = M\ddot{x}$$

$$M\ddot{x} + kx = P_m \sin \omega_f t$$

$$\ddot{x} + \frac{k}{M}x = \frac{P_m}{M}\sin \omega_f t$$

$$\omega_n^2 = \frac{k}{M}$$

From Equation (19.33):

$$x_m = \frac{\frac{P_m}{k}}{1-\left(\frac{\omega_f}{\omega_n}\right)^2}$$

But

$$\frac{P_m}{k} = \frac{mr\omega_f^2}{k} \qquad k = M\omega_n^2$$

$$\frac{P_m}{k} = r\left(\frac{m}{M}\right)\left(\frac{\omega_f}{\omega_n}\right)^2$$

Thus,

$$x_m = \frac{r\left(\frac{m}{M}\right)\left(\frac{\omega_f}{\omega_n}\right)^2}{1-\left(\frac{\omega_f}{\omega_n}\right)^2} \quad \text{Q.E.D.} \blacktriangleleft$$

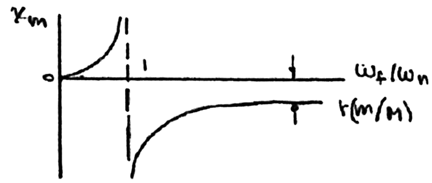
PROBLEM 19.114

As the rotational speed of a spring-supported 100-kg motor is increased, the amplitude of the vibration due to the unbalance of its 15-kg rotor first increases and then decreases. It is observed that as very high speeds are reached, the amplitude of the vibration approaches 3.3 mm. Determine the distance between the mass center of the rotor and its axis of rotation. (*Hint:* Use the formula derived in Problem 19.113.)

SOLUTION

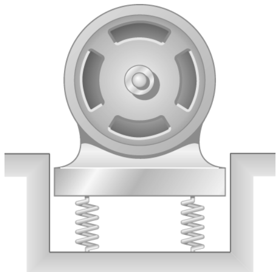
Use the equation derived in Problem 19.113.

$$x_m = \frac{r\left(\frac{m}{M}\right)\left(\frac{\omega_f}{\omega_n}\right)^2}{1-\left(\frac{\omega_f}{\omega_n}\right)^2} = \frac{r\left(\frac{m}{M}\right)}{\frac{1}{\left(\frac{\omega_f}{\omega_n}\right)^2}-1}$$



For very high speeds, $\frac{1}{\left(\frac{\omega_f}{\omega_n}\right)^2} \longrightarrow 0$ and $x_m \longrightarrow \frac{rm}{M}$,

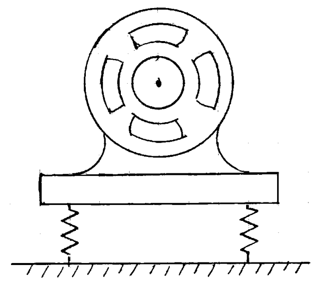
thus, $3.3 \text{ mm} = r\left(\frac{15}{100}\right)$ $r = 22 \text{ mm}$ ◀



PROBLEM 19.115

A motor of weight 40 lb is supported by four springs, each of constant 225 lb/in. The motor is constrained to move vertically, and the amplitude of its motion is observed to be 0.05 in. at a speed of 1200 rpm. Knowing that the weight of the rotor is 9 lb, determine the distance between the mass center of the rotor and the axis of the shaft.

SOLUTION



W = 40 lb

Four springs each of constant 225 lb/in.

We note that the motor is constrained to move vertically.

$$4(225 \text{ lb/in.}) = 900 \text{ lb/in.} = 10800 \text{ lb/ft}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{10800 \text{ lb/ft}}{(40 \text{ lb}/32.2)}} = 93.242 \text{ rad/s}$$

For $\omega = 1200 \text{ rpm} = 125.664 \text{ rad/s}$ we have

$$x_m = 0.05 \text{ in.} = 4.1667 \times 10^{-3} \text{ ft}$$

Eq. (19.33):

$$x_m = \frac{\frac{P_m}{k}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}$$

Thus:

$$\frac{P_m}{k} = x_m\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]$$

$$= (4.1667 \times 10^{-3} \text{ ft})\left[1 - \left(\frac{125.664}{93.242}\right)^2\right] = -3.4015 \times 10^{-3} \text{ ft} \text{ (out of phase)}$$

$$P_m = (10800 \text{ lb/ft})(3.4015 \times 10^{-3} \text{ ft}) = 36.736 \text{ lb}$$

We have found: $P_m = 36.736$ lb

For an unbalanced rotor of weight $W_R = 9$ lb, rotating at $\omega = 1200 \text{ rpm} = 125.664 \text{ rad/s}$, with the mass center at a distance $\bar{r}$ from the axis of rotation, we have,

$$P_m = m_R \bar{r} \omega^2$$

$$\bar{r} = \frac{P_m}{m_R \omega^2} = \frac{36.736 \text{ lb}}{(9 \text{ lb}/32.2)(125.664 \text{ rad/s})^2} = 8.3231 \times 10^{-3} \text{ ft}$$

$\bar{r} = 0.0999$ in. ◀

PROBLEM 19.116

A motor weighing 400 lb is supported by springs having a total constant of 1200 lb/in. The unbalance of the rotor is equivalent to a 1-oz weight located 8 in. from the axis of rotation. Determine the range of allowable values of the motor speed if the amplitude of the vibration is not to exceed 0.06 in.

SOLUTION

Let M = mass of motor, m = unbalance mass, r = eccentricity

$$M = \frac{400}{32.2} = 12.4224 \text{ lb}\cdot\text{s}^2/\text{ft}$$

$$m = \left(\frac{1}{16}\right)\left(\frac{1}{32.2}\right) = 0.001941 \text{ lb}\cdot\text{s}^2/\text{ft}$$

$$r = 8 \text{ in.} = 0.66667 \text{ ft} \quad k = 1200 \text{ lb/in.} = 14,400 \text{ lb/ft}$$

Natural circular frequency: $\omega_n = \sqrt{\dfrac{k}{M}} = \sqrt{\dfrac{14,400}{12.4224}} = 34.047 \text{ rad/s}$

$$\frac{rm}{M} = \frac{(0.66667)(0.001941)}{12.4224}$$

$$= 0.00014017 \text{ ft} = 0.00125 \text{ in.}$$

From the derivation given in Problem 19.113,

$$x_m = \frac{\left(\frac{rm}{M}\right)\left(\frac{\omega_f}{\omega_n}\right)^2}{1-\left(\frac{\omega_f}{\omega_n}\right)^2} = \frac{0.00125\left(\frac{\omega_f}{\omega_n}\right)^2}{1-\left(\frac{\omega_f}{\omega_n}\right)^2} \text{ in.}$$

In phase motion with $|x_m| < 0.06$ in.

$$\frac{0.00125\left(\frac{\omega_f}{\omega_n}\right)^2}{1-\left(\frac{\omega_f}{\omega_n}\right)^2} < 0.06$$

$$0.00125\left(\frac{\omega_f}{\omega_n}\right)^2\left(\frac{\omega_f}{\omega_n}\right)^2 < 0.06 - 0.06\left(\frac{\omega_f}{\omega_n}\right)^2$$

$$0.06125\left(\frac{\omega_f}{\omega_n}\right)^2 < 0.06$$

$$\frac{\omega_f}{\omega_n} < \sqrt{\frac{0.06}{0.06125}} = 0.98974$$

$\omega_f < (0.98974)(34.047) = 33.698 \text{ rad/s}$ $\qquad \omega_f < 322 \text{ rpm}$ ◀

PROBLEM 19.116 (Continued)

Out of phase motion with $|x_m| = 0.06$ in.

$$\frac{0.00125\left(\frac{\omega_f}{\omega_n}\right)^2}{\left(\frac{\omega_f}{\omega_n}\right)^2 - 1} < 0.06$$

$$0.00125\left(\frac{\omega_f}{\omega_n}\right)^2 < 0.06\left(\frac{\omega_f}{\omega_n}\right)^2 - 0.06$$

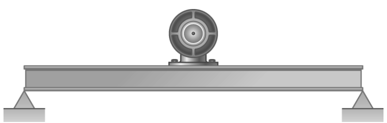
$$0.06 < 0.05875\left(\frac{\omega_f}{\omega_n}\right)^2$$

$$\frac{\omega_f}{\omega_n} > \sqrt{\frac{0.06}{0.05875}} = 1.01058$$

$$\omega_f > (1.01058)(34.047) = 34.407 \text{ rad/s}$$

$\omega_f > 329$ rpm ◀

PROBLEM 19.117



A 180-kg motor is bolted to a light horizontal beam. The unbalance of its rotor is equivalent to a 28-g mass located 150 mm from the axis of rotation, and the static deflection of the beam due to the weight of the motor is 12 mm. The amplitude of the vibration due to the unbalance can be decreased by adding a plate to the base of the motor. If the amplitude of vibration is to be less than 60 μm for motor speeds above 300 rpm, determine the required mass of the plate.

SOLUTION

Before the plate is added, $M_1 = 180\text{ kg},\quad m = 28\times10^{-3}\text{ kg}$

$$r = 150\text{ mm} = 0.150\text{ m}$$

Equivalent spring constant: $k = \dfrac{W_1}{\delta_{st}} = \dfrac{M_1 g}{\delta_{st}}$

$$k = \frac{(180)(9.81)}{12\times10^{-3}} = 147.15\times10^3\text{ N/m}$$

Let M_2 be the mass of motor plus the plate.

Natural circular frequency. $\omega_n = \sqrt{\dfrac{k}{M_2}}$

Forcing frequency: $\omega_f = 300\text{ rpm} = 31.416\text{ rad/s}$

$$\left(\frac{\omega_f}{\omega_n}\right)^2 = \frac{\omega_f^2 M_2}{k} = \frac{(31.416)^2 M_2}{147.15\times10^3} = 0.006707 M_2$$

From the derivation in Problem 19.113,

$$x_m = \frac{\left(\frac{rm}{M_2}\right)\left(\frac{\omega_f}{\omega_n}\right)^2}{1-\left(\frac{\omega_f}{\omega_n}\right)^2}$$

For out of phase motion with $x_m = -60\times10^{-6}$ m,

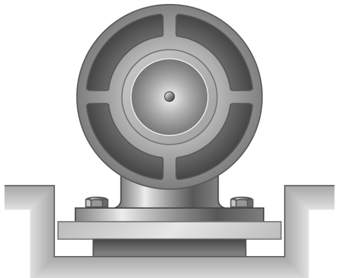
$$-60\times10^{-6} = \frac{\left[\frac{(0.150)(28\times10^{-3})}{M_2}\right](0.006707M_2)}{1-0.006707M_2}$$

$$-60\times10^{-6} + (60\times10^{-6})(0.006707)M_2 = 28.170\times10^{-6}$$

$$402.49\times10^{-9} M_2 = 88.170\times10^{-6}$$

$$M_2 = 219.10\text{ kg}$$

Added mass: $\Delta M = M_2 - M_1 = 219.10 - 180$ $\qquad \Delta M = 39.1\text{ kg}$ ◀



PROBLEM 19.118

The unbalance of the rotor of a 400-lb motor is equivalent to a 3-oz weight located 6 in. from the axis of rotation. In order to limit to 0.2 lb the amplitude of the fluctuating force exerted on the foundation when the motor is run at speeds of 100 rpm and above, a pad is to be placed between the motor and the foundation. Determine (*a*) the maximum allowable spring constant *k* of the pad, (*b*) the corresponding amplitude of the fluctuating force exerted on the foundation when the motor is run at 200 rpm.

SOLUTION

Mass of motor. $M = \dfrac{400}{32.2} = 12.422 \text{ lb}\cdot\text{s}^2/\text{ft}$

Unbalance mass. $m = \left(\dfrac{3}{16}\right)\left(\dfrac{1}{32.2}\right) = 0.005823 \text{ lb}\cdot\text{s}^2/\text{ft}$

Eccentricity. $r = 6 \text{ in.} = 0.5 \text{ ft}$

Equation of motion: $M\ddot{x} + kx = P_m \sin\omega_f t = mr\omega_f^2 \sin\omega_f t$

$$(-M\omega_f^2 + k)x_m = mr\omega_f^2$$

$$x_m = \frac{mr\omega_f^2}{k - M\omega_f^2}$$

Transmitted force. $F_m = kx_m = \dfrac{kmr\omega_f^2}{k - M\omega_f^2}$

For out of phase motion, $|F_m| = \dfrac{kmr\omega_f^2}{M\omega_f^2 - k}$ (1)

(*a*) Required value of *k*.

Solve Eq. (1) for *k*. $|F_m|(M\omega_f^2 - k) = kmr\omega_f^2$

$$k(mr\omega_f^2 + |F_m|) = |F_m| M\omega_f^2$$

$$k = \frac{|F_m| M\omega_f^2}{mr\omega_f^2 + |F_m|}$$

Data: $|F_m| = 0.2 \text{ lb}$ $\omega_f = 100 \text{ rpm} = 10.472 \text{ rad/s}$

$$k = \frac{(0.2)(12.422)(10.472)^2}{(0.005823)(0.5)(10.472)^2 + 0.2} = 524.65$$

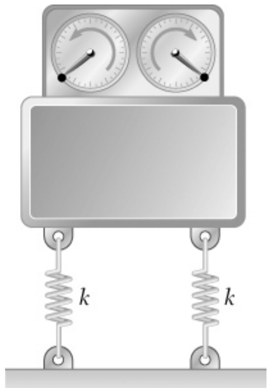
$k = 525 \text{ lb/ft}$ ◀

(*b*) <u>Force amplitude at 200 rpm</u>. $\omega_f = 20.944$ rad/s

From Eq. (1), $$|F_m| = \frac{(524.65)(0.005823)(0.5)(20.944)^2}{(12.422)(20.944)^2 - 524.65}$$

$$|F_m| = 0.1361 \text{ lb} \blacktriangleleft$$

PROBLEM 19.119



A counter-rotating eccentric mass exciter consisting of two rotating 100-g masses describing circles of radius r at the same speed but in opposite senses is placed on a machine element to induce a steady-state vibration of the element. The total mass of the system is 300 kg, the constant of each spring is $k = 600$ kN/m, and the rotational speed of the exciter is 1200 rpm. Knowing that the amplitude of the total fluctuating force exerted on the foundation is 160 N, determine the radius r.

SOLUTION

$$P_m = 2mr\omega_f^2, \qquad x_m = \frac{\frac{2mr\omega_f^2}{2k}}{1-\left(\frac{\omega_f}{\omega_n}\right)^2}, \qquad \omega_n^2 = \frac{2k}{M}$$

With

$$2kx_m = 160 \text{ N} = \pm\frac{2mr\omega_f^2}{1-\frac{M\omega_f^2}{2k}}, \qquad \omega_f = 40\pi \text{ rad/s}$$

Solving for r,

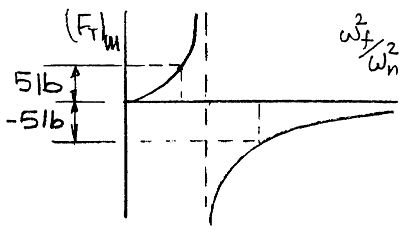
$$r = \pm\frac{160 \text{ N}\left[1-\frac{(300 \text{ kg})(40\pi \text{ s}^{-1})^2}{1200000 \text{ N/m}}\right]}{2(0.1 \text{ kg})(40\pi \text{ s}^{-1})^2} = 0.1493 \text{ m}$$

$r = 149.3$ mm ◀

PROBLEM 19.120

A 360-lb motor is supported by springs of total constant 12.5 kips/ft. The unbalance of the rotor is equivalent to a 0.9-oz weight located 7.5 in. from the axis of rotation. Determine the range of speeds of the motor for which the amplitude of the fluctuating force exerted on the foundation is less than 5 lb.

SOLUTION



From Problem 19.113
$$x_m = \frac{r\left(\frac{m}{M}\right)\left(\frac{\omega_f}{\omega_n}\right)^2}{1-\left(\frac{\omega_f}{\omega_n}\right)^2}$$

And $(F_T)_m = kx_m, \quad \frac{k}{M} = \omega_n^2, \quad (F_T)_m = \frac{rm\omega_f^2}{\left[1-\left(\frac{\omega_f}{\omega_n}\right)^2\right]}$

Then
$$rm = \left(\frac{7.5}{12}\text{ ft}\right)\left(\frac{\left(\frac{0.9}{16}\text{ lb}\right)}{32.2\text{ ft/s}^2}\right) = 0.0010918\text{ lb}\cdot\text{s}^2$$

$$\omega_n^2 = \frac{k}{M} = \frac{12500\text{ lb/ft}}{\frac{360\text{ lb}}{32.2\text{ ft/s}^2}} = 1118.1\text{ s}^{-2}$$

$$(F_T)_m = (0.0010918\text{ lb}\cdot\text{s}^2)\frac{\omega_f^2}{1-\frac{\omega_f^2}{1118.1}}$$

or
$$(F_T)_m\left[1-\frac{\omega_f^2}{1118.1}\right] = (0.0010918\text{ lb}\cdot\text{s}^2)\omega_f^2$$

$$(F_T)_m = \left[\frac{(F_T)_m}{1118.1} + (0.0010918)\right]\omega_f^2$$

Then
$$\omega_f^2 = \frac{1118.1(F_T)_m}{(F_T)_m + 1.2207}$$

(*a*) $(F_T)_m = +5: \quad \omega_f^2 = \frac{1118.1(5)}{5+1.2207} = 898.69\text{ s}^{-2},$

$$\omega_f \le 29.978\text{ rad/s}$$

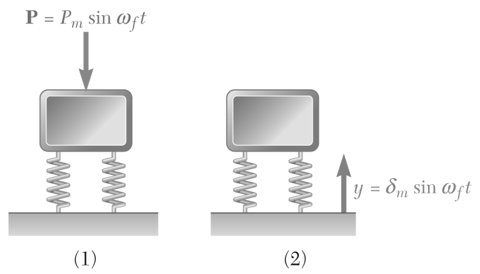
≤ 286.26 rpm $\qquad \omega_f \le 286$ rpm ◀

(*b*) $(F_T)_m = -5: \quad \omega_f^2 = \frac{1118.1\,(-5)}{-5+1.2207} = 1479.2\text{ s}^{-2},$

$$\omega_f > 38.461\text{ rad/s}$$

> 367.27 rpm $\qquad \omega_f > 367$ rpm ◀

PROBLEM 19.121



Figures (1) and (2) show how springs can be used to support a block in two different situations. In Figure (1), they help decrease the amplitude of the fluctuating force transmitted by the block to the foundation. In Figure (2), they help decrease the amplitude of the fluctuating displacement transmitted by the foundation to the block. The ratio of the transmitted force to the impressed force or the ratio of the transmitted displacement to the impressed displacement is called the *transmissibility*. Derive an equation for the transmissibility for each situation. Give your answer in terms of the ratio ω_f / ω_n of the frequency ω_f of the impressed force or impressed displacement to the natural frequency ω_n of the spring-mass system. Show that in order to cause any reduction in transmissibility, the ratio ω_f / ω_n must be greater than $\sqrt{2}$.

SOLUTION

(1) From Equation (19.33):

$$x_m = \frac{\frac{P_m}{k}}{1 - \left(\frac{\omega_f}{\omega_n}\right)^2}$$

Force transmitted:

$$(P_T)_m = kx_m = k\left[\frac{\frac{P_m}{k}}{1 - \left(\frac{\omega_f}{\omega_n}\right)^2}\right]$$

Thus,

$$\text{Transmissibility} = \frac{(P_T)_m}{P_m} = \frac{1}{1 - \left(\frac{\omega_f}{\omega_n}\right)^2} \blacktriangleleft$$

(2) From Equation (19.33′):

Displacement transmitted:

$$x_m = \frac{\delta_m}{1 - \left(\frac{\omega_f}{\omega_n}\right)^2}$$

$$\text{Transmissibility} = \frac{x_m}{\delta_m} = \frac{1}{1 - \left(\frac{\omega_f}{\omega_n}\right)^2} \blacktriangleleft$$

For $\frac{(P_T)_m}{P_m}$ or $\frac{x_m}{\delta_m}$ to be less than 1,

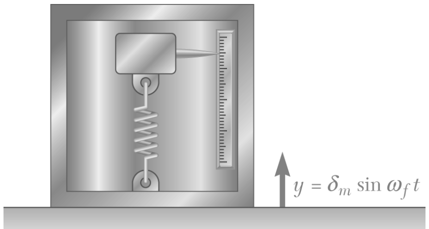
$$\frac{1}{\left|1 - \left(\frac{\omega_f}{\omega_n}\right)^2\right|} < 1$$

$$1 < \left|1 - \left(\frac{\omega_f}{\omega_n}\right)^2\right|$$

$$\left(\frac{\omega_f}{\omega_n}\right)^2 > 2$$

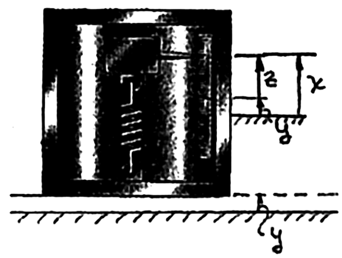
$$\frac{\omega_f}{\omega_n} > \sqrt{2} \quad \text{Q.E.D.} \blacktriangleleft$$

PROBLEM 19.122



A vibrometer used to measure the amplitude of vibrations consists essentially of a box containing a mass-spring system with a known natural frequency of 120 Hz. The box is rigidly attached to a surface, which is moving according to the equation $y = \delta_m \sin \omega_f t$. If the amplitude z_m of the motion of the mass relative to the box is used as a measure of the amplitude δ_m of the vibration of the surface, determine (*a*) the percent error when the frequency of the vibration is 600 Hz, (*b*) the frequency at which the error is zero.

