# CHAPTER 1

**P. E. 1.1**

(a) $\boldsymbol{A}+\boldsymbol{B}=(1,0,3)+(5,2,-6)=(6,2,-3)$

$$|\boldsymbol{A}+\boldsymbol{B}|=\sqrt{36+4+9}=\underline{\underline{7}}$$

(b) $5\boldsymbol{A}-\boldsymbol{B}=(5,0,15)-(5,2,-6)=\underline{\underline{(0,-2,21)}}$

(c) The component of $\boldsymbol{A}$ along $\mathbf{a}_y$ is $A_y = \underline{\underline{0}}$

(d) $3\boldsymbol{A}+\boldsymbol{B}=(3,0,9)+(5,2,-6)=(8,2,3)$

A unit vector parallel to this vector is

$$\boldsymbol{a}_{11}=\frac{(8,2,3)}{\sqrt{64+4+9}}$$

$$=\pm\underline{\underline{(0.9117\boldsymbol{a}_x+0.2279\boldsymbol{a}_y+0.3419\boldsymbol{a}_z)}}$$

**P. E. 1.2** (a) $\boldsymbol{r}_p=\underline{\underline{\boldsymbol{a}_x-3\boldsymbol{a}_y+5\boldsymbol{a}_z}}$

$$\boldsymbol{r}_R=\underline{\underline{3\boldsymbol{a}_y+8\boldsymbol{a}_z}}$$

(b) The distance vector is

$$\boldsymbol{r}_{QR}=\boldsymbol{r}_R-\boldsymbol{r}_Q=(0,3,8)-(2,4,6)=\underline{\underline{-2\boldsymbol{a}_x-\boldsymbol{a}_y+2\boldsymbol{a}_z}}$$

(c) The distance between Q and R is

$$|r_{QR}|=\sqrt{4+1+4}=\underline{\underline{3}}$$

**P. E. 1.3** Consider the figure shown on the next page:

$$\boldsymbol{u}_Z=\boldsymbol{u}_P+\boldsymbol{u}_W=-350\boldsymbol{a}_x+\frac{40}{\sqrt{2}}(-\boldsymbol{a}_x+\boldsymbol{a}_y)$$

$$=-378.28\boldsymbol{a}_x+28.28a_y \quad \text{km/hr}$$

or

$$\boldsymbol{u}_z=379.3\angle 175.72^\circ \quad \text{km/hr}$$

Where $\mathbf{u}_p$ = velocity of the airplane in the absence of wind

$\mathbf{u}_w$ = wind velocity

$\mathbf{u}_z$ = observed velocity

# CHAPTER 2

**P. E. 2.1**

(a) At P(1,3,5), x = 1, y = 3, z =5,

$$\rho = \sqrt{x^2 + y^2} = \sqrt{10}, \qquad z = 5, \qquad \phi = \tan^{-1} y/x = \tan^{-1} 3 = 71.6^o$$

$$P(\rho, \phi, z) = P(\sqrt{10}, \tan^{-1} 3, 5) = \underline{\underline{P(3.162, 71.6^o, 5)}}$$

Spherical system:

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{35} = 5.916$$

$$\theta = \tan^{-1} \sqrt{x^2 + y^2} \Big/ z = \tan^{-1} \sqrt{10} \Big/ 5 = \tan^{-1} 0.6325 = 32.31^\circ$$

$$P(r, \theta, \varphi) = \underline{\underline{P(5.916, 32.31^\circ, 71.57^\circ)}}$$

At T(0,-4,3), x = 0 y =-4, z =3;

$$\rho = \sqrt{x^2 + y^2} = 4, z = 3, \varphi = \tan^{-1} y/x = \tan^{-1} -4/0 = 270^\circ$$

$$\underline{\underline{T(\rho, \varphi, z) = T(4, 270^\circ, 3).}}$$

Spherical system:

$$r = \sqrt{x^2 + y^2 + z^2} = 5, \theta = \tan^{-1} \rho/z = \tan^{-1} 4/3 = 53.13^\circ.$$

$$\underline{\underline{T(r, \theta, \varphi) = T(5, 53.13^\circ, 270^\circ).}}$$

At S(-3-4-10), x =-3, y =-4, z =-10;

$$\rho = \sqrt{x^2 + y^2} = 5, \phi = \tan^{-1}\left(\frac{-4}{-3}\right) = 233.1^\circ$$

$$\underline{\underline{S(\rho, \phi, z) = S(5, 233.1, -10).}}$$

Spherical system:

$$r = \sqrt{x^2 + y^2 + z^2} = 5\sqrt{5} = 11.18.$$

$$\theta = \tan^{-1} \rho \big/ z = \tan^{-1} \frac{5}{-10} = 153.43^\circ;$$

$$\underline{\underline{S(r, \theta, \phi) = S(11.18, 153.43^\circ, 233.1^\circ).}}$$

(b) In Cylindrical system, $\rho = \sqrt{x^2 + y^2}$; $yz = z\rho \sin\phi$,

$$Q_x = \frac{\rho}{\sqrt{\rho^2 + z^2}}; \qquad Q_y = 0; \qquad Q_z = -\frac{z\rho \sin\phi}{\sqrt{\rho^2 + z^2}};$$

$$\begin{bmatrix} Q_\rho \\ Q_\phi \\ Q_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Q_x \\ 0 \\ Q_z \end{bmatrix};$$

$$Q_\rho = Q_x \cos\phi = \frac{\rho\cos\phi}{\sqrt{\rho^2+z^2}}, \qquad Q_\phi = -Q_x \sin\phi = \frac{-\rho\sin\phi}{\sqrt{\rho^2+z^2}}$$

Hence,

$$\boldsymbol{Q} = \frac{\rho}{\sqrt{\rho^2+z^2}}(\cos\phi\boldsymbol{a}_\rho - \sin\phi\,\boldsymbol{a}_\phi - z\sin\phi\,\boldsymbol{a}_z).$$

In Spherical coordinates:

$$Q_x = \frac{r\sin\theta}{r} = \sin\theta;$$

$$Q_z = -r\sin\phi\sin\theta\, r\cos\theta\,\frac{1}{r} = -r\sin\theta\cos\theta\sin\phi.$$

$$\begin{bmatrix} Q_r \\ Q_\theta \\ Q_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} Q_x \\ 0 \\ Q_z \end{bmatrix};$$

$$Q_r = Q_x \sin\theta\cos\phi + Q_z\cos\theta = \sin^2\theta\cos\phi - r\sin\theta\cos^2\theta\sin\phi.$$

$$Q_\theta = Q_x\cos\theta\cos\phi - Q_z\sin\theta = \sin\theta\cos\theta\cos\phi + r\sin^2\theta\cos\theta\sin\phi.$$

$$Q_\phi = -Q_x\sin\phi = -\sin\theta\sin\phi.$$

$$\therefore\ \boldsymbol{Q} = \sin\theta\left(\sin\theta\cos\phi - r\cos^2\theta\sin\phi\right)\boldsymbol{a}_r + \sin\theta\cos\theta(\cos\phi + r\sin\theta\sin\phi)\boldsymbol{a}_\theta - \sin\theta\sin\phi\,\boldsymbol{a}_\phi\ .$$

At T :

$$\boldsymbol{Q}(x,y,z) = \frac{4}{5}\boldsymbol{a}_x + \frac{12}{5}\boldsymbol{a}_z = 0.8\boldsymbol{a}_x + 2.4\boldsymbol{a}_z;$$

$$\boldsymbol{Q}(\rho,\phi,z) = \frac{4}{5}(\cos 270°\,\boldsymbol{a}_\rho - \sin 270°\,\boldsymbol{a}_\phi - 3\sin 270°\boldsymbol{a}_z$$

$$= 0.8\boldsymbol{a}_\phi + 2.4\boldsymbol{a}_z;$$

$$\boldsymbol{Q}(r,\theta,\phi) = \frac{4}{5}(0 - \frac{45}{25}(-1))\boldsymbol{a}_r + \frac{4}{5}(\frac{3}{5})(0 + \frac{20}{5}(-1))\boldsymbol{a}_\theta - \frac{4}{5}(-1)\boldsymbol{a}_\phi$$

$$= \frac{36}{25}\boldsymbol{a}_r - \frac{48}{25}\boldsymbol{a}_\theta + \frac{4}{5}\boldsymbol{a}_\phi = 1.44\boldsymbol{a}_r - 1.92\boldsymbol{a}_\theta + 0.8\boldsymbol{a}_\phi;$$

Note, that the magnitude of vector Q = 2.53 in all 3 cases above.

**P.E. 2.2 (a)**

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho z\sin\phi \\ 3\rho\cos\phi \\ \rho\cos\phi\sin\phi \end{bmatrix}$$

$$\boldsymbol{A} = (\rho z\cos\phi\sin\phi - 3\rho\cos\phi\sin\phi)\boldsymbol{a}_x + (\rho z\sin^2\phi + 3\rho\cos^2\phi)\boldsymbol{a}_y + \rho\cos\phi\sin\phi\,\boldsymbol{a}_z.$$

But $\rho = \sqrt{x^2+y^2}$, $\tan\phi = \dfrac{y}{x}$, $\cos\phi = \dfrac{x}{\sqrt{x^2+y^2}}$, $\sin\phi = \dfrac{y}{\sqrt{x^2+y^2}}$;

Substituting all this yields:

$$\boldsymbol{A} = \frac{1}{\sqrt{x^2+y^2}}[(xyz - 3xy)\boldsymbol{a}_x + (zy^2 + 3x^2)\boldsymbol{a}_y + xy\boldsymbol{a}_z].$$

$$\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} r^2 \\ 0 \\ \sin\theta \end{bmatrix}$$

Since $r = \sqrt{x^2+y^2+z^2}$, $\tan\theta = \dfrac{\sqrt{x^2+y^2}}{z}$, $\tan\phi = \dfrac{y}{z}$;

and $\sin\theta = \dfrac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}$, $\cos\theta = \dfrac{z}{\sqrt{x^2+y^2+z^2}}$;

and $\sin\phi = \dfrac{y}{\sqrt{x^2+y^2}}$, $\cos\phi = \dfrac{x}{\sqrt{x^2+y^2}}$;

$$B_x = r^2\sin\theta\cos\phi - \sin\theta\sin\phi = rx - \frac{y}{r} = \frac{1}{r}(r^2x - y).$$

$$B_y = r^2\sin\theta\sin\phi + \sin\theta\cos\phi = ry + \frac{x}{r} = \frac{1}{r}(r^2y + x).$$

$$B_z = r^2\cos\theta = r\,z = \frac{1}{r}(r^2z).$$

Hence,

$$\boldsymbol{B} = \frac{1}{\sqrt{x^2+y^2+z^2}}[\{x(x^2+y^2+z^2) - y\}\boldsymbol{a}_x + \{y(x^2+y^2+z^2) + x\}\boldsymbol{a}_y + z(x^2+y^2+z^2)\boldsymbol{a}_z].$$

**P.E.2.3** (a) At:

$(1, \pi/3, 0)$, $\boldsymbol{H} = (0, 0.06767, 1)$

$$\boldsymbol{a}_x = \cos\phi\,\boldsymbol{a}_\rho - \sin\phi\,\boldsymbol{a}_\phi = \frac{1}{2}(\boldsymbol{a}_\rho - \sqrt{3}\,\boldsymbol{a}_\phi)$$

$$\boldsymbol{H} \bullet \boldsymbol{a}_x = \underline{\underline{-0.0586.}}$$

(b) At:

$(1, \pi/3, 0)$, $\boldsymbol{a}_\theta = \cos\theta\,\boldsymbol{a}_\rho - \sin\theta\,\boldsymbol{a}_z = -\boldsymbol{a}_z$.

$$H \times \boldsymbol{a}_z = \begin{vmatrix} \boldsymbol{a}_\rho & a_\phi & a_z \\ 0 & 0.06767 & 1 \\ 0 & 0 & 1 \end{vmatrix} = \underline{\underline{-0.06767\, a_{\rho.}}}$$

(c) $(\boldsymbol{H} \bullet \boldsymbol{a}_\rho)\boldsymbol{a}_\rho = \underline{\underline{0\boldsymbol{a}_\rho}}.$

(d)

$$\boldsymbol{H} \times \boldsymbol{a}_z = \begin{vmatrix} \boldsymbol{a}_\rho & \boldsymbol{a}_\phi & \boldsymbol{a}_z \\ 0 & 0.06767 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 0.06767\ \boldsymbol{a}_\rho.$$

$$|\boldsymbol{H} \times \boldsymbol{a}_z| = \underline{\underline{0.06767}}$$

**P.E. 2.4**

(a)

$$A \cdot \boldsymbol{B} = (3, 2, -6) \bullet (4, 0, 3) = \underline{\underline{-6.}}$$

(b)

$$|\boldsymbol{A} \times B| = \begin{vmatrix} 3 & 2 & -6 \\ 4 & 0 & 3 \end{vmatrix} = |6\,\boldsymbol{a}_r - 33\boldsymbol{a}_\theta - 8\boldsymbol{a}_\phi|.$$

Thus the magnitude of $A \times B = \underline{\underline{34.48.}}$

(c )

$At$ $(1,\ \pi/3,\ 5\pi/4)$, $\theta = \pi/3$,

$$\boldsymbol{a}_z = \cos\theta\,\boldsymbol{a}_r - \sin\theta\,\boldsymbol{a}_\theta = \frac{1}{2}\boldsymbol{a}_r - \frac{\sqrt{3}}{2}\boldsymbol{a}_\theta.$$

$$(A \cdot \boldsymbol{a}_z)\boldsymbol{a}_z = \left(\frac{3}{2} - \sqrt{3}\right)\left(\frac{1}{2}\boldsymbol{a}_r - \frac{\sqrt{3}}{2}\boldsymbol{a}_\theta\right) = \underline{\underline{-0.116\boldsymbol{a}_r + 0.201\boldsymbol{a}_\theta}}$$

**Prob. 2.1**

(a)

$$\rho = \sqrt{x^2+y^2} = \sqrt{4+25} = 5.3852, \qquad \phi = \tan^{-1}\frac{y}{x} = \tan^{-1} 2.5 = 68.2^o$$

$$r = \sqrt{x^2+y^2+z^2} = \sqrt{4+25+1} = 5.477, \qquad \theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z} = \tan^{-1}\frac{5.3852}{1} = 79.48^o$$

$$P(\rho,\phi,z) = \underline{\underline{P(5.3852, 68.2^o, 1)}}, \qquad P(r,\theta,\phi) = \underline{\underline{P(5.477, 79.48^o, 68.2^o)}}$$

(b)

$$\rho = \sqrt{x^2+y^2} = \sqrt{9+16} = 5, \qquad \phi = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{4}{-3} = 360^o - 53.123^o = 306.88^o$$

$$r = \sqrt{x^2+y^2+z^2} = 5, \qquad \theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z} = \tan^{-1}\infty = 90^o$$

$$\mathrm{Q}(\rho,\phi,z) = \underline{\underline{\mathrm{Q}(5, 306.88^o, 0)}}, \qquad P(r,\theta,\phi) = \underline{\underline{P(5, 90^o, 306.88^o)}}$$

(c )

$$\rho = \sqrt{x^2+y^2} = \sqrt{36+4} = 6.325, \qquad \phi = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{2}{6} = 18.43^o$$

$$r = \sqrt{x^2+y^2+z^2} = \sqrt{36+4+16} = 7.483,$$

$$\theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z} = \tan^{-1}\frac{6.325}{-4} = 180^o - 57.69^o = 122.31^o$$

$$\mathrm{R}(\rho,\phi,z) = \underline{\underline{\mathrm{R}(6.325, 18.43^o, -4)}}, \qquad \mathrm{R}(r,\theta,\phi) = \underline{\underline{\mathrm{R}(7.483, 122.31^o, 18.43^o)}}$$

**Prob. 2.2**

(a)

$$x = \rho\cos\phi = 2\cos 30^\circ = 1.732;$$

$$y = \rho\sin\phi = 2\sin 30^\circ = 1;$$

$$z = 5;$$

$$P_1(x,y,z) = \underline{\underline{P_1(1.732, 1, 5).}}$$

(b)

$$x = 1\cos 90^\circ = 0; \qquad y = 1\sin 90^\circ = 1; \qquad z = -3.$$

$$P_2(x,y,z) = \underline{\underline{P_2(0, 1, -3).}}$$

(c)

$$x = r\sin\theta\cos\phi = 10\sin(\pi/4)\cos(\pi/3) = 3.535;$$
$$y = r\sin\theta\sin\phi = 10\sin(\pi/4)\sin(\pi/3) = 6.124;$$
$$z = r\cos\theta = 10\cos(\pi/4) = 7.0711$$
$$P_3(x,y,z) = \underline{\underline{P_3(3.535,\ 6.124, 7.0711).}}$$

(d)

$$x = 4\sin 30^\circ \cos 60^\circ = 1$$
$$y = 4\sin 30^\circ \sin 60^\circ = 1.7321$$
$$z = r\cos\theta = 4\cos 30^\circ = 3.464$$
$$P_4(x,y,z) = \underline{\underline{P_4(1, 1.7321, 3.464).}}$$

**Prob. 2.3**

$$\rho = \sqrt{x^2+y^2} = \sqrt{4+36} = 6.324$$

(a) $\phi = \tan^{-1}\dfrac{y}{x} = \tan^{-1}\dfrac{6}{2} = 71.56^o$

P is $\underline{\underline{(6.324, 71.56^o, -4)}}$

$$r = \sqrt{x^2+y^2+z^2} = \sqrt{4+36+16} = 7.485$$

(b) $\theta = \tan^{-1}\dfrac{\sqrt{x^2+y^2}}{z} = \tan^{-1}\dfrac{6.324}{-4} = 90^o + \tan^{-1}\dfrac{4}{6.324} = 122.3^o$

P is $\underline{\underline{(7.483, 122.3^o, 71.56^o)}}$

**Prob. 2.4**

(a)

$$x = \rho\cos\phi = 5\cos 120^o = -2.5$$
$$y = \rho\sin\phi = 5\sin 120^o = 4.33$$
$$z = 1$$

Hence Q $= \underline{\underline{(-2.5, 4.33, 1)}}$

(b)

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{\rho^2 + z^2} = \sqrt{25+1} = 5.099$$

$$\theta = \tan^{-1}\frac{\sqrt{x^2 + y^2}}{z} = \tan^{-1}\frac{\rho}{z} = \tan^{-1}\frac{5}{1} = 78.69^o$$

$$\phi = 120^o$$

Hence Q $= \underline{\underline{(5.099, 78.69^o, 120^o)}}$

**Prob. 2.5**

$$T(r,\theta,\phi) \longrightarrow r = 10, \theta = 60^o, \phi = 30^o$$

$$x = r\sin\theta\cos\phi = 10\sin 60^o \cos 30^o = 7.5$$

$$y = r\sin\theta\sin\phi = 10\sin 60^o \sin 30^o = 4.33$$

$$z = r\cos\theta = 10\cos 60^o = 5$$

$$T(x,y,z) = \underline{\underline{(7.5, 4.33, 5)}}$$

$$\rho = r\sin\theta = 10\sin 60^o = 8.66$$

$$T(\rho,\phi,z) = \underline{\underline{(8.66, 30^o, 5)}}$$

**Prob. 2.6**

**(a)**

$$x = \rho\cos\phi, \qquad y = \rho\sin\phi,$$

$$V = \underline{\underline{\rho z\cos\phi - \rho^2\sin\phi\cos\phi + \rho z\sin\phi}}$$

**(b)**

$$U = x^2 + y^2 + z^2 + y^2 + 2z^2$$

$$= r^2 + r^2\sin^2\theta\sin^2\phi + 2r^2\cos^2\theta$$

$$= \underline{\underline{r^2[1 + \sin^2\theta\sin^2\phi + 2\cos^2\theta]}}$$

**Prob. 2.7**
(a)

$$\begin{bmatrix} F_\rho \\ F_\phi \\ F_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dfrac{x}{\sqrt{\rho^2+z^2}} \\ \dfrac{y}{\sqrt{\rho^2+z^2}} \\ \dfrac{4}{\sqrt{\rho^2+z^2}} \end{bmatrix}$$

$$F_\rho = \frac{1}{\sqrt{\rho^2+z^2}}[\rho\cos^2\phi + \rho\sin^2\phi] = \frac{\rho}{\sqrt{\rho^2+z^2}};$$

$$F_\phi = \frac{1}{\sqrt{\rho^2+z^2}}[-\rho\cos\phi\sin\phi + \rho\cos\phi\sin\phi] = 0;$$

$$F_z = \frac{4}{\sqrt{\rho^2+z^2}};$$

$$\bar{F} = \frac{1}{\sqrt{\rho^2+z^2}}(\rho\boldsymbol{a}_\rho + 4\boldsymbol{a}_z)$$

In Spherical:

$$\begin{bmatrix} F_r \\ F_\theta \\ F_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} \dfrac{x}{r} \\ \dfrac{y}{r} \\ \dfrac{4}{r} \end{bmatrix}$$

$$F_r = \frac{r}{r}\sin^2\theta\cos^2\phi + \frac{r}{r}\sin^2\theta\sin^2\phi + \frac{4}{r}\cos\theta = \sin^2\theta + \frac{4}{r}\cos\theta;$$

$$F_\theta = \sin\theta\cos\theta\cos^2\phi + \sin\theta\cos\theta\sin^2\phi - \frac{4}{r}\sin\theta = \sin\theta\cos\theta - \frac{4}{r}\sin\theta;$$

$$F_\phi = -\sin\theta\cos\phi\sin\phi + \sin\theta\sin\phi\cos\phi = 0;$$

$$\therefore \bar{F} = (\sin^2\theta + \frac{4}{r}\cos\theta)\boldsymbol{a}_r + \sin\theta(\cos\theta - \frac{4}{r})\boldsymbol{a}_\theta$$

