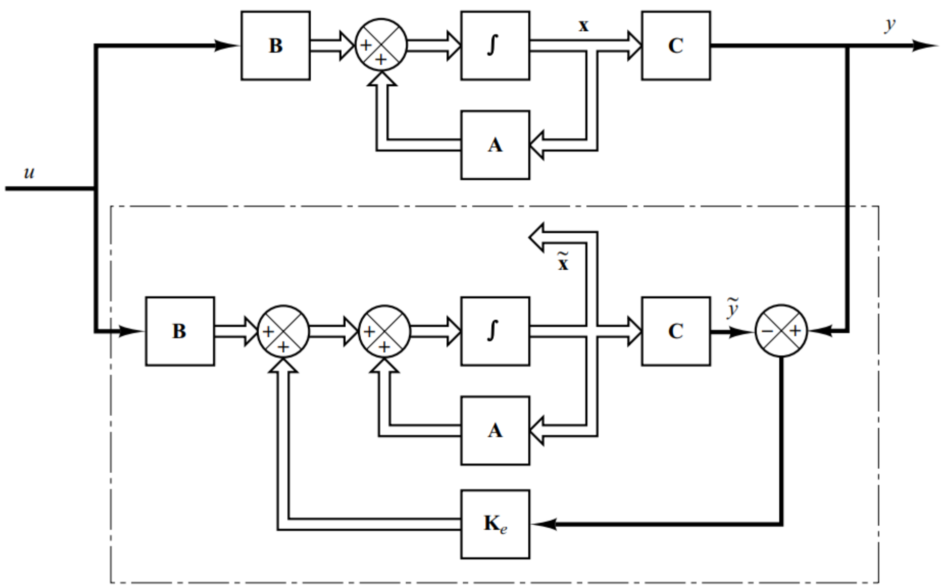
Design Observer

In control theory, a **state observer** is a system that provides an estimate of the internal state of a given real system, from measurements of the input and output of the real system. It is typically computer-implemented, and provides the basis of many practical applications.

State Observer: A state observer estimates the state variables based on the measurements of the output and control variables.

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$
$$y = \mathbf{C}\mathbf{x} \qquad (1)$$



Full-order state observer

- The observer is a subsystem to reconstruct the state vector of the plant. The mathematical model of the observer is basically the same as that of the plant, except that we include an additional term that includes the estimation error to compensate for inaccuracies in matrices **A** and **B** and the lack of the initial error.
- The estimation error or observation error is the difference between the measured output and the estimated output.
- The initial error is the difference between the initial state and the initial estimated state. Thus, we define the mathematical model of the observer to be

$$\dot{\tilde{\mathbf{x}}} = \mathbf{A}\tilde{\mathbf{x}} + \mathbf{B}u + \mathbf{K}_e(y - \mathbf{C}\tilde{\mathbf{x}})$$
$$= (\mathbf{A} - \mathbf{K}_e\mathbf{C})\tilde{\mathbf{x}} + \mathbf{B}u + \mathbf{K}_e y \qquad 2$$

Where:
$\tilde{\boldsymbol{x}}(t)$: is the estimated states vector.
$\tilde{\boldsymbol{y}}(t)$: is the estimated outputs vector.
$\boldsymbol{K_e}$: is the observer gain matrix.
$\boldsymbol{x}(t)$: is the real/actual states vector.
$\boldsymbol{y}(t)$: is the real/actual outputs vector.

To obtain the observer error equation, let us subtract Equation (1) from Equation (2):

$$\dot{\mathbf{x}} - \dot{\tilde{\mathbf{x}}} = \mathbf{A}\mathbf{x} - \mathbf{A}\tilde{\mathbf{x}} - \mathbf{K}_e(\mathbf{C}\mathbf{x} - \mathbf{C}\tilde{\mathbf{x}})$$
$$= (\mathbf{A} - \mathbf{K}_e\mathbf{C})(\mathbf{x} - \tilde{\mathbf{x}}) \qquad 3$$

Define the difference between $\boldsymbol{x}$ and $\tilde{\boldsymbol{x}}$ as the error vector **e**, or

$$\mathbf{e} = \mathbf{x} - \tilde{\mathbf{x}} \qquad 4$$

Then Equation (4) becomes

$$\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{K}_e\mathbf{C})\mathbf{e} \qquad 5$$

- From Equation (5), we see that the dynamic behavior of the error vector is determined by the eigenvalues of matrix $(\boldsymbol{A} - \boldsymbol{K_e}\,\boldsymbol{C})$. If matrix $(\boldsymbol{A} - \boldsymbol{K_e}\,\boldsymbol{C})$ is asymptotically stable matrix, the error vector will converge to zero for any initial error vector $\boldsymbol{e}(0)$. That is, $\tilde{\boldsymbol{x}}$(t) will converge to $\boldsymbol{x}$(t) regardless of the values of $\boldsymbol{x}(0)$ and $\tilde{\boldsymbol{x}}(0)$.