SOLUTION



$$x = \left(\frac{\delta_m}{1 - \frac{\omega_f^2}{\omega_n^2}} \right) \sin \omega_f t$$

$$y = \delta_m \sin \omega_f t$$

$z =$ relative motion

$$z = x - y = \left[\frac{\delta_m}{1 - \frac{\omega_f^2}{\omega_n^2}} - \delta_m \right] \sin \omega_f t$$

$$z_m = \delta_m \left[\frac{1}{1 - \frac{\omega_f^2}{\omega_n^2}} - 1 \right] = \frac{\delta_m \frac{\omega_f^2}{\omega_n^2}}{1 - \frac{\omega_f^2}{\omega_n^2}}$$

(*a*) $$\frac{z_m}{\delta_m} = \frac{\frac{\omega_f^2}{\omega_n^2}}{1 - \frac{\omega_f^2}{\omega_n^2}} = \frac{\left(\frac{600}{120}\right)^2}{1 - \left(\frac{600}{120}\right)^2} = \frac{25}{24} = 1.0417$$ ◀

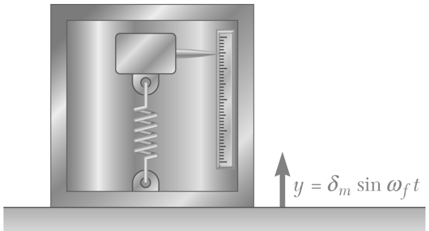
Error $= 4.17\%$

(*b*) $$\frac{z_m}{\delta_m} = 1 = \frac{\frac{\omega_f^2}{\omega_n^2}}{1 - \frac{\omega_f^2}{\omega_n^2}}$$

$$1 = 2 \frac{\omega_f^2}{\omega_n^2}$$

$$f_f = \frac{\sqrt{2}}{2} f_n = \frac{\sqrt{2}}{2}(120) = 84.853 \text{ Hz}$$

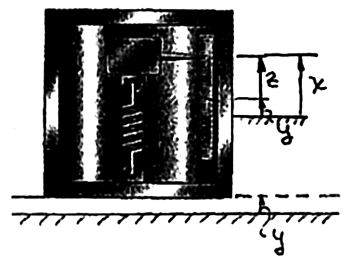
$f_n = 84.9$ Hz ◀



PROBLEM 19.123

A certain accelerometer consists essentially of a box containing a mass-spring system with a known natural frequency of 2200 Hz. The box is rigidly attached to a surface, which is moving according to the equation $y = \delta_m \sin \omega_f t$. If the amplitude z_m of the motion of the mass relative to the box times a scale factor ω_n^2 is used as a measure of the maximum acceleration $\alpha_m = \delta_m \omega_f^2$ of the vibrating surface, determine the percent error when the frequency of the vibration is 600 Hz.

SOLUTION



$$x = \left(\frac{\delta_m}{1 - \frac{\omega_f^2}{\omega_n^2}} \right) \sin \omega_f t$$

$$y = \delta_m \sin \omega_f t$$

$$z = \text{relative motion}$$

$$z = x - y = \left[\frac{\delta_m}{1 - \frac{\omega_f^2}{\omega_n^2}} - \delta_m \right] \sin \omega_f t$$

$$z_m = \delta_m \left[\frac{1}{1 - \frac{\omega_f^2}{\omega_n^2}} - 1 \right] = \frac{\delta_m \frac{\omega_f^2}{\omega_n^2}}{1 - \frac{\omega_f^2}{\omega_n^2}}$$

The actual acceleration is $a_m = -\omega_f^2 \delta_m$

The measurement is proportional to $z_m \omega_n^2$.

Then

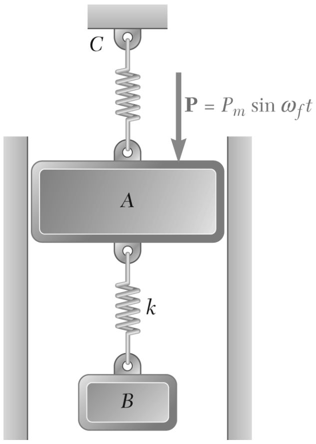
$$\frac{z_m \omega_n^2}{a_m} = \frac{z_m}{\delta_m} \left(\frac{\omega_n}{\omega_f} \right)^2$$

$$= \frac{1}{1 - \left(\frac{\omega_f}{\omega_n} \right)^2}$$

$$= \frac{1}{1 - \left(\frac{600}{2200} \right)^2}$$

$$= 1.0804$$

Error $= 8.04\%$ ◀



PROBLEM 19.124

Block A can move without friction in the slot as shown and is acted upon by a vertical periodic force of magnitude $P = P_m \sin \omega_f t$, where $\omega_f = 2$ rad/s and $P_m = 20$ N. A spring of constant k is attached to the bottom of block A and to a 22-kg block B. Determine (*a*) the value of the constant k which will prevent a steady-state vibration of block A, (*b*) the corresponding amplitude of the vibration of block B.

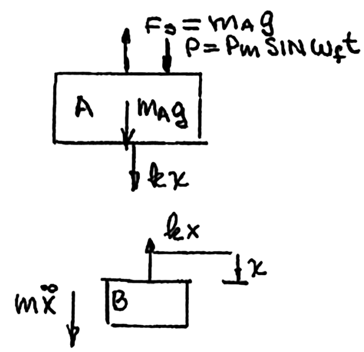
SOLUTION

In steady state vibration, block A does not move and therefore, remains in its original equilibrium position.

Block A: $+\downarrow \Sigma F = 0$

$$kx = -P_m \sin \omega_f t \quad (1)$$

Block B: $+\downarrow \Sigma F = m_B \ddot{x}$



$$m_B \ddot{x} + kx = 0$$

$$x = x_m \sin \omega_n t$$

$$\omega_n^2 = k/m_B$$

From Eq. (1): $kx_m \sin \omega_n t = -P_m \sin \omega_f t$

$$\omega_n = \omega_f = 2 \text{ rad/s}$$

$$kx_m = -P_m$$

$$\omega_n = \sqrt{\frac{k}{m_B}}$$

$$k = m_B \omega_n^2$$

(*a*) Required spring constant. $k = (22)(2)^2$ $\quad k = 88.0$ N/m ◀

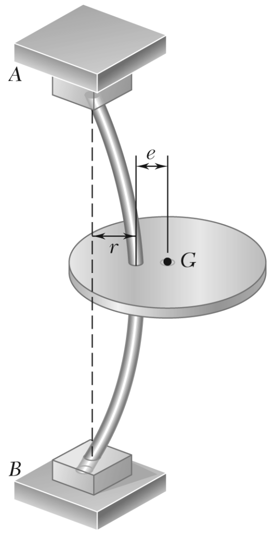
(*b*) Corresponding amplitude of vibration of B.

$$kx_m = -P_m$$

$$x_m = -\frac{P_m}{k}$$

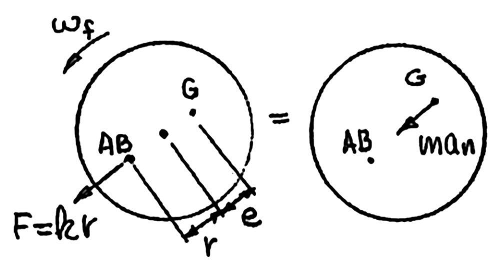
$$x_m = -\frac{20 \text{ N}}{88 \text{ N/m}} \qquad x_m = -0.227 \text{ m} ◀$$

PROBLEM 19.125



A 60-lb disk is attached with an eccentricity $e = 0.006$ in. to the midpoint of a vertical shaft AB, which revolves at a constant angular velocity ω_f. Knowing that the spring constant k for horizontal movement of the disk is 40,000 lb/ft, determine (*a*) the angular velocity ω_f at which resonance will occur, (*b*) the deflection r of the shaft when $\omega_f = 1200$ rpm.

SOLUTION



G describes a circle about the axis AB of radius $r + e$.

Thus, $$a_n = (r+e)\omega_f^2$$

Deflection of the shaft is $$F = kr$$

Thus, $$F = ma_n$$

$$kr = m(r+e)\omega_f^2$$

$$\omega_n^2 = \frac{k}{m} \qquad m = \frac{k}{\omega_n^2}$$

$$\not{k}r = \frac{\not{k}}{\omega_n^2}(r+e)\omega_f^2$$

$$r = \frac{e\frac{\omega_f^2}{\omega_n^2}}{1 - \frac{\omega_f^2}{\omega_n^2}}$$

PROBLEM 19.125 (Continued)

(*a*) Resonance occurs when $\omega_f = \omega_n$, i.e., $r \longrightarrow \infty$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{kg}{W}}$$

$$= \sqrt{\frac{(40{,}000)(32.2)}{60}}$$

$$= 146.52 \text{ rad/s}$$

$$= 1399.1 \text{ rpm}$$

$\omega_n = \omega_f = 1399 \text{ rpm}$ ◀

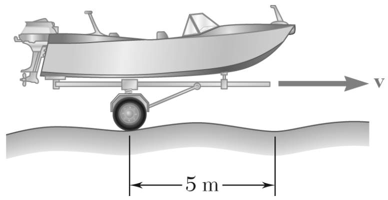
(*b*)

$$r = \frac{(0.006 \text{ in.})\left(\frac{1200}{1399.1}\right)^2}{1-\left(\frac{1200}{1399.1}\right)^2}$$

$$= 0.01670 \text{ in.}$$

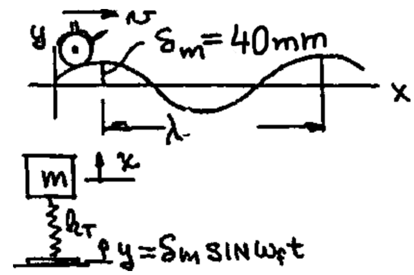
$r = 0.01670 \text{ in.}$ ◀

PROBLEM 19.126



A small trailer and its load have a total mass of 250-kg. The trailer is supported by two springs, each of constant 10 kN/m, and is pulled over a road, the surface of which can be approximated by a sine curve with an amplitude of 40 mm and a wavelength of 5 m (i.e., the distance between successive crests is 5 m and the vertical distance from crest to trough is 80 mm). Determine (*a*) the speed at which resonance will occur, (*b*) the amplitude of the vibration of the trailer at a speed of 50 km/h.

SOLUTION



Total spring constant $k = 2(10\times10^3 \text{ N/m})$

$$= 20\times10^3 \text{ N/m}$$

(*a*) $$\omega_n^2 = \frac{k}{m} = \frac{20\times10^3 \text{ N/m}}{250 \text{ kg}} = 80 \text{ s}^{-2}$$

$$\lambda = 5 \text{ m}$$

$$\delta_m = 40 \text{ mm} = 40\times10^{-3} \text{ m}$$

$$y = \delta_m \sin\frac{2\pi x}{\lambda} \quad \text{where} \quad x = vt$$

$$y = \delta_m \sin\omega_f t$$

$$\omega_f = \frac{2\pi v}{\lambda}$$

$$\delta_m = 40 \text{ mm} = 40\times10^{-3} \text{ m}$$

From Equation (19.33′): $$x_m = \frac{\delta_m}{\left(1 - \frac{\omega_f^2}{\omega_n^2}\right)}$$

Resonance: $$\omega_f = \frac{2\pi v}{5} = \omega_n = \sqrt{80} \text{ s}^{-1},$$

$$v = 7.1176 \text{ m/s}$$

$v = 25.6$ km/h ◀

(*b*) Amplitude at $v = 50 \text{ km/h} = 13.8889 \text{ m/s}$

$$\omega_f = \frac{2\pi(13.8889)}{5} = 17.4533 \text{ rad/s}$$

$$\omega_f^2 = 304.60 \text{ s}^{-2}$$

$$x_m = \frac{40\times10^{-3}}{1 - \frac{304.62}{80}} = -14.246\times10^{-3} \text{ m}$$

$x_m = -14.25$ mm ◀

PROBLEM 19.127

Show that in the case of heavy damping $(c > c_c)$, a body never passes through its position of equilibrium O (*a*) if it is released with no initial velocity from an arbitrary position or (*b*) if it is started from O with an arbitrary initial velocity.

SOLUTION

Since $c > c_c$, we use Equation (19.42), where

$$\lambda_1 < 0, \ \lambda_2 < 0$$

$$x = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \quad (1)$$

$$v = \frac{dx}{dt} = C_1 \lambda_1 e^{\lambda_1 t} + C_2 \lambda_2 e^{\lambda_2 t} \quad (2)$$

(*a*) $t = 0, \ x = x_0, \quad v = 0$:

From Eqs. (1) and (2):

$$x_0 = C_1 + C_2$$
$$0 = C_1 \lambda_1 + C_2 \lambda_2$$

Solving for c_1 and c_2,

$$C_1 = \frac{\lambda_2}{\lambda_2 - \lambda_1} x_0$$

$$C_2 = \frac{-\lambda_1}{\lambda_2 - x_1} x_0$$

Substituting for C_1 and C_2 in Eq. (1), $x = \dfrac{x_2}{\lambda_2 - \lambda_1}\left[\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t}\right]$

For $x = 0$: when $t \neq \infty$, we must have

$$\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t} = 0 \qquad \frac{\lambda_2}{\lambda_1} = e^{(\lambda_2 - \lambda_1)t} \quad (3)$$

Recall that

$\lambda_1 < 0, \ \lambda_2 < 0$. Choosing λ_1 and λ_2 so that $\lambda_1 < \lambda_2 < 0$, we have

$$0 < \frac{\lambda_2}{\lambda_1} < 1 \quad \text{and} \quad \lambda_2 - \lambda_1 > 0$$

Thus a positive solution for $t > 0$ for Equation (3) cannot exist, since it would require that e raised to a positive power be less than 1, which is impossible. Thus, x is never 0.

The $x - t$ curve for this case is as shown.

◀

PROBLEM 19.127 (Continued)

(*b*) $t = 0, \; x = 0, \; v = v_0$: Equations (1) and (2) yield

$$0 = C_1 + C_2$$
$$v_0 = C_1\lambda_1 + C_2\lambda_2$$

Solving for C_1 and C_2,
$$C_1 = -\frac{v_0}{\lambda_2 - \lambda_1}$$
$$C_2 = \frac{v_2}{\lambda_2 - \lambda_1}$$

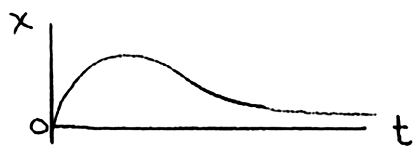
Substituting into Eq. (1),
$$x = \frac{v_0}{\lambda_2 - \lambda_1}[e^{\lambda_2 t} - e^{\lambda_1 t}]$$

For $x = 0$, and $t > 0$

$$e^{\lambda_2 t} = e^{\lambda_1 t}$$

For $c > c_c$, $\lambda_1 \neq \lambda_2$; thus, no solution can exist for t, and x is never 0 when $t > 0$.

The $x-t$ curve for this motion is as shown.



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PROBLEM 19.128

Show that in the case of heavy damping $(c > c_c)$, a body released from an arbitrary position with an arbitrary initial velocity cannot pass more than once through its equilibrium position.

SOLUTION

Substitute the initial conditions, $t = 0$, $x = x_0$, $v = v_0$ in Equations (1) and (2) of Problem 19.127.

$$x_0 = C_1 + C_2 \qquad v_0 = C_1\lambda_1 + C_2\lambda_2$$

Solving for C_1 and C_2,

$$C_1 = -\frac{(v_0 - \lambda_2 x_0)}{\lambda_2 - \lambda_1}$$

$$C_2 = \frac{(v_0 - \lambda_1 x_0)}{\lambda_2 - \lambda_1}$$

And substituting in Eq. (1)

$$x = \frac{1}{\lambda_2 - \lambda_1}\left[(v_0 - \lambda_1 x_0)e^{\lambda_2 t} - (v_0 - \lambda_2 x_0)e^{\lambda_1 t}\right]$$

For $x = 0$, $t \neq \infty$:

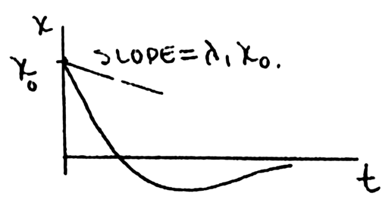
$$(v_0 - \lambda_1 x_0)e^{\lambda_2 t} = (v_0 - \lambda_2 x_0)e^{\lambda_1 t}$$

$$e^{(\lambda_2 - \lambda_1)t} = \frac{(v_0 - \lambda_2 x_0)}{(v_0 - \lambda_1 x_0)}$$

$$t = \frac{1}{(\lambda_2 - \lambda_1)}\ln\frac{v_0 - \lambda_2 x_0}{v_0 - \lambda_1 x_0}$$

This defines one value of t only for $x = 0$, which will exist if the argument of the natural logarithm is positive, i.e., if $\frac{v_0 - \lambda_2 x_0}{v_0 - \lambda_1 x_0} > 1$. Assuming $\lambda_1 < \lambda_2 < 0$,

this occurs if $v_0 < \lambda_1 x_0$.



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PROBLEM 19.129

In the case of light damping, the displacements x_1, x_2, x_3, shown in Figure 19.11 may be assumed equal to the maximum displacements. Show that the ratio of any two successive maximum displacements x_n and x_{n+1} is a constant and that the natural logarithm of this ratio, called the *logarithmic decrement*, is

$$\ln \frac{x_n}{x_{n+1}} = \frac{2\pi(c/c_c)}{\sqrt{1-(c/c_c)^2}}$$

SOLUTION

For light damping,

Equation (19.46): $$x = x_0 e^{-\left(\frac{c}{2m}\right)t} \sin(\omega_0 t + \phi)$$

At given maximum displacement, $$t = t_n, \quad x = x_n$$

$$\sin(\omega_0 t_n + \phi) = 1$$

$$x_n = x_0 e^{-\left(\frac{c}{2m}\right)t_n}$$

At next maximum displacement, $$t = t_{n+1}, \quad x = x_{n+1}$$

$$\sin(\omega_0 t_{n+1} + \phi) = 1$$

$$x_{n+1} = x_0 e^{-\left(\frac{c}{2m}\right)t_{n+1}}$$

But $$\omega_D t_{n+1} - \omega_D t_n = 2\pi$$

$$t_{n+1} - t_n = \frac{2\pi}{\omega_D}$$

Ratio of successive displacements: $$\frac{x_n}{x_{n+1}} = \frac{x_0 e^{-\frac{c}{2m}t_n}}{x_0 e^{-\frac{c}{2m}t_{n+1}}}$$

$$= e^{-\frac{c}{2m}(t_n - t_{n+1})} = e^{+\frac{c}{2m}\frac{2\pi}{\omega_D}}$$

Thus, $$\ln \frac{x_n}{x_{n+1}} = \frac{c\pi}{m\omega_D} \qquad (1)$$

From Equations (19.45) and (19.41): $$\omega_D = \omega_n \sqrt{1 - \left(\frac{c}{c_c}\right)^2}$$

$$\omega_D = \frac{c_c}{2m}\sqrt{1 - \left(\frac{c}{c}\right)^2}$$

Thus, $$\ln \frac{x_n}{x_{n+1}} = \frac{c\pi}{m}\frac{2m}{c_c}\frac{1}{\sqrt{1 - \left(\frac{c}{c_c}\right)^2}} \qquad \ln \frac{x_n}{x_{n+1}} = \frac{2\pi\left(\frac{c}{c_c}\right)}{\sqrt{1 - \left(\frac{c}{c_c}\right)^2}}$$ Q.E.D. ◀

PROBLEM 19.130

In practice, it is often difficult to determine the logarithmic decrement of a system with light damping defined in Problem 19.129 by measuring two successive maximum displacements. Show that the logarithmic decrement can also be expressed as $(1/k)\ln(x_n/x_{n+k})$, where k is the number of cycles between readings of the maximum displacement.