(b)

$$\begin{bmatrix} G_\rho \\ G_\phi \\ G_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dfrac{x\rho^2}{\sqrt{\rho^2+z^2}} \\ \dfrac{y\rho^2}{\sqrt{\rho^2+z^2}} \\ \dfrac{z\rho^2}{\sqrt{\rho^2+z^2}} \end{bmatrix}$$

$$G_\rho = \frac{\rho^2}{\sqrt{\rho^2+z^2}}[\rho\cos^2\phi + \rho\sin^2\phi] = \frac{\rho^3}{\sqrt{\rho^2+z^2}};$$

$$G_\phi = 0;$$

$$G_z = \frac{z\rho^2}{\sqrt{\rho^2+z^2}};$$

$$\boldsymbol{G} = \frac{\rho^2}{\sqrt{\rho^2+z^2}}(\rho\,\boldsymbol{a}_\rho + z\,\boldsymbol{a}_z)$$

Spherical :

$$\boldsymbol{G} = \frac{\rho^2}{r}(x\boldsymbol{a}_x + y\boldsymbol{a}_y + z\boldsymbol{a}_z) = \frac{r^2\sin^2\theta}{r} r\boldsymbol{a}_r = r^2\sin^2\theta\boldsymbol{a}_r$$

**Prob. 2.8**

$$\boldsymbol{B} = \rho\boldsymbol{a}_x + \frac{y}{\rho}\boldsymbol{a}_y + z\boldsymbol{a}_z$$

$$\begin{bmatrix} B_\rho \\ B_\phi \\ B_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho \\ y/\rho \\ z \end{bmatrix}$$

$$B_\rho = \rho\cos\phi + \frac{y}{\rho}\sin\phi$$

$$B_\phi = -\rho\sin\phi + \frac{y}{\rho}\cos\phi$$

$$B_z = z$$

But $y = \rho\sin\phi$

$$B_\rho = \rho\cos\phi + \sin^2\phi, B_\phi = -\rho\sin\phi + \sin\phi\cos\phi$$

Hence,

$$\boldsymbol{B} = (\rho\cos\phi + \sin^2\phi)\boldsymbol{a}_\rho + \sin\phi(\cos\phi - \rho)\boldsymbol{a}_\phi + z\boldsymbol{a}_z$$

**Prob. 2.9**

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

At P, $\rho = 2, \quad \phi = \pi/2, \quad z = -1$

$$A_x = 2\cos\phi - 3\sin\phi = 2\cos 90^o - 3\sin 90^o = -3$$

$$A_y = 2\sin\phi + 3\cos\phi = 2\sin 90^o + 3\cos 90^o = 2$$

$$A_z = 4$$

Hence, $\underline{\underline{\boldsymbol{A} = -3\boldsymbol{a}_x + 2\boldsymbol{a}_y + 4\boldsymbol{a}_z}}$

**Prob. 2.10**

(a)

$$\begin{bmatrix} A_x \\ A_y \\ A_y \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho\sin\phi \\ \rho\cos\phi \\ -2z \end{bmatrix}$$

$$A_x = \rho\sin\phi\cos\phi - \rho\cos\phi\sin\phi = 0$$

$$A_y = \rho\sin^2\phi + \rho\cos^2\phi = \rho = \sqrt{x^2 + y^2}$$

$$A_z = -2z$$

Hence,

$$\underline{\underline{\boldsymbol{A} = \sqrt{x^2 + y^2}\,\boldsymbol{a}_y - 2z\boldsymbol{a}_z}}$$

(b)

$$\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} 4r\cos\phi \\ r \\ 0 \end{bmatrix}$$

$$B_x = 4r\sin\theta\cos^2\phi + r\cos\theta\cos\phi$$

$$B_y = 4r\sin\theta\sin\phi\cos\phi + r\cos\theta\sin\phi$$

$$B_z = 4r\cos\theta\cos\phi - r\sin\theta$$

But $\quad r = \sqrt{x^2 + y^2 + z^2}, \quad \sin\theta = \dfrac{\sqrt{x^2 + y^2}}{r}, \quad \cos\theta = \dfrac{z}{r}$

$$\sin\phi = \frac{y}{\sqrt{x^2 + y^2}}, \quad \cos\phi = \frac{x}{\sqrt{x^2 + y^2}}$$

$$B_x = 4\sqrt{x^2+y^2}\,\frac{x^2}{x^2+y^2} + \frac{zx}{\sqrt{x^2+y^2}}$$

$$B_y = 4\sqrt{x^2+y^2}\,\frac{xy}{x^2+y^2} + \frac{zy}{\sqrt{x^2+y^2}}$$

$$B_z = 4z\frac{x}{\sqrt{x^2+y^2}} - \sqrt{x^2+y^2}$$

$$\boldsymbol{B} = \frac{1}{\sqrt{x^2+y^2}}\left[x(4x+z)\boldsymbol{a}_x + y(4x+z)\boldsymbol{a}_y + (4xz - x^2 - y^2)\boldsymbol{a}_z\right]$$

**Prob. 2.11**

Method 1:

$$\begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix}\begin{bmatrix} 4/r^2 \\ 0 \\ 0 \end{bmatrix}$$

$$F_x = \frac{4}{r^2}\sin\theta\cos\phi, \qquad F_y = \frac{4}{r^2}\sin\theta\sin\phi, \qquad F_z = \frac{4}{r^2}\cos\theta$$

$$r^2 = x^2+y^2+z^2, \qquad \sin\theta = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}, \qquad \cos\theta = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$\sin\phi = \frac{y}{\sqrt{x^2+y^2}}, \qquad \cos\phi = \frac{x}{\sqrt{x^2+y^2}}$$

$$F_x = \frac{4}{x^2+y^2+z^2}\,\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}\,\frac{x}{\sqrt{x^2+y^2}} = \frac{4x}{(x^2+y^2+z^2)^{3/2}}$$

$$F_y = \frac{4}{x^2+y^2+z^2}\,\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}\,\frac{y}{\sqrt{x^2+y^2}} = \frac{4y}{(x^2+y^2+z^2)^{3/2}}$$

$$F_x = \frac{4}{x^2+y^2+z^2}\,\frac{z}{(x^2+y^2+z^2)} = \frac{4z}{(x^2+y^2+z^2)^{3/2}}$$

Thus,

$$\boldsymbol{F} = \frac{4}{(x^2+y^2+z^2)^{3/2}}\left[x\boldsymbol{a}_x + y\boldsymbol{a}_y + z\boldsymbol{a}_z\right]$$

Method 2:

$$\boldsymbol{F} = \frac{4\boldsymbol{a}_r}{r^2} \cdot \frac{\boldsymbol{r}}{r} = \frac{4r\boldsymbol{a}_r}{r^3}$$

$$\boldsymbol{F} = \frac{4}{(x^2+y^2+z^2)^{3/2}}\left[ x\boldsymbol{a}_x + y\boldsymbol{a}_y + z\boldsymbol{a}_z \right]$$

**Prob. 2.12**

(a) $r = 2, \quad \theta = \pi/2, \quad \phi = 3\pi/2$

$$\boldsymbol{B} = 2\sin(\pi/2)\boldsymbol{a}_r - 4\cos(3\pi/2)\boldsymbol{a}_\phi = 2\boldsymbol{a}_r$$

(b)

$$\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \cos\theta\cos\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} r\sin\theta \\ 0 \\ -r^2\cos\phi \end{bmatrix}$$

$$B_x = r\sin^2\theta\cos\phi - r^2\sin\phi\cos\phi, \qquad B_y = r\sin\theta\cos\theta\cos\phi - r^2\cos^2\phi$$

$$B_z = r\sin\theta\cos\theta$$

But $\quad r = \sqrt{x^2+y^2+z^2}, \cos\theta = \dfrac{z}{r}, \sin\theta = \dfrac{\rho}{r} = \dfrac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}$

$$\cos\phi = \frac{x}{\rho} = \frac{x}{\sqrt{x^2+y^2}}, \qquad \sin\phi = \frac{y}{\rho} = \frac{y}{\sqrt{x^2+y^2}}$$

$$B_x = \sqrt{x^2+y^2+z^2}\,\frac{x^2+y^2}{x^2+y^2+z^2}\frac{x}{\sqrt{x^2+y^2}} - (x^2+y^2+z^2)\frac{xy}{x^2+y^2}$$

$$= \frac{x\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}} - \frac{xy(x^2+y^2+z^2)}{x^2+y^2}$$

$$B_y = \sqrt{x^2+y^2+z^2}\,\frac{z\sqrt{x^2+y^2}}{x^2+y^2+z^2}\frac{x}{\sqrt{x^2+y^2}} - (x^2+y^2+z^2)\frac{x^2}{x^2+y^2}$$

$$= \frac{xz}{\sqrt{x^2+y^2+z^2}} - \frac{x^2(x^2+y^2+z^2)}{x^2+y^2}$$

$$B_z = \sqrt{x^2+y^2+z^2}\,\frac{z\sqrt{x^2+y^2}}{x^2+y^2+z^2} = \frac{z\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}$$

$$\boldsymbol{B} = B_x\boldsymbol{a}_x + B_y\boldsymbol{a}_y + B_z\boldsymbol{a}_z$$

**Prob. 2.13**

(a)
$$x = \rho\cos\phi$$
$$\underline{\underline{\boldsymbol{B} = \rho\cos\phi\boldsymbol{a}_z}}$$

$$x = r\sin\theta\cos\phi$$
(b)
$$\boldsymbol{B} = r\sin\theta\cos\phi\boldsymbol{a}_z, \qquad B_x = 0 = B_y, B_z = r\sin\theta\cos\phi$$
$$\begin{bmatrix} B_r \\ B_\theta \\ B_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ r\sin\theta\cos\phi \end{bmatrix}$$
$$B_r = r\sin\theta\cos\theta\cos\phi = 0.5r\sin(2\theta)\cos\phi$$
$$B_\theta = -r\sin^2\theta\cos\phi, \qquad B_\phi = 0$$
$$\underline{\underline{\boldsymbol{B} = 0.5r\sin(2\theta)\cos\phi\boldsymbol{a}_r - r\sin^2\theta\cos\phi\boldsymbol{a}_\theta}}$$

**Prob. 2.14**

(a)
$$\boldsymbol{a}_x \times \boldsymbol{a}_\rho = (\cos\phi\,\boldsymbol{a}_\rho - \sin\phi\boldsymbol{a}_\phi)\times\boldsymbol{a}_\rho = \cos\phi$$
$$\boldsymbol{a}_x \times \boldsymbol{a}_\phi = (\cos\phi\boldsymbol{a}_\rho - \sin\phi\,\boldsymbol{a}_\phi)\times\boldsymbol{a}_\phi = -\sin\phi$$
$$\boldsymbol{a}_y \times \boldsymbol{a}_\rho = (\sin\phi\,\boldsymbol{a}_\rho + \cos\phi\boldsymbol{a}_\phi)\times\boldsymbol{a}_\rho = \sin\phi$$
$$\bar{a}_y \times \bar{a}_\phi = (\sin\phi\,\boldsymbol{a}_\rho + \sin\phi\boldsymbol{a}_\phi)\times\boldsymbol{a}_\phi = \cos\phi$$

(b) and (c)

In spherical system:
$$\boldsymbol{a}_x = \sin\theta\cos\phi\boldsymbol{a}_r + \cos\theta\cos\phi\,\boldsymbol{a}_\theta - \sin\phi\boldsymbol{a}_\phi.$$
$$\boldsymbol{a}_y = \sin\theta\sin\phi\,\boldsymbol{a}_r + \cos\theta\sin\phi\boldsymbol{a}_\theta - \cos\phi\boldsymbol{a}_\phi.$$
$$\boldsymbol{a}_z = \cos\theta\,\boldsymbol{a}_x - \sin\theta\boldsymbol{a}_\theta.$$

Hence,

$$\boldsymbol{a}_x \times \boldsymbol{a}_r = \sin\theta\cos\phi;$$

$$\boldsymbol{a}_x \times \boldsymbol{a}_\theta = \cos\theta\cos\phi;$$

$$\boldsymbol{a}_y \times \boldsymbol{a}_r = \sin\theta\sin\phi;$$

$$\boldsymbol{a}_y \times \boldsymbol{a}_\theta = \cos\theta\sin\phi;$$

$$\bar{a}_z \times \bar{a}_r = \cos\theta;$$

$$\bar{a}_z \times \bar{a}_\theta = -\sin\theta;$$

**Prob. 2.15**

(a)

$$\boldsymbol{a}_\rho = \cos\phi\boldsymbol{a}_x + \sin\phi\boldsymbol{a}_y, \qquad \boldsymbol{a}_\phi = -\sin\phi\boldsymbol{a}_x + \cos\phi\boldsymbol{a}_y$$

$$\boldsymbol{a}_\rho \times \boldsymbol{a}_\phi = \begin{vmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \end{vmatrix} = (\cos^2\phi + \sin^2\phi)\boldsymbol{a}_z = \boldsymbol{a}_z$$

$$\boldsymbol{a}_z \times \boldsymbol{a}_\rho = \begin{vmatrix} 0 & 0 & 1 \\ \cos\phi & \sin\phi & 0 \end{vmatrix} = -\sin\phi\boldsymbol{a}_x + \cos\phi\boldsymbol{a}_y = \boldsymbol{a}_\phi$$

$$\boldsymbol{a}_\phi \times \boldsymbol{a}_z = \begin{vmatrix} -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} = \cos\phi\boldsymbol{a}_x + \sin\phi\boldsymbol{a}_y = \boldsymbol{a}_\rho$$

(b)

$$\boldsymbol{a}_r = \sin\theta\cos\phi\boldsymbol{a}_x + \sin\theta\sin\phi\boldsymbol{a}_y + \cos\theta\boldsymbol{a}_z$$

$$\boldsymbol{a}_\theta = \cos\theta\cos\phi\boldsymbol{a}_x + \cos\theta\sin\phi\boldsymbol{a}_y - \sin\theta\boldsymbol{a}_z$$

$$\boldsymbol{a}_\phi = -\sin\phi\boldsymbol{a}_x + \cos\phi\boldsymbol{a}_y$$

$$\boldsymbol{a}_r \times \boldsymbol{a}_\theta = \begin{vmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \end{vmatrix}$$

$$= (-\sin^2\theta\sin\phi - \cos^2\theta\sin\phi)\boldsymbol{a}_x + (\cos^2\theta\cos\phi + \sin^2\theta\cos\phi)\boldsymbol{a}_y$$

$$+(\sin\theta\cos\theta\sin\phi\cos\phi - \sin\theta\cos\theta\sin\phi\cos\phi)\boldsymbol{a}_z$$

$$= -\sin\phi\boldsymbol{a}_x + \cos\phi\boldsymbol{a}_y = \boldsymbol{a}_\phi$$

$$\boldsymbol{a}_\phi \times \boldsymbol{a}_r = \begin{vmatrix} -\sin\phi & \cos\phi & 0 \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{vmatrix}$$

$$= \cos\theta\cos\phi\ \boldsymbol{a}_x + \cos\theta\sin\phi\boldsymbol{a}_y + (-\sin\theta\sin^2\phi - \sin\theta\cos^2\phi)\boldsymbol{a}_z$$

$$= \cos\theta\cos\phi\boldsymbol{a}_x + \cos\theta\sin\phi\boldsymbol{a}_y - \sin\theta\boldsymbol{a}_z = \boldsymbol{a}_\theta$$

$$\boldsymbol{a}_\theta \times \boldsymbol{a}_\phi = \begin{vmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{vmatrix}$$

$$= \sin\theta\cos\phi\ \boldsymbol{a}_x + \sin\theta\sin\phi\boldsymbol{a}_y + (\cos\theta\cos^2\phi + \cos\theta\sin^2\phi)\boldsymbol{a}_z$$

$$= \sin\theta\cos\phi\boldsymbol{a}_x + \sin\theta\sin\phi\boldsymbol{a}_y + \cos\theta\boldsymbol{a}_z = \boldsymbol{a}_r$$

**Prob. 2.16**

(a)

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{\rho^2 + z^2}.$$

$$\theta = \tan^{-1}\frac{\rho}{z}; \qquad \phi = \phi.$$

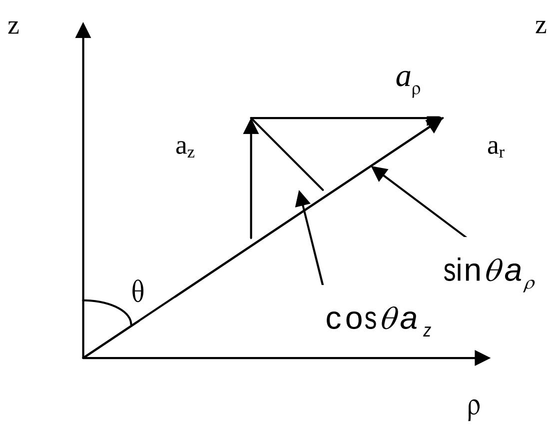
*or*

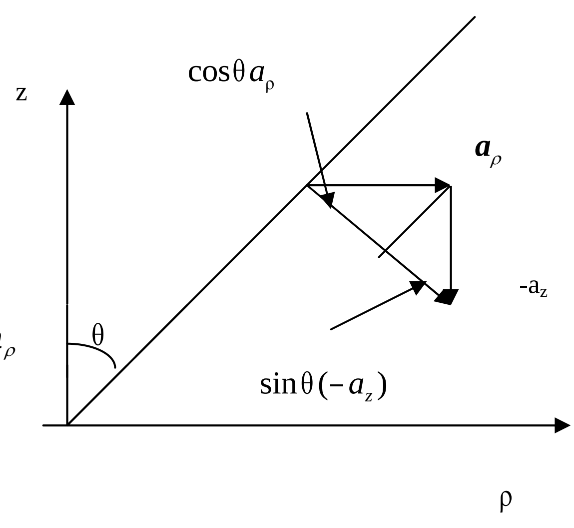
$$\rho = \sqrt{x^2 + y^2} = \sqrt{r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi}.$$

$$= r\sin\theta;$$

$$z = r\cos\theta; \qquad \phi = \phi.$$

(b) From the figures below,



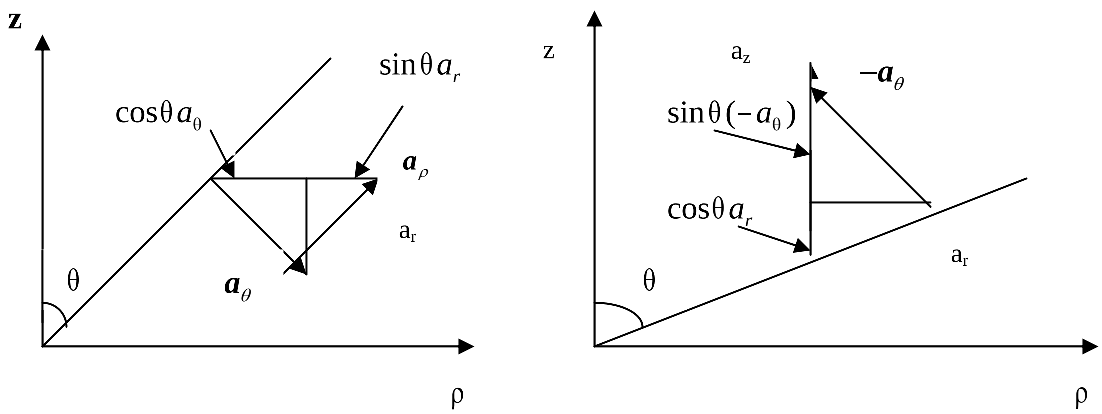
$\mathbf{a}_r = \sin\theta\,\mathbf{a}_\rho + \cos\theta\,\mathbf{a}_z$; $\quad \mathbf{a}_\theta = \cos\theta\,\mathbf{a}_\rho - \sin\theta\,\mathbf{a}_z$; $\quad \mathbf{a}_\phi = \mathbf{a}_\phi$.

Hence,

$$\begin{bmatrix} \mathbf{a}_r \\ \mathbf{a}_\theta \\ \mathbf{a}_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{a}_\rho \\ \mathbf{a}_\phi \\ \mathbf{a}_z \end{bmatrix}$$

From the figures below,

$\mathbf{a}_\rho = \cos\theta\,\mathbf{a}_\theta + \sin\theta\,\mathbf{a}_r; \mathbf{a}_z = \cos\theta\,\mathbf{a}_r - \sin\theta\,\mathbf{a}_\theta; \mathbf{a}_\phi = \mathbf{a}_\phi.$




$$\begin{bmatrix} \mathbf{a}_\rho \\ \mathbf{a}_\phi \\ \mathbf{a}_z \end{bmatrix} = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} \mathbf{a}_r \\ \mathbf{a}_\theta \\ \mathbf{a}_z \end{bmatrix}$$

**Prob. 2.17**

At $P(2,0,-1)$, $\quad \phi = 0, \quad \theta = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) = \cos^{-1}\left(\frac{-1}{\sqrt{5}}\right) = 116.56^o$

(a) $\mathbf{a}_\rho \bullet \mathbf{a}_x = \cos\phi = \underline{\underline{1}}$

(b) $\mathbf{a}_\phi \bullet \mathbf{a}_y = \cos\phi = \underline{\underline{1}}$

(c) $\mathbf{a}_r \bullet \mathbf{a}_z = \cos\theta = \underline{\underline{-0.4472}}$

**Prob. 2.18**

If **A** and **B** are perpendicular to each other, $\mathbf{A}\cdot\mathbf{B} = 0$

$$\boldsymbol{A}\cdot\boldsymbol{B} = \rho^2 \sin^2\phi + \rho^2\cos^2\phi - \rho^2$$
$$= \rho^2(\sin^2\phi + \cos^2\phi) - \rho^2$$
$$= \rho^2 - \rho^2$$
$$= 0$$

As expected.