- Usually the eigenvalues for the observer are two to three times faster than the eigenvalues for the controller.

Duality Problem:

To design the full state observer firstly convert the system to the duality problem form then use it to find the gain matrix $\boldsymbol{K_e}$

Consider the system defined be

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \qquad (6)$$
$$y = \mathbf{C}\mathbf{x}$$

In designing the full-order state observer, we may solve the dual problem, that is, solve the pole-placement problem for the dual system:

$$\dot{\boldsymbol{z}} = \boldsymbol{A_n}\boldsymbol{z} + \boldsymbol{B_n}v \qquad (7)$$
$$n = \boldsymbol{C_n}\boldsymbol{z}$$

Where:

$\boldsymbol{A_n} = \boldsymbol{A^T}$**:** the new dynamic matrix for the duality problem.

$\boldsymbol{B_n} = \boldsymbol{C^T}$**:** the new input matrix / vector for the duality problem.

$\boldsymbol{C_n} = \boldsymbol{B^T}$**:** the new output matrix / vector for the duality problem.

Based on that, the state space representation for the duality problem is shown below:

$$\dot{\boldsymbol{z}} = \boldsymbol{A^T}\boldsymbol{z} + \boldsymbol{C^T}v$$

$$n = \boldsymbol{B^T}\boldsymbol{z}$$

Assuming the control signal v to be:

$$v = -\mathbf{K}\mathbf{z} \qquad (8)$$

- The controllability matrix for the duality problem it is the observability matrix for the original system see Equation (6).
- If the system defined by Equation (1) is completely state observable, then, by specifying the desired eigenvalues $\mu_1, \mu_2, \ldots, \mu_n$ for the matrix $(\boldsymbol{A^T} - \boldsymbol{K}\boldsymbol{C^T})$, matrix K can be determined by the pole-placement technique or Linear Quadratic Regulator (LQR).

- Finally, to find the gain matrix $\boldsymbol{K_e}$ use Equation (9).

$$\boldsymbol{K_e} = \boldsymbol{K^T} \qquad 9$$

EXAMPLE 10–6 Consider the system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$
$$y = \mathbf{C}\mathbf{x}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 20.6 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{C} = [0 \quad 1]$$

We use the observed state feedback such that

Design a full-order state observer, assuming that the desired eigenvalues for the system are $\mu_{1,2} = -10$.

Let us examine the observability matrix.

$$O=\begin{bmatrix} C \\ A \end{bmatrix}=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \; |O|= -1.$$

Therefore, the system is fully states observable.

1. Convert the system to the duality problem:

$$\dot{\boldsymbol{z}} = \boldsymbol{A^T z} + \boldsymbol{C^T} v$$

$$n = \boldsymbol{B^T z}$$

$$\dot{\boldsymbol{z}} = \begin{bmatrix} 0 & 1 \\ 20.6 & 0 \end{bmatrix} \boldsymbol{z} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v$$

$$n = [0 \quad 1]\boldsymbol{z}$$

Note: the duality problem is in the form of the first companion form.

1. Find the real characteristic equation

$$|sI - A_n| = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 20.6 & 0 \end{bmatrix} = \begin{bmatrix} s & -1 \\ -20.6 & s \end{bmatrix}$$

$$|sI - A_n| = s^2 - 20.6$$

By the way the general form is:

$$s^2 + a_1\, s + a_2$$

$$a_1 = 0 \quad a_2 = -20.6$$

2. Find the desired characteristic equation

$$(s + 10)(s + 10) = s^2 + 20\, s + 100$$

By the way the general form is:

$$s^2 + \alpha_1\, s + \alpha_2$$

$$\alpha_1 = 20 \quad \alpha_2 = 100$$

3. Find the gain matrix:

$$\boldsymbol{K} = [(\alpha_2 - a_2) \quad (\alpha_1 - a_1)] = [(100 - -20.6) \quad (20 - 0)]$$

$$K = [120.6 \quad 20]$$

$$\boldsymbol{K_e} = \boldsymbol{K^T} = \begin{bmatrix} 120.6 \\ 20 \end{bmatrix}$$