SOLUTION

As in Problem 19.129, for maximum displacements x_n and x_{n+k} at t_n and t_{n+k}, $\sin(\omega_0 t_n + \phi) = 1$

and $\sin(\omega_n t_{n+k} + \phi) = 1$.

$$x_n = x_0 e^{-\left(\frac{c}{2m}\right)t_n}$$

$$x_{n+k} = x_0 e^{-\left(\frac{c}{2m}\right)(t_{n+k})}$$

Ratio of maximum displacements:

$$\frac{x_n}{x_{n+k}} = \frac{x_0 e^{\left(\frac{-c}{2m}\right)t_n}}{x_0 e^{\left(\frac{-c}{2m}\right)t_{n+k}}} = e^{\frac{-c}{2m}(t_n - t_{n+k})}$$

But

$$\omega_D t_{n+k} - \omega_D t_n = k(2\pi)$$

$$t_n - t_{n+k} = k\frac{2\pi}{\omega_D}$$

Thus,

$$\frac{x_n}{x_{n+k}} = +\frac{c}{2m}\left(\frac{2k\pi}{\omega_D}\right)$$

$$\ln\frac{x_n}{x_{n+k}} = k\frac{c\pi}{m\omega_D} \qquad (2)$$

But from Problem 19.129, Equation (1):

$$\log \text{ decrement} = \ln\frac{x_n}{x_{n+1}} = \frac{c\pi}{m\omega_D}$$

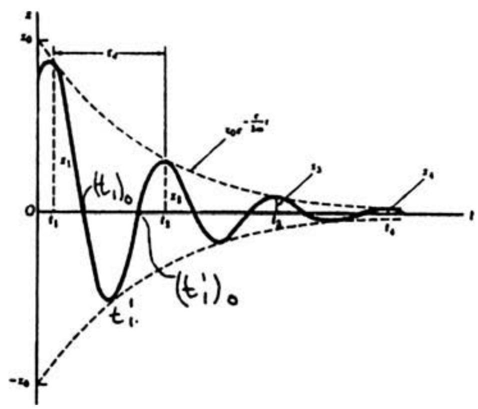
Comparing with Equation (2),

$$\log \text{ decrement} = \frac{1}{k}\ln\frac{x_n}{x_{n+k}} \quad \text{Q.E.D.} \blacktriangleleft$$

PROBLEM 19.131

In a system with light damping ($c < c_c$), the period of vibration is commonly defined as the time interval $\tau_d = 2\pi/\omega_d$ corresponding to two successive points where the displacement-time curve touches one of the limiting curves shown in Figure 19.11. Show that the interval of time (*a*) between a maximum positive displacement and the following maximum negative displacement is $\frac{1}{2}\tau_d$, (*b*) between two successive zero displacements is $\frac{1}{2}\tau_d$, (*c*) between a maximum positive displacement and the following zero displacement is greater than $\frac{1}{4}\tau_d$.

SOLUTION



Equation (19.46): $$x = x_0 e^{-\left(\frac{c}{2m}\right)t}\sin(\omega_d t + \phi)$$

(*a*) Maxima (positive or negative) when $\dot{x} = 0$:

$$\dot{x} = x_0\left(\frac{-c}{2m}\right)e^{-\left(\frac{c}{2m}\right)t}\sin(\omega_d t + \phi) + x_0\omega_d e^{-\left(\frac{c}{2m}\right)t}\cos(\omega_d t + \phi)$$

Thus, zero velocities occur at times when

$$\dot{x} = 0, \quad \text{or} \quad \tan(\omega_d t + \phi) = \frac{2m\omega_d}{c} \qquad (1)$$

The time to the first zero velocity, t_1, is

$$t_1 = \frac{\left[\tan^{-1}\left(\frac{2m\omega_d}{c}\right) - \phi\right]}{\omega_d} \qquad (2)$$

The time to the next zero velocity where the displacement is negative is

$$t_1' = \frac{\left[\tan^{-1}\left(\frac{2m\omega_d}{c}\right) - \phi + \pi\right]}{\omega_d} \qquad (3)$$

Subtracting Eq. (2) from Eq. (3),

$$t_1' - t_1 = \frac{\pi}{\omega_d} = \frac{\pi \cdot \tau_d}{2\pi} = \frac{\tau_d}{2} \quad \text{Q.E.D.}$$

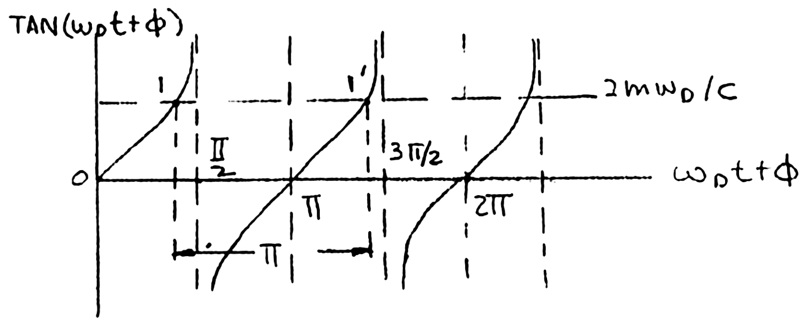
(*b*) Zero displacements occur when

$$\sin(\omega_\theta t + \phi) = 0 \quad \text{or at intervals of}$$

$$\omega_d t + \phi = \pi,\ 2\pi\ n\pi$$

Thus, $$(t_1)_0 = \frac{(\pi - \phi)}{\omega_d} \quad \text{and} \quad (t_1')_0 = \frac{(2\pi - \phi)}{\omega_d}$$

Time between $$0_s' = (t_1')_0 - (t_1)_0 = \frac{2\pi - \pi}{\omega_d} = \frac{\pi\tau_d}{2\pi} = \frac{\tau_d}{2} \quad \text{Q.E.D.}$$



Plot of Equation (1)

(*c*) The first maximum occurs at 1: $(\omega_d t_1 + \phi)$

The first zero occurs at $(\omega_d (t_1)_0 + \phi) = \pi$

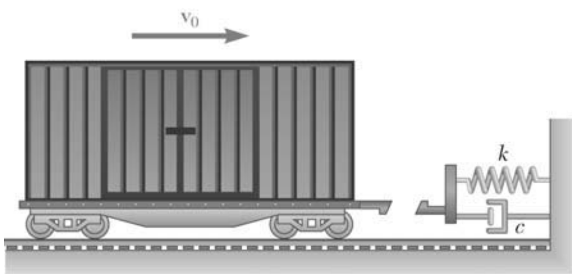
From the above plot, $$(\omega_d (t_1)_0 + \phi) - (\omega_D t_1 + \phi) > \frac{\pi}{2}$$

or $$(t_1)_0 - t_1 > \frac{\pi}{2\omega_d} \quad (t_1)_0 - t_1 > \frac{\tau_d}{4} \quad \text{Q.E.D.}$$

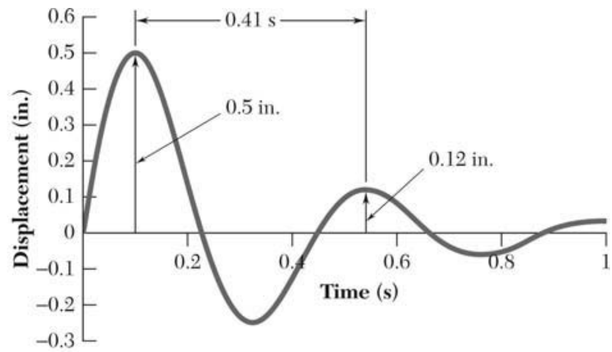
Similar proofs can be made for subsequent maximum and minimum.

PROBLEM 19.132

A loaded railroad car weighing 30,000 lb is rolling at a constant velocity $\mathbf{v}_0$ when it couples with a spring and dashpot bumper system (Figure 1). The recorded displacement-time curve of the loaded railroad car after coupling is as shown (Figure 2). Determine (*a*) the damping constant, (*b*) the spring constant. (*Hint:* Use the definition of logarithmic decrement given in Problem 19.129.)



(1)



(2)

SOLUTION

Mass of railroad car: $$m = \frac{W}{g} = \frac{30,000}{32.2}$$

$$= 931.67 \text{ lb}\cdot\text{s}^2/\text{ft}$$

The differential equation of motion for the system is

$$m\ddot{x} + c\dot{x} + kx = 0$$

For light damping, the solution is given by Eq. (19.44):

$$x = e^{-\left(\frac{c}{2m}\right)t}(C_1 \sin \omega_d t + C_2 \cos \omega_d t)$$

From the displacement versus time curve,

$$\tau_d = 0.41 \text{ s}$$

$$\omega_d = \frac{2\pi}{\tau_d} = \frac{2\pi}{0.41} = 15.325 \text{ rad/s}$$

At the first peak, $x_1 = 0.5$ in. and $t = t_1$.

At the second peak, $x_2 = 0.12$ in. and $t = t_1 + \tau_d$.

Forming the ratio $\frac{x_2}{x_1}$, $$\frac{x_2}{x_1} = \frac{e^{-\left(\frac{c}{2m}\right)(t_1+\tau_d)}}{e^{-\left(\frac{c}{2m}\right)t_1}} = e^{-\left(\frac{c}{2m}\right)\tau_d} \quad (1)$$

$$\frac{x_1}{x_2} = e^{\frac{c\tau_d}{2m}}$$

PROBLEM 19.132 (Continued)

(*a*) Damping constant.

From Eq. (1):

$$\frac{c\tau_d}{2m} = \ln\left(\frac{x_1}{x_2}\right)$$

$$c = \frac{2m}{\tau_d}\ln\frac{x_1}{x_2}$$

$$= \frac{(2)(931.67)}{0.41}\ln\frac{0.5}{0.12}$$

$$= 6485.9 \text{ lb}\cdot\text{s/ft} \qquad c = 6.49 \text{ kip}\cdot\text{s/ft} ◀$$

(*b*) Spring constant.

Equation for ω_d:

$$\omega_d^2 = \frac{k}{m} - \left(\frac{c}{2m}\right)^2$$

$$k = m\omega_d^2 + \frac{c^2}{4m}$$

$$= (931.67)(15.325)^2 + \frac{(6485.9)^2}{(4)(931.67)}$$

$$= 230\times10^3 \text{ lb/ft} \qquad k = 230 \text{ kips/ft} ◀$$

PROBLEM 19.133

A torsional pendulum has a centroidal mass moment of inertia of 0.3 kg-m^2 and when given an initial twist and released is found to have a frequency of oscillation of 200 rpm. Knowing that when this pendulum is immersed in oil and given the same initial condition it is found to have a frequency of oscillation of 180 rpm, determine the damping constant for the oil.

SOLUTION

Let the mass be rotated through the small angle θ from the equilibrium position.

Couples acting on the mass: Shaft: $-K\theta$

Oil: $-C\dot{\theta}$

Equation of motion: $\Sigma M = \bar{I}\ddot{\theta}: -K\theta - C\dot{\theta} = \bar{I}\ddot{\theta}$

$$\bar{I}\ddot{\theta} + C\dot{\theta} + K\theta = 0$$

Solution for light damping: $\theta = e^{-\lambda t}(C_1 \sin \omega_d t + C_2 \sin \omega_d t)$

where $\lambda = \dfrac{C}{2\bar{I}}$

$$\omega_n = \sqrt{\frac{K}{\bar{I}}}$$

$$\omega_d = \sqrt{\omega_n^2 - \lambda^2}$$

When there is no oil, assume $C \approx 0$.

$$\omega_n = 200 \text{ rpm} = 20.944 \text{ rad/s}$$

When oil is present,

$$\omega_d = 180 \text{ rpm} = 18.8496 \text{ rad/s}$$

$$\lambda = \sqrt{\omega_n^2 - \omega_d^2} = 9.1293 \text{ s}^{-1}$$

Damping constant for oil.

$$C = 2I\lambda = (2)(0.3 \text{ kg} \cdot \text{m}^2)(9.1293 \text{ s}^{-1})$$

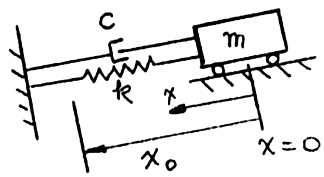
$C = 5.48 \text{ N} \cdot \text{m} \cdot \text{s}$ ◀

PROBLEM 19.134

The barrel of a field gun weighs 1500 lb and is returned into firing position after recoil by a recuperator of constant $c = 1100$ lb·s/ft. Determine (*a*) the constant k which should be used for the recuperator to return the barrel into firing position in the shortest possible time without any oscillation, (*b*) the time needed for the barrel to move back two-thirds of the way from its maximum-recoil position to its firing position.

SOLUTION

(*a*) A critically damped system regains its equilibrium position in the shortest time.



$$c = c_c = 1100 = 2m\sqrt{\frac{k}{m}}$$

Then $k = \dfrac{\left(\frac{c_c}{2}\right)^2}{m} = \dfrac{\left(\frac{1100}{2}\right)^2}{\frac{1500\text{ lb}}{32.2\text{ ft/s}^2}} = 6494$ lb/ft $\qquad k = 6490$ lb/ft ◀

(*b*) For a critically damped system, Equation (19.43):

$$x = (C_1 + C_2 t)e^{-\omega_n t}$$

We take $t = 0$ at maximum deflection x_0.

Thus, $\dot{x}(0) = 0$

$x(0) = x_0$

Using the initial conditions, $x(0) = x_0 = (C_1 + 0)e^0$, so $C_1 = x_0$

$$x = (x_0 + C_2 t)e^{-\omega_n t}$$

and $\dot{x} = -\omega_n (x_0 + C_2 t)e^{-\omega_n t} + C_2 e^{-\omega_n t}$

$\dot{x}(0) = 0 = -\omega_n x_0 + C_2$, so $C_2 = \omega_n x_0$

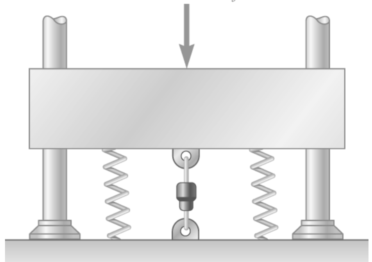
Thus, $x = x_0(1 + \omega_n t)e^{-\omega_n t}$

For $x = \dfrac{x_0}{3}$, $\dfrac{1}{3} = (1 + \omega_n t)e^{-\omega_n t}$

Solving by trial for $\omega_n t$ gives: $\omega_n t = 2.289$

But $\omega_n = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{6494\text{ lb/ft}}{\left(\frac{1500\text{ lb}}{32.2\text{ ft/s}^2}\right)}} = 11.807\text{ s}^{-1}$

Then $t = \dfrac{\omega_n t}{\omega_n} = \dfrac{2.289}{11.807} = 0.19387$ $\qquad t = 0.1939$ s ◀



PROBLEM 19.135

A platform of weight 200 lb, supported by two springs each of constant k = 250 lb/in., is subjected to a periodic force of maximum magnitude equal to 125 lb. Knowing that the coefficient of damping is 12 lb·s/in., determine (*a*) the natural frequency in rpm of the platform if there were no damping, (*b*) the frequency in rpm of the periodic force corresponding to the maximum value of the magnification factor, assuming damping, (*c*) the amplitude of the actual motion of the platform for each of the frequencies found in parts *a* and *b*.

SOLUTION

(*a*) *No Damping*:

$$k = 2(250 \text{ lb/in.}) = 500 \text{ lb/in.} = 6000 \text{ lb/ft}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{6000 \text{ lb/ft}}{(2000 \text{ lb})/(32.2 \text{ ft/s}^2)}} = 31.08 \text{ rad/s}$$

$$f = 4.947 \text{ Hz}$$

$f = 297$ rpm ◀

(*b*) *Damped Motion*:

$$c = 12 \text{ lb}\cdot\text{s/in.} = 144 \text{ lb}\cdot\text{s/ft}$$

$$c_c = 2m\omega_n = (2)\left(\frac{200}{32.2}\right)(31.08) = 386.09 \text{ lb}\cdot\text{s/ft}$$

$$\frac{c}{c_c} = \frac{144 \text{ lb}\cdot\text{s/ft}}{386.09 \text{ lb}\cdot\text{s/ft}} = 0.37297$$

From Eq. (19.53):

$$\frac{x_m}{\delta_m} = \frac{1}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right] + \left[2\frac{c}{c_c}\left(\frac{\omega}{\omega_n}\right)\right]^2}}$$

For maximum amplitude we set equal to zero the derivative with respect to $\left(\dfrac{\omega}{\omega_n}\right)$ of the square of the denominator.

$$2\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]\left[-2\left(\frac{\omega}{\omega_n}\right)\right] + 8\left(\frac{c}{c_c}\right)^2\left(\frac{\omega}{\omega_n}\right) = 0$$

Rearranging, we obtain

$$4\left(\frac{\omega}{\omega_n}\right)\left[\left(\frac{\omega}{\omega_n}\right)^2 - 1 + 2\left(\frac{c}{c_c}\right)^2\right] = 0$$

$$\left(\frac{\omega}{\omega_n}\right)^2 = 1 - 2\left(\frac{c}{c_c}\right)^2$$

$$\left(\frac{\omega}{\omega_n}\right)^2 = 1 - 2\left(\frac{c}{c_c}\right)^2 = 1 - 2(0.37297)^2 = 0.72179$$

$$\frac{\omega}{\omega_n} = 0.84958 \qquad \omega = (0.84958)\omega_n = (0.84958)(31.08 \text{ rad/s})$$

$$\omega = 26.405 \text{ rad/s} \qquad f = 4.2025 \text{ Hz}$$

$f = 252$ rpm ◀

(*c*) *Amplitude*:

From Eq. (19.53):

$$x_m = \frac{\frac{P_m}{k}}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\left(\frac{c}{c_c}\right)\left(\frac{\omega}{\omega_n}\right)\right]^2}}$$

For part (*a*) with $P_m = 125$ lb and $\omega = \omega_n$

$$x_m = \frac{\frac{125 \text{ lb}}{6000 \text{ lb/ft}}}{\sqrt{[1-1]^2 + [2(0.37297)(1)]^2}} = 0.02793 \text{ ft}$$

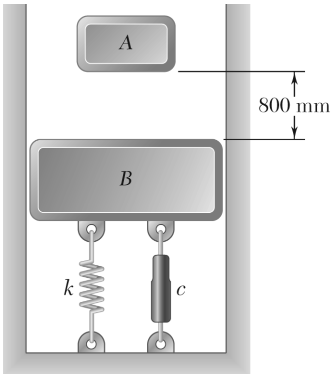
$x_m = 0.335$ in. ◀

For part (*b*) with $P_m = 125$ lb and $\left(\frac{\omega}{\omega_n}\right) = 0.84958$

$$x_m = \frac{\frac{125 \text{ lb}}{6000 \text{ lb/ft}}}{\sqrt{[1-(0.84958)^2]^2 + [2(0.84958)(0.37297)]^2}} = 0.0301 \text{ ft}$$

$x_m = 0.361$ in. ◀

PROBLEM 19.136



A 4-kg block A is dropped from a height of 800 mm onto a 9-kg block B which is at rest. Block B is supported by a spring of constant $k = 1500$ N/m and is attached to a dashpot of damping coefficient $c = 230$ N·s/m. Knowing that there is no rebound, determine the maximum distance the blocks will move after the impact.

SOLUTION

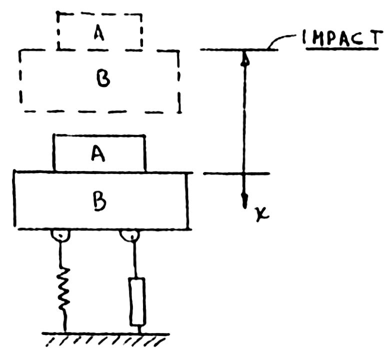
Velocity of Block A just before impact.

$$v_A = \sqrt{2gh}$$
$$= \sqrt{2(9.81)(0.8)}$$
$$= 3.962 \text{ m/s}$$

Velocity of Blocks A and B immediately after impact.

Conservation of momentum.

$$m_A v_A + m_B v_B = (m_A + m_B)v'$$
$$(4)(3.962) + 0 = (4+9)v'$$
$$v' = 1.219 \text{ m/s}$$
$$\dot{x}_0 = +1.219 \text{ m/s} \downarrow = \dot{x}_0$$



Static deflection (Block A):

$$x_0 = -\frac{m_A g}{k}$$
$$= -\frac{(4)(9.82)}{1500}$$
$$= -0.02619 \text{ m}$$

$x = 0$, Equilibrium position for both blocks:

$$c_c = 2\sqrt{km}$$
$$= 2\sqrt{(1500)(13)}$$
$$= 279.3 \text{ N·s/m}$$

Since $c < c_c$, Equation (19.44):

$$x = e^{-\left(\frac{c}{2m}\right)t}[C_1 \sin \omega_d t + C_2 \cos \omega_d t]$$

$$\frac{c}{2m} = \frac{230}{(2)(13)}$$
$$= 8.846 \text{ s}^{-1}$$

PROBLEM 19.136 (Continued)

Expression for ω_d:

$$\omega_d^2 = \frac{k}{m} - \left(\frac{c}{2m}\right)^2$$

$$\omega_d = \sqrt{\frac{1500}{13} - \left(\frac{230}{(2)(13)}\right)^2}$$

$$= 6.094 \text{ rad/s}$$

$$x = e^{-8.846t}(C_1 \sin 6.094t + C_2 \cos 6.094t)$$

Initial conditions:

$$x_0 = -0.02619 \text{ m}$$

$$(t = 0) \quad \dot{x}_0 = +1.219 \text{ m/s}$$

$$x_0 = -0.02619 = e^0[C_1(0) + C_2(1)]$$

$$C_2 = -0.02619$$

$$\dot{x}(0) = -8.846e^{(-8.846)0}[C_1(0) + (-0.02619)(1)]$$

$$+e^{(-8.846)(0)}[6.094C_1(1) + C_2(0)] = 1.219$$

$$1.219 = (-8.846)(-0.02619) + 6.094C_1$$

$$C_1 = 0.16202$$

$$x = e^{-8.846t}(0.16202 \sin 6.094t - 0.02619 \cos 6.094t)$$

Maximum deflection occurs when $\dot{x} = 0$

$$\dot{x} = 0 = -8.846e^{-8.846t_m}(0.16202 \sin 6.094t_m - 0.02619 \cos 6.094t_m)$$

$$+ e^{-8.846t_m}[6.094][0.1620 \cos 6.094t_m + 0.02619 \sin 6.094t_m]$$

$$0 = [(-8.846)(0.16202) + (6.094)(0.02619)] \sin 6.094t_m$$

$$+[(-8.846)(-0.02619) + (6.094)(0.1620)] \cos 6.094t_m$$

$$0 = -1.274 \sin 6.094t_m + 1.219 \cos 6.094t_m$$

$$\tan 6.094t_m = \frac{1.219}{1.274} = 0.957$$

$$\text{Time at maximum deflection} = t_m = \frac{\tan^{-1} 0.957}{6.094} = 0.1253 \text{ s}$$

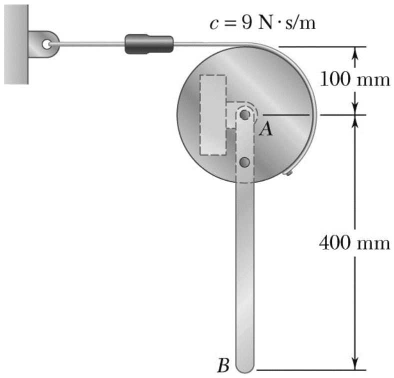
$$x_m = e^{-(8.846)(0.1253)}[0.1620 \sin(6.094)(0.1253)$$

$$-0.02619 \cos(6.094)(0.1253)]$$

$$x_m = (0.3301)(0.1120 - 0.0189) = 0.0307 \text{ m}$$

Blocks move, static deflection $+ x_m$ Total distance $= 0.02619 + 0.0307 = 0.0569 \text{ m} = 56.9 \text{ mm}$ ◀

PROBLEM 19.137



A 3-kg slender rod AB is bolted to a 5-kg uniform disk. A dashpot of damping coefficient $c = 9\ \text{N}\cdot\text{s/m}$ is attached to the disk as shown. Determine (*a*) the differential equation of motion for small oscillations, (*b*) the damping factor c/c_c.