**Prob. 2.19**

$$(a)\ \boldsymbol{A} + \boldsymbol{B} = \underline{\underline{8\boldsymbol{a}_\rho + 2\boldsymbol{a}_\phi - 7\boldsymbol{a}_z}}$$

$$(b)\ \boldsymbol{A}\cdot\boldsymbol{B} = 15 + 0 - 8 = \underline{\underline{7}}$$

$$(c)\ \boldsymbol{A}\times\boldsymbol{B} = \begin{vmatrix} 3 & 2 & 1 \\ 5 & 0 & -8 \end{vmatrix}$$
$$= -16\mathbf{a}_\rho + (5+24)\boldsymbol{a}_\phi - 10\boldsymbol{a}_z$$
$$= \underline{\underline{-16\mathbf{a}_\rho + 29\boldsymbol{a}_\phi - 10\boldsymbol{a}_z}}$$

$$(d)\ \cos\theta_{AB} = \frac{\boldsymbol{A}\cdot\boldsymbol{B}}{AB} = \frac{7}{\sqrt{9+4+1}\sqrt{25+64}} = \frac{7}{\sqrt{14}\sqrt{89}}$$
$$= 0.19831$$
$$\underline{\underline{\theta_{\text{AB}} = 78.56^o}}$$

**Prob. 2.20**

$$\begin{bmatrix} G_x \\ G_y \\ G_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} G_\rho \\ G_\phi \\ G_z \end{bmatrix}$$

$$G_x = G_\rho\cos\phi - G_\phi\sin\phi = 3\rho\cos\phi - \rho\cos\phi\sin\phi$$
$$= 3x - x\sin\phi = 3(3) - (3)\sin(306.87^o) = 11.4$$
$$\boldsymbol{G}_x = G_x\boldsymbol{a}_x = \underline{\underline{11.4\boldsymbol{a}_x}}$$

**Prob. 2.21**

$$\begin{bmatrix} G_\rho \\ G_\phi \\ G_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} yz \\ xz \\ xy \end{bmatrix}$$

$$G_\rho = yz\cos\phi + xz\sin\phi$$

$$x = \rho\cos\phi,\ y = \rho\sin\phi, \qquad yz = \rho z\sin\phi,\ xz = \rho z\cos\phi$$

$$G_\rho = \rho z\sin\phi\cos\phi + \rho z\cos\phi\sin\phi = 2\rho z\sin\phi\cos\phi = \rho z\sin 2\phi$$

$$G_\phi = -yz\sin\phi + xz\cos\phi = \rho z(\cos^2\phi - \sin^2\phi) = \rho z\cos 2\phi$$

$$G_z = xy = \rho^2\cos\phi\sin\phi = 0.5\rho^2\sin 2\phi$$

$$\underline{\underline{\boldsymbol{G} = \rho z\sin 2\phi \boldsymbol{a}_y + \rho z\cos 2\phi \boldsymbol{a}_\phi + 0.5\rho^2\sin 2\phi \boldsymbol{a}_z}}$$

**Prob. 2.22**

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

$$= \underline{\underline{\begin{bmatrix} \dfrac{x}{\sqrt{x^2+y^2}} & -\dfrac{y}{\sqrt{x^2+y^2}} & 0 \\ \dfrac{y}{\sqrt{x^2+y^2}} & \dfrac{x}{\sqrt{x^2+y^2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}}}$$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

$$= \underline{\underline{\begin{bmatrix} \dfrac{x}{\sqrt{x^2+y^2+z^2}} & \dfrac{xz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}} & \dfrac{-y}{\sqrt{x^2+y^2}} \\ \dfrac{y}{\sqrt{x^2+y^2+z^2}} & \dfrac{yz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}} & \dfrac{x}{\sqrt{x^2+y^2}} \\ \dfrac{z}{\sqrt{x^2+y^2+z^2}} & -\dfrac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}} & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\phi \\ A_\phi \end{bmatrix}}}$$

**Prob. 2.23** (a) Using the results in Prob.2.14,

$$A_\rho = \rho z \sin\phi = r^2 \sin\theta\cos\theta\sin\phi$$
$$A_\phi = 3\rho\cos\phi = 3r\sin\theta\cos\phi$$
$$A_z = \rho\cos\phi\sin\phi = r\sin\theta\cos\phi\sin\phi$$

Hence,

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} r^2\sin\theta\cos\theta\sin\phi \\ 3r\sin\theta\cos\phi \\ r\sin\theta\cos\phi\sin\phi \end{bmatrix}$$

$$\mathbf{A}(r,\theta,\phi) = \underline{\underline{r\sin\theta\left[\sin\phi\cos\theta\left(r\sin\theta+\cos\phi\right)\mathbf{a}_r + \sin\phi\left(r\cos^2\theta - \sin\theta\cos\phi\right)\mathbf{a}_\theta + 3\cos\phi\,\mathbf{a}_\phi\right]}}$$

At $(10, \pi/2, 3\pi/4)$, $r = 10, \theta = \pi/2, \phi = 3\pi/4$

$$\mathbf{A} = 10(0\mathbf{a}_r + 0.5\mathbf{a}_\theta - \frac{3}{\sqrt{2}}\mathbf{a}_\phi) = \underline{\underline{5\mathbf{a}_\theta - 21.21\mathbf{a}_\phi}}$$

(b) $B_r = r^2 = (\rho^2 + z^2)$, $B_\theta = 0$, $B_\phi = \sin\theta = \dfrac{\rho}{\sqrt{\rho^2 + z^2}}$

$$\begin{bmatrix} B_\rho \\ B_\phi \\ B_z \end{bmatrix} = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} B_r \\ B_\theta \\ B_\phi \end{bmatrix}$$

$$\mathbf{B}(\rho,\phi,z) = \underline{\underline{\sqrt{\rho^2+z^2}\left(\rho\mathbf{a}_\rho + \frac{\rho}{\rho^2+z^2}\mathbf{a}_\phi + z\mathbf{a}_z\right)}}$$

At $(2, \pi/6, 1)$, $\rho = 2, \phi = \pi/6, z = 1$

$$\mathbf{B} = \sqrt{5}(2\mathbf{a}_\rho + 0.4\mathbf{a}_\phi + \mathbf{a}_z) = \underline{\underline{4.472\mathbf{a}_\rho + 0.8944\mathbf{a}_\phi + 2.236\mathbf{a}_z}}$$

**Prob. 2.24**

(a) $d = \sqrt{(6-2)^2 + (-1-1)^2 + (2-5)^2} = \sqrt{29} = \underline{\underline{5.385}}$

(b)
$$d^2 = 3^2 + 5^2 - 2(3)(5)\cos\pi + (-1-5)^2 = 100$$
$$d = \sqrt{100} = \underline{\underline{10}}$$

(c)

$$d^2 = 10^2 + 5^2 - 2(10)(5)\cos\frac{\pi}{4}\cos\frac{\pi}{6} - 2(10)(5)\sin\frac{\pi}{4}\sin\frac{\pi}{6}\cos(7\frac{\pi}{4} - \frac{3\pi}{4})$$
$$= 125 - 100(0.7071)(0.866) - 100(0.7071)(0.5)(-0.2334)$$
$$= 125 - 61.23 + 35.33 = 99.118$$
$$d = \sqrt{99.118} = \underline{\underline{9.956.}}$$

**Prob. 2.25**

Using eq. (2.33),

$$d^2 = r_1^2 + r_2^2 - 2r_1r_2\cos\theta_1\cos\theta_2 - 2r_1r_2\sin\theta_1\sin\theta_2\cos(\phi_2 - \phi_1)$$
$$= 16 + 36 - 2(4)(6)\cos 30^o\cos 90^o - 2(4)(6)\sin 30^o\sin 90^o\cos(180^o)$$
$$= 16 + 36 - 0 - 48(0.5)(1)(-1) = 52 + 24 = 76$$
$$d = 8.718$$

**Prob. 2.26**

$$\begin{bmatrix} \boldsymbol{a}_\rho \\ \boldsymbol{a}_\phi \\ \boldsymbol{a}_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \boldsymbol{a}_x \\ \boldsymbol{a}_y \\ \boldsymbol{a}_z \end{bmatrix}$$

$$\boldsymbol{a}_\rho = \cos\phi\boldsymbol{a}_x + \sin\phi\boldsymbol{a}_y, \qquad \boldsymbol{a}_\phi = -\sin\phi\boldsymbol{a}_x + \cos\phi\boldsymbol{a}_y$$

At (0, 4, -1), $\phi = 90^o$

$$\boldsymbol{a}_\rho = \sin 90^o\boldsymbol{a}_y = \underline{\underline{\boldsymbol{a}_y}}$$
$$\boldsymbol{a}_\phi = -\sin 90^o\boldsymbol{a}_x = \underline{\underline{-\boldsymbol{a}_x}}$$

**Prob. 2.27**

At $(1, 60^o, -1)$, $\rho = 1, \phi = 60^o, z = -1,$

(a) $\boldsymbol{A} = (-2 - \sin 60^o)\boldsymbol{a}_\rho + (4 + 2\cos 60^o)\boldsymbol{a}_\phi - 3(1)(-1)\boldsymbol{a}_z$

$$= -2.866\boldsymbol{a}_\rho + 5\boldsymbol{a}_\phi + 3\boldsymbol{a}_z$$

$$\boldsymbol{B} = 1\cos 60^o\boldsymbol{a}_\rho + \sin 60^o\boldsymbol{a}_\phi + \boldsymbol{a}_z = 0.5\boldsymbol{a}_\phi + 0.866\boldsymbol{a}_\phi + \boldsymbol{a}_z$$
$$\boldsymbol{A}\cdot\boldsymbol{B} = -1.433 + 4.33 + 3 = 5.897$$
$$AB = \sqrt{2.866^2 + 26 + 9}\sqrt{0.25 + 1 + 0.866^2} = 9.1885$$
$$\cos\theta_{AB} = \frac{\boldsymbol{A}\cdot\boldsymbol{B}}{AB} = \frac{5.897}{9.1885} = 0.6419 \longrightarrow \theta_{AB} = \underline{\underline{50.07^o}}$$

Let $\mathbf{D} = \mathbf{A} \times \mathbf{B}$. At $(1, 90^o, 0)$, $\rho = 1, \phi = 90^o, z = 0$

(b) $\boldsymbol{A} = -\sin 90^o \boldsymbol{a}_\rho + 4\boldsymbol{a}_\phi = -\boldsymbol{a}_\rho + 4\boldsymbol{a}_\phi$

$\boldsymbol{B} = 1\cos 90^o \boldsymbol{a}_\rho + \sin 90^o \boldsymbol{a}_\phi + \boldsymbol{a}_z = \boldsymbol{a}_\phi + \boldsymbol{a}_z$

$$\boldsymbol{D} = \boldsymbol{A} \times \boldsymbol{B} = \begin{vmatrix} \boldsymbol{a}_\rho & \boldsymbol{a}_\phi & \boldsymbol{a}_z \\ -1 & 4 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 4\boldsymbol{a}_\rho + \boldsymbol{a}_\phi - \boldsymbol{a}_z$$

$$\boldsymbol{a}_D = \frac{\boldsymbol{D}}{D} = \frac{(4, 1, -1)}{\sqrt{16+1+1}} = \underline{\underline{0.9428\boldsymbol{a}_\rho + 0.2357\boldsymbol{a}_\phi - 0.2357\boldsymbol{a}_z}}$$

**Prob.2.28**

$At\ P(0, 2, -5), \quad \phi = 90°;$

$$\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} B_\rho \\ B_\phi \\ B_z \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -5 \\ 1 \\ -3 \end{bmatrix}$$

$\boldsymbol{B} = -\boldsymbol{a}_x - 5\boldsymbol{a}_y - 3\boldsymbol{a}_z$

$(a)$ $\boldsymbol{A} + \boldsymbol{B} = (2, 4, 10) + (-1, -5, -3)$

$= \underline{\underline{\boldsymbol{a}_x - \boldsymbol{a}_y + 7\boldsymbol{a}_z}}.$

$(b)$ $\cos\theta_{AB} = \dfrac{\boldsymbol{A} \bullet \boldsymbol{B}}{AB} = \dfrac{-52}{\sqrt{4200}}$

$\theta_{AB} = \cos^{-1}\left(\dfrac{-52}{\sqrt{4200}}\right) = \underline{\underline{143.36°.}}$

$(c)$ $A_B = \boldsymbol{A} \bullet \boldsymbol{a}_B = \dfrac{\boldsymbol{A} \bullet \boldsymbol{B}}{B} = -\dfrac{52}{\sqrt{35}} = \underline{\underline{-8.789.}}$

**Prob. 2.29**

$\mathbf{B} \bullet \mathbf{a}_x = B_x$

$$\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} B_\rho \\ B_\phi \\ B_z \end{bmatrix}$$

$$B_x = B_\rho \cos\phi - B_\phi \sin\phi = \rho^2 \sin\phi \cos\phi - (z-1)\cos\phi \sin\phi$$

$$= 16(0.5) - (-2)(0.5) = 8 + 1 = \underline{\underline{9}}$$

**Prob. 2.30**

$$\bar{G} = \cos^2\phi \mathbf{a}_x + \frac{2r\cos\theta\sin\phi}{r\sin\theta}\bar{a}_y + (1-\cos^2\phi)\bar{a}_z$$

$$= \cos^2\phi\,\bar{a}_x + 2\cot\theta\sin\phi\,\bar{a}_y + \sin^2\phi\,\bar{a}_z$$

$$\begin{pmatrix} G_r \\ G_\theta \\ G_\phi \end{pmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} \cos^2\phi \\ 2\cot\theta\sin\phi \\ \sin^2\phi \end{bmatrix}$$

$$G_r = \sin\theta\cos^3\phi + 2\cos\theta\sin^2\phi + \cos\theta\sin^2\phi$$

$$= \sin\theta\cos^3\phi + 3\cos\theta\sin^2\phi$$

$$G_\theta = \cos\theta\cos^3\phi + 2\cot\theta\cos\theta\sin^2\phi - \sin\theta\sin^2\phi$$

$$G_\phi = -\sin\phi\cos^2\phi + 2\cot\theta\sin\phi\cos\phi$$

$$\bar{G} = [\sin\theta\cos^3\phi + 3\cos\theta\sin^2\phi]\mathbf{a}_r$$

$$+ [\cos\theta\cos^3\phi + 2\cot\theta\cos\theta\sin^2\phi - \sin\theta\sin^2\phi]\mathbf{a}_\theta$$

$$\underline{\underline{+ \sin\phi\cos\phi(2\cot\theta - \cos\phi)\mathbf{a}_\phi}}$$

**Prob. 2.31**

(a) An infinite line parallel to the z-axis.

(b) Point (2,-1,10).

(c) A circle of radius $r\sin\theta = 5$ , i.e. the intersection of a cone and a sphere.

(d) An infinite line parallel to the z-axis.

(e) A semi-infinite line parallel to the x-y plane.

(f) A semi-circle of radius 5 in the y-z plane.

**Prob. 2.32**

(a) $\boldsymbol{J}_z = (\boldsymbol{J} \bullet \boldsymbol{a}_z)\boldsymbol{a}_z.$

$At\,(2,\ \ \pi/2,\ 3\pi/2),\ \ \boldsymbol{a}_z = \cos\theta\boldsymbol{a}_r - \sin\theta\boldsymbol{a}_\theta = -\boldsymbol{a}_\theta.$

$\boldsymbol{J}_z = -\cos 2\theta \sin\phi \boldsymbol{a}_\theta = -\cos\pi \sin(3\pi/2)\,\boldsymbol{a}_\theta = -\boldsymbol{a}_\theta.$

$(b)\boldsymbol{J}_\phi = \tan\frac{\theta}{2}\ln r\,\boldsymbol{a}_\phi = \tan\frac{\pi}{4}\ln 2\,\boldsymbol{a}_\phi = \ln 2\boldsymbol{a}_\phi = 0.6931\boldsymbol{a}_\phi.$

$(c)\boldsymbol{J}_t = \boldsymbol{J} - \boldsymbol{J}_n = \boldsymbol{J} - \boldsymbol{J}_r = -\boldsymbol{a}_\theta + \ln 2\,\boldsymbol{a}_\phi = \underline{\underline{-\boldsymbol{a}_\theta + 0.6931\boldsymbol{a}_\phi}}$

$(d)\quad \boldsymbol{J}_P = (\boldsymbol{J} \bullet \boldsymbol{a}_x)\boldsymbol{a}_x$

$\boldsymbol{a}_x = \sin\theta\cos\phi\boldsymbol{a}_r + \cos\theta\cos\phi\boldsymbol{a}_\theta - \sin\phi\boldsymbol{a}_\phi = \boldsymbol{a}_\phi.$

At $(2,\ \ \pi/2,\ \ 3\pi/2),$

$\boldsymbol{J}_P = \underline{\underline{\ln 2\boldsymbol{a}_\phi.}}$

**Prob. 2.33**

$\boldsymbol{H} \cdot \boldsymbol{a}_x = H_x$

$$\begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho^2\cos\phi \\ -\rho\sin\phi \\ 0 \end{bmatrix}$$

$H_x = \rho^2\cos^2\phi + \rho\sin^2\phi$

At P, $\rho = 2, \phi = 60^o, z = -1$

$H_x = 4(1/4) + 2(3/4) = 1 + 1.5 = \underline{\underline{2.5}}$

**Prob. 2.34**

(a) $5 = \boldsymbol{r}\cdot\boldsymbol{a}_x + \boldsymbol{r}\cdot\boldsymbol{a}_y = x + y$ $\quad\quad$ $\underline{\underline{\text{a plane}}}$

(b) $10 = |\boldsymbol{r}x\boldsymbol{a}_z| = \begin{vmatrix} x & y & z \\ 0 & 0 & 1 \end{vmatrix} = |\,y\boldsymbol{a}_x - x\boldsymbol{a}_y\,| = \sqrt{x^2 + y^2} = \rho$

$\underline{\underline{\text{a cylinder of infinite length}}}$

# CHAPTER 3

**P. E. 3.1**

(a) $DH = \int_{\phi=45°}^{\phi=60°} r\sin\theta d\phi \Big|_{r=3,90^o} = 3(1)[\frac{\pi}{3} - \frac{\pi}{4}] = \frac{\pi}{4} = \underline{\underline{0.7854.}}$

(b) $FG = \int_{\theta=60°}^{\theta=90°} rd\theta \Big|_{r=5} = 5(\frac{\pi}{2} - \frac{\pi}{3}) = \frac{5\pi}{6} = \underline{\underline{2.618.}}$

(c)

$$AEHD = \int_{\theta=60°}^{\theta=90°} \int_{\phi=45°}^{\phi=60°} r^2 \sin\theta \, d\theta \, d\phi \Big|_{r=3} = 9(-\cos\theta)\Big|_{\theta=60°}^{\theta=90°} \phi\Big|_{\phi=45°}^{\phi=60°}$$

$$= 9(\frac{1}{2})(\frac{\pi}{12}) = \frac{3\pi}{8} = \underline{\underline{1.178.}}$$

(d)

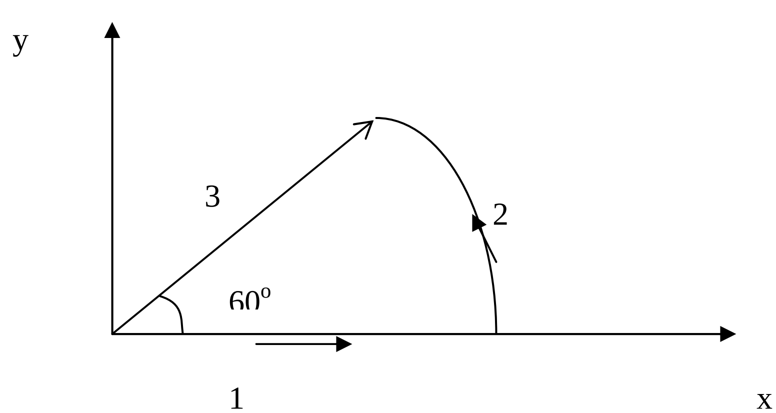
$$ABCD = \int_{r=3}^{r=5} \int_{\theta=60}^{\theta=90} rd\theta\, dr = \frac{r^2}{2}\Big|_{r=3}^{r=5}(\frac{\pi}{2} - \frac{\pi}{3}) = \frac{4\pi}{3} = \underline{\underline{4.189.}}$$

(e)

$$\text{Volume} = \int_{r=3}^{r=5} \int_{\phi=45°}^{\phi=60°} \int_{\theta=60}^{\theta=90} r^2 \sin\theta dr\, d\theta\, d\phi = \frac{r^3}{3}\Big|_{r=3}^{r=5} (-\cos\theta)\Big|_{\theta=60°}^{\theta=90°} \phi\Big|_{\phi=45°}^{\phi=60°} = \frac{1}{3}(98)(\frac{1}{2})\frac{\pi}{12}$$

$$= \underline{\underline{\frac{49\pi}{36} = 4.276.}}$$

**P.E. 3.2**



$$\oint_L \boldsymbol{A}\bullet d\boldsymbol{l} = (\int_1 + \int_2 + \int_3)\boldsymbol{A}\bullet d\boldsymbol{l} = C_1 + C_2 + C_3$$

Along (1), $C_1 = \int \boldsymbol{A}\bullet d\boldsymbol{l} = \int_0^2 \rho\cos\phi\, d\rho\,|_{\phi=0} = \left.\frac{\rho^2}{2}\right|_0^2 = 2.$

Along (2), $d\boldsymbol{l} = \rho d\phi\, \boldsymbol{a}_\phi,\ \boldsymbol{A}\bullet d\boldsymbol{l} = 0,\quad C_2 = 0$

Along (3), $C_3 = \int_2^0 \rho\cos\phi\, d\rho_{\phi=60°} = -\left.\frac{\rho^2}{2}\right|_0^2 (\frac{1}{2}) = -1$

$$\oint_l \boldsymbol{A}\bullet d\boldsymbol{l} = C_1 + C_2 + C_3 = 2 + 0 - 1 = \underline{\underline{1}}$$

**P.E. 3.3**

(*a*) $\nabla U = \frac{\partial U}{\partial x}\boldsymbol{a}_x + \frac{\partial U}{\partial y}\boldsymbol{a}_y + \frac{\partial U}{\partial z}\boldsymbol{a}_z$

$$= \underline{\underline{y(2x+z)\boldsymbol{a}_x + x(x+z)\boldsymbol{a}_y + xy\,\boldsymbol{a}_z}}$$

(b) $\nabla V = \frac{\partial V}{\partial \rho}\boldsymbol{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\boldsymbol{a}_\phi + \frac{\partial V}{\partial z}\boldsymbol{a}_z$

$$= \underline{\underline{(z\sin\phi + 2\rho)\boldsymbol{a}_\rho + (z\cos\phi - \frac{z^2}{\rho}\sin 2\phi)\boldsymbol{a}_\phi + (\rho\sin\phi + 2z\cos^2\phi)\boldsymbol{a}_z}}$$

**(c)**

$$\nabla f = \frac{\partial f}{\partial r}\boldsymbol{a}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\boldsymbol{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\boldsymbol{a}_\phi$$

$$= (\frac{\cos\theta\sin\phi}{r} + 2r\phi)\boldsymbol{a}_r - \frac{\sin\theta\sin\phi\ln r}{r}\boldsymbol{a}_\theta + \frac{(\cos\theta\cos\phi\ln r + r^2)}{r\sin\theta}\boldsymbol{a}_\phi$$

$$= \underline{\underline{\left(\frac{\cos\theta\sin\phi}{r} + 2r\phi\right)\boldsymbol{a}_r - \frac{\sin\theta\sin\phi\ln r}{r}\boldsymbol{a}_\theta + \left(\frac{\cot\theta\cos\phi\ln r}{r} + r\,\mathrm{cosec}\theta\right)\boldsymbol{a}_\phi}}$$

**P.E. 3.4**

$\nabla\Phi = (y+z)\boldsymbol{a}_x + (x+z)\boldsymbol{a}_y + (x+y)\boldsymbol{a}_z$

At $(1,2,3)$, $\nabla\Phi = \underline{\underline{(5,4,3)}}$

$\nabla\Phi\bullet\boldsymbol{a}_1 = (5,4,3)\bullet\frac{(2,2,1)}{3} = \frac{21}{3} = \underline{\underline{7}},$

where $(2,2,1) = (3,4,4) - (1,2,3)$

**P.E. 3.5**

Let $f = x^2 y + z - 3, \quad g = x \log z - y^2 + 4,$

$$\nabla f = 2xy\,\boldsymbol{a}_x + x^2\,\boldsymbol{a}_y + \boldsymbol{a}_z,$$

$$\nabla g = \log z\,\boldsymbol{a}_x - 2y\,\boldsymbol{a}_y + \frac{x}{z}\boldsymbol{a}_z$$

At P(−1, 2,1),

$$\boldsymbol{n}_f = \pm\frac{\nabla f}{|\nabla f|} = \pm\frac{(-4\,\boldsymbol{a}_x + \boldsymbol{a}_y + \boldsymbol{a}_z)}{\sqrt{18}}, \qquad \boldsymbol{n}_g = \pm\frac{\nabla g}{|\nabla g|} = \pm\frac{(-4\,\boldsymbol{a}_y - \boldsymbol{a}_z)}{\sqrt{17}}$$

$$\cos\theta = \boldsymbol{n}_f\,.\,\boldsymbol{n}_g = \pm\frac{(-5)}{\sqrt{18\times 17}}$$

Take positive value to get acute angle.

$$\theta = \cos^{-1}\ \frac{5}{17.493} = 73.39^\circ$$

**P.E. 3.6**

(*a*) $\nabla \bullet \boldsymbol{A} = \dfrac{\partial A_x}{\partial x} + \dfrac{\partial A_y}{\partial y} + \dfrac{\partial A_z}{\partial z} = 0 + 4x + 0 = 4x.$

At $(1, -2, 3)$, $\nabla \bullet \boldsymbol{A} = 4.$

**(b)**

$$\nabla \bullet \boldsymbol{B} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho B_\rho) + \frac{1}{\rho}\frac{\partial B_\phi}{\partial\phi} + \frac{\partial B_z}{\partial\rho}$$

$$= \frac{1}{\rho}2\rho z\sin\phi - \frac{1}{\rho}3\rho z^2\sin\phi = 2z\sin\phi - 3z^2\sin\phi$$

$$= (2 - 3z)z\sin\phi.$$

$At\,(5, \frac{\pi}{2}, 1), \quad \nabla \bullet \boldsymbol{B} = (2-3)(1) = -1.$

(c )

$$\nabla \bullet \boldsymbol{C} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 C_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(C_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial C_\phi}{\partial\phi}$$

$$= \frac{1}{r^2}6r^2\cos\theta\cos\phi$$

$$= 6\cos\theta\cos\phi$$

$$At\,(1,\frac{\pi}{6},\frac{\pi}{3}),\quad \nabla\bullet\boldsymbol{C} = 6\cos\frac{\pi}{6}\,\cos\frac{\pi}{3} = \underline{\underline{2.598.}}$$

**P.E. 3.7** This is similar to Example 3.7.

$$\Psi = \oint_S \boldsymbol{D}\bullet d\boldsymbol{S} = \Psi_t + \Psi_b + \Psi_c$$

$\Psi_t = 0 = \Psi_b$ since $\boldsymbol{D}$ has no z-component

$$\Psi_c = \iint \rho^2\cos^2\phi\rho d\phi dz = \rho^3\int_{\phi=0}^{\phi=2\pi}\cos^2\phi d\phi\int_{z=0}^{z=1}dz\Big|_{\rho=4}$$

$$= (4)^3\pi(1) = 64\pi$$

$$\Psi = 0+0+64\pi = \underline{\underline{64\pi}}$$

By the divergence theorem,

$$\oint_S \boldsymbol{D}\bullet d\boldsymbol{S} = \oint_V \nabla\bullet\boldsymbol{D}dv$$

$$\nabla\bullet\boldsymbol{D} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho^3\cos^2\phi) + \frac{1}{\rho}\frac{\partial}{\partial\phi}z\sin\phi + \frac{\partial A_z}{dz}$$

$$= 3\rho\cos^2\phi + \frac{z}{\rho}\cos\phi.$$

$$\Psi = \int_V \nabla\bullet\boldsymbol{D}dv = \int_V (3\rho\cos^2\phi + \frac{z}{\rho}\cos\phi)\rho d\phi dz d\rho$$

$$= 3\int_0^4 \rho^2 d\rho\int_0^{2\pi}\cos^2\phi\, d\phi\int_0^1 dz + \int_0^4 d\rho\int_0^{2\pi}\cos\phi d\phi\int_0^1 zdz$$

$$= 3(\frac{4^3}{3})\pi(1) = \underline{\underline{64\pi.}}$$

**P.E. 3.8**

(a)

$$\nabla\times\boldsymbol{A} = \boldsymbol{a}_x(1-0)+\boldsymbol{a}_y(y-0)+\boldsymbol{a}_z(4y-z)$$

$$= \underline{\underline{\boldsymbol{a}_x + y\boldsymbol{a}_y + (4y-z)\boldsymbol{a}_z}}$$

At $(1,-2,3), \nabla\times\boldsymbol{A} = \underline{\underline{\boldsymbol{a}_x - 2\boldsymbol{a}_y - 11\boldsymbol{a}_z}}$

(b)

$$\nabla\times\boldsymbol{B} = \boldsymbol{a}_\rho(0-6\rho z\cos\phi)+\boldsymbol{a}_\phi(\rho\sin\phi-0)+\boldsymbol{a}_z\frac{1}{\rho}(6\rho z^2\cos\phi-\rho z\cos\phi)$$

$$= \underline{\underline{-6\rho z\cos\phi\,\boldsymbol{a}_\rho+\rho\sin\phi\,\boldsymbol{a}_\phi+(6z-1)z\cos\phi\,\boldsymbol{a}_z}}$$

At $(5,\frac{\pi}{2},-1)$, $\nabla\times\boldsymbol{B} = \underline{\underline{5\boldsymbol{a}_\phi}}$

**(c)**

$$\nabla\times\boldsymbol{C} = \boldsymbol{a}_r\frac{1}{r\sin\theta}(r^{-1/2}\cos\theta-0)+\frac{\boldsymbol{a}_\theta}{r}\left(-\frac{2r\cos\theta\sin\phi}{\sin\theta}-\frac{3}{2}r^{1/2}\right)+\frac{\boldsymbol{a}_\phi}{r}(0+2r\sin\theta\cos\phi)$$

$$= \underline{\underline{r^{-1/2}\cot\theta\,\boldsymbol{a}_r-(2\cot\theta\sin\phi+\frac{3}{2}r^{-1/2})\boldsymbol{a}_\theta+2\sin\theta\cos\phi\,\boldsymbol{a}_\phi}}$$

At $(1,\frac{\pi}{6},\frac{\pi}{3})$, $\nabla\times\boldsymbol{C} = \underline{\underline{1.732\boldsymbol{a}_r-4.5\boldsymbol{a}_\theta+0.5\boldsymbol{a}_\phi}}$

**P.E. 3.9**

$$\oint_L \boldsymbol{A}\bullet d\boldsymbol{l} = \int_S(\nabla\times\boldsymbol{A})\bullet d\boldsymbol{S}$$

But $(\nabla\times\boldsymbol{A}) = \sin\phi\,\boldsymbol{a}_z+\frac{z\cos\phi}{\rho}\boldsymbol{a}_\rho$ and $d\,S = \rho d\phi d\rho\,\boldsymbol{a}_z$

$$\int_S(\nabla\times\boldsymbol{A})\bullet d\boldsymbol{S} = \iint\rho\sin\phi\;d\phi\,d\rho$$

$$= \frac{\rho^2}{2}\Big|_0^2\,(-\cos\phi)\Big|_0^{60^\circ}$$

$$= 2(-\frac{1}{2}+1) = \underline{\underline{1.}}$$

**P.E. 3.10**

$$\nabla\times\nabla V = \begin{vmatrix}\boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z\\ \frac{\partial}{\partial \boldsymbol{x}} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{\partial V}{\partial \boldsymbol{x}} & \frac{\partial V}{\partial y} & \frac{\partial V}{\partial z}\end{vmatrix} =$$

$$= \left(\frac{\partial^2 V}{\partial y\partial z}-\frac{\partial^2 V}{\partial y\partial z}\right)\boldsymbol{a}_x+\left(\frac{\partial^2 V}{\partial x\partial z}-\frac{\partial^2 V}{\partial z\partial x}\right)\boldsymbol{a}_y+\left(\frac{\partial^2 V}{\partial x\partial y}-\frac{\partial^2 V}{\partial y\partial x}\right)\boldsymbol{a}_z = 0$$

**P.E. 3.11**

(a)

$$\nabla^2 U = \frac{\partial}{\partial x}(2xy + yz) + \frac{\partial}{\partial y}(x^2 + xz) + \frac{\partial}{\partial z}(xy)$$

$$= \underline{\underline{2y}}.$$

**(b)**

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho(z\sin\phi + 2\rho) + \frac{1}{\rho^2}(-\rho z\sin\phi - 2z^2\frac{\partial}{\partial\rho}\sin\phi\cos\phi) + \frac{\partial}{\partial z}(\rho\sin\phi + 2z\cos^2\phi)$$

$$= \frac{1}{\rho}(z\sin\phi + 4\rho) - \frac{1}{\rho^2}(z\rho\sin\phi + 2z^2\cos 2\phi) + 2\cos^2\phi.$$

$$= \underline{\underline{4 + 2\cos^2\phi - \frac{2z^2}{\rho^2}\cos 2\phi.}}$$

**(c)**

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}[r^2\frac{1}{r}\cos\theta\sin\phi + 2r^3\phi] + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}[-\sin^2\theta\sin\phi\ln r]$$

$$+ \frac{1}{r^2\sin^2\theta}[-\cos\theta\sin\phi\ln r]$$

$$= \underline{\underline{\frac{1}{r^2}\cos\theta\sin\phi(1 - 2\ln r - \csc^2\theta\ln r) + 6\phi}}$$

**P.E. 3.12**

If $\boldsymbol{B}$ is conservative, $\nabla\times\boldsymbol{B} = \boldsymbol{0}$ must be satisfied.

$$\nabla\times\boldsymbol{B} = \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ \frac{\partial}{\partial \boldsymbol{x}} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y + z\cos xz & x & x\cos xz \end{vmatrix}$$

$$= 0\boldsymbol{a}_x + (\cos xz - xz\sin xz - \cos xz + xz\sin xz)\boldsymbol{a}_y + (1-1)\boldsymbol{a}_z = \boldsymbol{0}$$

Hence <u>**B** is a conservative field</u>.

**Prob. 3.1**

(a)

$$dl = \rho d\phi; \qquad \rho = 3$$

$$L = \int dl = 3\int_{\pi/4}^{\pi/2} d\phi = 3(\frac{\pi}{2} - \frac{\pi}{4}) = \frac{3\pi}{4} = \underline{\underline{2.356}}$$

(b)

$$dl = r\sin\theta d\phi; \qquad r = 1, \quad \theta = 30^\circ;$$

$$L = \int dl = r\sin\theta \int_0^{\pi/3} d\phi = (1)\sin 30^\circ [(\frac{\pi}{3}) - 0] = \underline{\underline{0.5236.}}$$

**(c)**

$$dl = r d\theta$$

$$L = \int dl = r\int_{\pi/6}^{\pi/2} d\theta = 4(\frac{\pi}{2} - \frac{\pi}{6}) = \frac{4\pi}{3} = \underline{\underline{4.189}}$$

**Prob. 3.2**

(a)

$$dS = \rho d\phi dz$$

$$S = \int dS = \rho\iint d\phi dz = 2\int_0^5 dz \int_{\pi/3}^{\pi/2} d\phi = 2(5)[\frac{\pi}{2} - \frac{\pi}{3}] = \frac{10\pi}{6} = \underline{\underline{5.236}}$$

(b)

In cylindrical, $dS = \rho\, d\rho\, d\phi$

$$S = \int dS = \int_1^3 \rho\, d\rho \int_0^{\pi/4} d\phi = \frac{\pi}{4}(\frac{\rho^2}{2})\Big|_1^3 = \underline{\underline{3.142}}$$

$(c)$ In spherical, $dS = r^2 \sin\theta\, d\phi\, d\theta$

$$S = \int dS = 100\int_{\pi/4}^{2\pi/3} \sin\theta\, d\theta \int_0^{2\pi} d\phi = 100(2\pi)(-\cos\theta)\Big|_{\pi/4}^{2\pi/3} = 200\pi(0.5 + 0.7071) = \underline{\underline{758.4}}$$

(d)

$$dS = r\,dr\,d\theta$$

$$S = \int dS = \int_0^4 r dr \int_{\pi/3}^{\pi/2} d\theta = \frac{r^2}{2}\Big|_0^4 (\frac{\pi}{2} - \frac{\pi}{3}) = \frac{8\pi}{6} = \underline{\underline{4.189}}$$

**Prob.3.3**

(a) $dV = dxdydz$

$$V = \int dxdydz = \int_0^1 dx \int_1^2 dy \int_{-3}^3 dz = (1)\ (2-1)(3--3) = \underline{\underline{6}}$$

(b) $dV = \rho d\phi d\rho dz$

$$V = \int_2^5 \rho d\rho \int_{-1}^4 dz \int_{\pi/3}^{\pi} d\phi = \frac{\rho^2}{2}\Big|_2^5 (4--1)(\pi - \frac{\pi}{3}) = \frac{1}{2}(25-4)(5)(\frac{2\pi}{3}) = 35\pi = \underline{\underline{110}}$$

(c) $dV = r^2 \sin\theta\, drd\theta d\phi$

$$V = \int_1^3 r^2 dr \int_{\pi/2}^{2\pi/3} \sin\theta \int_{\pi/6}^{\pi/2} d\phi = \frac{r^3}{3}\Big|_1^3 (-\cos\theta)\Big|_{\pi/2}^{\pi/3} (\frac{\pi}{2} - \frac{\pi}{6})$$

$$= \frac{1}{3}(27-1)(\frac{1}{2})(\frac{\pi}{3}) = \frac{26\pi}{18} = \underline{\underline{4.538}}$$

**Prob.3.4**

$$L = \int_L dl = \int_{\phi=0}^{\pi/6} \rho d\phi \Big|_{\rho=4} = 4(\pi/6) = \underline{\underline{2.094}}$$

**Prob. 3.5**

$$dS = r^2 \sin\theta d\theta d\phi$$

$$S = r^2 \int_0^{\pi/2} d\phi \int_0^{\pi/4} \sin\theta d\theta \Big|_{r=5} = 25(\pi/2)(-\cos\theta)\Big|_0^{\pi/4}$$

$$= \frac{25\pi}{2}(-\cos(\pi/4)+1) = \underline{\underline{11.502}}$$

**Prob. 3.6**

$$dv = \rho d\rho d\phi dz$$

$$V = \int_{z=0}^{10} dz \int_{\phi=0}^{30} d\phi \int_{\rho=2}^{5} \rho d\rho = 10(\pi/6)\left(\frac{\rho^2}{2}\Big|_2^5\right) = \frac{5\pi}{6}(25-4) = \underline{\underline{54.98}}$$

**Prob. 3.7**

$d\mathbf{l} = dx\mathbf{a}_x + dy\mathbf{a}_y$

$$I = \int_L \mathbf{H} \bullet d\mathbf{l} = \int (xy^2 dx + x^2 ydy)$$

But on L, $x = y^2 \quad \rightarrow \quad dx = 2ydy$

$$I = \int y^4(2ydy) + y^3 dy = \int_{y=1}^{4} (2y^5 dy + y^3 dy) = \left( \frac{2y^6}{6} + \frac{y^4}{4} \right) \Bigg|_1^4$$

$$= \frac{1}{3}(4096) + 64 - \frac{2}{6} - \frac{1}{4} = \underline{\underline{1428.75}}$$

**Prob. 3.8**

The line joining P and Q is $y = x - 2, dy = dx$

$$I = \int_L (2x^2 - 4xy)dx + (3xy - 2x^2 y)dy = \int_{x=1}^{3} \left[ 2x^2 - 4x(x-2) \right] dx + \left[ 3x(x-2) - 2x^2(x-2) \right] dx$$

$$= \int_{x=1}^{3} \left[ 3x^2 - 2x - 2x^3 \right] dx = \left( x^3 + x^2 - \frac{x^4}{4} \right) \Bigg|_1^3 = 27 + 9 - 81/2 = \underline{\underline{-4.5}}$$

**Prob. 3.9**

**(a)**

$$\int \mathbf{F} \bullet d\mathbf{l} = \int_{y=0}^{1} (x^2 - z^2) dy \Big|_{x=0, z=0} + \int_{x=0}^{x=2} 2xydx \Big|_{y=1, z=0} + \int_{z=0}^{z=3} (-3xz^2) dz \Big|_{x=2, y=1}$$

$$= 0 + 2(1) \frac{x^2}{2} \Big|_0^2 - 3(2) \frac{z^3}{3} \Big|_0^3$$

$$= 0 + 4 - 54 = \underline{\underline{-50}}$$

**(b)**

$$Let \quad x = 2t. \quad y = t, \quad z = 3t$$

$$dx = 2dt, \quad dy = dt, \quad dz = 3dt;$$

$$\int \mathbf{F} \bullet d\mathbf{l} = \int_0^1 (8t^2 - 5t^2 - 162t^3) \; dt$$

$$= (t^3 - 40.5t^4) \Big|_0^1 = \underline{\underline{-39.5}}$$

**Prob.3.10**

$$W=\int_L \boldsymbol{F}\bullet d\boldsymbol{l}=\int_{\phi=0}^{\pi/4} z\rho d\phi\ \Big|_{z=0,\rho=2}+\int_{z=0}^{3}\cos\phi dz\Big|_{\phi=\pi/4}$$

$$=0+\cos(\pi/4)(3)=3\cos 45^o\ \underline{\underline{=2.121\text{ J}}}$$

**Prob. 3.11**

$$\int \boldsymbol{H}\bullet d\boldsymbol{l}\ =\int_{x=1}^{0}(x-y)dx\Big|_{y=0,z=0}+\int_{z=0}^{1}5yzdz\Big|_{x=0,y=0}$$

$$+\int(x^2+zy)dy+5yzdz\Big|_{x=0,z=1-y/2}$$

$$=\int_1^0 xdx+\int_0^2(y-\frac{y^2}{2})dy+\int_1^0(10z-10z^2)dz$$

$$=\underline{\underline{-1.5}}$$

**Prob. 3.12**

*Method 1:*

$$\oint_L \boldsymbol{B}\cdot d\boldsymbol{l}=-\int_{y=0}^{1}yzdy\Big|_{z=0}+\int_{z=0}^{1}xzdz\Big|_{x=1}+\int(-yzdy+xzdz)\Big|_{x=1}$$

But $z=y \longrightarrow dz=dy$ on the last segment (or integral).

$$\oint_L \boldsymbol{B}\cdot d\boldsymbol{l}\ =\ 0+\frac{z^2}{2}\Big|_0^1+\int_{y=1}^{0}(-y^2+y)dy=\frac{1}{2}+(-\frac{y^3}{3}+\frac{y^2}{2})\Big|_1^0$$

$$=\frac{1}{2}+\frac{1}{3}-\frac{1}{2}=\frac{1}{3}=\underline{\underline{0.333}}$$

*Method 2:*

$$\oint_L \boldsymbol{B}\cdot d\boldsymbol{l}=\int_S \nabla\times\boldsymbol{B}\cdot d\boldsymbol{S}$$

$$\nabla\times\boldsymbol{B}=\begin{vmatrix}\frac{\partial}{\partial \boldsymbol{x}} & \frac{\partial}{\partial \boldsymbol{y}} & \frac{\partial}{\partial \boldsymbol{z}}\\ xy & -yz & xz\end{vmatrix}=y\boldsymbol{a}_x-z\boldsymbol{a}_y-x\boldsymbol{a}_z,\qquad d\boldsymbol{S}=dydz\boldsymbol{a}_x$$

$$\int_S \nabla\times\boldsymbol{B}\cdot d\boldsymbol{S}=\int_{y=0}^{1}\int_{z=0}^{y}ydzdy=\int_0^1 y^2dy=\frac{y^3}{3}\Big|_0^1=\frac{1}{3}=\underline{\underline{0.333}}$$

**Prob. 3.13**

$$\psi = \int_S \boldsymbol{A} \cdot d\boldsymbol{S} = \int\int z dx dz = \int_0^1 dx \int_0^2 z dz = (1)\frac{z^2}{2}\Big|_0^2 = \underline{\underline{2}}$$