SOLUTION

Data: $r = 100\ \text{mm} = 0.100\ \text{m},\ \ l = 400\ \text{mm} = 0.400\ \text{m}$

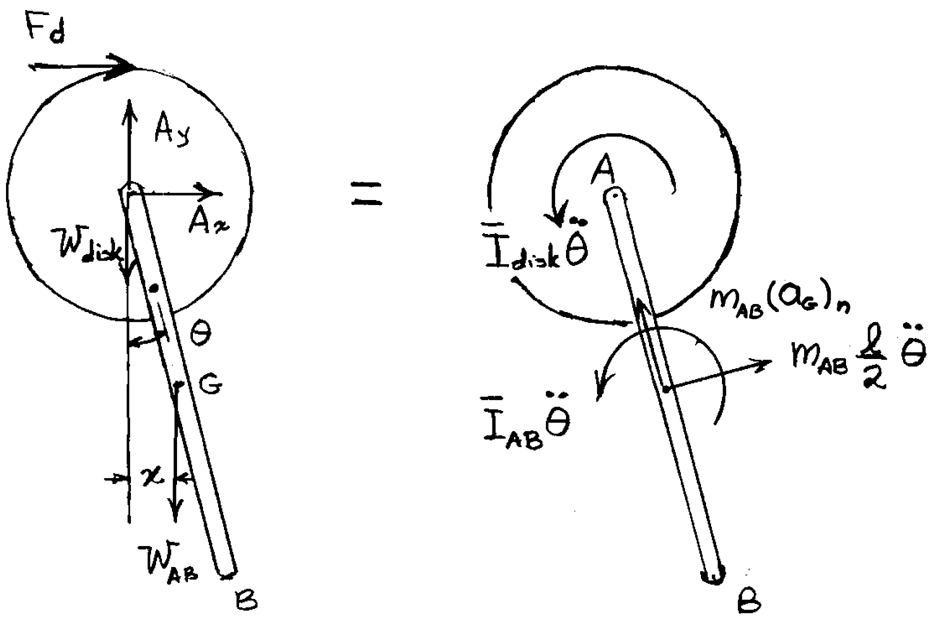
$$\bar{I}_{\text{disk}} = \frac{1}{2}m_{\text{disk}}r^2 = \frac{1}{2}(5\ \text{kg})(0.100\ \text{m})^2 = 0.025\ \text{kg}\cdot\text{m}^2$$

$$\bar{I}_{AB} = \frac{1}{2}m_{AB}l^2 = \frac{1}{12}(3\ \text{kg})(0.400\ \text{m})^2 = 0.040\ \text{kg}\cdot\text{m}^2$$

$$W_{AB} = m_{AB}g = (3\ \text{kg})(9.81\ \text{m/s}^2) = 29.43\ \text{N}$$

$$c = 9\ \text{N}\cdot\text{s/m}$$

Equation of motion: Let the disk and rod assembly be rotated through a small counterclockwise angle θ



$$\Sigma M_A = \Sigma(M_A)_{\text{eff}}:$$

$$-W_{AB}x - F_d r = \bar{I}_{\text{disk}}\ddot{\theta} + \bar{I}_{AB}\ddot{\theta} + m_{AB}\left(\frac{l}{2}\ddot{\theta}\right)\frac{l}{2}$$

PROBLEM 19.137 (Continued)

where $$W_{AB}x = -W_{AB}\frac{l}{2}\sin\theta$$
$$= (29.43 \text{ N})(0.2 \text{ m}) \sin\,\theta$$
$$= 5.886 \sin\theta \text{ N} \cdot \text{m}$$
$$\approx -5.886\,\theta$$

Damping force: $$F_d = cr\dot{\theta}$$

$$F_d r = cr^2\dot{\theta} = (9 \text{ N}\cdot\text{s}\cdot\text{m})(0.100 \text{ m})^2\dot{\theta} = 0.09\,\dot{\theta} = C\dot{\theta}$$

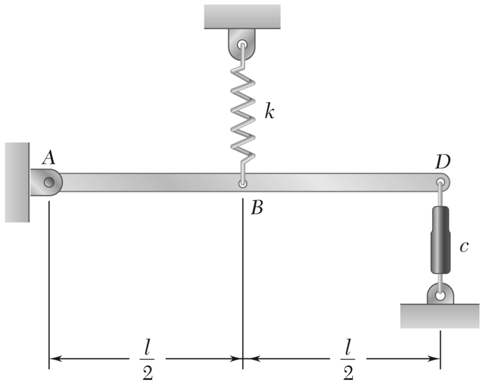
Inertia: $$\bar{I}_{\text{disk}} + \bar{I}_{AB} + m_{AB}\left(\frac{l}{2}\right)^2 = 0.025 + 0.040 + (3)(0.2)^2 = 0.185 \text{ kg}\cdot\text{m}^2$$

$$-5.886\theta - 0.09\dot{\theta} = 0.185\ddot{\theta}$$
$$0.185\ddot{\theta} + 0.09\dot{\theta} + 5.886\theta = 0$$
$$M\ddot{\theta} + C\dot{\theta} + K\theta = 0$$

$$\omega_n = \sqrt{\frac{K}{M}} = \sqrt{\frac{5.886}{0.185}} = 5.6406 \text{ rad/s}$$

$$C_c = 2\,M\,\omega_2 = (2)(0.185)(5.6406) = 2.087$$

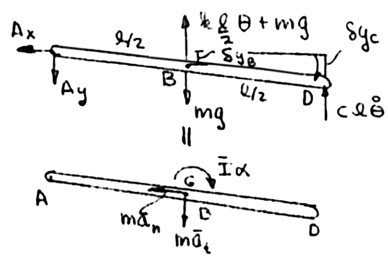
$$\frac{c}{c_c} = \frac{C}{C_c} = \frac{0.09}{2.087}$$ $$\frac{c}{c_c} = 0.0431$$ ◀



PROBLEM 19.138

A uniform rod of mass m is supported by a pin at A and a spring of constant k at B and is connected at D to a dashpot of damping coefficient c. Determine in terms of m, k, and c, for small oscillations, (*a*) the differential equation of motion, (*b*) the critical damping coefficient c_c.

SOLUTION



In equilibrium, the force in the spring is mg.

For small angles, $\sin\theta \approx \theta \qquad \cos\theta \approx 1$

$$\delta y_B = \frac{l}{2}\theta$$

$$\delta y_C = l\theta$$

(*a*) Newton's Law: $\Sigma M_A = (\Sigma M_A)_{\text{eff}}$

$$\frac{mgl}{2} - \left(k\frac{l}{2}\theta + mg\right)\frac{l}{2} - cl\dot{\theta}l = \bar{I}\alpha + m\bar{a}_t\frac{l}{2}$$

Kinematics: $\alpha = \ddot{\theta}$

$$\bar{a}_t = \frac{l}{2}\alpha = \frac{l}{2}\ddot{\theta}$$

$$\left[\bar{I} + m\left(\frac{l}{2}\right)^2\right]\ddot{\theta} + cl^2\dot{\theta} + k\left(\frac{l}{2}\right)^2\theta = 0$$

$$\bar{I} + m\left(\frac{l}{2}\right)^2 = \frac{1}{3}ml^2$$

$$\ddot{\theta} + \left(\frac{3c}{m}\right)\dot{\theta} + \left(\frac{3k}{4m}\right)\theta = 0 \blacktriangleleft$$

(*b*) Substituting $\theta = e^{\lambda t}$ into the differential equation obtained in (*a*), we obtain the characteristic equation,

$$\lambda^2 + \left(\frac{3c}{m}\right)\lambda + \frac{3k}{4m} = 0$$

and obtain the roots

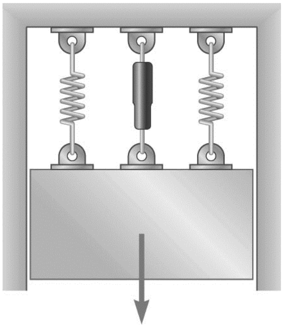
$$\lambda = \frac{\frac{-3c}{m} \mp \sqrt{\left(\frac{3c}{m}\right)^2 - \left(\frac{3k}{m}\right)}}{2}$$

The critical damping coefficient, c_c, is the value of c, for which the radicand is zero.

Thus, $$\left(\frac{3c_c}{m}\right)^2 = \frac{3k}{m}$$

$$c_c = \sqrt{\frac{km}{3}} \blacktriangleleft$$

PROBLEM 19.139



A machine element weighing 800 lb is supported by two springs, each having a constant of 200 lb/in. A periodic force of maximum value 30 lb is applied to the element with a frequency of 2.5 cycles per second. Knowing that the coefficient of damping is $8 \text{ lb}\cdot\text{s/in.}$, determine the amplitude of the steady-state vibration of the element.

SOLUTION

Equivalent spring: $k = 2(200) = 400 \text{ lb/in.} = 4800 \text{ lb/ft}$

Undamped natural frequency: $\omega_n = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{4800}{800/32.2}} = 13.90 \text{ rad/s}$

$$f_n = \frac{\omega_n}{2\pi} = \frac{13.90}{2\pi} = 2.212 \text{ Hz}$$

Critical damping coefficient: $c_c = 2\,m\omega_n = 2\left(\dfrac{800}{32.2}\right)(13.90) = 691 \text{ lb}\cdot\text{s/ft}$

Damping coefficient: $c = 8 \text{ lb}\cdot\text{s/in.} = 96 \text{ lb}\cdot\text{s/ft}$

Damping ratio: $\dfrac{c}{c_c} = \dfrac{96}{691} = 0.1390$

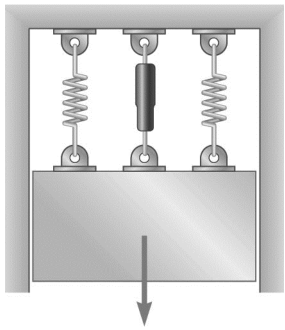
Amplitude:

$$x_m = \frac{P_m/k}{\sqrt{\left[1-\left(\frac{\omega_f}{\omega_n}\right)^2\right]^2 + \left[2\frac{c}{c_c}\frac{\omega_f}{\omega_n}\right]^2}} \qquad (1)$$

where $\dfrac{\omega_f}{\omega_n} = \dfrac{f_f}{f_n} = \dfrac{2.5 \text{ Hz}}{2.212 \text{ Hz}} = 1.130$

Substituting into Eq. (1) with $P_m = 30$ lb, we have

$$x_m = \frac{30 \text{ lb}/400 \text{ lb/in.}}{\sqrt{[1-(1.130)^2]^2 + [2(0.1390)(1.130)]^2}} \qquad x_m = 0.1791 \text{ in.} \blacktriangleleft$$



PROBLEM 19.140

In Problem 19.139, determine the required value of the coefficient of damping if the amplitude of the steady-state vibration of the element is to be 0.15 in.

PROBLEM 19.139 A machine element weighing 800 lb is supported by two springs, each having a constant of 200 lb/in. A periodic force of maximum value 30 lb is applied to the element with a frequency of 2.5 cycles per second. Knowing that the coefficient of damping is 8 lb · s/in., determine the amplitude of the steady-state vibration of the element.

SOLUTION

Equivalent spring: $k = 2(200) = 400 \text{ lb/in.} = 4800 \text{ lb/ft}$

Undamped natural frequency: $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4800}{800/32.2}} = 13.90 \text{ rad/s}$

$$f_n = \frac{\omega_n}{2\pi} = \frac{13.90}{2\pi} = 2.212 \text{ Hz}$$

Critical damping coefficient: $c_c = 2\,m\omega_n = 2\left(\frac{800}{32.2}\right)(13.90) = 691 \text{ lb} \cdot \text{s/ft}$

Amplitude:

$$x_m = \frac{P_m/k}{\sqrt{\left[1-\left(\frac{\omega_f}{\omega_n}\right)^2\right]^2 + \left[2\frac{c}{c_c}\frac{\omega_f}{\omega_n}\right]^2}} \qquad (1)$$

where $\frac{\omega_f}{\omega_n} = \frac{f_f}{f_n} = \frac{2.5 \text{ Hz}}{2.212 \text{ Hz}} = 1.130$

Using $x_m = 0.15$ in., $P_m = 30$ lb, and $k = 400$ lb/in.

$$0.15 \text{ in.} = \frac{30 \text{ lb}/400 \text{ lb/in.}}{\sqrt{[1-(1.130)^2]^2 + \left[2\frac{c}{c_c}(1.130)\right]^2}}$$

Solving for $\frac{c}{c_c}$, we find $\frac{c}{c_c} = 0.1842$.

Since $c_c = 691 \text{ lb} \cdot \text{s/ft}$, we have

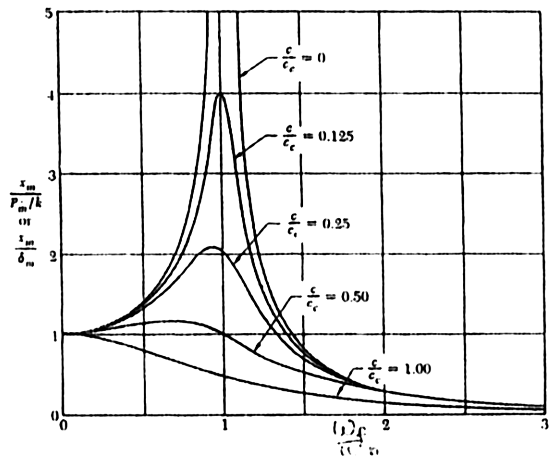
$$c = (0.1842)(691) \qquad c = 1273 \text{ lb} \cdot \text{s/ft}$$

or $c = 10.61 \text{ lb} \cdot \text{s/in.}$ ◀

PROBLEM 19.141

In the case of the forced vibration of a system, determine the range of values of the damping factor c/c_c for which the magnification factor will always decrease as the frequency ratio ω_f/ω_n increases.

SOLUTION



From Eq. (19.53)′:

Magnification factor:
$$\frac{x_m}{\frac{P_m}{k}} = \frac{1}{\sqrt{\left[1-\left(\frac{\omega_f}{\omega_n}\right)^2\right]^2 + \left[2\left(\frac{c}{c_c}\right)\left(\frac{\omega_f}{\omega_n}\right)\right]^2}}$$

Find value of $\frac{c}{c_c}$ for which there is no maximum for $\frac{x_m}{\frac{P_m}{k}}$ as $\frac{\omega_f}{\omega_n}$ increases.

$$\frac{d\left(\frac{x_m}{\frac{P_m}{k}}\right)^2}{d\left(\frac{\omega_f}{\omega_n}\right)^2} = \frac{-\left[2\left(1-\left(\frac{\omega_f}{\omega_n}\right)^2\right)(-1) + 4\frac{c^2}{c_c^2}\right]}{\left\{\left[1-\left(\frac{\omega_f}{\omega_n}\right)^2\right]^2 + \left[2\left(\frac{c}{c_c}\right)\left(\frac{\omega_f}{\omega_n}\right)^2\right]\right\}^2} = 0$$

$$-2 + 2\left(\frac{\omega_f}{\omega_n}\right)^2 + 4\frac{c^2}{c_c^2} = 0$$

$$\left(\frac{\omega_f}{\omega_n}\right)^2 = 1 - 2\frac{c^2}{c_c^2}$$

For $\frac{c^2}{c_c^2} \geq \frac{1}{2}$, there is no maximum for $\frac{x_m}{\left(\frac{P_m}{k}\right)}$ and the magnification factor will decrease as $\frac{\omega_f}{\omega_n}$ increases.

$$\frac{c}{c_c} \geq \frac{1}{\sqrt{2}}$$

$\frac{c}{c_c} \geq 0.707$ ◀

PROBLEM 19.142

Show that for a small value of the damping factor c/c_c, the maximum amplitude of a forced vibration occurs when $\omega_f \approx \omega_n$ and that the corresponding value of the magnification factor is $\frac{1}{2}(c/c_c)$.

SOLUTION

From Eq. (19.53′):

$$\text{Magnification factor} = \frac{x_m}{\frac{P_m}{k}} = \frac{1}{\sqrt{\left[1-\left(\frac{\omega_f}{\omega_n}\right)^2\right]^2 + \left[2\left(\frac{c}{c_c}\right)\left(\frac{\omega_f}{\omega_n}\right)\right]^2}}$$

Find value of $\dfrac{\omega_f}{\omega_n}$ for which $\dfrac{x_m}{\frac{P_m}{k}}$ is a maximum.

$$0 = \frac{d\left(\frac{x_m}{\frac{P_m}{k}}\right)^2}{d\left(\frac{\omega_f}{\omega_n}\right)^2} = -\frac{\left[2\left[1-\left(\frac{\omega_f}{\omega_n}\right)^2\right](-1) + 4\left(\frac{c^2}{c_c^2}\right)\right]}{\left\{\left[1-\left(\frac{\omega_f}{\omega_n}\right)^2\right]^2 + \left[2\left(\frac{c}{c_c}\right)\left(\frac{\omega_f}{\omega_n}\right)\right]^2\right\}^2}$$

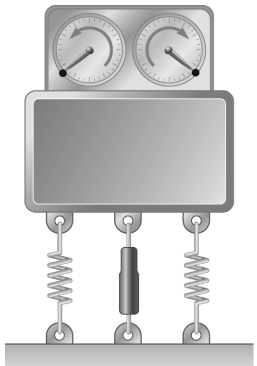
$$-2 + 2\left(\frac{\omega_f}{\omega_n}\right)^2 + 4\left(\frac{c}{c_c}\right)^2 = 0$$

For small $\dfrac{c}{c_c}$, $\qquad \dfrac{\omega_f}{\omega_n} \approx 1 \qquad \omega_f \approx \omega_n$

For $\qquad \dfrac{\omega_f}{\omega_n} = 1$

$$\frac{x_m}{\frac{P_m}{k}} = \frac{1}{\sqrt{[1-1]^2 + \left[2\left(\frac{c}{c_c}\right)1\right]^2}} \qquad \frac{x_m}{\left(\frac{P_m}{k}\right)} = \frac{1}{2}\frac{c_c}{c} \blacktriangleleft$$

PROBLEM 19.143



A counter-rotating eccentric mass exciter consisting of two rotating 14-oz masses describing circles of 6-in. radius at the same speed but in opposite senses is placed on a machine element to induce a steady-state vibration of the element and to determine some of the dynamic characteristics of the element. At a speed of 1200 rpm a stroboscope shows the eccentric masses to be exactly under their respective axes of rotation and the element to be passing through its position of static equilibrium. Knowing that the amplitude of the motion of the element at that speed is 0.6 in. and that the total mass of the system is 300 lb, determine (*a*) the combined spring constant *k*, (*b*) the damping factor c/c_c.