**Prob. 3.14**

$$\Psi = \oint_S \boldsymbol{D} \bullet dS = \int_v \nabla \bullet \boldsymbol{D} dv$$

$$\nabla \bullet D = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = 2xz + 3y^2 + 2yz$$

$$\Psi = \int_v \nabla \bullet \boldsymbol{D} dv = \iiint (2xz + 3y^2 + 2yz) dx dy dz$$

$$= 2\int_{-1}^{1} x dx \int_0^4 dy \int_1^3 z dz + 3\int_{-1}^{1} dx \int_0^4 y^2 dy \int_1^3 dz + 2\int_{-1}^{1} dx \int_0^4 y dy \int_1^3 z dz$$

$$= 0 + 3(2)\left(\frac{y^3}{3}\Big|_0^4\right)(3) + 2(2)\left(\frac{y^2}{2}\Big|_0^4\right)\left(\frac{z^2}{2}\Big|_0^4\right) = 6(64) + 16(9-1) = 384 + 128$$

$$\underline{\underline{\Psi \quad = 512}}$$

**Prob. 3.15**

$$\Psi = \oint_S \boldsymbol{A} \bullet dS = \int_v \nabla \bullet \boldsymbol{A} dv$$

$$\nabla \bullet A = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(-3\sin\theta) + \frac{1}{r\sin}(5) = 3 - \frac{3\cos\theta}{r\sin\theta} + \frac{5}{r\sin\theta}$$

$$dv = r^2 \sin\theta d\theta dr d\phi$$

$$\Psi = 3\iiint r^2 \sin\theta \, d\theta \, d\phi dr - 3\iiint r\cos\theta \, d\theta \, d\phi dr + \iiint 5 \, d\theta \, d\phi dr$$

$$= 3\int_0^4 r^2 dr \int_0^{\pi/2} \sin\theta d\theta \int_0^{3\pi/2} d\phi - 3\int_0^4 r dr \int_0^{\pi/2} \cos\theta d\theta \int_0^{3\pi/2} d\phi + 5\int_0^4 dr \int_0^{\pi/2} d\theta \int_0^{3\pi/2} d\phi$$

$$= 3\left(\frac{r^3}{3}\Big|_0^4\right)(-\cos\theta)\Big|_0^{\pi/2}(3\pi/2) - 3\left(\frac{r^2}{2}\Big|_0^4\right)(\sin\theta)\Big|_0^{\pi/2}(3\pi/2) + 5(4)(\pi/2)(3\pi/2)$$

$$= 96\pi - 36\pi + 15\pi^2 = \underline{\underline{336.54}}$$

**Prob. 3.16**

(a) dv = dxdydz

$$\int_v xydv = \int_{z=0}^{2}\int_{y=0}^{1}\int_{z=0}^{1} xydxdydz = \int_0^1 xdx\int_0^1 ydy\int_0^z dz$$

$$= \frac{x^2}{2}\Big|_0^1 \frac{y^2}{2}\Big|_0^1 z\Big|_0^2 = (1/2)(1/2)(2) = \underline{\underline{0.5}}$$

(b)

$$dv = \rho d\rho d\phi dz$$

$$\int_v \rho z dv = \int_{\phi=0}^{\pi}\int_{z=0}^{2}\int_{\rho=1}^{3} \rho z \; \rho d\rho d\phi dz = \int_1^3 \rho^2 d\rho \int_0^2 zdz \int_0^\pi d\phi$$

$$= \frac{\rho^3}{3}\Big|_1^3 \frac{z^2}{2}\Big|_0^2 (\pi) = (9-\frac{1}{3})(2\pi) = \underline{\underline{54.45}}$$

**Prob. 3.17**

$$(a) \quad \nabla V_1 = \frac{\partial V_1}{\partial x}\boldsymbol{a_x} + \frac{\partial V_1}{\partial y}\boldsymbol{a_y} + \frac{\partial V_1}{\partial z}\boldsymbol{a_z}$$

$$= \underline{\underline{(6y-2z)\boldsymbol{a_x} + 6x\boldsymbol{a_y} + (1-2x)\boldsymbol{a_z}}}$$

$$(b) \quad \nabla V_2 = \frac{\partial V_2}{\partial \rho}\boldsymbol{a}_\rho + \frac{1}{\rho}\frac{\partial V_2}{\partial \phi}\boldsymbol{a}_\phi + \frac{\partial V_2}{\partial z}\boldsymbol{a_z}$$

$$= \underline{\underline{(10\cos\phi - z)\boldsymbol{a}_\rho - 10\sin\phi\boldsymbol{a}_\phi - \rho\boldsymbol{a_z}}}$$

$$\nabla V_3 = \frac{\partial V_3}{\partial r}\boldsymbol{a_r} + \frac{1}{r}\frac{\partial V_3}{\partial \theta}\boldsymbol{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V_3}{\partial \phi}\boldsymbol{a}_\phi$$

(c )

$$= -\frac{2}{r^2}\cos\phi\boldsymbol{a_r} + 0 + \frac{1}{r\sin\theta}\left(-\frac{2}{r}\sin\phi\right)\boldsymbol{a}_\phi$$

$$= \underline{\underline{-\frac{2}{r^2}\cos\phi\boldsymbol{a_r} - \frac{2\sin\phi}{r^2\sin\theta}\boldsymbol{a}_\phi}}$$

**Prob. 3.18**

(a)

$$\nabla V = (10yz - 4xz)\boldsymbol{a}_x + 10xz\boldsymbol{a}_y + (10xy - 2x^2)\boldsymbol{a}_z$$

At $P$, $x = -1, y = 4, z = 3$. Hence,

$$\nabla V = (120+12)\boldsymbol{a}_x - 30\boldsymbol{a}_y + (-40-2)\boldsymbol{a}_z = \underline{\underline{132\boldsymbol{a}_x - 30\boldsymbol{a}_y - 42\boldsymbol{a}_z}}$$

(b)

$$\nabla U = (2\sin\phi + z)\boldsymbol{a}_\rho + 2\cos\phi\boldsymbol{a}_\phi + \rho\boldsymbol{a}_z$$

At Q, $\rho = 2, \phi = 90^o, z = -1$

$$\nabla U = (2-1)\boldsymbol{a}_\rho + 0\boldsymbol{a}_\phi + 2\boldsymbol{a}_z = \boldsymbol{a}_\rho + 2\boldsymbol{a}_z$$

(c )

$$\nabla W = -\frac{8}{r^3}\sin\theta\cos\phi\boldsymbol{a}_r + \frac{1}{r}\frac{4\cos\theta\cos\phi}{r^2}\boldsymbol{a}_\theta + \frac{1}{r\sin\theta}\left(\frac{-4\sin\theta\sin\phi}{r^2}\right)\boldsymbol{a}_\phi$$

At R, $r = 1, \theta = \pi/6, \phi = \pi/2$

$$\nabla W = -\frac{4}{(1)^3}\boldsymbol{a}_\phi = \underline{\underline{-4\boldsymbol{a}_\phi}}$$

**Prob. 3.19**

$$r = \sqrt{x^2 + y^2 + z^2}, \qquad r^n = (x^2 + y^2 + z^2)^{n/2}$$

<u>Method 1</u>:

$$\nabla r^n = \frac{\partial r^n}{\partial x}\boldsymbol{a}_x + \frac{\partial r^n}{\partial y}\boldsymbol{a}_y + \frac{\partial r^n}{\partial z}\boldsymbol{a}_z = \frac{n}{2}(x^2 + y^2 + z^2)^{n/2-1}(2x)\boldsymbol{a}_x + \cdots$$

$$= n(x^2 + y^2 + z^2)^{\frac{n-2}{2}}(x\boldsymbol{a}_x + y\boldsymbol{a}_y + z\boldsymbol{a}_z) = \underline{\underline{nr^{n-2}\boldsymbol{r}}}$$

<u>Method 2:</u>

$$\nabla r^n = \frac{\partial r^n}{\partial r}\boldsymbol{a}_r = nr^{n-1}\frac{\boldsymbol{r}}{r} = nr^{n-2}\boldsymbol{r}$$

**Prob. 3.20**

$$\nabla T = 2x\boldsymbol{a}_x + 2y\boldsymbol{a}_y - \mathrm{a}_z$$

At $(1,1,2), \nabla T = (2,2,-1)$. The mosquito should move in the direction of

$$\underline{\underline{2\boldsymbol{a}_x + 2\boldsymbol{a}_y - \boldsymbol{a}_z}}$$

**Prob. 3.21**

$$\nabla \boldsymbol{F} = \boldsymbol{a}_x - 2\boldsymbol{a}_y + \boldsymbol{a}_z$$

$$a_n = \frac{\nabla \boldsymbol{F}}{|\nabla \boldsymbol{F}|} = \frac{\boldsymbol{a}_x - 2\boldsymbol{a}_y + \boldsymbol{a}_z}{\sqrt{1+4+1}} = \underline{\underline{0.4082\boldsymbol{a}_x - 0.8165\boldsymbol{a}_y + 0.4082\boldsymbol{a}_z}}$$

**Prob. 3.22**

Method 1:

$$\nabla T = \frac{\partial T}{\partial r}\boldsymbol{a}_r + \frac{1}{r}\frac{\partial T}{\partial \theta}\boldsymbol{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\boldsymbol{a}_\phi$$

$$= \sin\theta\cos\phi\boldsymbol{a}_r + \cos\theta\cos\phi\boldsymbol{a}_\theta - \sin\phi\boldsymbol{a}_\phi$$

At P, $r = 2, \theta = 60^o, \phi = 30^o$

$$\nabla T = \sin 60^o \cos 30^o \boldsymbol{a}_r + \cos 60^o \cos 30^o \boldsymbol{a}_\theta - \sin 30^o \boldsymbol{a}_\phi$$

$$= 0.75\boldsymbol{a}_r \text{ - } 0.433\boldsymbol{a}_\theta - 0.5\boldsymbol{a}_\phi$$

$$|\nabla T| = \sqrt{0.75^2 + 0.433^2 + 0.5^2} = 1$$

The magnitude of T is 1 and its direction is along ∇T.

Method 2:

$T = r\sin\theta\cos\phi = x$

$\nabla T = \boldsymbol{a}_x$

$|\nabla T| = 1$

**Prob. 3.23**

$$\nabla f = \frac{\partial f}{\partial x}\boldsymbol{a}_x + \frac{\partial f}{\partial y}\boldsymbol{a}_y + \frac{\partial f}{\partial z}\boldsymbol{a}_z = (2xy - 2y^2)\boldsymbol{a}_x + (x^2 - 4xy)\boldsymbol{a}_y + 3z^2\boldsymbol{a}_z$$

At point (2,4,-3), $x = 2, y = 4, z = -3$

$$\nabla f = (16 - 32)\boldsymbol{a}_x + (4 - 32)\boldsymbol{a}_y + 27\boldsymbol{a}_z = -16\boldsymbol{a}_x - 28\boldsymbol{a}_y + 27\boldsymbol{a}_z$$

$$\boldsymbol{a} = \frac{\boldsymbol{a}_x + 2\boldsymbol{a}_y - \boldsymbol{a}_z}{\sqrt{1+4+1}} = \frac{1}{\sqrt{6}}(1, 2, -1)$$

The directional derivative is

$$\nabla f \cdot \boldsymbol{a} = (-16, -28, 27) \cdot \frac{1}{\sqrt{6}}(1, 2, -1) = -\frac{99}{\sqrt{6}} = \underline{\underline{-40.42}}$$

**Prob. 3.24**

(a) Let f = ax + by + cz – d = 0

$\nabla f = a\mathbf{a}_x + b\mathbf{a}_y + c\mathbf{a}_z$

$$\mathbf{a}_{n1} = \frac{\nabla f}{|\nabla f|} = \frac{a\mathbf{a}_x + b\mathbf{a}_y + c\mathbf{a}_z}{\sqrt{a^2+b^2+c^2}}$$

Let $g = \alpha x + \beta y + \gamma z - \delta$

$$\mathbf{a}_{n2} = \frac{\nabla g}{|\nabla g|} = \frac{\alpha\mathbf{a}_x + \beta\mathbf{a}_y + \gamma\mathbf{a}_z}{\sqrt{\alpha^2+\beta^2+\gamma^2}}$$

$$\cos\theta = \mathbf{a}_{n1} \cdot \mathbf{a}_{n2} = \frac{a\alpha + b\beta + c\gamma}{\sqrt{a^2+b^2+c^2}\sqrt{\alpha^2+\beta^2+\gamma^2}}$$

$$\theta = \cos^{-1}\frac{a\alpha + b\beta + c\gamma}{\sqrt{(a^2+b^2+c^2)(\alpha^2+\beta^2+\gamma^2)}}$$

$a = 1, b = 2, c = 3$

(b) $\alpha = 1, \beta = 1, \gamma = 0$

$$\theta = \cos^{-1}\frac{1+2+0}{\sqrt{(1^2+2^2+3^2)(1^2+1^2+0^2)}} = \cos^{-1}\frac{3}{\sqrt{28}} = \cos^{-1}0.5669 = \underline{\underline{55.46^o}}$$

**Prob. 3.25**

$$\nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z = 4ye^z\mathbf{a}_x + 4xe^z\mathbf{a}_y + 4xye^z\mathbf{a}_z$$

At (3,1,-2), $x = 3$, $y = 1$, $z = -2$

$$\nabla V = 4e^{-2}\mathbf{a}_x + 12e^{-2}\mathbf{a}_y + 12e^{-2}\mathbf{a}_z = \underline{\underline{0.5413\mathbf{a}_x + 1.624\mathbf{a}_y + 1.624\mathbf{a}_z}}$$

This is the direction. The maximum rate of change is

$$|\nabla V| = e^{-2}\sqrt{4^2+12^2+12^2} = 0.1353 \times 17.44 = \underline{\underline{2.36}}$$

**Prob. 3.26**

(a)

$$\nabla UV = \frac{\partial}{\partial x}(UV)\mathbf{a}_x + \frac{\partial}{\partial y}(UV)\mathbf{a}_y + \frac{\partial}{\partial z}(UV)\mathbf{a}_z$$

$$= \left(U\frac{\partial V}{\partial x} + V\frac{\partial U}{\partial x}\right)\mathbf{a}_x + \left(U\frac{\partial V}{\partial y} + V\frac{\partial U}{\partial y}\right)\mathbf{a}_y + \left(U\frac{\partial V}{\partial z} + V\frac{\partial U}{\partial z}\right)\mathbf{a}_z$$

$$= U\nabla V + V\nabla U$$

(b)

$\nabla V = 10xy\boldsymbol{a}_x + (5x^2 + 2z)\boldsymbol{a}_y + 2y\boldsymbol{a}_z$

$U\nabla V = 30x^2y^2z\boldsymbol{a}_x + (15x^3yz + 6xyz^2)\boldsymbol{a}_y + 6xy^2z\boldsymbol{a}_z$

$\nabla U = 3yz\boldsymbol{a}_x + 3xz\boldsymbol{a}_y + 3xy\boldsymbol{a}_z$

$V\nabla U = (15x^2y^2z + 6y^2z^2)\boldsymbol{a}_x + (15x^3yz + 6xyz^2)\boldsymbol{a}_y + (15x^3y + 6xy^2z)\boldsymbol{a}_z$

$U\nabla V + V\nabla U = (45x^2y^2z + 6y^2z^2)\boldsymbol{a}_x + (30x^3yz + 12xyz^2)\boldsymbol{a}_y + (15x^3y + 12xy^2z)\boldsymbol{a}_z - (1)$

$UV = 15x^3y^2z + 6xy^2z^2$

$\nabla(UV) = (45x^2y^2z + 6y^2z^2)\boldsymbol{a}_x + (30x^3yz + 12xyz^2)\boldsymbol{a}_y + (15x^3y + 12xy^2z)\boldsymbol{a}_z - (2)$

From (1) and (2), the formula is verified.

**Prob. 3.27**

(a) $\nabla \cdot \boldsymbol{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \underline{\underline{3y - x}}$

(b) $\nabla \cdot \boldsymbol{B} = \frac{1}{\rho} 2\rho z^2 + \frac{1}{\rho} \rho 2 \sin\phi \cos\phi + 2\rho \sin^2\phi$

$= \underline{\underline{2z^2 + \sin 2\phi + 2\rho \sin^2\phi}}$

(c ) $\nabla \cdot \boldsymbol{C} = \frac{1}{r^2} 3r^2 + 0 = \underline{\underline{3}}$

**Prob. 3.28**

$\nabla \cdot \boldsymbol{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 2xy + 0 + 2y = 2y(1+x)$

**(a)** At $(-3, 4, 2), x = -3, y = 4$

$\nabla \cdot \boldsymbol{A} = 2(4)(1-3) = \underline{\underline{-16}}$

$\nabla \cdot \boldsymbol{B} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho B_\rho\right) + \frac{1}{\rho}\frac{\partial B_\phi}{\partial \phi} + \frac{\partial B_z}{\partial z} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(3\rho^2 \sin\phi\right) + 0 + 8z\cos^2\phi$

**(b)** $= 6\sin\phi + 8z\cos^2\phi$

At $(5, 30^o, 1), z = 1, \phi = 30^o$

$\nabla \cdot \boldsymbol{B} = 6\sin 30^o + 8(1)\cos^2 30^o = 3 + 6 = \underline{\underline{9}}$

$$\nabla \bullet \boldsymbol{C} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 C_r) + 0 + \frac{1}{r\sin\theta}\frac{\partial C_\phi}{\partial \phi} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^4\cos\theta) + 0 = 4r\cos\theta$$

**(c )** At $(2,\pi/3,\pi/2), r = 2, \theta = \pi/3$

$\nabla \bullet \boldsymbol{C}$ = 4(2)cos($\pi$/3) = $\underline{\underline{4}}$

**Prob. 3.29**

$$\nabla \bullet \boldsymbol{H} = k\nabla \bullet \nabla T = k\nabla^2 T$$

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 50\sin\frac{\pi x}{2}\cosh\frac{\pi y}{2}\left(-\frac{\pi^2}{4} + \frac{\pi^2}{4}\right) = 0$$

Hence, $\nabla \bullet \boldsymbol{H} = 0$

**Prob. 3.30**

**(a)**

$$\nabla \bullet (V\boldsymbol{A}) = \frac{\partial}{\partial x}(VA_x) + \frac{\partial}{\partial y}(VA_y) + \frac{\partial}{\partial z}(VA_z)$$

$$= \left(A_x\frac{\partial V}{\partial x} + V\frac{\partial A_x}{\partial x}\right) + \left(A_y\frac{\partial V}{\partial y} + V\frac{\partial A_y}{\partial y}\right) + \left(A_z\frac{\partial V}{\partial z} + V\frac{\partial A_z}{\partial z}\right)$$

$$= V\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) + A_x\frac{\partial V}{\partial x} + A_y\frac{\partial V}{\partial y} + A_z\frac{\partial V}{\partial z}$$

$$= \underline{\underline{V\nabla \bullet \boldsymbol{A} + \boldsymbol{A} \bullet \nabla V}}$$

**(b)**

$$\nabla \bullet \boldsymbol{A} = 2 + 3 - 4 = 1; \quad \nabla V = yz\,\boldsymbol{a}_x + xz\,\boldsymbol{a}_y + xy\,\boldsymbol{a}_z$$

$$\nabla \bullet (V\boldsymbol{A}) = V\nabla \bullet A + \boldsymbol{A} \bullet \nabla V$$

$$= xyz + 2xyz + 3xyz - 4xyz = \underline{\underline{2xyz}}$$

**Prob. 3.31**

(a)

$$(\nabla \bullet \boldsymbol{r})\boldsymbol{T} = 3\boldsymbol{T} = \underline{\underline{6yz\,\boldsymbol{a}_y + 3xy^2\,\boldsymbol{a}_y + 3x^2yz\,\boldsymbol{a}_z}}$$

(b)

$$x\frac{\partial \mathbf{T}}{\partial x}+y\frac{\partial \mathbf{T}}{\partial y}+z\frac{\partial \mathbf{T}}{\partial z}=x\ (y^2\,\mathbf{a}_y+2xyz\,\mathbf{a}_z)+y(2z\,\mathbf{a}_x+2xy\,\mathbf{a}_y+x^2z\,\mathbf{a}_z)$$

$$+z(2y\,\mathbf{a}_x+x^2y\,\mathbf{a}_z)$$

$$=\underline{\underline{4yz\,\mathbf{a}_x+3xy^2\,\mathbf{a}_y+4x^2yz\,\mathbf{a}_z}}$$

(c)

$$\nabla \bullet \boldsymbol{r}(\boldsymbol{r}\bullet\boldsymbol{T})=3\ (2xyz+xy^3+x^2yz^2)$$

$$=\underline{\underline{6xyz+3xy^3+3x^2yz^2}}$$

(d)

$$(\boldsymbol{r}\bullet\nabla)r^2=(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z})(x^2+y^2+z^2)$$

$$=x(2x)+y(2y)+z(2z)$$

$$=\underline{\underline{2(x^2+y^2+z^2)=2r^2}}$$

**Prob. 3.32**

We convert A to cylindrical coordinates; only the ρ-component is needed.

$$A_\rho=A_x\cos\phi+A_y\sin\phi=2x\cos\phi-z^2\sin\phi$$

But $x=\rho\cos\phi$,

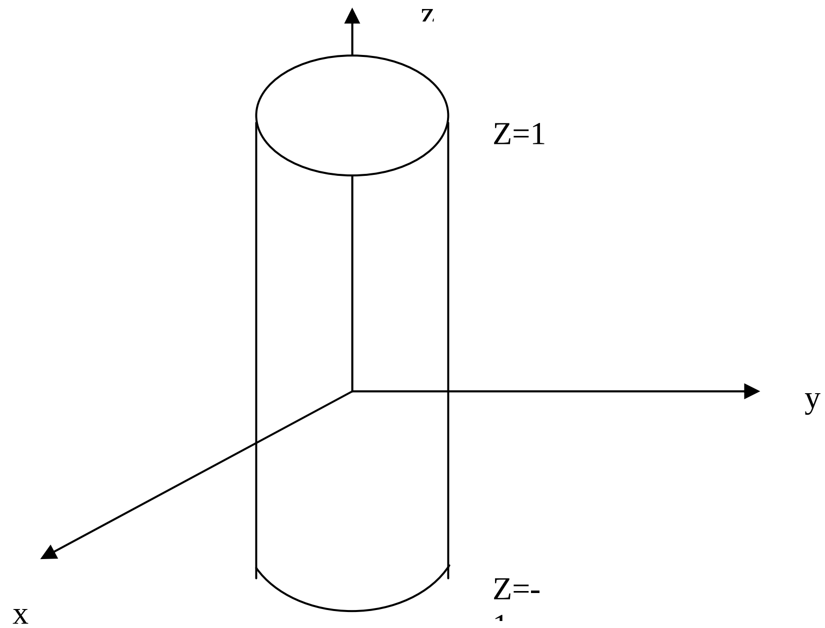
$$A_\rho=2\rho\cos^2\phi-z^2\sin\phi$$

$$\Psi=\int_S \boldsymbol{A}\cdot d\boldsymbol{S}=\iint A_\rho\rho d\phi dz=\iint\left[2\rho^2\cos^2\phi-\rho z^2\sin\phi\right]d\phi dz$$

$$=2(2)^2\int_0^{\pi/2}\frac{1}{2}(1+\cos 2\phi)d\phi\int_0^1 dz-2\int_0^1 z^2dz\int_0^{\pi/2}\sin\phi d\phi$$

$$=4(\phi+\frac{1}{2}\sin 2\phi)\Big|_0^{\pi/2}-2\frac{z^3}{3}\Big|_0^1(-\cos\phi)\Big|_0^{\pi/2}=2\pi-2/3=\underline{\underline{5.6165}}$$

**Prob. 3.33**




**(a)**

$$\oint \mathbf{D} \bullet d\mathbf{S} = [\iint_{z=-1} + \iint_{z=1} + \iint_{\rho=5}] \mathbf{D} \bullet d\mathbf{S}$$

$$= -\iint \rho^2 \cos^2\phi \, d\phi d\rho + \iint \rho^2 \cos^2\phi \, d\phi d\rho + \iint 2\rho^2 z^2 d\phi dz \Big|_{\rho=5}$$

$$= 2(5)^2 \int_0^{2\pi} d\phi \int_{-1}^{1} z^2 dz = +50(2\pi)(\frac{z^3}{3} \Big|_{-1}^{1})$$

$$= \frac{200\pi}{3} = \underline{\underline{209.44}}$$

(b) $\nabla \bullet \mathbf{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(2\rho^2 z^2) = 4z^2$

$$\int \nabla \bullet \mathbf{D} dv = \iiint 4z^2 \rho d\rho d\phi dz = 4\int_{-1}^{1} z^2 dz \int_0^5 \rho d\rho \int_0^{2\pi} d\phi$$

$$= 4x \frac{z^3}{3}\Big|_{-1}^{1} \quad \frac{\rho^2}{2}\Big|_0^5 (2\pi) = \frac{200\pi}{3} = \underline{\underline{209.44}}$$

**Prob. 3.34**

$$\int_S \boldsymbol{H} \cdot d\boldsymbol{S} = \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} 10\cos\theta r^2 \sin\theta d\theta d\phi \Big|_{r=1}$$

$$= 10(1)^2 \int_0^{2\pi} d\phi \int_0^{\pi/2} \sin\theta\cos\theta d\theta = 10(2\pi)\int_0^{\pi/2} \sin\theta d(\sin\theta)$$

$$= 20\pi\left(\frac{\sin^2\theta}{2}\right)\Big|_0^{\pi/2} = 10\pi = \underline{\underline{31.416}}$$

**Prob. 3.35**

$$\oint_S \boldsymbol{H}\cdot d\boldsymbol{S} = \int_v \nabla\cdot\boldsymbol{H}dv$$

$$\oint_S \boldsymbol{H}\cdot d\boldsymbol{S} = -\iint 2xydydz\Big|_{x=0} + \iint 2xydydz\Big|_{x=1} - \iint (x^2+z^2)dxdz\Big|_{y=1}$$

$$+\iint (x^2+z^2)dxdz\Big|_{y=2} - \iint 2yzdxdy\Big|_{z=-1} + \iint 2yzdxdy\Big|_{z=3}$$

$$= 0 + 2\int_1^2 ydy\int_{-1}^3 dz + 2\int_0^1 dx\int_1^2 ydy + 6\int_0^1 dx\int_1^2 ydy$$

$$= 12 + 3 + 9 = \underline{\underline{24}}$$

$$\nabla\cdot\boldsymbol{H} = \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = 2y + 0 + 2y = 4y$$

$$\int_V \nabla\cdot\boldsymbol{H}dv = \iiint 4ydxdydz = 4\int_0^1 dx\int_1^2 ydy\int_{-1}^3 dz$$

$$= 4(1)\frac{y^2}{2}\Big|_1^2 (3+1) = \underline{\underline{24}}$$

**Prob. 36**

$$\Psi = \oint_S \boldsymbol{H}\bullet d\boldsymbol{S} = \int_v \nabla\bullet\boldsymbol{H}dv$$

To find $\oint_S \boldsymbol{H}\bullet d\boldsymbol{S}$, let

$\Psi = \Psi_t + \Psi_b + \Psi_s$

wher $\Psi_t, \Psi_b,$ and $\Psi_s$ are the fluxes from the top, bottom, and side of the cylinder.

For $\Psi_t$, $d\boldsymbol{S} = \rho d\rho d\phi \boldsymbol{a}_z$ ,

$$\Psi_t = -2z\iint \rho d\rho d\phi \Big|_{z=3} = -6\int_0^{10} \rho d\rho \int_0^{2\pi} d\phi = -12\pi \frac{\rho^2}{2}\Big|_0^{10} = -600\pi$$

For $\Psi_t$, $d\boldsymbol{S} = \rho d\rho d\phi(-\boldsymbol{a}_z)$,

$$\Psi_b = 2z\iint \rho d\rho d\phi \Big|_{z=0} = 0$$

For $\Psi_s$, $d\boldsymbol{S} = \rho d\phi dz \boldsymbol{a}_\rho$ ,

$$\Psi_t = 4\rho^3 \iint d\phi dz \Big|_{\rho=10} = 4000\int_0^3 dz \int_0^{2\pi} d\phi = 4000(3)(2\pi) = 24000\pi$$

$$\Psi = -600\pi + 0 + 24000\pi = 23400\pi = \underline{\underline{73,513.27}}$$

To get $\int_v \nabla \bullet \boldsymbol{H} dv$, $dv = \rho d\phi d\rho dz$

$$\nabla \bullet \boldsymbol{H} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(4\rho^3) - 2 = 12\rho - 2$$

$$\Psi = \iiint (12\rho - 2)\rho d\phi d\rho dz = 12\int_0^{10} \rho^2 d\rho \int_0^{2\pi} d\phi \int_0^3 dz - 2\int_0^{10} \rho d\rho \int_0^{2\pi} d\phi \int_0^3 dz$$

$$= 12\frac{(1000)}{3}(2\pi)(3) - 2\frac{(100)}{2}(2\pi)(3) = 24000\pi - 600\pi = \underline{\underline{73,513.27}}$$

**Prob. 3.37**

Let $\psi = \oint_S \mathbf{A} \cdot d\mathbf{S} = \int_v \nabla \cdot \boldsymbol{A} dv$

$$\nabla \cdot \boldsymbol{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 2(x + y + z)$$

$$\nabla \cdot \boldsymbol{A} = 2(\rho\cos\phi + \rho\sin\phi + z), \qquad dv = \rho d\rho d\phi dz$$

$$\psi = \iiint 2(\rho\cos\phi + \rho\sin\phi + z)\rho d\rho d\phi dz$$

$$= 0 + 0 + 2\int_0^1 \rho d\rho \int_2^4 z dz \int_0^{2\pi} d\phi = \frac{2\rho^2}{2}\Big|_0^1 \frac{z^2}{2}\Big|_2^4 (2\pi) = \frac{1}{2}(16-4)(2\pi)$$

$$= 12\pi = \underline{\underline{37.7}}$$

**Prob. 3.38**

$$\nabla \bullet \boldsymbol{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^4) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(r\sin^2\theta\cos\phi)$$

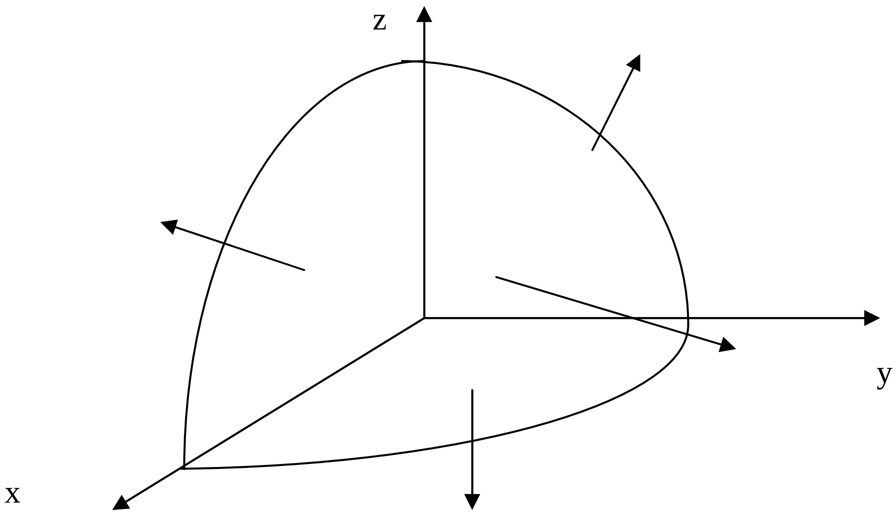
$$= 4r + 2\cos\theta\cos\phi$$

$$\int \nabla \bullet \boldsymbol{A} dv = \iiint 4r^3 \sin\theta d\theta d\phi dr + \iiint 2r^2 \sin\theta \cos\theta \cos\phi d\theta d\phi dr$$

$$= 4\frac{r^4}{4}\Big|_0^3 (-\cos\theta)\Big|_0^{\pi/2} (\frac{\pi}{2}) + \frac{2r^3}{3}\Big|_0^3 (-\frac{\cos^2\theta}{2})\Big|_0^{\pi/2} \sin\phi\Big|_0^{\frac{\pi}{2}}$$

$$= 81(1)(\frac{\pi}{2}) + 18(0 + \frac{1}{2})(1-0)$$

$$= \frac{81\pi}{2} + 9 = \underline{\underline{136.23}}$$




$$\int \boldsymbol{A} \bullet d\boldsymbol{S} = [\iint_{\phi=0} + \iint_{\phi=\pi/2} + \iint_{r=3} + \iint_{\theta=\pi/2}] \boldsymbol{A} \bullet d\boldsymbol{S}$$

Since $\boldsymbol{A}$ has no $\phi-$component, the first two integrals on the right hand side vanish.

$$\int \boldsymbol{A} \bullet d\boldsymbol{S} = \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{\pi/2} r^4 \sin\theta \, d\theta \, d\phi \Big|_{r=3} + \int_{r=0}^{3} \int_{\phi=0}^{\pi/2} r^2 \sin^2\theta \cos\phi dr d\phi \Big|_{\theta=\pi/2}$$

$$= 81\,(\frac{\pi}{2})\,(-\cos\theta)\Big|_0^{\pi/2} + 9(1)\sin\phi\Big|_0^{\pi/2}$$

$$= \frac{81\pi}{2} + 9 = \underline{\underline{136.23}}$$

**Prob. 3.39**

Let $\psi = \oint \boldsymbol{F} \bullet d\boldsymbol{S} = \psi_t + \psi_b + \psi_o + \psi_i$

where $\psi_t, \psi_b, \psi_o, \psi_i$ are the fluxes through the top surface, bottom surface, outer surface ($\rho = 3$), and inner surface respectively.

For the top surface, $d\boldsymbol{S} = \rho\, d\phi\, d\rho\, \boldsymbol{a}_z, \qquad z = 5;$

$\boldsymbol{F} \bullet d\boldsymbol{S} = \rho^2\ z\ d\phi\ dz.$ Hence:

$$\psi_t = \int_{\rho=2}^{3} \int_{\phi=0}^{2\pi} \rho^2\ z\ d\phi\ dz\Big|_{z=5} = \frac{190\ \pi}{3} = 198.97$$

For the bottom surface, $z = 0,\ d\boldsymbol{S} = \rho\, d\phi\, d\rho(-\boldsymbol{a}_z)$

$\boldsymbol{F} \bullet d\boldsymbol{S} = -\rho^2 z\, d\phi\ d\rho = 0.$ Hence, $\psi_b = 0.$

For the outer curved surface, $\rho = 3, \quad d\boldsymbol{S} = \rho d\phi dz\, \mathrm{a}_\rho$

$\boldsymbol{F} \bullet d\boldsymbol{S} = \rho^2 \sin\phi\ d\phi\ dz.$ Hence,

$$\psi_a = \int_{z=0}^{5} dz\ \rho^3 \int_{\phi=0}^{2\pi} \sin\phi\, d\phi\Big|_{\rho=3} = 0$$

For the inner curved surface, $\rho = 2, \quad d\,\boldsymbol{S} = \rho\, d\phi\, dz\, (-\boldsymbol{a}_\rho)$

$\boldsymbol{F} \bullet d\boldsymbol{S} = -\rho^3 \sin\phi\ d\phi\ dz.$ Hence,

$$\psi_a = -\int_{z=0}^{5} dz\ \rho^3 \int_{\phi=0}^{2\pi} \sin\phi\, d\phi\Big|_{\rho=2} = 0$$

$$\psi = \frac{190\,\pi}{3} + 0 + 0 + 0 = \frac{190\ \pi}{3} = \underline{\underline{198.97}}$$

$$\psi = \oint \boldsymbol{F} \bullet d\boldsymbol{S} = \int \nabla \bullet \boldsymbol{F} dV$$

$$\nabla \bullet \boldsymbol{F} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\ \rho^3 \sin\phi) + \frac{1}{\rho}\frac{\partial}{\partial\phi}(z\cos\phi) + \rho$$

$$= 3\rho\sin\phi - \frac{z}{\rho}\sin\phi + \rho$$

$$\int_V \nabla \bullet \boldsymbol{F} dv = \iiint (3\rho \sin\phi - \frac{z}{\rho}\sin\phi + \rho)\ \rho\, d\phi\, d\rho\, dz$$

$$= 0 + 0 + \int_0^5 dz \int_0^{2\pi} d\phi \int_2^3 \rho^2 d\rho$$

$$= \frac{190\ \pi}{3} = 198.97$$

**Prob. 3.40**

(a) $$\nabla x\boldsymbol{A} = \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ \frac{\partial}{\partial \boldsymbol{x}} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & y^2 & -xz \end{vmatrix} = z\boldsymbol{a}_y - x\boldsymbol{a}_z$$

$$\nabla x\boldsymbol{B} = \left(\frac{1}{\rho} 2\rho z 2\sin\phi\cos\phi - 0\right)\boldsymbol{a}_\rho + (2\rho z - 2z\sin^2\phi)\boldsymbol{a}_\phi + \frac{1}{\rho}\left(2\rho\sin^2\phi - 0\right)\boldsymbol{a}$$

(b) $$= 4z\sin\phi\cos\phi\boldsymbol{a}_\rho + 2(\rho z - z\sin^2\phi)\boldsymbol{a}_\phi + 2\sin^2\phi\boldsymbol{a}_z$$

$$= 2z\sin 2\phi\boldsymbol{a}_\rho + 2z(\rho - \sin^2\phi)\boldsymbol{a}_\phi + 2\sin^2\phi\boldsymbol{a}_z$$

$$\nabla x\boldsymbol{C} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(r\cos^2\theta\sin\theta\right]\boldsymbol{a}_r - \frac{1}{r}\left[\frac{\partial}{\partial r}(r^2\cos^2\theta\right]\boldsymbol{a}_\theta$$

(c ) $$= \frac{r}{r\sin\theta}\left[(2\cos\theta)(-\sin\theta)\sin\theta + \cos\theta(\cos^2\theta)\right]\boldsymbol{a}_r - \frac{\cos^2\theta}{r}(2r)\boldsymbol{a}_\theta$$

$$= \frac{(\cos^3\theta - 2\sin^2\theta\cos\theta)}{\sin\theta}\boldsymbol{a}_r - 2\cos^2\theta\boldsymbol{a}_\theta$$

**Prob. 3.41**

(a)

$$\nabla \times A = \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ \frac{\partial}{\partial \boldsymbol{x}} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y & y^2 z & -2xz \end{vmatrix} = -y^2\,\mathrm{a}_x + 2z\,\boldsymbol{a}_y - x^2\,\mathrm{a}_z$$

$$\nabla \bullet \nabla \times \boldsymbol{A} = 0$$

(b)

$$\nabla \times A = \left(\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\boldsymbol{a}_\rho + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right)\boldsymbol{a}_\phi + \frac{1}{\rho}\left(\frac{\partial(\rho A_\rho)}{\partial \rho} - \frac{\partial A_\rho}{\partial \phi}\right)\boldsymbol{a}_z$$

$$= (0-0)\boldsymbol{a}_\rho + (\rho^2 - 3z^2)\boldsymbol{a}_\phi + \frac{1}{\rho}(4\rho^3 - 0)\boldsymbol{a}_z$$

$$= \underline{\underline{(\rho^2 - 3z^2)\boldsymbol{a}_\phi + 4\rho^2\,\boldsymbol{a}_z}}$$

$$\nabla \bullet \nabla \times A = \underline{\underline{0}}$$

(c)

$$\nabla \times \boldsymbol{A} = \frac{1}{r\sin\theta}\left[0 - \frac{\sin\phi}{r^2}\right]\boldsymbol{a}_r + \frac{1}{r}\left[\frac{-1}{\sin\theta}\frac{\cos\phi}{r^2} - 0\right]\boldsymbol{a}_\theta + \frac{1}{r}\left[\frac{\partial}{\partial r}\left(\frac{\cos\phi)}{r} - 0\right]\boldsymbol{a}_\phi$$

$$\underline{\underline{= -\frac{\sin\phi}{r^3\sin\theta}\boldsymbol{a}_r + \frac{\cos\phi}{r^3\sin\theta}\boldsymbol{a}_\theta + \frac{\cos\phi}{r^3}\boldsymbol{a}_\phi}}$$

$$\nabla \bullet \nabla \times \boldsymbol{A} = \frac{-\sin\phi}{r^4\sin\theta} + 0 + \frac{\sin\phi}{r^4\sin\theta} = 0$$

$$\nabla \bullet \nabla \times \boldsymbol{A} = \underline{\underline{0}}$$

**Prob. 3.42**

$$\nabla \times \boldsymbol{H} = 0\boldsymbol{a}_\rho + 1\boldsymbol{a}_\phi + \frac{1}{\rho}(2\rho\cos\phi - \rho\cos\phi)\boldsymbol{a}_z = \underline{\underline{\boldsymbol{a}_\phi + \cos\phi\boldsymbol{a}_z}}$$

$$\nabla \times \nabla \times \boldsymbol{H} = \left(-\frac{1}{\rho}\sin\phi - 0\right)\boldsymbol{a}_\rho + 0\boldsymbol{a}_\phi + \frac{1}{\rho}(1-0)\boldsymbol{a}_z = \underline{\underline{-\frac{1}{\rho}\sin\phi\boldsymbol{a}_\rho + \frac{1}{\rho}\boldsymbol{a}_z}}$$

**Prob. 3.43**
*Method 1:* We can express **A** in spherical coordinates.

$$\boldsymbol{A} = \frac{r}{r^3}\boldsymbol{a}_r = \frac{\boldsymbol{a}_r}{r^2},$$

$$\nabla \times \boldsymbol{A} = \nabla \times \left(\frac{\boldsymbol{a}_r}{r^2}\right) = \nabla\left(\frac{1}{r^2}\right) \times \boldsymbol{a}_r = \frac{-2}{r^3}\boldsymbol{a}_r \times \boldsymbol{a}_r = \mathbf{0}$$

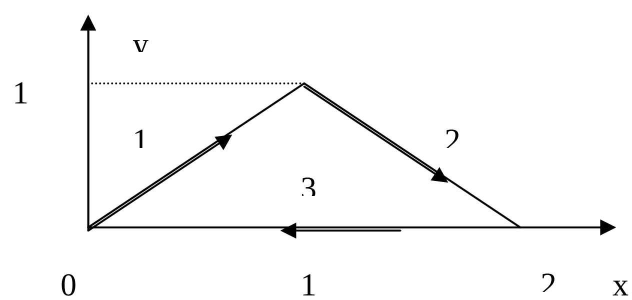
*Method 2:*

$$\boldsymbol{A} = \frac{x}{r^3}\boldsymbol{a}_x + \frac{y}{r^3}\boldsymbol{a}_y + \frac{z}{r^3}\boldsymbol{a}_z$$

$$\nabla \times \boldsymbol{A} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{r^3} & \frac{y}{r^3} & \frac{z}{r^3} \end{vmatrix} = \left\{-\frac{3}{2}z(x^2+y^2+z^2)^{-5/2}(2y) - -\frac{3}{2}y(x^2+y^2+z^2)^{-5/2}(2z)\right\}\boldsymbol{a}_x + \ldots$$

$$= \boldsymbol{0}$$

**Prob. 3.44**




(a)

$$\oint_L \boldsymbol{F} \bullet d\boldsymbol{l} = \left(\int_1 + \int_2 + \int_3\right) \boldsymbol{F} \bullet d\boldsymbol{l}$$

For 1, $y = x \quad dy = dx, \bar{d}l = dx\,\bar{a}_x + dy\,\bar{a}_y,$

$$\int_1 \boldsymbol{F} \bullet d\boldsymbol{l} = \int_0^1 x^3 dx - x dx = -\frac{1}{4}$$

For 2, $y = -x + 2, dy = -dx, \bar{d}l = dx\,\bar{a}_x + dy\,\bar{a}_y,$

$$\int_2 \boldsymbol{F} \bullet d\boldsymbol{l} = \int_1^2 (-x^3 + 2x^2 - x + 2)dx = \frac{17}{12}$$

For 3,

$$\int_3 \boldsymbol{F} \bullet d\boldsymbol{l} = \int_2^0 x^2 y dx \Big|_{y=0} = 0$$

$$\oint_L \boldsymbol{F} \bullet d\boldsymbol{l} = -\frac{1}{4} + \frac{17}{12} + 0 = \underline{\underline{\frac{7}{6}}}$$

(b)

$$\nabla \times \boldsymbol{F} = -x^2\,\boldsymbol{a}_z \;; \qquad d\boldsymbol{S} = dxdy(-\boldsymbol{a}_z)$$

$$\int (\nabla \times \boldsymbol{F}) \bullet d\mathrm{S} = -\iint (-x^2) dxdy = \int_0^1 \int_0^x x^2 dydx + \int_1^2 \int_{y=0}^{-x+2} x^2 dydx$$

$$= \int_0^1 x^2 y \Big|_0^x dx + \int_1^2 x^2 y \Big|_0^{-x+2} dx = \frac{x}{4}\Big|_0^1 + \int_1^2 x^2(-x+2)dx = \underline{\underline{\frac{7}{6}}}$$