SOLUTION

Forcing frequency: $\omega_f = 1200 \text{ rpm} = 125.664 \text{ rad/s}$

Unbalance of one mass: $w = 14 \text{ oz} = 0.875 \text{ lb}$

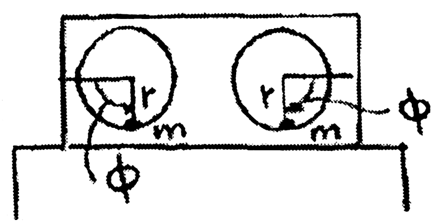
$r = 6 \text{ in.} = 0.5 \text{ ft}$

Shaking force: $P = 2\, mr\omega_f^2 \sin \omega_f t$

$= (2)\left(\frac{0.875}{32.2}\right)(0.5)(125.664)^2 \sin \omega_f t$

$= 429.11 \sin \omega_f t$

$P_m = 429 \text{ lb}$



Total weight: $W = 300 \text{ lb}$

By Eqs. (19.48) and (19.52), the vibratory response of the system is

$$x = x_m \sin(\omega_f t - \varphi)$$

where
$$x_m = \frac{P_m}{\sqrt{\left(k - M\omega_f^2\right)^2 + (c\omega_f)^2}} \qquad (1)$$

and
$$\tan \varphi = \frac{c\omega_f}{k - M\omega_f^2} \qquad (2)$$

Since $\varphi = 90° = \frac{\pi}{2}$, $\quad \tan \varphi = \infty \quad$ and $\quad k - M\omega_f^2 = 0.$

(*a*) Combined spring constant.

$$k = M\omega_f^2$$

$$= \left(\frac{300}{32.2}\right)(125.664)^2$$

$$= 147.12 \times 10^3 \text{ lb/ft}$$

$k = 147 \text{ kip/ft}$ ◄

PROBLEM 19.143 (Continued)

The observed amplitude is $x_m = 0.6 \text{ in.} = 0.05 \text{ ft}$

From Eq. (1):

$$c = \frac{1}{\omega_f}\sqrt{\left(\frac{P_m}{x_m}\right)^2 - (k - M\omega_f)^2} = \frac{P_m}{\omega_f x_m}$$

$$= \frac{429.11}{(125.664)(0.05)}$$

$$= 68.296 \text{ lb} \cdot \text{s/ft}$$

Critical damping coefficient:

$$c_c = 2\sqrt{kM}$$

$$= 2\sqrt{(147.12 \times 10^3)\left(\tfrac{300}{32.2}\right)}$$

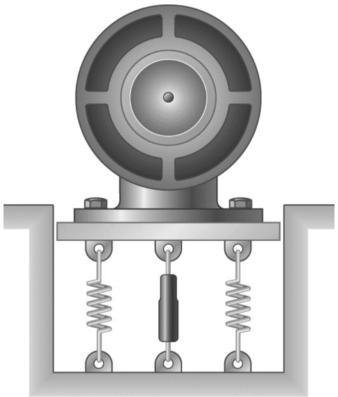
$$= 2.3416 \times 10^3 \text{ lb} \cdot \text{s/ft}$$

(*b*) Damping factor.

$$\frac{c}{c_c} = \frac{68.296}{2.3416 \times 10^3}$$

$\dfrac{c}{c_c} = 0.0292$ ◀

PROBLEM 19.144



A 15-kg motor is supported by four springs, each of constant 40 kN/m. The unbalance of the motor is equivalent to a mass of 20 kg located 125 mm from the axis of rotation. Knowing that the motor is constrained to move vertically and that the damping factor c/c_c is equal to 0.4, determine the range of frequencies for which the amplitude of the steady-state vibration of the motor is less than 0.2 mm.

SOLUTION

Equivalent spring: $k = (4)(40\times10^3 \text{ N/m}) = 160\times10^3 \text{ N/m}$

Mass: $m = 15 \text{ kg}$

Natural frequency: $\omega_n = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{160\times10^3}{15}} = 103.280 \text{ rad/s}$

Unbalanced force:

$$P_m = m_{\text{eq}} r\omega_f^2 = m_{\text{eq}} r\omega_n^2 \left(\frac{\omega_f}{\omega_n}\right)^2$$

$$= (0.020 \text{ kg})(0.125 \text{ m})(130.280 \text{ rad/s})^2$$

$$= 26.667\left(\frac{\omega_f}{\omega_n}\right)^2 \text{ N}$$

At steady state,

$$x_m = \frac{P_m/k}{\left[\left(1-\left(\frac{\omega_f}{\omega_n}\right)^2\right)^2 + \left(2\frac{c}{c_c}\frac{\omega_f}{\omega_n}\right)^2\right]^{1/2}}$$

$$\left[\left(1-\left(\frac{\omega_f}{\omega_n}\right)^2\right)^2 + \left(2\frac{c}{c_c}\frac{\omega_f}{\omega_n}\right)^2\right]^{1/2} = \frac{P_m}{kx_m}$$

$$1-2\left(\frac{\omega_f}{\omega}\right)^2 + \left(\frac{\omega_f}{\omega_n}\right)^4 + \left(2\frac{c}{c_c}\frac{\omega_f}{\omega_n}\right)^2 = \left(\frac{P_m}{kx_m}\right)^2 \qquad (1)$$

Amplitude: $x_m = 0.2 \text{ mm} = 0.2\times10^{-3} \text{ m}$

$$\frac{P_m}{kx_m} = \frac{26.667}{(160\times10^3)(0.2\times10^{-3})}\left(\frac{\omega_f}{\omega_n}\right)^2 = 0.83333\left(\frac{\omega_f}{\omega_n}\right)^2$$

Damping factor: $\dfrac{c}{c_c} = 0.4$

PROBLEM 19.144 (Continued)

Substituting into Eq. (1),

$$1-2\left(\frac{\omega_f}{\omega_n}\right)^2+\left(\frac{\omega_f}{\omega_n}\right)^4+\left[(2)(0.4)\frac{\omega_f}{\omega_n}\right]^2=\left(0.83333\frac{\omega_f^2}{\omega_n^2}\right)^2$$

$$[(1-(0.83333)^2]\left(\frac{\omega_f}{\omega_n}\right)^4-[2-(2)^2(0.4)^2]\left(\frac{\omega_f}{\omega_n}\right)^2+1=0$$

$$0.30556\left(\frac{\omega_f}{\omega_n}\right)^4-1.36\left(\frac{\omega_f}{\omega_n}\right)^2+1=0$$

Solving the quadratic equation for $\left(\frac{\omega_f}{\omega_n}\right)^2$,

$$\left(\frac{\omega_f}{\omega_n}\right)^2=3.5216 \quad \text{and} \quad 0.92934$$

$$\frac{\omega_f}{\omega_n}=1.8766 \quad \text{and} \quad 0.96402$$

$$\omega_f=(1.8766)(103.280 \text{ rad/s})=193.815 \text{ rad/s}$$

and

$$\omega_f=(0.96402)(103.280 \text{ rad/s})=99.564 \text{ rad/s}$$

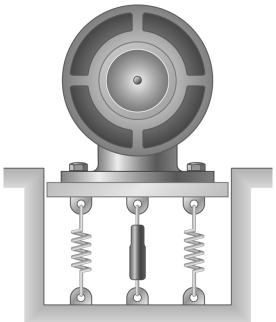
For $x_m < 0.2$ m, the forcing frequency must satisfy

$$\omega_f > 193.8 \text{ rad/s} \quad \text{and} \quad \omega_f < 99.6 \text{ rad/s}$$

Since

$$f_f=\frac{\omega_f}{2\pi},$$

$$f_f > 30.8 \text{ Hz} \quad \text{and} \quad f_f < 15.85 \text{ Hz} \blacktriangleleft$$



PROBLEM 19.145

A 220-lb motor is supported by four springs, each of constant 500 lb/in., and is connected to the ground by a dashpot having a coefficient of damping $c = 35 \text{ lb}\cdot\text{s/in.}$ The motor is constrained to move vertically, and the amplitude of its motion is observed to be 0.08 in. at a speed of 1200 rpm. Knowing that the weight of the rotor is 30 lb, determine the distance between the mass center of the rotor and the axis of the shaft.

SOLUTION

Forcing frequency: $\omega_f = 1200 \text{ rpm} = 125.664 \text{ rad/s}$

Equivalent spring: $k = (4)(500) = 2000 \text{ lb/in.} = 24000 \text{ lb/ft}$

Mass: $m = \dfrac{W}{g} = \dfrac{220 \text{ lb}}{32.2 \text{ ft/s}^2} = 6.8323 \text{ lb}\cdot\text{s}^2/\text{ft}$

Natural frequency: $\omega_n = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{24000}{6.8323}} = 59.268 \text{ rad/s}$

$$\frac{\omega_f}{\omega_n} = \frac{125.664 \text{ rad/s}}{59.268 \text{ rad/s}} = 2.12026$$

Critical damping coefficient: $c_c = 2m\omega_n$

$$c_c = (2)(6.8323)(59.268) = 809.87 \text{ lb}\cdot\text{s/ft}$$

Damping coefficient: $c = 35 \text{ lb}\cdot\text{s/in.} = 420 \text{ lb}\cdot\text{s/ft}$

Damping factor: $\dfrac{c}{c_c} = \dfrac{420}{809.87} = 0.51860$

Amplitude: $x_m = 0.08 \text{ in.} = 6.6667\times10^{-3} \text{ ft}$

$$x_m = \frac{P_m/k}{\left[\left(1-\left(\frac{\omega_f}{\omega_n}\right)^2\right)^2 + \left(2\frac{c}{c_c}\frac{\omega_f}{\omega_n}\right)^2\right]^{1/2}}$$

Unbalanced force: $P_m = kx_m\left[\left(1-\left(\frac{\omega_f}{\omega_n}\right)^2\right)^2 + \left(2\frac{c}{c_c}\frac{\omega_f}{\omega_n}\right)^2\right]^{1/2}$

$$= kx_m[(1-4.4955)^2 + ((2)(0.51860)(2.12026))^2]^{1/2}$$

$$= kx_m\ [12.2185 + 4.8362]^{1/2}$$

$$= 4.1297\, kx_m = (4.1297)(24000)(6.6667\times10^{-3})$$

$$= 660.76 \text{ lb}$$

But, $P_m = m' e \omega_f^2$

where m' is the mass of the rotor and e is the distance between the mass center of the rotor and the axis of the shaft.

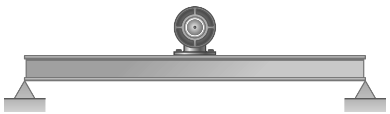
$$m' = \frac{30 \text{ lb}}{32.2 \text{ ft/s}^2} = 0.93168 \text{ lb} \cdot \text{s}^2/\text{ft}$$

$$e = \frac{P_m}{m' \omega_f^2} = \frac{660.76 \text{ lb}}{(0.93168 \text{ lb} \cdot \text{s}^2/\text{ft})(125.664 \text{ rad/s})^2}$$

$$= 0.044911 \text{ ft}$$

$e = 0.539$ in. ◀

PROBLEM 19.146



A 100-lb motor is directly supported by a light horizontal beam which has a static deflection of 0.2 in. due to the weight of the motor. The unbalance of the rotor is equivalent to a mass of 3.5 oz located 3 in. from the axis of rotation. Knowing that the amplitude of the vibration of the motor is 0.03 in. at a speed of 400 rpm, determine (*a*) the damping factor c/c_c, (*b*) the coefficient of damping c.

SOLUTION

Spring constant: $$k = \frac{W}{\delta_{\text{st}}} = \frac{100}{\frac{0.2}{12}} = 6000 \text{ lb/ft}$$

Natural undamped circular frequency: $$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{6000}{\frac{100}{32.2}}} = 43.955 \text{ rad/s}$$

Unbalance: $$m' = \frac{w}{g} = \frac{\frac{3.5}{16}}{32.2} 6.7935\times 10^{-3} \text{ slug}$$

$$r = 3 \text{ in.} = 0.25 \text{ ft}$$

Forcing frequency: $$\omega_f = 400 \text{ rpm} = 41.888 \text{ rad/s}$$

Unbalance force: $$P_m = m' r \omega_f^2 = (6.7935\times 10^{-3})(0.25)(41.888)^2 = 2.98 \text{ lb}$$

Static deflection: $$\delta_{\text{st}} = \frac{P_m}{k} = \frac{2.98}{6000} = 0.49666\times 10^{-3} \text{ ft}$$

Amplitude: $$x_m = 0.03 \text{ in.} = 2.5\times 10^{-3} \text{ ft}$$

Frequency ratio: $$\frac{\omega_f}{\omega_n} = \frac{41.888}{43.955} = 0.95298$$

Eq. (19.53): $$x_m = \frac{\delta_{\text{st}}}{\sqrt{\left[1-\left(\frac{\omega_f}{\omega_n}\right)^2\right]^2 + \left[2\left(\frac{c}{c_c}\right)\left(\frac{\omega_f}{\omega_n}\right)\right]^2}}$$

$$\left[1-\left(\frac{\omega_f}{\omega_n}\right)^2\right]^2 + \left[2\left(\frac{c}{c_c}\right)\left(\frac{\omega_f}{\omega_n}\right)\right]^2 = \left(\frac{\delta_{\text{st}}}{x_m}\right)^2$$

$$[1-(0.95298)^2]^2 + \left[2\left(\frac{c}{c_c}\right)(0.95298)\right]^2 = \left[\frac{0.49666\times 10^{-3}}{2.5\times 10^{-3}}\right]^2$$

$$0.0084326 + 3.6327\left(\frac{c}{c_c}\right)^2 = 0.039467$$

$$\left(\frac{c}{c_c}\right)^2 = 0.0085431$$

PROBLEM 19.146 (Continued)

(*a*) Damping factor. $\frac{c}{c_c} = 0.092429$ $\frac{c}{c_c} = 0.0924$ ◀

Critical damping factor. $c_c = 2\sqrt{km}$

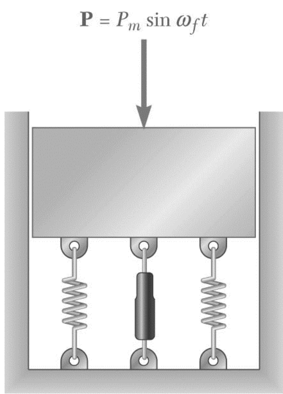
$$= 2\sqrt{(6000)\left(\frac{100}{32.2}\right)}$$

$$= 273.01 \text{ lb} \cdot \text{s/ft}$$

(*b*) Coefficient of damping. $c = \left(\frac{c}{c_c}\right)c_c$

$$= (0.092429)(273.01)$$

$c = 25.2 \text{ lb} \cdot \text{s/ft}$ ◀



PROBLEM 19.147

A machine element is supported by springs and is connected to a dashpot as shown. Show that if a periodic force of magnitude $P = P_m \sin \omega_f t$ is applied to the element, the amplitude of the fluctuating force transmitted to the foundation is

$$F_m = P_m \sqrt{\frac{1 + \left[2\left(\frac{c}{c_c}\right)\left(\frac{\omega_f}{\omega_n}\right)\right]^2}{\left[1 - \left(\frac{\omega_f}{\omega_n}\right)^2\right]^2 + \left[2\left(\frac{c}{c_c}\right)\left(\frac{\omega_f}{\omega_n}\right)^2\right]^2}}$$

SOLUTION

From Equation (19.48), the motion of the machine is $x = x_m \sin(\omega_f t - \phi)$

The force transmitted to the foundation is

Springs: $F_s = kx = kx_m \sin(\omega_f t - \phi)$

Dashpot: $F_d = c\dot{x} = cx_m\omega_f \cos(\omega_f t - \phi)$

$$F_t = x_m[k \sin(\omega_f t - \phi) + c\omega_f \cos(\omega_f t - \phi)]$$

or recalling the identity,

$$A \sin y + B \cos y = \sqrt{A^2 + B^2} \sin(y + \psi)$$

$$\sin\psi = \frac{B}{\sqrt{A^2 + B^2}}$$

$$\cos\psi = \frac{A}{\sqrt{A^2 + B^2}}$$

$$F_t = \left[x_m\sqrt{k^2 + (c\omega_f)^2}\right]\sin(\omega_f t - \phi + \psi)$$

Thus, the amplitude of F_t is $F_m = x_m\sqrt{k^2 + (c\omega_f)^2}$ (1)

From Equation (19.53):

$$x_m = \frac{\frac{P_m}{k}}{\sqrt{\left[1 - \left(\frac{\omega_f}{\omega_n}\right)^2\right]^2 + \left[2\left(\frac{c}{c_c}\right)\frac{\omega_f}{\omega_n}\right]^2}}$$

Substituting for x_m in Equation (1),

$$F_m = \frac{P_m\sqrt{1 + \left(\frac{c\omega_f}{k}\right)^2}}{\sqrt{\left[1 - \left(\frac{\omega_f}{\omega_n}\right)^2\right]^2 + \left[2\left(\frac{c}{c_c}\right)\left(\frac{\omega_f}{\omega_n^2}\right)\right]}} \quad (2)$$

$$\omega_n^2 = \frac{k}{m}$$

PROBLEM 19.147 (Continued)

and Equation (19.41),

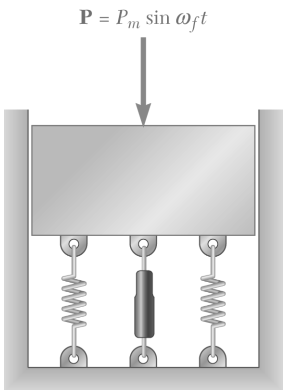
$$c_c = 2m\omega_n$$

$$m = \frac{c\omega_n}{2}$$

$$\frac{c\omega_f}{k} = \frac{c\omega_f}{m\omega_n^2} = 2\left(\frac{c}{c_c}\right)\left(\frac{\omega_f}{\omega_n}\right)$$

Substituting in Eq. (2),

$$F_m = \frac{P_m\sqrt{1+\left[2\left(\frac{c}{c_c}\right)\left(\frac{\omega_f}{\omega_n}\right)\right]^2}}{\sqrt{\left[1-\left(\frac{\omega_f}{\omega_n}\right)^2\right]^2+\left[2\left(\frac{c}{c_c}\right)\left(\frac{\omega_f}{\omega_n}\right)\right]^2}}$$ Q.E.D. ◀



PROBLEM 19.148

A 91-kg machine element supported by four springs, each of constant $k = 175$ N/m, is subjected to a periodic force of frequency 0.8 Hz and amplitude 89 N. Determine the amplitude of the fluctuating force transmitted to the foundation if (*a*) a dashpot with a coefficient of damping $c = 365$ N · s/m is connected to the machine element and to the ground, (*b*) the dashpot is removed.

SOLUTION

Forcing frequency: $\omega_f = 2\pi f_f = (2\pi)(0.8) = 1.6\pi$ rad/s

Exciting force amplitude: $P_m = 89$ N

Equivalent spring constant: $k = (4)(175\text{ N/m}) = 700$ N/m

Natural frequency:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{700}{91}}$$
$$= 2.7735 \text{ rad/s}$$

Frequency ratio:

$$\frac{\omega_f}{\omega_n} = \frac{1.6\pi}{2.7735}$$
$$= 1.8123$$

Critical damping coefficient:

$$c_c = 2\sqrt{km}$$
$$= 2\sqrt{(700)(91)}$$
$$= 504.78 \text{ N}\cdot\text{s/m}$$

From the derivation given in Problem 19.147, the amplitude of the force transmitted to the foundation is

$$F_m = \frac{P_m\sqrt{1+\left[2\left(\frac{c}{c_c}\right)\left(\frac{\omega_f}{\omega_n}\right)\right]^2}}{\sqrt{\left[1-\left(\frac{\omega_f}{\omega_n}\right)^2\right]^2+\left[2\left(\frac{c}{c_c}\right)\left(\frac{\omega_f}{\omega_n}\right)\right]^2}} \qquad (1)$$

$$1-\left(\frac{\omega_f}{\omega_n}\right)^2 = 1-(1.8123)^2 = -2.2844$$

PROBLEM 19.148 (Continued)

(*a*) F_m when $c = 365\ \text{N}\cdot\text{s/m}$:

$$\frac{c}{c_c} = \frac{365}{504.78}$$

$$= 0.72309$$

$$2\left(\frac{c}{c_c}\right)\left(\frac{\omega_f}{\omega_n}\right) = (2)(0.72309)(1.8123)$$

$$= 2.6209$$

From Eq. (1):

$$F_m = \frac{89\sqrt{1+(2.6209)^2}}{\sqrt{(-2.2844)^2+(2.6209)^2}}$$

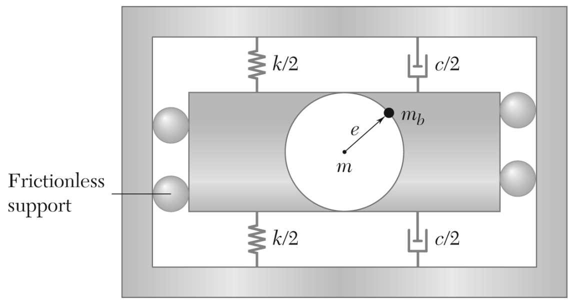
$$= \frac{89\sqrt{7.8692}}{\sqrt{12.088}}$$

$F_m = 71.8\ \text{N}$ ◀

(*b*) F_m when $c = 0$:

$$F_m = \frac{P_m}{\left|1-\left(\frac{\omega_f}{\omega_n}\right)^2\right|} = \frac{89}{2.2844}$$

$F_m = 39.0\ \text{N}$ ◀



PROBLEM 19.149

A simplified model of a washing machine is shown. A bundle of wet clothes forms a weight w_b of 20 lb in the machine and causes a rotating unbalance. The rotating mass is 40 lb (including m_b) and the radius of the washer basket e is 9 in. Knowing the washer has an equivalent spring constant k = 70 lb/ft and damping ratio $\zeta = c/c_c = 0.05$ and during the spin cycle the drum rotates at 250 rpm, determine the amplitude of the motion and the magnitude of the force transmitted to the sides of the washing machine.