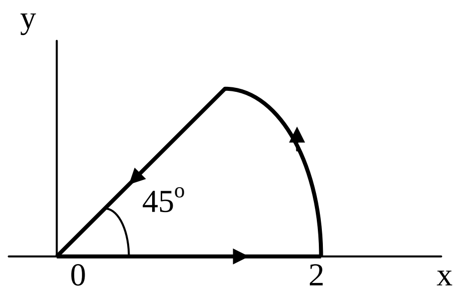
(c) <u>Yes</u>

**Prob. 3.45**

$$\oint \boldsymbol{A} \bullet d\boldsymbol{l} = \int_{\rho=2}^{1} \rho \sin\phi d\rho \Big|_{\phi=0} + \int_{\phi=0}^{\pi/2} \rho^2 \rho d\phi \Big|_{\rho=1} + \int_{\rho=1}^{2} \rho \sin\phi d\rho \Big|_{\phi=90^o} + \int_{\phi=\pi/2}^{0} \rho^3 d\phi \Big|_{\rho=2}$$

$$= \frac{\pi}{2} + \frac{1}{2}(4-1) + 8\left(-\frac{\pi}{2}\right) = \underline{\underline{-9.4956}}$$

**Prob. 3.46**




$$\oint \boldsymbol{F} \bullet dl = \int_0^2 2\rho z d\rho \Big|_{z=1} + \int_0^{\pi/4} 3z \sin\phi\, \rho d\phi \Big|_{\rho=2, z=1} + \int_2^0 2\rho z d\rho \Big|_{z=1}$$

$$= \rho^2 \Big|_0^2 + (-6\cos\phi) \Big|_0^{\pi/4} + \rho^2 \Big|_2^0 = (4-0) + 6(-\cos\pi/4 + 1) + (0-4) = \underline{\underline{1.757}}$$

$$\nabla x \boldsymbol{F} = \frac{1}{\rho}\left[3z\sin\phi - 0\right]\boldsymbol{a}_z + \ldots$$

$$\int (\nabla x \boldsymbol{F}) \cdot d\boldsymbol{S} = \int_{\rho=0}^{2} \int_{\phi=0}^{\pi/4} \frac{3z}{\rho} \sin\phi\, \rho d\phi d\rho \Big|_{z=} \qquad = 3(2)(-\cos\phi) \Big|_0^{\pi/4}$$

$$= 6(-\cos\pi + 1) = \underline{\underline{1.757}}$$

**Prob. 3.47**

$$\nabla \cdot A = 8xe^{-y} + 8xe^{-y} = 16xe^{-y}$$
$$\nabla(\nabla \cdot \boldsymbol{A}) = 16e^{-y}\boldsymbol{a}_x - 16xe^{-y}\boldsymbol{a}_y$$

$$\nabla x \nabla(\nabla \cdot \boldsymbol{A}) = \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ \frac{\partial}{\partial \boldsymbol{x}} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 16e^{-y} & -16xe^{-y} & 0 \end{vmatrix} = (-16e^{-y} + 16e^{-y})\boldsymbol{a}_z = \underline{\underline{0}}$$

Should be expected since $\nabla x \nabla V = 0$.

**Prob. 3.48**

(a) $\nabla V = \underline{\underline{-\frac{\sin\theta\cos\phi}{r^2}\boldsymbol{a}_r + \frac{\cos\theta\cos\phi}{r^2}\boldsymbol{a}_\theta - \frac{\sin\phi}{r^2}\boldsymbol{a}_\phi}}$

(b) $\nabla x \nabla V = \underline{\underline{0}}$

(c)

$$\nabla \cdot \nabla V = \nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}(-\sin\theta\cos\phi) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\cos\theta\cos\phi}{r}) + \frac{1}{r^2\sin^2\theta}(-\frac{\sin\theta\cos\phi}{r})$$

$$= 0 + \frac{\cos\phi}{r^3\sin\theta}(1-2\sin^2\theta) - \frac{\cos\phi}{r^3\sin\theta}$$

$$= \underline{\underline{-\frac{2\sin\theta\cos\phi}{r^3}}}$$

**Prob.3.49**

$$\boldsymbol{Q} = \frac{r}{r\sin\theta} r\sin\theta[(\cos\phi - \sin\phi)\boldsymbol{a}_x + (\cos\phi + \sin\phi)\boldsymbol{a}_y]$$

$$= r(\cos\phi - \sin\phi)\boldsymbol{a}_x + r(\cos\phi + \sin\phi)\boldsymbol{a}_y$$

$$\begin{bmatrix} Q_r \\ Q_\theta \\ Q_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} Q_x \\ Q_y \\ Q_z \end{bmatrix}$$

$$\boldsymbol{Q} = r\sin\theta\,\boldsymbol{a}_r + r\cos\theta\,\boldsymbol{a}_\theta + r\,\boldsymbol{a}_\phi$$

(a)

$$d\boldsymbol{l} = \rho\, d\phi\, \boldsymbol{a}_\phi, \quad \rho = r\sin 30° = 2(\frac{1}{2}) = 1$$

$$z = r\cos 30° = \sqrt{3}$$

$$Q_\phi = r = \sqrt{\rho^2 + z^2}$$

$$\oint \boldsymbol{Q} \bullet d\boldsymbol{l} = \int_0^{2\pi} \sqrt{\rho^2 + z^2}\,\rho\, d\phi = 2(1)(2\pi) = \underline{\underline{4\pi}}$$

(b)

$$\nabla \times \boldsymbol{Q} = \cot\theta\,\boldsymbol{a}_r - 2\boldsymbol{a}_\theta + \cos\theta\,\boldsymbol{a}_\phi$$

For $S_1$, $d\boldsymbol{S} = r^2 \sin\theta\, d\theta\, d\phi\, \boldsymbol{a}_r$

$$\int_{S_1} (\nabla \times \boldsymbol{Q}) \bullet d\boldsymbol{S} = \int r^2 \sin\theta\cot\theta\, d\theta\, d\phi \Big|_{r=2}$$

$$= 4\int_0^{2\pi} d\phi \int_0^{30°} \cos\theta\, d\theta = \underline{\underline{4\pi}}$$

(c)

For $S_2$, $d\boldsymbol{S} = r\sin\theta d\theta dr\, \boldsymbol{a}_\theta$

$$\int_{S_2} (\nabla\times\boldsymbol{Q})\bullet d\boldsymbol{S} = -2\int r\sin\theta\, d\phi\, dr\Big|_{\theta=30°}$$

$$= -2\sin 30\int_0^2 rdr \int_0^{2\pi} d\phi$$

$$= \underline{\underline{-4\pi}}$$

(d)

For $S_1$, $d\boldsymbol{S} = r^2\sin\theta\, d\phi\, d\theta\, \boldsymbol{a}_r$

$$\int_{S_1} \boldsymbol{Q}\bullet d\,\boldsymbol{S} = r^3\int \sin^2\theta\, d\theta\, d\phi\Big|_{r=2}$$

$$= 8\int_0^{2\pi} d\phi \int_0^{30^\circ} \sin^2\theta\, d\theta$$

$$= 4\pi[\frac{\pi}{3} - \frac{\sqrt{3}}{2}] = \underline{\underline{2.2767}}$$

(e)

For $S_2$, $d\boldsymbol{S} = r\sin\theta\, d\phi\, dr\, \boldsymbol{a}_\theta$

$$\int_{S_2} \boldsymbol{Q}\bullet d\boldsymbol{S} = \int r^2\sin\theta\cos\theta\, d\phi\, dr\Big|_{\theta=30°}$$

$$= \frac{4\pi\sqrt{3}}{3} = \underline{\underline{7.2552}}$$

(f)

$$\nabla\bullet\boldsymbol{Q} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^3\sin\theta) + \frac{r}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\cos\theta) + 0$$

$$= 2\sin\theta + \cos\theta\cot\theta$$

$$\int\nabla\bullet\boldsymbol{Q}dv = \int(2\sin\theta + \cos\theta\cot\theta)r^2\sin\theta\, d\theta\, d\phi\, dr$$

$$= \frac{r^3}{3}\Big|_0^2 (2\pi)\int_0^{30}(1+\sin^2\theta)d\theta$$

$$= \frac{4\pi}{3}(\pi - \frac{\sqrt{3}}{2}) = \underline{\underline{9.532}}$$

$$Check: \int \nabla \bullet \mathbf{Q} dv = (\int_{S_1} + \int_{S_2} \mathbf{Q} \bullet d\mathbf{S}$$

$$= 4\pi[\frac{\pi}{3} - \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{3}]$$

$$= \frac{4\pi}{3}[\pi - \frac{\sqrt{3}}{2}] \quad \text{(It checks!)}$$

**Prob. 3.50**

Since $\mathbf{u} = \boldsymbol{\omega} \times \mathbf{r}$, $\nabla \times \mathbf{u} = \nabla \times (\boldsymbol{\omega} \times \mathbf{r})$. From Appendix A.10,

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \bullet \mathbf{B}) - \mathbf{B}(\nabla \bullet \mathbf{A}) + (\mathbf{B} \bullet \nabla)\mathbf{A} - (\mathbf{A} \bullet \nabla)\mathbf{B}$$

$$\nabla \times \mathbf{u} = \nabla \times (\boldsymbol{\omega} \times \mathbf{r})$$

$$\nabla \times (\boldsymbol{\omega} \times \mathbf{r}) = \boldsymbol{\omega}(\nabla \bullet \mathbf{r}) - \mathbf{r}(\nabla \bullet \boldsymbol{\omega}) + (\mathbf{r} \bullet \nabla)\boldsymbol{\omega} - (\bar{\boldsymbol{\omega}} \bullet \nabla)\mathbf{r}$$

$$= \boldsymbol{\omega}(3) - \boldsymbol{\omega} = 2\boldsymbol{\omega}$$

or $\boldsymbol{\omega} = \frac{1}{2}\nabla \times \mathbf{u}$.

Alternatively, let $x = r\cos\omega t$, $y = r\sin\omega t$

$$\mathbf{u} = \frac{\partial x}{\partial t}\mathbf{a}_x + \frac{\partial y}{\partial t}\mathbf{a}_y$$

$$= -\omega r\sin\omega t\,\mathbf{a}_x + \omega r\cos\omega t\,\mathbf{a}_y$$

$$= -\omega y\,\mathbf{a}_x + \omega x\,\mathbf{a}_y$$

$$\nabla \times \mathbf{u} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\omega y & \omega x & 0 \end{vmatrix} = 2\omega\mathbf{a}_z = 2\boldsymbol{\omega}$$

*i.e.*, $\boldsymbol{\omega} = \frac{1}{2}\nabla \times \mathbf{u}$

Note that we have used the fact that $\nabla \bullet \boldsymbol{\omega} = 0$, $(\mathbf{r} \bullet \nabla)\boldsymbol{\omega} = 0$, $(\boldsymbol{\omega} \bullet \nabla)\mathbf{r} = \boldsymbol{\omega}$

**Prob. 3.51**

(a) $\nabla \bullet \boldsymbol{H} = \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = \underline{\underline{2z + 5x + 8}}$

(b) $\nabla \times \boldsymbol{H} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy & 5xy & 8(y+z) \end{vmatrix} = \underline{\underline{8\boldsymbol{a}_x + 2x\boldsymbol{a}_y + 5y\boldsymbol{a}_z}}$

**Prob. 3.52**

$$\nabla \bullet \boldsymbol{B} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 B_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(B_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial B_\phi}{\partial \phi}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(r^4) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(4\,\mathrm{r}\sin\theta\cos 2\theta) + 0$$

$$= 4r + \frac{1}{\sin\theta}\left[4r\cos\theta\cos 2\theta + 4r\sin\theta(-2\sin 2\theta)\right]$$

$$\underline{\underline{\nabla \bullet \boldsymbol{B} = 4r + 4r\cot\theta\cos 2\theta - 8r\sin 2\theta}}$$

$$\nabla \times \boldsymbol{B} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(B_\phi \sin\theta) - \frac{\partial B_\phi}{\partial \phi}\right]\boldsymbol{a}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial B_r}{\partial \phi} - \frac{\partial}{\partial r}(rB_\phi)\right]\boldsymbol{a}_\theta + \frac{1}{r}\left[\frac{\partial}{\partial r}(rB_\theta) - \frac{\partial B_r}{\partial \theta}\right]\boldsymbol{a}_\phi$$

All terms are zero except one.

$$\nabla \times \boldsymbol{B} = 0\boldsymbol{a}_r + 0\boldsymbol{a}_\theta + \frac{1}{r}\left[\frac{\partial}{\partial r}(rB_\theta) - 0\right]\boldsymbol{a}_\phi = \frac{1}{r}\frac{\partial}{\partial r}(4r^2\cos 2\theta)\boldsymbol{a}_\phi = \underline{\underline{8\cos 2\theta\,\boldsymbol{a}_\phi}}$$

**Prob. 3.53**

(a) $\nabla\bullet(\mathrm{V}\nabla\mathrm{V}) = \nabla\bullet\left(\mathrm{V}\frac{\partial V}{\partial x}\boldsymbol{a}_x + \mathrm{V}\frac{\partial V}{\partial y}\boldsymbol{a}_y + \mathrm{V}\frac{\partial V}{\partial z}\boldsymbol{a}_z\right)$

$$= \frac{\partial}{\partial x}\left(\mathrm{V}\frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mathrm{V}\frac{\partial V}{\partial y}\right) + \frac{\partial}{\partial z}\left(\mathrm{V}\frac{\partial V}{\partial z}\right)$$

$$= V\frac{\partial^2 V}{\partial x^2} + V\frac{\partial^2 V}{\partial y^2} + V\frac{\partial^2 V}{\partial z^2} + \left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2$$

$$= V\nabla^2 V + |\nabla V|^2$$

(b)

$$\nabla \times V\boldsymbol{A} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ VA_x & VA_y & VA_z \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(VA_z) - \frac{\partial}{\partial z}(VA_y)\right]\boldsymbol{a}_x + \left[\frac{\partial}{\partial z}(VA_x) - \frac{\partial}{\partial x}(VA_z)\right]\boldsymbol{a}_y + \left[\frac{\partial}{\partial x}(VA_y) - \frac{\partial}{\partial y}(VA_x)\right]\boldsymbol{a}_z$$

$$= \left[A_z\frac{\partial V}{\partial y} + V\frac{\partial A_z}{\partial y} - A_y\frac{\partial V}{\partial z} - V\frac{\partial A_y}{\partial z}\right]\boldsymbol{a}_x$$

$$+ \left[A_x\frac{\partial V}{\partial z} + V\frac{\partial A_x}{\partial z} - A_z\frac{\partial V}{\partial x} - V\frac{\partial A_z}{\partial x}\right]\boldsymbol{a}_y$$

$$+ \left[A_y\frac{\partial V}{\partial x} + V\frac{\partial A_y}{\partial x} - A_x\frac{\partial V}{\partial y} - V\frac{\partial A_x}{\partial y}\right]\boldsymbol{a}_z$$

$$\nabla \times V\boldsymbol{A} = V\left[\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\boldsymbol{a}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\boldsymbol{a}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\boldsymbol{a}_z\right]$$

$$+ \left(A_z\frac{\partial V}{\partial y} - A_y\frac{\partial V}{\partial z}\right)\boldsymbol{a}_x + \left(A_x\frac{\partial V}{\partial z} - A_z\frac{\partial V}{\partial x}\right)\boldsymbol{a}_y + \left(A_y\frac{\partial V}{\partial x} - A_x\frac{\partial V}{\partial y}\right)\boldsymbol{a}_z$$

$$= V\nabla \times \boldsymbol{A} + \nabla V \times \boldsymbol{A}$$

**Prob. 3.54**

(a)

$$\nabla \cdot \boldsymbol{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 2xy + 1 + 1 = \underline{\underline{2 + 2xy}}$$

(b)

$$\nabla \times \boldsymbol{B} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y & (2x^2 + y) & (z - y) \end{vmatrix} = (-1 + 0)\boldsymbol{a}_x + (0 - 0)\boldsymbol{a}_y + (4x + x^2)\boldsymbol{a}_z$$

$$= \underline{\underline{-\boldsymbol{a}_x + x(4 - x)\boldsymbol{a}_z}}$$

(c )

$$\nabla(\nabla \cdot \boldsymbol{B}) = \nabla(2 + 2xy) = \underline{\underline{2y\boldsymbol{a}_x + 2x\boldsymbol{a}_y}}$$

(d)

$$\nabla \times \nabla \times \boldsymbol{B} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -1 & 0 & (4x - x^2) \end{vmatrix} = 0\boldsymbol{a}_x - (4 - 2\mathrm{x})\boldsymbol{a}_y + 0\boldsymbol{a}_z$$

$$= \underline{\underline{2(x-2)\boldsymbol{a}_y}}$$

**Prob. 3.55**

(a)

$$V_1 = x^3 + y^3 + z^3$$

$$\nabla^2 V_1 = \frac{\partial^2 V_1}{\partial x^2} + \frac{\partial^2 V_1}{\partial y^2} + \frac{\partial^2 V_1}{\partial z^2}$$

$$= \frac{\partial}{\partial x}(3x^2) + \frac{\partial}{\partial y}\left(3y^2\right) + \frac{\partial}{\partial x}(3z^2)$$

$$= 6x + 6y + 6z = \underline{\underline{6(x + y + z)}}$$

(b)

$$V_2 = \rho z^2 \sin 2\phi$$

$$\nabla^2 V_2 = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho z^2 \sin 2\phi) - \frac{4z^2}{\rho}\sin 2\phi + \frac{\partial}{\partial z}(2\rho z \sin 2\phi)$$

$$= \frac{z^2}{\rho}\sin 2\phi - \frac{4z^2}{\rho}\sin 2\phi + 2\rho \sin 2\phi$$

$$= \underline{\underline{(\frac{-3z^2}{\rho} + 2\rho)\sin 2\phi}}$$

(c)

$$V_3 = r^2(1 + \cos\theta\sin\phi)$$

$$\nabla^2 V_3 = \frac{1}{r^2}\frac{\partial}{\partial r}[2r^3(1 + \cos\theta\sin\phi)]$$

$$+ \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}(-\sin^2\theta\sin\phi)r^2 + \frac{1}{r^2\sin^2\theta}r^2(-\cos\theta\sin\phi)$$

$$= 6(1 + \cos\theta\sin\phi) - \frac{2\sin\theta}{\sin\theta}\cos\theta\sin\phi - \frac{\cos\theta\sin\phi}{\sin^2\theta}$$

$$= \underline{\underline{6 + 4\cos\theta\sin\phi - \frac{\cos\theta\sin\phi}{\sin^2\theta}}}$$

**Prob. 3.56**

(a)

$$U = x^3 y^2 e^{xz}$$

$$\nabla^2 U = \frac{\partial}{\partial x}(3x^2y^2e^{xz} + x^3y^2ze^{xz}) + \frac{\partial}{\partial y}(2x^3y\,e^{xz}) + \frac{\partial}{\partial z}(x^4y^2e^{xz})$$

$$= 6xy^2e^{xz} + 3x^2yze^{xz} + 3x^2y^2ze^{xz} + x^3y^2z^2e^{xz} + 2x^3e^{xz} + x^5y^2e^{xz}$$

$$= e^{xz}(6xy^2 + 3x^2y^2z + 3x^2y^2z + x^3y^2z^2 + 2x^3 + x^5y^2)$$

At $(1,-1,1)$,

$$\nabla^2 U = e^1(6+3+3+1+2+1) = 16e = 43.493$$

(b)

$$V = \rho^2 z(\cos\phi + \sin\phi)$$

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial\rho}[2\rho^2 z(\cos\phi + \sin\phi)] - z(\cos\phi + \sin\phi) + 0$$

$$= 4z(\cos\phi + \sin\phi) - z(\cos\phi + \sin\phi)$$

$$= 3z(\cos\phi + \sin\phi)$$

At $(5, \frac{\pi}{6}, -2)$, $\nabla^2 V = -6(0.866 + 0.5) = -8.196$

(c )

$$W = e^{-r}\sin\theta\cos\phi$$

$$\nabla^2 W = \frac{1}{r^2}\frac{\partial}{\partial r}\ (-r^2e^{-r}\sin\theta\cos\phi) + \frac{e^{-r}}{r^2\sin\theta}\cos\phi\frac{\partial}{\partial\theta}(\sin\theta\cos\theta)$$

$$-\frac{e^{-r}\sin\theta\cos\phi}{r^2\sin^2\theta}$$

$$= \frac{1}{r^2}(-2re^{-r}\sin\theta\cos\phi) + e^{-r}\sin\theta\cos\phi$$

$$+\frac{e^{-r}\cos\phi}{r^2\sin\theta}(1 - 2\sin^2\theta) - \frac{e^{-r}\cos\phi}{r^2\sin\theta}$$

$$\nabla^2 W = e^{-r}\sin\theta\cos\phi(1 - \frac{2}{r} - \frac{2}{r^2})$$

At $(1, 60°, 30°)$,

$$\nabla^2 W = e^{-1}\sin 60\cos 30(1 - 2 - 2) = -2.25e^{-1} = -0.8277$$

**Prob. 3.57**

(*a*) Let $V = 1nr = 1n\sqrt{x^2 + y^2 + z^2}$

$$\frac{\partial V}{\partial x} = \frac{1}{r}\ \frac{1}{2}\left(2x\right)(x^2 + y^2 + z^2)^{-1/2} = \frac{x}{r^2}$$

$$\nabla V = \frac{\partial V}{ox}\boldsymbol{a}_x + \frac{\partial V}{oy}\boldsymbol{a}_y + \frac{\partial V}{oz}\boldsymbol{a}_z = \frac{x\boldsymbol{a}_x + y\boldsymbol{a}_y + z\boldsymbol{a}_z}{r^2} = \underline{\underline{\frac{\boldsymbol{r}}{r^2}}}$$