SOLUTION

Forced circular frequency: $\omega_f = \dfrac{(2\pi)(250)}{60} = 26.18 \text{ rad/s}$

System mass: $m = \dfrac{W}{g} = \dfrac{40 \text{ lb}}{32.2}$

Spring constant: $k = 70 \text{ lb/ft}$

Natural circular frequency: $\omega_n = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{70}{\frac{40}{32.2}}} = 7.5067 \text{ rad/s}$

Critical damping constant: $c_c = 2\sqrt{km} = 2\sqrt{(70)\left(\frac{40}{32.2}\right)} = 18.650 \text{ lb} \cdot \text{s/ft}$

Damping constant: $c = \left(\dfrac{c}{c_c}\right)c_c = (0.05)(18.650) = 0.9325 \text{ lb} \cdot \text{s/ft}$

Unbalance force: $m_b = \dfrac{w_b}{g}$

$$P_m = m_b e \omega_f^2$$

$$P_m = \left(\frac{20 \text{ lb}}{32.2}\right)\left(\frac{9}{12}\text{ft}\right)(26.18 \text{ rad/s})^2 = 319.28 \text{ lb}$$

The differential equation of motion is

$$m\ddot{x} + c\dot{x} + kx = P_m \sin \omega_f t$$

The steady state response is

$$x = x_m \sin(\omega_f t - \varphi) \qquad \dot{x} = \omega_f x_m \cos(\omega_f t - \varphi)$$

PROBLEM 19.149 (Continued)

where
$$x_m = \frac{P_m}{\sqrt{\left(k - m\omega_f^2\right)^2 + (c\omega_f)^2}}$$
$$= \frac{319.28}{\sqrt{[70 - \left(\frac{40}{32.2}\right)(26.18)^2]^2 + [(0.9325)(26.18)]^2}}$$
$$= \frac{319.28}{\sqrt{(-781.42)^2 + (24.413)^2}} = \frac{319.28}{781.796} = 0.40839 \text{ ft}$$

(*a*) <u>Amplitude of vibration</u>. $x_m = 4.90$ in. ◀

$$x = 0.40839 \sin(\omega_f t - \varphi)$$
$$\dot{x} = (26.18)(0.40839)\cos(\omega_f t - \varphi)$$
$$= 10.6917 \cos(\omega_f t - \varphi)$$

Spring force:
$$kx = (70)(0.40839)\sin(\omega_f t - \varphi)$$
$$= 28.588 \sin(\omega_f t - \varphi)$$

Damping force:
$$c\dot{x} = (0.9325)(10.6917)\cos(\omega_f t - \varphi)$$
$$= 9.9701 \cos(\omega_f t - \varphi)$$

(*b*) Total force:
$$F = 28.588 \sin(\omega_f t - \varphi) + 9.9701 \cos(\omega_f t - \varphi)$$

Let
$$F = F_m \cos\psi \sin(\omega_f t - \varphi) + F_m \sin\psi \sin(\omega_f t - \varphi)$$
$$= F_m \sin(\omega_f t - \varphi + \psi)$$

<u>Maximum force</u>.
$$F_m^2 = F_m^2 \cos^2\psi + F_m^2 \sin^2\psi$$
$$= (28.588)^2 + (9.9701)^2$$
$$= 916.65$$

$F_m = 30.3$ lb ◀

PROBLEM 19.150*

For a steady-state vibration with damping under a harmonic force, show that the mechanical energy dissipated per cycle by the dashpot is $E = \pi c x_m^2 \omega_f$, where c is the coefficient of damping, x_m is the amplitude of the motion, and ω_f is the circular frequency of the harmonic force.

SOLUTION

Energy is dissipated by the dashpot.

From Equation (19.48), the deflection of the system is

$$x = x_m \sin(\omega_f t - \varphi)$$

The force on the dashpot.

$$F_d = c\dot{x}$$

$$F_d = c x_m \omega_f \cos(\omega_f t - \varphi)$$

The work done in a complete cycle with

$$\tau_f = \frac{2\pi}{\omega_f}$$

$$E = \int F_d dx \text{ (i.e., force} \times \text{distance)}$$

$$dx = x_m \omega_f \cos(\omega_f t - \varphi) dt$$

$$E = \int_0^{2\pi/\omega_f} c x_m^2 \omega_f^2 \cos^2(\omega_f t - \varphi) dt$$

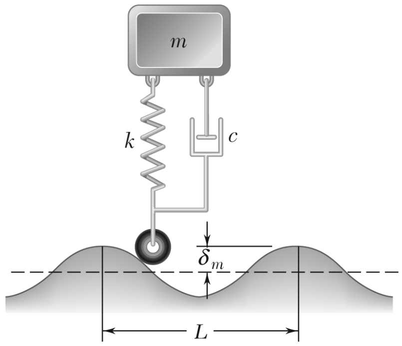
$$\cos^2(\omega_D t - \varphi) = \frac{[1 - 2\cos(\omega_f t - \phi)]}{2}$$

$$E = c x_m^2 \omega_f^2 \int_0^{2\pi/\omega_f} \frac{1 - 2\cos(\omega_f t - \phi)}{2} dt$$

$$E = \frac{c x_m^2 \omega_f^2}{2} \left[t - \frac{2\sin(\omega_f t - \phi)}{\omega_f} \right]_0^{2\pi/\omega_f}$$

$$E = \frac{c x_m^2 \omega_f^2}{2} \left[\frac{2\pi}{\omega_f} - \frac{2}{\omega_f} (\sin(2\pi - \varphi) + \sin\varphi) \right]$$

$E = \pi c x_m^2 \omega_f$ Q.E.D. ◀

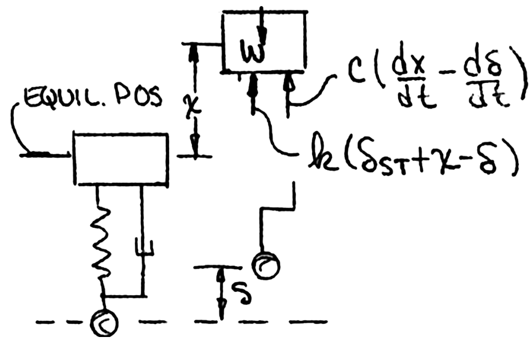


PROBLEM 19.151*

The suspension of an automobile can be approximated by the simplified spring-and-dashpot system shown. (*a*) Write the differential equation defining the vertical displacement of the mass *m* when the system moves at a speed *v* over a road with a sinusoidal cross section of amplitude δ_m and wave length *L*. (*b*) Derive an expression for the amplitude of the vertical displacement of the mass *m*.

SOLUTION

(*a*)



$$+\downarrow \Sigma F = ma: \quad W - k(\delta_{st} + x - \delta) - c\left(\frac{dx}{dt} - \frac{d\delta}{dt}\right) = m\frac{d^2x}{dt^2}$$

Recalling that $W = k\delta_{st}$, we write

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = k\delta + c\frac{d\delta}{dt} \tag{1}$$

Motion of wheel is a sine curve, $\delta = \delta_m \sin\omega_f t$. The interval of time needed to travel a distance *L* at a speed *v* is $t = \frac{L}{v}$.

Thus,
$$\omega_f = \frac{2\pi}{\tau_f} = \frac{2\pi}{\frac{L}{v}} = \frac{2\pi v}{L}$$

and
$$\delta = \delta_m \sin\omega_f t \qquad \frac{d\delta}{dt} = \frac{\delta_m 2\pi}{\frac{L}{v}}\cos\omega_f t$$

Thus, Equation (1) is

$$m\frac{d^2x}{dt} + c\frac{dx}{dt} + kx = (k\sin\omega_f t + c\omega_f\cos\omega_f t)\delta_m \blacktriangleleft$$

PROBLEM 19.151* (Continued)

(*b*) From the identity $$A\sin y + B\cos y = \sqrt{A^2 + B^2}\,\sin(y + \psi)$$

$$\sin\psi = \frac{B}{\sqrt{A^2 + B^2}}$$

$$\cos\psi = \frac{A}{\sqrt{A^2 + B^2}}$$

We can write the differential equation

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = \delta_m\sqrt{k^2 + (c\omega_f)^2}\,\sin(\omega_f t + \psi)$$

$$\psi = \tan^{-1}\frac{c\omega_f}{k}$$

The solution to this equation is analogous to Equations 19.47 and 19.48, with

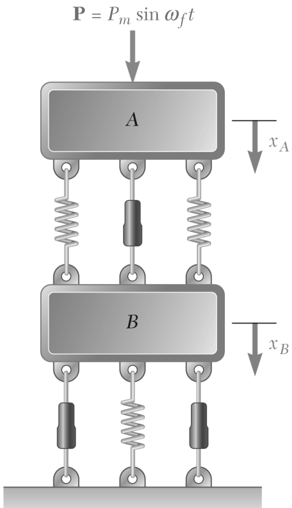
$$P_m = \delta_m\sqrt{k^2 + (c\omega_f)^2}$$

$$x = x_m\sin(\omega_f t - \varphi + \psi) \text{ (where analogous to Equations (19.52))} \blacktriangleleft$$

$$x_m = \frac{\delta_m\sqrt{k^2 + (c\omega_f)^2}}{\sqrt{\left(k - m\omega_f^2\right)^2 + (c\omega_f)^2}} \blacktriangleleft$$

$$\tan\varphi = \frac{c\omega_f}{k - m\omega_f^2} \blacktriangleleft$$

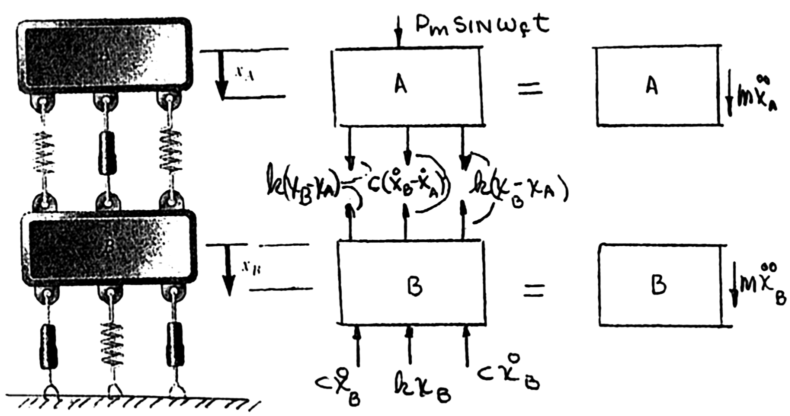
$$\tan\psi = \frac{c\omega_f}{k} \blacktriangleleft$$



PROBLEM 19.152*

Two blocks A and B, each of mass m, are supported as shown by three springs of the same constant k. Blocks A and B are connected by a dashpot, and block B is connected to the ground by two dashpots, each dashpot having the same coefficient of damping c. Block A is subjected to a force of magnitude $P = P_m \sin \omega_f t$. Write the differential equations defining the displacements x_A and x_B of the two blocks from their equilibrium positions.

SOLUTION



Since the origins of coordinates are chosen from the equilibrium position, we may omit the initial spring compressions and the effect of gravity

For load A,

$$+\downarrow \Sigma F = ma_A: \quad P_m \sin \omega_f t + 2k(x_B - x_A) + c(\dot{x}_B - \dot{x}_A) = m\ddot{x}_A \qquad (1)$$

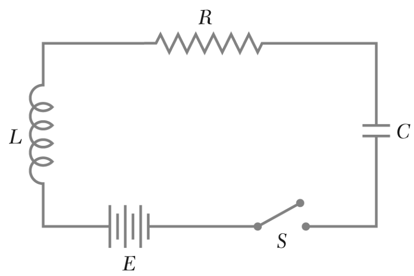
For load B,

$$+\downarrow \Sigma F = ma_B: \quad -2k(x_B - x_A) - c(\dot{x}_B - \dot{x}_A) - kx_B - 2c\dot{x}_B = m\ddot{x}_B \qquad (2)$$

Rearranging Equations (1) and (2), we find:

$$m\ddot{x}_A + c(\dot{x}_A - \dot{x}_B) + 2k(x_A - x_B) = P_m \sin \omega_f t \blacktriangleleft$$

$$m\ddot{x}_B + 3c\dot{x}_B - c\dot{x}_A + 3kx_B - 2kx_A = 0 \blacktriangleleft$$



PROBLEM 19.153

Express in terms of L, C, and E the range of values of the resistance R for which oscillations will take place in the circuit shown when switch S is closed.

SOLUTION

For a mechanical system, oscillations take place if $c < c_c$ (lightly damped).

But from Equation (19.41),

$$c_c = 2m\sqrt{\frac{k}{m}} = 2\sqrt{km}$$

Therefore,

$$c < 2\sqrt{km} \qquad (1)$$

From Table 19.2:

$$c \longrightarrow R$$

$$m \longrightarrow L$$

$$k \longrightarrow \frac{1}{C} \qquad (2)$$

Substituting in Eq. (1) the analogous electrical values in Eq. (2), we find that oscillations will take place if

$$R < 2\sqrt{\left(\frac{1}{C}\right)(L)}$$

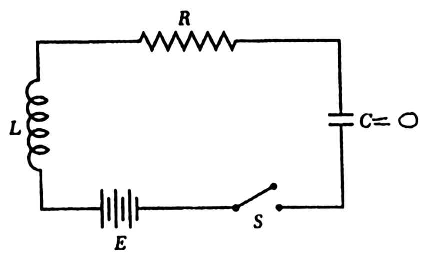
$$R < 2\sqrt{\frac{L}{C}} \blacktriangleleft$$

PROBLEM 19.154

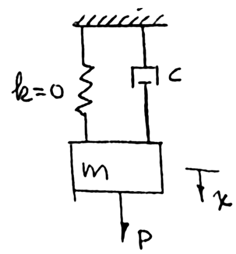
Consider the circuit of Problem 19.153 when the capacitor C is removed. If switch S is closed at time $t = 0$, determine (a) the final value of the current in the circuit, (b) the time t at which the current will have reached $(1 - 1/e)$ times its final value. (The desired value of t is known as the *time constant* of the circuit.)

SOLUTION

Electrical system

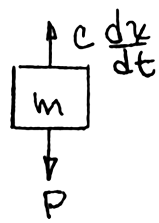


Mechanical system



The mechanical analogue of closing a switch S is the sudden application of a constant force of magnitude P to the mass.

(a) Final value of the current corresponds to the final velocity of the mass, and since the capacitance is zero, the spring constant is also zero



$$+\!\downarrow \Sigma F = ma: \quad P - c\frac{dx}{dt} = m\frac{d^2x}{dt^2} \qquad (1)$$

Final velocity occurs when $\dfrac{d^2x}{dt^2} = 0$

$$P - c\left.\frac{dx}{dt}\right|_{\text{final}} = 0 \qquad \left.\frac{dx}{dt}\right|_{\text{final}} = v_{\text{final}}$$

$$v_{\text{final}} = \frac{P}{c}$$

From Table 19.2: $v \longrightarrow i, \quad P \longrightarrow E, \quad c \longrightarrow R$

Thus, $$i_{\text{final}} = \frac{E}{R} \blacktriangleleft$$

PROBLEM 19.154 (Continued)

(*b*) Rearranging Equation (1), we have

$$m\frac{d^2x}{dt^2}+c\frac{dx}{dt}=P$$

Substitute $\quad \frac{dx}{dt}=Ae^{-\lambda t}+\frac{P}{c}; \qquad \frac{d^2x}{dt}=-A\lambda e^{-\lambda t}$

$$m\left[-A\lambda e^{-\lambda t}\right]+c\left[Ae^{-\lambda t}+\frac{P}{c}\right]=P$$

$$-m\lambda+c=0 \qquad \lambda=\frac{c}{m}$$

Thus, $\quad \frac{dx}{dt}=Ae^{-(c/m)t}+\frac{P}{c}$

At $t=0$, $\quad \frac{dx}{dt}=0 \quad 0=A+\frac{P}{c} \qquad A=-\frac{P}{c}$

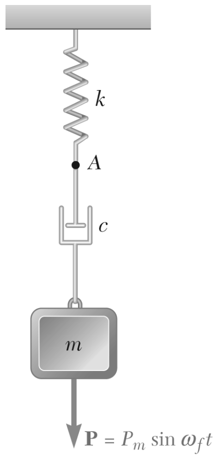
$$v=\frac{dx}{dt}=\frac{p}{c}\left[1-e^{-(c/m)t}\right]$$

From Table 19.2: $\quad v \longrightarrow i, \; P \longrightarrow E, \; c \longrightarrow R, \; m \longrightarrow L$

$$L=\frac{E}{R}\left[1-e^{-(R/L)t}\right]$$

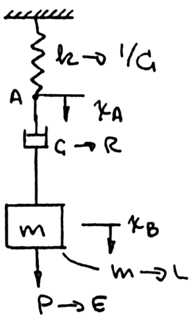
For $i=\left(\frac{E}{R}\right)\left(1-\frac{1}{e}\right)$, $\quad \left(\frac{R}{L}\right)t=1 \qquad t=\frac{L}{R}$ ◀

PROBLEM 19.155



Draw the electrical analogue of the mechanical system shown. (*Hint:* Draw the loops corresponding to the free bodies *m* and *A*.)

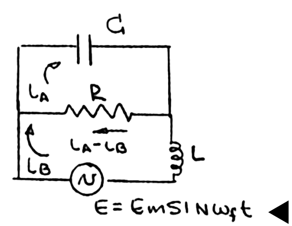
SOLUTION

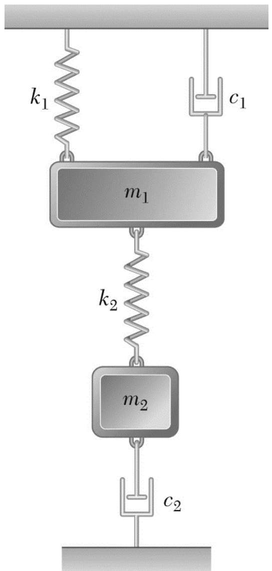


We note that both the spring and the dashpot affect the motion of Point *A*. Thus, one loop in the electrical circuit should consist of a capacitor $\left(k \Rightarrow \frac{1}{C}\right)$ and a resistance $(c \Rightarrow R)$.

The other loop consists of $(P_m \sin \omega_f t \longrightarrow E_m \sin \omega_f t)$, an inductor $(m \longrightarrow L)$ and the resistor $(c \longrightarrow R)$.

Since the resistor is common to both loops, the circuit is

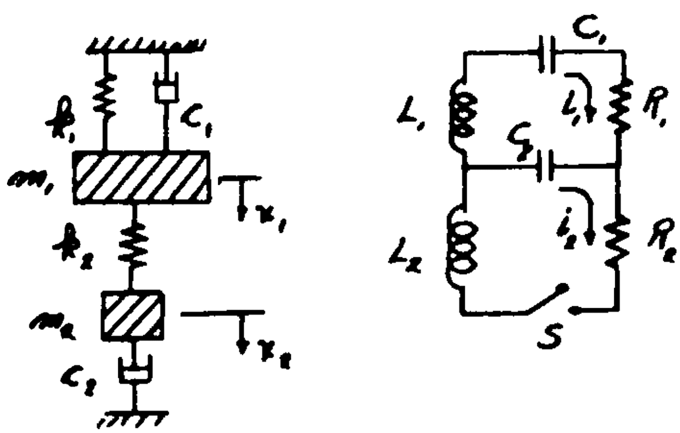




PROBLEM 19.156

Draw the electrical analogue of the mechanical system shown. (*Hint:* Draw the loops corresponding to the free bodies *m* and *A*.)

SOLUTION



Loop 1 (Mass 1)

$k_1 \longrightarrow 1/C_1$

$c_1 \longrightarrow R_1$

$m_1 \longrightarrow L_1$

$x_1 \longrightarrow q_1$

$\dot{x}_1 \longrightarrow \dot{x}_1$

Loop 2 (Mass 2)

$k_2 \longrightarrow 1/C_2$

$c_2 \longrightarrow R_2$

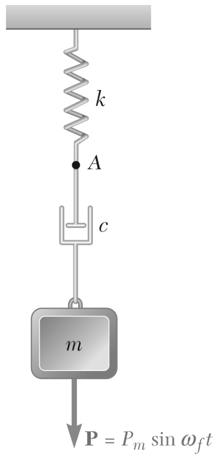
$m_2 \longrightarrow L_2$

$x_2 \longrightarrow q_2$

$\dot{x}_2 \longrightarrow \dot{x}_2$

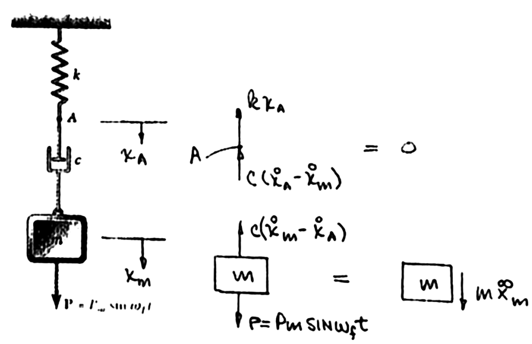
k_2 is connected to both masses, so C_2 is common to both loops.

PROBLEM 19.157



Write the differential equations defining (*a*) the displacements of the mass *m* and of the Point *A*, (*b*) the charges on the capacitors of the electrical analogue.

SOLUTION



(*a*) Mechanical system.

Point *A*: $+\uparrow \Sigma F = 0$: $c\dfrac{d}{dt}(x_A - x_m) + kx_A = 0$ ◀

Mass *m*:

$$+\uparrow \Sigma F = ma: \quad c\frac{d}{dt}(x_m - x_A) - P_m \sin \omega_f t = -m\frac{d^2 x_m}{dt^2}$$

$$m\frac{d^2 x_m}{dt^2} + c\frac{d}{dt}(x_m - x_A) = P_m \sin \omega_f t \quad ◀$$

(*b*) Electrical analogue.
From Table 19.2:

$$m \longrightarrow L$$
$$c \longrightarrow R$$
$$k \longrightarrow \frac{1}{C}$$
$$x \longrightarrow q$$
$$P \longrightarrow E$$

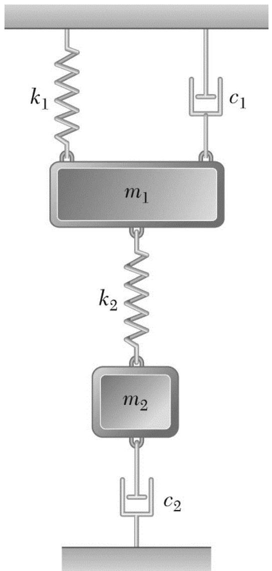
Substituting into the results from Part (*a*), the analogous electrical characteristics,

$$R\frac{d}{dt}(q_A - q_m) + \left(\frac{1}{C}\right)q_n = 0 \blacktriangleleft$$

$$L\frac{d^2q_m}{dt^2} + R\frac{d}{dt}(q_m - q_A) = E_m \sin \omega_f t \blacktriangleleft$$

Note: These equations can also be obtained by summing the voltage drops around the loops in the circuit of Problem 19.155.

PROBLEM 19.158



Write the differential equations defining (*a*) the displacements of the masses m_1 and m_2, (*b*) the charges on the capacitors of the electrical analogue.

SOLUTION

(*a*) Displacements at masses m_1 and m_2

$$m_1 \frac{d^2x_1}{dt^2} + c_1 \frac{dx_1}{dt} + k_1 x_1 + k_2(x_1 - x_2) = 0 \quad \blacktriangleleft$$

$$m_2 \frac{d^2x_2}{dt^2} + c_2 \frac{dx_2}{dt} + k_2(x_2 - x_1) = 0 \quad \blacktriangleleft$$

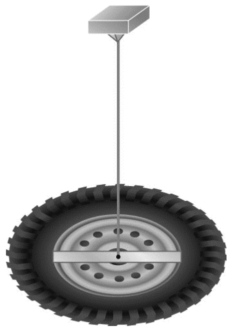
(*b*) Electrical analogues.

We let: $q_1 = \int i_1 dt \qquad q_2 = \int i_2 dt$

Thus, $i_1 = \dfrac{dq_1}{dt} \qquad i_2 = \dfrac{dq_2}{dt}$

$$L_1 \frac{d^2q_1}{dt^2} + R_1 \frac{dq_1}{dt} + \frac{q_1}{C_1} + \frac{(q_1 - q_2)}{C_2} = 0 \quad \blacktriangleleft$$

$$L_2 \frac{d^2q_2}{dt^2} + R_2 \frac{dq_2}{dt} + \frac{q_2 - q_1}{C_2} = 0 \quad \blacktriangleleft$$



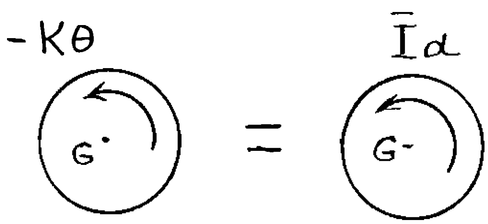
PROBLEM 19.159

An automobile wheel-and-tire assembly of total weight 47 lb is attached to a mounting plate of negligible weight which is suspended from a steel wire. The torsional spring constant of the wire is known to be $K = 0.40\ \text{lb}\cdot\text{in./rad}$. The wheel is rotated through 90° about the vertical and then released. Knowing that the period of oscillation is observed to be 30 s, determine the centroidal mass moment of inertia and the centroidal radius of gyration of the wheel-and-tire assembly.

SOLUTION

Torsional spring constant: $K = 0.40\ \text{lb}\cdot\text{in/rad} = 33.333\times10^{-3}\ \text{lb}\cdot\text{ft/rad}$

Let the wheel-and-tire assembly be rotated through the small angle θ. The moment that the wire exerts on the assembly is



$$M = -K\theta$$

$$\Sigma M = \Sigma M_{\text{eff}} = \bar{I}\alpha: \quad -K\theta = \bar{I}\alpha = \bar{I}\ddot{\theta}$$

$$\ddot{\theta} + \frac{K}{\bar{I}}\theta = 0$$

$$\omega_n^2 = \frac{K}{\bar{I}} \qquad (1)$$

Frequency: $f = \dfrac{1}{\tau} = \dfrac{1}{30\ \text{s}} = 0.033333\ \text{H}_z$

$$\omega_n = 2\pi f = (2\pi)(0.033333) = 0.20944\ \text{rad/s}$$

From Eq. (1), $\bar{I} = \dfrac{K}{\omega_n^2} = \dfrac{33.333\times10^{-3}\ \text{lb}\cdot\text{ft/rad}}{(0.20944\ \text{rad/s})^2}$

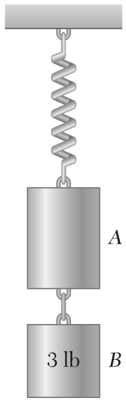
Centroidal mass moment of inertia: $\bar{I} = 0.75990\ \text{lb}\cdot\text{s}^2\cdot\text{ft}$ $\qquad \bar{I} = 0.760\ \text{lb}\cdot\text{s}^2\cdot\text{ft}$ ◀

Mass: $m = \dfrac{W}{g} = \dfrac{47\ \text{lb}}{32.2\ \text{ft/s}^2} = 1.4596\ \text{lb}\cdot\text{s}^2/\text{ft}$

Centroidal radius of gyration: $\bar{k}^2 = \dfrac{\bar{I}}{m} = \dfrac{0.75990\ \text{lb}\cdot\text{s}^2\cdot\text{ft}}{1.4596\ \text{lb}\cdot\text{s}^2/\text{ft}} = 0.52061\ \text{ft}^2$

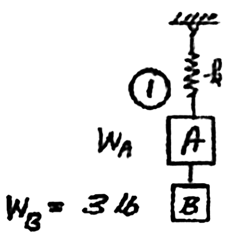
$\bar{k} = 0.7215\ \text{ft}$ $\qquad \bar{k} = 8.66\ \text{in.}$ ◀

PROBLEM 19.160



The period of vibration of the system shown is observed to be 0.6 s. After cylinder *B* has been removed, the period is observed to be 0.5 s. Determine (*a*) the weight of cylinder *A*, (*b*) the constant of the spring.

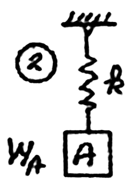
SOLUTION



$$m_1 = \frac{W_A + 3}{g} \qquad \tau_1 = 0.6 \text{ s}$$

$$\tau_1 = \frac{2\pi}{\omega_1} \qquad \omega_1 = \frac{2\pi}{\tau_1} = \frac{2\pi}{0.6} = 3.333\pi \text{ rad/s}$$

$$\omega_1^2 = \frac{k}{m_1} \qquad k = m_1\omega_1^2 = \left(\frac{W_A + 3}{g}\right)(3.333\pi)^2 \qquad (1)$$



$$m_2 = \frac{W_A}{g} \qquad \tau_2 = 0.5 \text{ s}$$

$$\tau_2 = \frac{2\pi}{\omega_2} \qquad \omega_2 = \frac{2\pi}{\tau_2} = \frac{2\pi}{0.5} = 4\pi \text{ rad/s}$$

$$\omega_2^2 = \frac{k}{m_2} \qquad k = m_2\omega_2^2 = \frac{W_A}{g}(4\pi)^2 \qquad (2)$$

(*a*) Equating the expressions found for *k* in Eqs. (1) and (2):

$$\frac{W_A + 3}{g}(3.333\pi)^2 = \frac{W_A}{g}(4\pi)^2$$

$$(11.111)(W_A + 3) = 16W_A$$

$$4.889W_A = 33.333$$

$$W_A = 6.818 \text{ lb}$$

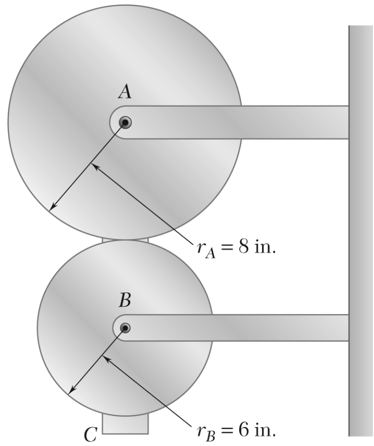
$W_A = 6.82$ lb ◀

(*b*) Eq. (1):

$$k = \frac{6.818 \text{ lb} + 3 \text{ lb}}{32.2 \text{ ft/s}^2}(3.333\pi \text{ rad/s})^2$$

$$k = 33.44 \text{ lb/ft}$$

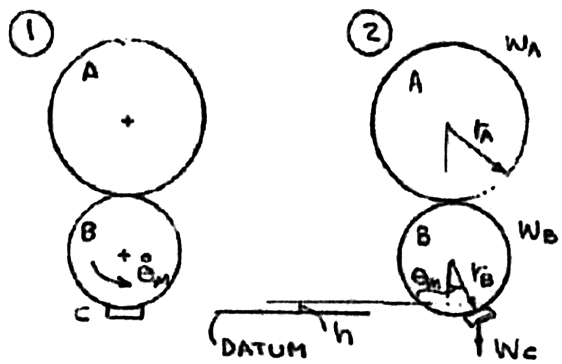
$k = 33.4$ lb/ft ◀



PROBLEM 19.161

Disks A and B weigh 30 lb and 12 lb, respectively, and a small 5-lb block C is attached to the rim of disk B. Assuming that no slipping occurs between the disks, determine the period of small oscillations of the system.

SOLUTION



Small oscillations: $$h = r_B(1-\cos\theta_m) \approx \frac{r_B\theta_B^2}{2}$$

Position ① $$r_B\dot{\theta}_B = r_A\dot{\theta}_A$$

$$T_1 = \frac{1}{2}m_C(r_B\dot{\theta}_m)^2 + \frac{1}{2}\bar{I}_B\dot{\theta}_m^2 + \frac{1}{2}\bar{I}_A\left(\frac{r_B}{r_A}\dot{\theta}_m\right)^2$$

$$\bar{I}_B = \frac{m_B r_B^2}{2}$$

$$\bar{I}_A = \frac{m_A r_A^2}{2}$$

$$T_1 = \frac{1}{2}\left[m_C r_B^2 + \frac{m_B r_B^2}{2} + \left(\frac{m_A r_A^2}{2}\right)\left(\frac{r_B}{r_A}\right)^2\right]\dot{\theta}_m^2$$

$$T_1 = \frac{1}{2}\left[\left(m_C + \frac{m_B}{2} + \frac{m_A}{2}\right)\right]r_B^2\dot{\theta}_m^2$$

$$V_1 = 0$$

Position ②

$$T_2 = 0$$

$$V_2 = m_C gh$$

$$= \frac{m_C g\dot{\theta}_m^2}{2}$$

Conservation of energy and simple harmonic motion.

$$T_1 + V_1 = T_2 + V_2$$

$$\dot{\theta}_m = \omega_n \theta_m$$

$$\frac{1}{2}\left[\left(m_C + \frac{m_B}{2} + \frac{m_A}{2}\right)\right] r_B^2 \omega_n^2 \theta_m^2 + 0 = 0 + \frac{m_C g r_B \theta_m^2}{2}$$

$$\omega_n^2 = \frac{m_C}{m_C + \frac{(m_B + m_A)}{2}} \frac{g}{r_B}$$

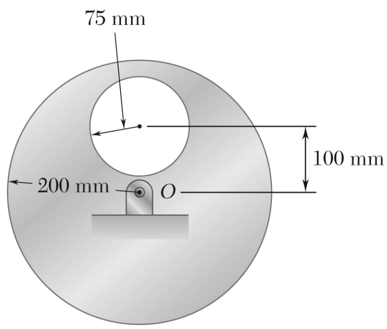
$$\omega_n^2 = \frac{5}{5 + \frac{(12+30)}{2}} \frac{(32.2\ \text{ft/s}^2)}{\left(\frac{6}{12}\right)\text{ft}}$$

$$\omega_n^2 = 12.39\ \text{s}^{-2}$$

Period of small oscillations.

$$\tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{\sqrt{12.39}}$$

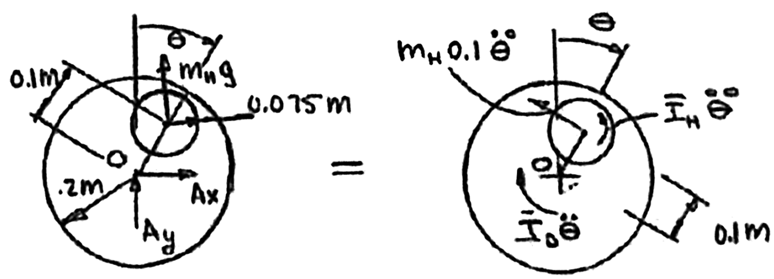
$\tau_n = 1.785\ \text{s}$ ◀



PROBLEM 19.162

A 75-mm-radius hole is cut in a 200-mm-radius uniform disk, which is attached to a frictionless pin at its geometric center O. Determine (a) the period of small oscillations of the disk, (b) the length of a simple pendulum which has the same period.

SOLUTION



Equation of motion.

$$\Sigma M_0 = (\Sigma M_0)_{\text{eff}}: \quad \circlearrowleft + -m_H g(0.1)\sin\theta = \bar{I}_D\theta - I_H\ddot{\theta} - (0.1)^2 m_H\ddot{\theta}$$

$$m_D = \rho t\pi R^2 = (\rho t\pi)(0.2)^2 = (0.04)\pi\rho t$$

$$m_H = \rho t\pi r^2 = (\rho t\pi)(0.075)^2 = (0.005625)\pi\rho t$$

$$I_D = \frac{1}{2}m_D R^2 = \frac{1}{2}(0.04\pi\rho t)(0.2)^2 = 800\times10^{-6}\pi\rho t$$

$$I_H = \frac{1}{2}m_H r^2 = \frac{1}{2}(0.005625\pi\rho t)(0.075)^2 = 15.82\times10^{-6}\pi\rho t$$

Small angles. $\sin\theta \approx \theta$

$$[(800\times10^{-6}\pi - 15.82\times10^{-6}\pi - (0.1)^2(0.005625\pi)]\rho t\ddot{\theta} + (0.005625)\pi\rho t(9.81)(0.1)\theta = 0$$

$$727.9\times10^{-6}\ddot{\theta} + 5.518\times10^{-3}\theta = 0$$

PROBLEM 19.162 (Continued)

(*a*) Natural frequency and period.

$$\omega_n^2 = \frac{5.518\times10^{-3}}{727.9\times10^{-6}}$$

$$= 7.581$$

$$\omega_n = 2.753 \text{ rad/s}$$

$$\tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{2.753}$$

$\tau_n = 2.28$ s ◀

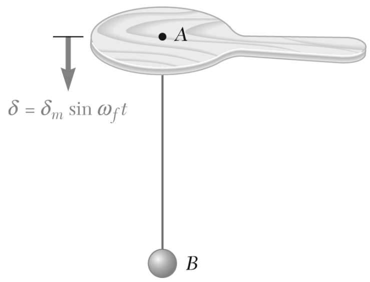
(*b*) Length and period of a simple pendulum.

$$\tau_n = 2\pi\sqrt{\frac{l}{g}}$$

$$l = \left(\frac{\tau_n}{2\pi}\right)^2 g$$

$$l = \left[\frac{(2.753)}{2\pi}\right]^2 (9.81 \text{ m/s}^2)$$

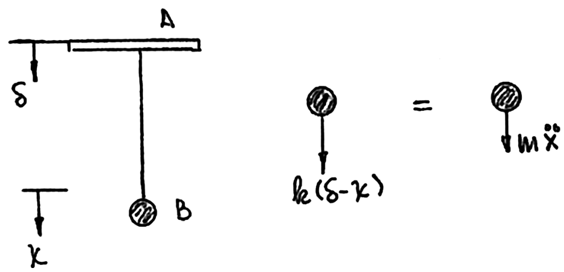
$l = 1.294$ m ◀



PROBLEM 19.163

An 0.8-lb ball is connected to a paddle by means of an elastic cord AB of constant $k = 5$ lb/ft. Knowing that the paddle is moved vertically according to the relation $\delta = \delta_m \sin \omega_f t$, where $\delta_m = 8$ in., determine the maximum allowable circular frequency ω_f if the cord is not to become slack.

SOLUTION



$$\Sigma F = ma \qquad k(\delta - x) = m\ddot{x} \qquad \ddot{x} + \left(\frac{k}{m}\right)x = \delta$$

From Equation (19.31 and 19.33′):

$$x_m = \frac{\delta_m}{\left(1 - \frac{\omega_f^2}{\omega_n^2}\right)}$$

Data:

$$m = \frac{W}{g} = \frac{0.8}{32.2} = 0.024845 \text{ lb} \cdot \text{s}^2/\text{ft}$$

$$k = 5 \text{ lb/ft} \qquad \delta_m = 8 \text{ in.} = 0.66667 \text{ ft}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{5}{0.024845}} = 14.186 \text{ rad/s}$$

The cord becomes slack if $x_m - \delta_m$ exceeds δ_{st}, where

$$\delta_{st} = \frac{W}{k} = \frac{0.8 \text{ lb}}{5 \text{ lb/ft}} = 0.16 \text{ ft}$$

PROBLEM 19.163 (Continued)

Then $$\frac{0.66667}{1-\left(\frac{\omega_f}{\omega_n}\right)^2}-0.66667<0.16$$

$$0.66667-0.66667+0.66667\left(\frac{\omega_f}{\omega_n}\right)^2<0.16-0.16\left(\frac{\omega_f}{\omega_n}\right)^2$$

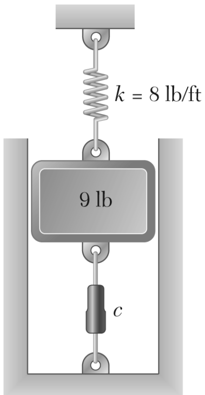
$$0.82667\left(\frac{\omega_f}{\omega_n}\right)^2<0.16$$

$$\frac{\omega_f}{\omega_n}<\sqrt{\frac{0.16}{0.82667}}=0.43994$$

Maximum allowable circular frequency.

$$\omega_f<(0.43994)(14.186\text{ rad/s})$$

$\omega_f < 6.24$ rad/s ◀



PROBLEM 19.164

The block shown is depressed 1.2 in. from its equilibrium position and released. Knowing that after 10 cycles the maximum displacement of the block is 0.5 in., determine (*a*) the damping factor *c/c*, (*b*) the value of the coefficient of viscous damping. (*Hint:* See Problems 19.129 and 19.130.)

SOLUTION

From Problems 19.130 and 19.129:

$$\left(\frac{1}{k}\right)\ln\left(\frac{x_n}{x_{n+k}}\right)=\frac{2\pi\frac{c}{c_c}}{\sqrt{1-\left(\frac{c}{c_c}\right)^2}}$$

where k = number of cycles = 10

(*a*) First maximum is $x_1 = 1.2$ in.

Thus, $n = 1$

$$\frac{x_1}{x_{1+10}}=\frac{1.2}{0.5}=2.4$$

$$\frac{1}{10}\ln 2.4 = 0.08755$$

$$=\frac{2\pi\frac{c}{c_c}}{\sqrt{1-\left(\frac{c}{c_c}\right)^2}}$$

Damping factor.

$$1-\left(\frac{c}{c_c}\right)^2=\left(\frac{2\pi}{0.08755}\right)^2\left(\frac{c}{c_c}\right)^2$$

$$\left(\frac{c}{c_c}\right)^2\left[\left(\frac{2\pi}{0.08755}\right)^2+1\right]=1$$

$$\left(\frac{c}{c_c}\right)^2=\frac{1}{(5150+1)}$$

$$=0.0001941$$

$$\frac{c}{c_c}=0.01393 \blacktriangleleft$$

PROBLEM 19.164 (Continued)

(*b*) Critical damping coefficient. $c_c = 2m\sqrt{\frac{k}{m}}$ (Eq. 19.41)

or $c_c = 2\sqrt{km}$

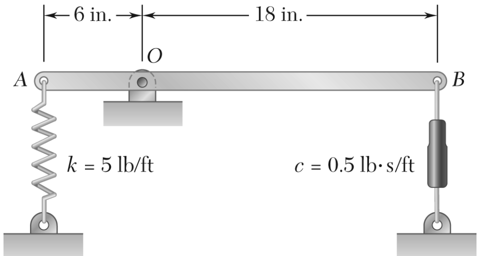
$$c_c = 2\sqrt{(8\ \text{lb/ft})\left(\frac{9\ \text{lb}}{32.2\ \text{ft/s}^2}\right)}$$

$$c_c = 2.991\ \text{lb}\cdot\text{s/ft}$$

From Part (*a*), $\frac{c}{c_c} = 0.01393$

$$c = (0.01393)(2.991)$$

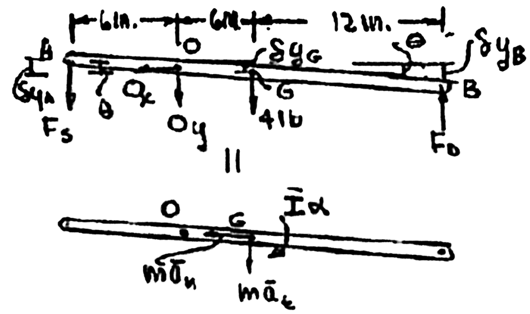
Coefficient of viscous damping. $c = 0.0417\ \text{lb}\cdot\text{s/ft}$ ◀



PROBLEM 19.165

A 4-lb uniform rod is supported by a pin at O and a spring at A, and is connected to a dashpot at B. Determine (a) the differential equation of motion for small oscillations, (b) the angle that the rod will form with the horizontal 5 s after end B has been pushed 0.9 in. down and released.

SOLUTION



Small angles: $\sin\theta \approx \theta, \quad \cos\theta \approx 1$

$$\delta y_A = \left(\frac{6}{12}\text{ ft}\right)\theta = \frac{\theta}{2}$$

$$\delta y_C = \left(\frac{6}{12}\text{ ft}\right)\theta = \frac{\theta}{2}$$

$$\delta y_B = \left(\frac{18}{12}\text{ ft}\right)\theta = \frac{3\theta}{2}$$

(a) Newton's Law: $\Sigma M_0 = (\Sigma M_0)_{\text{eff}}$

$$+\circlearrowleft\; -\left(\frac{6}{12}\text{ ft}\right)F_s + \left(\frac{6}{12}\text{ft}\right)(4) - \left(\frac{18}{12}\text{ ft}\right)F_D$$

$$= \bar{I}\alpha + \left(\frac{6}{12}\text{ ft}\right)m\bar{a}_t \qquad (1)$$

$$F_s = k(\delta y_A + (\delta_{\text{st}})_A) = k\left(\frac{\theta}{2} + (\delta_{\text{st}})_A\right)$$

$$F_D = c\delta\dot{y}_B = c\frac{3}{2}\dot{\theta}$$

$$\bar{I} = \frac{1}{12}ml^2 = \frac{1}{12}m\left(\frac{24}{12}\text{ft}\right)^2 = \frac{1}{3}m$$

Kinematics: $\alpha = \ddot{\theta}, \quad \bar{a}_t = \left(\frac{6}{12}\text{ft}\right)\alpha = \frac{\ddot{\theta}}{2}$

Thus, from Eq. (1),

$$\left[\frac{m}{3} + \frac{m}{4}\right]\ddot{\theta} + \left(\frac{3}{2}\right)^2 c\dot{\theta} + \left(\frac{k}{2}\right)\left(\frac{\theta}{2} + (\delta_{\text{st}})_A\right) - 2 = 0 \qquad (2)$$

But in equilibrium, $\Sigma M_0 = 0$

$$+\circlearrowleft\; k(\delta_{\text{st}})_A\left(\frac{6}{12}\right) - (4)\left(\frac{6}{12}\right) = 0, \quad \frac{k}{2}(\delta_{\text{st}})_A = 2$$

Equation (2) becomes

$$\left(\frac{7}{12}\right)m\ddot{\theta}+\left(\frac{9}{4}\right)\left(\ddot{\theta}+\frac{k}{4}\right)\theta=0$$

$$\frac{7}{12}m=\left(\frac{7}{12}\right)\left(\frac{4}{32.2}\right)=0.07246$$

$$\frac{9}{4}c=\left(\frac{9}{4}\right)(0.15)=0.3375$$

$$\frac{k}{4}=\frac{5}{4}=1.25$$

$$0.07246\ddot{\theta}+0.3375\dot{\theta}+1.25\theta=0 \blacktriangleleft$$

(*b*) Substituting $e^{\lambda t}$ into the above differential equation,

$$0.07246\lambda^2+0.3375\lambda+1.25=0$$

$$\lambda=\frac{(-0.3375\mp\sqrt{(0.3375)^2-4(.07246)(1.25)})}{(2)(0.07246)}$$

$$\lambda=\frac{(-0.3375\mp\sqrt{(-0.2484)})}{(2)(0.07246)}$$

$$\lambda=-2.329\pm3.439i$$

Since the roots are complex and conjugate (light damping), the solution to the differential equation is (Eq. 19.46):

$$\theta=\theta_0 e^{-2.329t}\sin(3.939t+\phi) \qquad (3)$$

Initial conditions.

$$(\delta y_B)(0)=0.9\text{ in.}$$

$$\theta(0)=\frac{(\delta y_B)}{18\text{ in.}}=\frac{0.9}{18}$$

$$\theta(0)=0.05\text{ rad}$$

$$\dot{\theta}(0)=0$$

From Eq. (3):

$$\theta(0)=0.05=\theta_0\sin\phi$$

$$\dot{\theta}(0)=0=-2.329\theta_0\sin\phi+3.439\theta_0\cos\phi$$

$$\tan\phi=\frac{3.439}{2.329}$$

$$\phi=0.9755\text{ rad}$$

$$\theta_0=\frac{0.05}{\sin(0.9755)}=0.06039\text{ rad}$$

PROBLEM 19.165 (Continued)

Substituting into Eq. (3), $\theta = 0.06039e^{-2.329t} \sin(3.439t + 0.9752)$

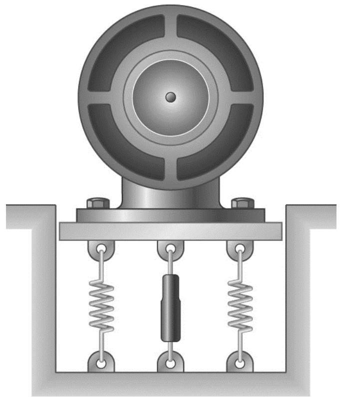
At $t = 5$ s, $\theta = 0.06039e^{-(2.329)(5)} \sin[(3.439)(5) + 0.9752]$

$= 0.06039e^{-11.645} \sin(18.1702)$

$= (0.06039)(8.7627 \times 10^{-6})(-0.6283)$

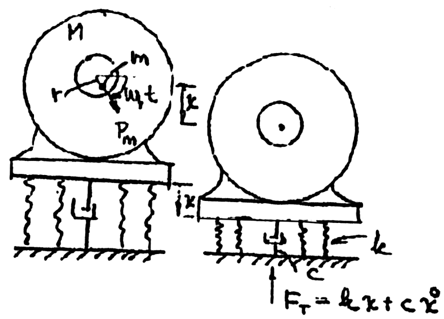
$= -0.332 \times 10^{-6}$ rad $\qquad \theta = -19.05 \times 10^{-6}$ degrees ◀

PROBLEM 19.166



A 400-kg motor supported by four springs, each of constant 150 kN/m, and a dashpot of constant $c = 6500\ \text{N}\cdot\text{s/m}$ is constrained to move vertically. Knowing that the unbalance of the rotor is equivalent to a 23-g mass located at a distance of 100 mm from the axis of rotation, determine for a speed of 800 rpm (*a*) the amplitude of the fluctuating force transmitted to the foundation, (*b*) the amplitude of the vertical motion of the motor.

SOLUTION



Total mass: $M = 400\ \text{kg}$

Unbalance: $m = 23\ \text{g} = 0.023\ \text{kg}$

$r = 100\ \text{mm} = 0.100\ \text{m}$

Forcing frequency: $\omega_f = 800\ \text{rpm}$

$= 83.776\ \text{rad/s}$

Spring constant: $(4)(150\times 10^3\ \text{N/m}) = 600\times 10^3\ \text{N/m}$

Natural frequency: $$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{600\times 10^3}{400}}$$

$= 38.730\ \text{rad/s}$

Frequency ratio: $$\frac{\omega_f}{\omega_n} = 2.1631$$

Viscous damping coefficient: $c = 6500\ \text{N}\cdot\text{s/m}$

Critical damping coefficient: $c_c = 2\sqrt{kM} = 2\sqrt{(600\times 10^3)(400)}$

$= 30{,}984\ \text{N}\cdot\text{s/m}$

Damping factor: $$\frac{c}{c_c} = 0.20978$$

Unbalance force: $P_m = mr\omega_f^2 = (0.023)(0.100)(83.776)^2$

$= 16.1424\ \text{N}$

Static deflection: $$\delta_{st} = \frac{P_m}{k} = \frac{16.1424}{600\times 10^3}$$

$= 26.904\times 10^{-6}\ \text{m}$

Amplitude of vibration. Use Eq. (19.53).

$$x_m = \frac{\frac{P_m}{k}}{\sqrt{\left[1-\left(\frac{\omega_f}{\omega_n}\right)^2\right]^2 + \left[2\left(\frac{c}{c_c}\right)\left(\frac{\omega_f}{\omega_n}\right)\right]^2}}$$

Where $\quad 1-\left(\frac{\omega_f}{\omega_n}\right)^2 = 1-(2.1631)^2 = -3.679$

and $\quad 2\left(\frac{c}{c_c}\right)\left(\frac{\omega_f}{\omega_n}\right) = (2)(0.20978)(2.1631) = 0.90755$

$$x_m = \frac{26.904\times 10^{-6}}{\sqrt{(-3.679)^2 + (0.90755)^2}}$$
$$= 7.1000\times 10^{-6}\text{ m}$$

Resulting motion: $\quad x = x_m \sin(\omega_f t - \varphi)$

$$\dot{x} = \omega_f x_m \cos(\omega_f t - \varphi)$$

Spring force: $\quad F_s = kx = kx_m \sin(\omega_f t - \varphi) = 4.26\sin(\omega_f t - \varphi)$

Damping force: $\quad F_d = c\dot{x} = c\omega_f x_m \cos(\omega_f t - \varphi) = 3.8663\cos(\omega_f t - \varphi)$

Let $\quad F_s = F_m \cos\psi \sin(\omega_f t - \varphi)$ and $F_d = F_m \sin\psi \cos(\omega_f t - \varphi)$

Total force: $\quad F = F_m \cos\psi \sin(\omega_f t - \varphi) + F_m \sin\psi \cos(\omega_f t - \varphi)$

$$= F_m \sin(\omega_f t - \varphi + \psi)$$

(*a*) Force amplitude.

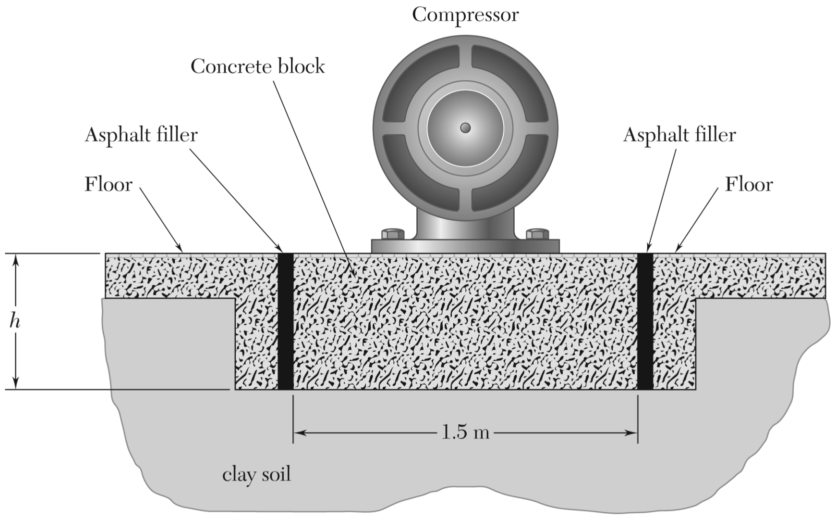
$$F_m = \sqrt{(F_m \cos\psi)^2 + (F_m \sin\psi)^2}$$
$$F_m = \sqrt{(kx_m)^2 + (c\omega_f x_m)^2}$$
$$= \sqrt{(4.26)^2 + (3.8663)^2}$$

$F_m = 5.75\text{ N}$ ◀

(*b*) Amplitude of vibration. $\quad x_m = 0.00710\text{ mm}$ ◀

PROBLEM 19.167

The compressor shown has a mass of 250 kg and operates at 2000 rpm. At this operating condition the force transmitted to the ground is excessively high and is found to be $mr\omega_f^2$ where mr is the unbalance and ω_f is the forcing frequency. To fix this problem, it is proposed to isolate the compressor by mounting it on a square concrete block separated from the rest of the floor as shown. The density of concrete is 2400 kg/m^3 and the spring constant for the soil is found to be 80×10^6 N/m. The geometry of the compressor leads to choosing a block that is 1.5 m by 1.5 m. Determine the depth h that will reduce the force transmitted to the ground by 75%.



SOLUTION

Forced circular frequency corresponding to 2000 rpm.

$$\omega_f = \frac{(2\pi)(2000)}{60} = 209.44 \text{ rad/s}$$

In the first case the natural frequency is very large so that the transmitted force is $mr\omega_f^2$.

After the problem is fixed, the transmitted force is

$$P = kx_m = \frac{P_m}{\left|1 - \left(\frac{\omega_f}{\omega_n}\right)^2\right|}$$

Since the motion is out-of-phase,

$$P = \frac{P_m}{\left(\frac{\omega_f}{\omega_n}\right)^2 - 1} = \frac{mr\omega_f^2}{\left(\frac{\omega_f}{\omega_n}\right)^2 - 1} \quad (1)$$

But $$P = (1 - 0.75)mr\omega_f^2 = 0.25\, mr\omega_f^2 \quad (2)$$

PROBLEM 19.167 (Continued)

Equating expressions (1) and (2) dividing by $mr\omega_f^2$,

$$\frac{1}{\left(\frac{\omega_f}{\omega_n}\right)^2 - 1} = 0.25$$

$$\left(\frac{\omega_f}{\omega_n}\right)^2 - 1 = 4$$

$$\frac{\omega_f}{\omega_n} = \sqrt{5}$$

$$\omega_n = \frac{1}{\sqrt{5}}\omega_f = \frac{1}{\sqrt{5}}(209.44) = 93.664 \text{ rad/s}$$

$$\sqrt{\frac{k}{m}} = \omega_n$$

$$m = \frac{k}{\omega_n^2} = \frac{80\times10^6 \text{ N/m}}{(93.664 \text{ rad/s})^2} = 9119 \text{ kg}$$

Required properties of the attached concrete block.

Mass: $m - 250 \text{ kg} = 8869 \text{ kg}$

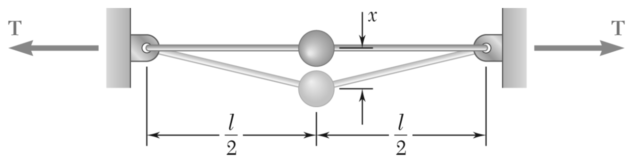
$$\text{volume} = \frac{\text{mass}}{\text{density}} = \frac{8869 \text{ kg}}{2400 \text{ kg/m}^3} = 3.6954 \text{ m}^3$$

$$\text{area} = 1.5 \text{ m}\times1.5 \text{ m} = 2.25 \text{ m}^2$$

$$\text{depth} = \frac{\text{volume}}{\text{area}} = \frac{3.6954 \text{ m}^3}{2.25 \text{ m}^2}$$

$h = 1.642$ m ◀

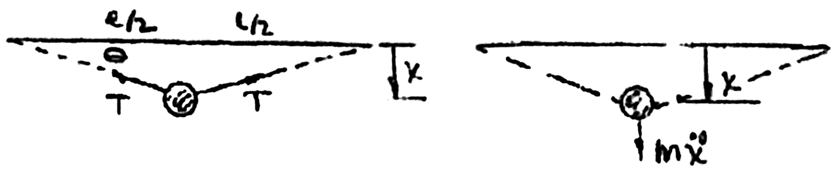
PROBLEM 19.168



A small ball of mass m attached at the midpoint of a tightly stretched elastic cord of length l can slide on a horizontal plane. The ball is given a small displacement in a direction perpendicular to the cord and released. Assuming the tension T in the cord to remain constant, (a) write the differential equation of motion of the ball, (b) determine the period of vibration.

SOLUTION

(a) Differential equation of motion.



$$+\uparrow \Sigma F = ma: \quad 2T\sin\theta = -m\ddot{x}$$

For small x,

$$\sin\theta \approx \tan\theta = \frac{x}{\left(\frac{l}{2}\right)} = \frac{2x}{l}$$

$$m\ddot{x} + (2T)\left(\frac{2x}{l}\right) = 0 \blacktriangleleft$$

$$m\ddot{x} + \left(\frac{4T}{l}\right)x = 0$$

Natural circular frequency.

$$\omega_n^2 = \frac{4T}{ml}$$

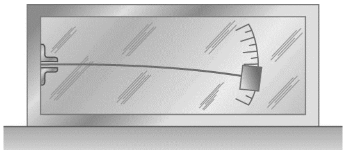
$$\omega_n = 2\sqrt{\frac{T}{ml}}$$

(b) Period of vibration.

$$\tau_n = \frac{2\pi}{\omega_n}$$

$$\tau_n = \pi\sqrt{\frac{ml}{T}} \blacktriangleleft$$

PROBLEM 19.169



A certain vibrometer used to measure vibration amplitudes consists essentially of a box containing a slender rod to which a mass m is attached; the natural frequency of the mass-rod system is known to be 5 Hz. When the box is rigidly attached to the casing of a motor rotating at 600 rpm, the mass is observed to vibrate with an amplitude of 0.06 m. relative to the box. Determine the amplitude of the vertical motion of the motor.

SOLUTION

Natural frequency:

$$f_n = 5 \text{ Hz}$$

$$\omega_n = 2\pi f_n = 31.416 \text{ rad/s}$$

Forcing frequency:

$$f_f = 600 \text{ rpm} = 10 \text{ Hz}$$

$$\omega_f = 2\pi f_f = 62.832 \text{ rad/s}$$

Ratio:

$$\frac{\omega_f}{\omega_n} = 2.000$$

$$x_m = \frac{\delta_m}{1-\left(\frac{\omega_f}{\omega_n}\right)^2} = \frac{\delta_m}{1-(2)^2} = \frac{\delta_m}{-3}$$

$$x_m = -\frac{1}{3}\delta_m$$

Relative motion:

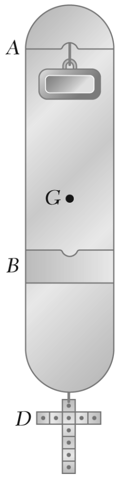
$$y_m = x_m - \delta_m = \frac{4}{3}x_m$$

The observed relative motion is

$$y_m = 0.06 \text{ in.}$$

$$x_m = \frac{3}{4}y_m = \frac{3}{4}(0.06 \text{ in.})$$

$x_m = 0.045$ in. ◀

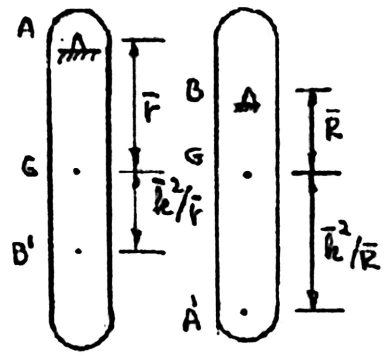


PROBLEM 19.170

If either a simple or a compound pendulum is used to determine experimentally the acceleration of gravity g, difficulties are encountered. In the case of the simple pendulum, the string is not truly weightless, while in the case of the compound pendulum, the exact location of the mass center is difficult to establish. In the case of a compound pendulum, the difficulty can be eliminated by using a reversible, or Kater, pendulum. Two knife edges A and B are placed so that they are obviously not at the same distance from the mass center G, and the distance l is measured with great precision. The position of a counterweight D is then adjusted so that the period of oscillation τ is the same when either knife edge is used. Show that the period τ obtained is equal to that of a true simple pendulum of length l and that $g = 4\pi^2 l/\tau^2$.

SOLUTION

From Problem 19.52, the length of an equivalent simple pendulum is:



$$l_A = \bar{r} + \frac{\bar{k}^2}{\bar{r}}$$

and
$$l_B = \bar{R} + \frac{\bar{k}^2}{\bar{R}}$$

But
$$\tau_A = \tau_B$$

$$2\pi\sqrt{\frac{l_A}{g}} = 2\pi\sqrt{\frac{l_B}{g}}$$

Thus,
$$l_A = l_B$$

For
$$l_A = l_B$$

$$\bar{r} + \frac{\bar{k}^2}{\bar{r}} = \bar{R} + \frac{\bar{k}^2}{\bar{R}}$$

$$\bar{r}^2\bar{R} + \bar{k}^2\bar{R} = \bar{r}\,\bar{R}^2 + \bar{k}^2\bar{r}$$

$$\bar{r}\,\bar{R}[\bar{r} - \bar{R}] = \bar{k}^2[\bar{r} - \bar{R}]$$

$$(\bar{r} - \bar{R}) \neq 0$$

Thus,
$$\bar{r}\,\bar{R} = \bar{k}^2$$

or
$$\bar{r} = \frac{\bar{k}^2}{\bar{R}}$$

$$\bar{R} = \frac{\bar{k}^2}{\bar{r}}$$

PROBLEM 19.170 (Continued)

Thus, $AG = GA'$ and $BG = GB'$

That is, $A = A'$ and $B = B'$

Noting that $l_A = l_B = l$

$$\tau = 2\pi\sqrt{\frac{l}{g}}$$

or
$$g = \frac{4\pi^2 l}{\tau^2}$$