(*b*) Let $\nabla V = \boldsymbol{A} = \frac{\boldsymbol{r}}{r^2} = \frac{1}{r}\boldsymbol{a}_r$ in spherical coordinates.

$$\nabla^2(1nr) = \nabla \cdot \nabla(1nr) = \nabla \cdot \boldsymbol{A} = \frac{1}{r^2}\ \frac{\partial}{\partial r}(r^2 A_r) = \frac{1}{r^2}\frac{\partial}{\partial r}(r)$$

$$= \underline{\underline{\frac{1}{r^2}}}$$

**Prob. 3.58**

(a)

$$\nabla U = \frac{\partial U}{\partial x}\boldsymbol{a}_x + \frac{\partial U}{\partial y}\boldsymbol{a}_y + \frac{\partial U}{\partial z}\boldsymbol{a}_z = \underline{\underline{y^2z^3\boldsymbol{a}_x + 2xyz^3\boldsymbol{a}_y + 3xy^2z^2\frac{\partial U}{\partial z}\boldsymbol{a}_z}}$$

$$\nabla^2 U = \frac{\partial}{\partial x}\left(\frac{\partial U}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial U}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial U}{\partial z}\right) = 0 + 2xz^3 + 6xy^2z = \underline{\underline{2xz^3 + 6xy^2z}}$$

(b)

$$\nabla V = \frac{\partial V}{\partial \rho}\boldsymbol{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\boldsymbol{a}_\phi + \frac{\partial V}{\partial z}\boldsymbol{a}_z = -\frac{\sin\phi}{\rho^2}\boldsymbol{a}_\rho + \frac{\cos\phi}{\rho^2}\boldsymbol{a}_\phi + 0 = \underline{\underline{\frac{1}{\rho^2}\left[-\sin\phi\boldsymbol{a}_\rho + \cos\phi\boldsymbol{a}_\phi\right]}}$$

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(-\rho^{-1}\sin\theta\right) + \frac{1}{\rho^2}\left(-\frac{\sin\phi}{\rho}\right) + 0$$

$$= \frac{\sin\phi}{\rho^3} - \frac{1}{\rho^3}\sin\phi = \underline{\underline{0}}$$

( c)

$$\nabla W = \frac{\partial W}{\partial r}\boldsymbol{a}_r + \frac{1}{r}\frac{\partial W}{\partial \theta}\boldsymbol{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial W}{\partial \phi}\boldsymbol{a}_\phi$$

$$\underline{\underline{= 2r\sin\theta\cos\phi\boldsymbol{a}_r + r\cos\theta\cos\phi\boldsymbol{a}_\theta - r\sin\phi\boldsymbol{a}_\phi}}$$

$$\nabla^2 W = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial W}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial W}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 W}{\partial\phi^2}$$

$$=\frac{1}{r^2}\frac{\partial}{\partial r}\left(2r^3\sin\theta\cos\phi\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(r^2\sin\theta\cos\theta\cos\phi\right)+\frac{1}{r^2\sin^2\theta}(-r^2\sin\theta\cos\phi)$$

$$=6\sin\theta\cos\phi+\frac{1}{\sin\theta}(\cos^2\theta\cos\phi-\sin^2\theta\cos\phi)-\frac{\cos\phi}{\sin\theta}$$

$$=6\sin\theta\cos\phi+\frac{\cos\phi}{\sin\theta}-2\sin\theta\cos\phi-\frac{\cos\phi}{\sin\theta}=\underline{\underline{4\sin\theta\cos\phi}}$$

**Prob. 3.59**

$$\nabla V=\frac{\partial V}{\partial\rho}\boldsymbol{a}_\rho+\frac{1}{\rho}\frac{\partial V}{\partial\phi}\boldsymbol{a}_\phi+\frac{\partial V}{\partial z}\boldsymbol{a}_z=\underline{\underline{2\rho z\cos\phi\boldsymbol{a}_\rho - \rho z\sin\phi\boldsymbol{a}_\phi+\rho^2\cos\phi\boldsymbol{a}_z}}$$

$$\nabla^2 V=\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 V}{\partial\phi^2}+\frac{\partial^2 V}{\partial z^2}=\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(2\rho^2 z\cos\phi\right)-\frac{1}{\rho^2}\rho^2 z\cos\phi+0$$

$$=\left(4-1\right)z\cos\phi=\underline{\underline{3z\cos\phi}}$$

**Prob.3.60**

(a)

$$\nabla V=\frac{\partial V}{\partial r}\boldsymbol{a}_r+\frac{1}{r}\frac{\partial V}{\partial\theta}\boldsymbol{a}_\theta+\frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi}\boldsymbol{a}_\phi$$

$$=\underline{\underline{-\frac{10}{r^3}\cos\phi\boldsymbol{a}_r-\frac{5\sin\phi}{r^3\sin\theta}\boldsymbol{a}_\phi}}$$

(b)

$$\nabla\cdot\nabla V=\nabla^2 V=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial V}{\partial r})+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial V}{\partial\theta})+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}$$

$$=\frac{1}{r^2}\frac{\partial}{\partial r}r^2(-\frac{10\cos\phi}{r^3})+0+\frac{1}{r^2\sin^2\theta}(-\frac{5\cos\phi}{r^2})$$

$$\underline{\underline{\nabla\cdot\nabla V=\frac{10\cos\phi}{r^4}-\frac{5\cos\phi}{r^4\sin^2\theta}}}$$

(c ) $\nabla\times\nabla V=0$, see Example 3.10.

**Prob. 3.61**

$$\nabla U = \frac{\partial U}{\partial x}\boldsymbol{a}_x + \frac{\partial U}{\partial y}\boldsymbol{a}_y + \frac{\partial U}{\partial z}\boldsymbol{a}_z = 4yz^2\boldsymbol{a}_x + (4xz^2 + 10z)\boldsymbol{a}_y + (8xyz + 10y)\boldsymbol{a}_z$$

$$\nabla \cdot \nabla U = \frac{\partial}{\partial x}(\nabla U_x) + \frac{\partial}{\partial y}(\nabla U_y) + \frac{\partial}{\partial z}(\nabla U_z) = 0 + 0 + 8xy = 8xy$$

$$\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0 + 0 + 8xy = 8xy$$

Hence, $\nabla^2 U = \nabla \cdot \nabla U$

**Prob. 3.62**

Method 1

$$\nabla^2 \boldsymbol{G}\Big|_\rho = \nabla^2 G_\rho - \frac{2}{\rho^2}\frac{\partial G_\phi}{\partial \phi} - \frac{G_\rho}{\rho^2}$$

$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(2\rho \sin\phi\right) - \frac{2\rho\sin\phi}{\rho^2} + 0 + \frac{8\rho\sin\phi}{\rho^2} - \frac{2\rho\sin\phi}{\rho^2}$$

$$= \frac{2\sin\phi}{\rho} - \frac{2\sin\phi}{\rho} + \frac{8\sin\phi}{\rho} - \frac{2\sin\phi}{\rho} \qquad = \frac{6\sin\phi}{\rho}$$

$$\nabla^2 \boldsymbol{G}\Big|_\phi = \nabla^2 G_\phi + \frac{2}{\rho^2}\frac{\partial G_\rho}{\partial \phi} - \frac{G_\phi}{\rho^2}$$

$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}(4\rho\cos\phi) - \frac{1}{\rho}4\rho\cos\phi + 0 + \frac{4\rho\cos\phi}{\rho^2} - \frac{4\rho\cos\phi}{\rho^2}$$

$$= \frac{4\cos\phi}{\rho} - \frac{4\cos\phi}{\rho} + \frac{4\cos\phi}{\rho} - \frac{4\cos\phi}{\rho} = 0$$

$$\nabla^2 \boldsymbol{G}\Big|_z = \nabla^2 G_z = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left[\rho(z^2+1)\right] + 0 + \frac{\partial}{\partial z}(2z\rho)$$

$$= \frac{1}{\rho}(z^2+1) + 2\rho$$

Adding the components together gives

$$\underline{\underline{\nabla^2 \boldsymbol{G} = \frac{6\sin\phi}{\rho}\boldsymbol{a}_\rho + \left[2\rho + \frac{1}{\rho}(z^2+1)\right]\boldsymbol{a}_z}}$$

Method 2:

$$\nabla^2 \boldsymbol{G} = \nabla(\nabla \cdot \boldsymbol{G}) - \nabla \times (\nabla \times \boldsymbol{G})$$

Let $V = \nabla \cdot \boldsymbol{G} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(2\rho^2 \sin\phi) + \frac{1}{\rho}(-4\rho \sin\phi) + 2z\rho = 2z\rho$

$$\nabla(\nabla \cdot \boldsymbol{G}) = \nabla V = 2z\boldsymbol{a}_\rho + 2\rho \boldsymbol{a}_z$$

Let $A = \nabla \times \boldsymbol{G} = \left[\frac{1}{\rho}0 - 0\right]\boldsymbol{a}_\rho + \left[0 - (z^2+1)\right]\boldsymbol{a}_\phi + \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(4\rho^2 \cos\phi) - 2\rho\cos\phi\right]\boldsymbol{a}_z$

$$= -(z^2+1)\boldsymbol{a}_\phi + 6\cos\phi \boldsymbol{a}_z$$

$$\nabla \times \nabla \times \boldsymbol{G} = \nabla \times \boldsymbol{A} = \left[-\frac{6}{\rho}\sin\phi + 2z\right]\boldsymbol{a}_\rho + (0-0)\boldsymbol{a}_\phi + \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(-\rho(z^2+1)) - 0\right]\boldsymbol{a}_z$$

$$= \left[2z - \frac{6}{\rho}\sin\phi\right]\boldsymbol{a}_\rho - \frac{1}{\rho}(z^2+1)\boldsymbol{a}_z$$

$$\nabla^2 \boldsymbol{G} = \nabla V - \nabla \times \boldsymbol{A}$$

$$= 2z\boldsymbol{a}_\rho + 2\rho\boldsymbol{a}_z - \left[2z - \frac{6}{\rho}\sin\phi\right]\boldsymbol{a}_\rho + \frac{1}{\rho}(z^2+1)\boldsymbol{a}_z$$

$$= \frac{6}{\rho}\sin\phi \boldsymbol{a}_\rho + \left[2\rho + \frac{1}{\rho}(z^2+1)\right]\boldsymbol{a}_z$$

**Prob. 3.63**

$$\nabla \cdot \boldsymbol{A} = \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(z^2) + \frac{\partial}{\partial z}(yz) = z + y$$

$$\nabla(\nabla \cdot \boldsymbol{A}) = \boldsymbol{a}_y + \boldsymbol{a}_z$$

$$\nabla^2 \boldsymbol{A} = \nabla^2 A_x \boldsymbol{a}_x + \nabla^2 A_y \boldsymbol{a}_y + \nabla^2 A_z \boldsymbol{a}_z = 0 + 2\boldsymbol{a}_y + 0 = 2\boldsymbol{a}_y$$

$$\nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A} = -\boldsymbol{a}_y + \boldsymbol{a}_z \qquad (1)$$

$$\nabla \times \boldsymbol{A} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & z^2 & yz \end{vmatrix} = -z\boldsymbol{a}_x + x\boldsymbol{a}_y$$

$$\nabla \times \nabla \times \boldsymbol{A} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -z & x & 0 \end{vmatrix} = -\boldsymbol{a}_y + \boldsymbol{a}_z \qquad (2)$$

From (1) and (2),

$$\nabla \times \nabla \times \boldsymbol{A} = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A}$$

**Prob. 3.64**

$$\nabla \cdot \boldsymbol{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 1+1+1 = 3 \neq 0$$

$$\nabla \times \boldsymbol{A} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \mathbf{0}$$

$$\nabla \cdot \boldsymbol{B} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho B_\rho) + \frac{1}{\rho}\frac{\partial B_\phi}{\partial \phi} + \frac{\partial B_z}{\partial z} = 4\cos\phi - 4\cos\phi = 0$$

$$\nabla \times \boldsymbol{B} = \left[\frac{1}{\rho}\frac{\partial B_z}{\partial \phi} - \frac{\partial B_\phi}{\partial z}\right]\boldsymbol{a}_\rho + \left[\frac{\partial B_\rho}{\partial z} - \frac{\partial B_z}{\partial \rho}\right]\boldsymbol{a}_\phi + \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho B_\rho) - \frac{\partial B_\rho}{\partial \phi}\right]\boldsymbol{a}_z$$

$$= 0\boldsymbol{a}_\rho + 0\boldsymbol{a}_\phi + \frac{1}{\rho}\left[-8\rho\sin\phi + 2\rho\sin\phi\right]\boldsymbol{a}_z = -6\sin\phi\boldsymbol{a}_z \neq \mathbf{0}$$

$$\nabla \cdot \boldsymbol{C} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\sin\theta) + 0 + 0 = \frac{2\sin\theta}{r} \neq 0$$

$$\nabla \times \boldsymbol{C} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(r\sin^2\theta) - 0\right]\boldsymbol{a}_r + \frac{1}{r}\left[0 - \frac{\partial}{\partial r}(r^2\sin\theta)\right]\boldsymbol{a}_\theta$$

$$+ \frac{1}{r}\left[0 - \cos\theta\right]\boldsymbol{a}_\phi$$

$$= 2\cos\theta\boldsymbol{a}_r - 2\sin\theta\boldsymbol{a}_\theta - \frac{\cos\theta}{r}\boldsymbol{a}_\phi \neq \mathbf{0}$$

**(a)** **B** is solenoidal.
**(b)** **A** is irrotational.

**Prob. 3.65** **(a)**

$$\nabla \times \boldsymbol{G} = \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ \frac{\partial}{\partial \mathbf{x}} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 16xy - z & 8x^2 & -x \end{vmatrix}$$

$$= 0\boldsymbol{a}_x + (-1+1)\boldsymbol{a}_y + (16x - 16x)\boldsymbol{a}_z = 0$$

Thus, ***G*** is irrotational.

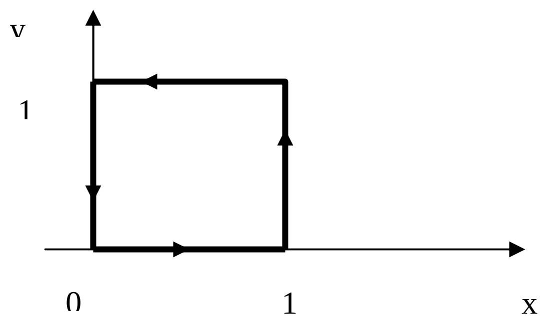
(b) Assume that $\psi$ represents the net flux.

$$\psi = \oint \boldsymbol{G} \bullet d\boldsymbol{S} = \int \nabla \bullet \boldsymbol{G} dv$$

$$\nabla \bullet \boldsymbol{G} = 16y + 0 + 0 = 16y$$

$$\psi = \iiint 16y dx dy dz = 16\int_0^1 dx \int_0^1 dz \int_0^1 y dy = 16(1)(1)\left(\frac{y^2}{2}\Big|_0^1\right) = \underline{\underline{8}}$$

**(c)**




$$\oint_L \boldsymbol{G} \bullet d\boldsymbol{l} = \int_{x=0}^{x=1} (16xy - z)dx\Big|_{\substack{y=0\\z=0}} + \int_{y=0}^{y=1} 8x^2 dy\Big|_{\substack{x=1\\z=0}} + \int_{x=1}^{x=0} (16xy - z)dx\Big|_{\substack{y=1\\z=0}} + \int_{y=1}^{y=0} 8x^2 dy\Big|_{\substack{x=0\\z=0}}$$

$$= 0 + 8(1)y\Big|_0^1 + 16(1)\frac{x^2}{2}\Big|_1^0 + 0$$

$$= 8 - 8 = \underline{\underline{0}}$$

This is expected since $\boldsymbol{G}$ is irrotational, i.e.

$$\oint \boldsymbol{G} \bullet d\boldsymbol{l} = \int (\nabla \times \boldsymbol{G}) \bullet d\boldsymbol{S} = 0$$

$$\nabla \bullet \boldsymbol{T} = -6 + 0 = \underline{\underline{-6}}$$

**Prob. 3.66**

$$\nabla \bullet \boldsymbol{E} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho E_\rho) = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\frac{\lambda}{2\pi\varepsilon}\right) = 0$$

Hence, **E** is solenoidal.

$$\nabla \times \boldsymbol{E} = \frac{\partial E_\rho}{\partial z}\boldsymbol{a}_\phi - \frac{\partial E_\rho}{\partial \phi}\boldsymbol{a}_z = \boldsymbol{0}$$

showing that **E** is conservative.

**Prob. 3.67**

$$\nabla \times \boldsymbol{H} = \mathbf{0}$$

$$\oint_L H \cdot dl = \int_S (\nabla \times \boldsymbol{H}) \cdot d\boldsymbol{S} = 0$$

**Prob. 3.68**

$$\nabla \times \boldsymbol{B} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2 z + y^2 & 2xy & x^3 \end{vmatrix} = (0-0)\boldsymbol{a}_x + (3x^2 - 3x^2)\boldsymbol{a}_y + (2y - 2y)\boldsymbol{a}_z = \mathbf{0}$$

showing that $\boldsymbol{B}$ is conservative.

**Prob. 3.69**

$$\nabla \bullet \boldsymbol{D} = \frac{1}{\rho}(\rho 0) + \frac{1}{\rho}(0) + 0 = 0$$

We conclude that $\boldsymbol{D}$ is solenoidal.

**Prob. 3.70**

$$\nabla \times \boldsymbol{E} = \frac{1}{r \sin\theta}(0-0)\boldsymbol{a}_r + \frac{1}{r}\left(\frac{1}{\sin\theta} 0 - 0\right)\boldsymbol{a}_\theta + \frac{1}{r}\left[\frac{\partial}{\partial r}(kr^{-2} \sin\theta) - \frac{2k(-\sin\theta)}{r^3}\right]\boldsymbol{a}_\phi$$

$$= 0 + \frac{1}{r}\left[-\frac{2k \sin\theta}{r^3} + \frac{2k \sin\theta}{r^3}\right]\boldsymbol{a}_\phi = \mathbf{0}$$

showing that $\boldsymbol{E}$ is conservative.

# CHAPTER 4

**P. E. 4.1**

$$(a)\ \boldsymbol{F} = \frac{1\times10^{-9}}{4\pi\left(\frac{10^{-9}}{36\pi}\right)}\left[\frac{5\times10^{-9}[(1,-3,7)-(2,0,4)]}{[(1,-3,7)-(2,0,4)]^3} + \frac{(-2\times10^{-9})[(1,-3,7)-(-3,0,5)]}{[(1,-3,7)-(-3,0,5)]^3}\right]$$

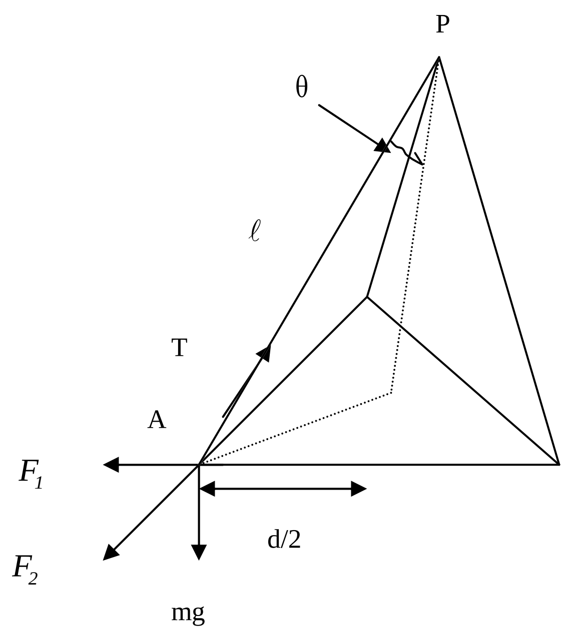
$$= [\frac{45(-1,-3,3)}{19^{3/2}} - \frac{18(4,-3,2)}{29^{3/2}}] \quad \text{nN}$$

$$= -1.004\boldsymbol{a}_x - 1.284\boldsymbol{a}_y + 1.4\boldsymbol{a}_z \ \text{nN}$$

$$(b) \qquad \boldsymbol{E} = \frac{\boldsymbol{F}}{Q} = -1.004\boldsymbol{a}_x - 1.284\boldsymbol{a}_y + 1.4\boldsymbol{a}_z \ \text{V/m}$$

**P. E. 4.2**

Let q be the charge on each sphere, i.e. q=Q/3. The free body diagram below helps us to establish the relationship between various forces.



At point $A$,

$$T\sin\theta\cos 30° = F_1 + F_2\cos 60°$$

$$= \frac{q^2}{4\pi\varepsilon_0 d^2} + \frac{q^2}{4\pi\varepsilon_0 d^2}\left(\frac{1}{2}\right)$$

$$= \frac{3q^2}{8\pi\varepsilon_0 d^2}$$

$$T\cos\theta = mg$$

Hence, $\tan\theta\cos 30° = \dfrac{3q^2}{8\pi\varepsilon_0 d^2 mg}$

But $\sin\theta = \dfrac{h}{l} = \dfrac{d}{\sqrt{3}\,l}$ $\tan\theta = \dfrac{\frac{d}{\sqrt{3}}}{\sqrt{l^2 - \frac{d^2}{3}}}$

Thus, $\dfrac{\frac{d}{\sqrt{3}}\left(\frac{\sqrt{3}}{2}\right)}{\sqrt{l^2 - \frac{d^2}{3}}} = \dfrac{3q^2}{8\pi\varepsilon_0 d^2 mg}$

or $q^2 = \dfrac{4\pi\varepsilon_0 d^3 mg}{3\sqrt{l^2 - \frac{d^2}{3}}}$

but $q = \dfrac{Q}{3}$ $\longrightarrow$ $q^2 = \dfrac{Q}{9}$. Hence,

$$Q^2 = \frac{12\pi\varepsilon_0 d^3 mg}{\sqrt{l^2 - \frac{d^2}{3}}}$$

**P.E. 4.3**

$$e\bar{E} = m\frac{d^2\bar{l}}{dt^2}$$

$$eE_0(-2\bar{a}_x + \bar{a}_y) = m(\frac{d^2x}{dt^2}\bar{a}_x + \frac{d^2y}{dt^2}\bar{a}_y + \frac{d^2z}{dt^2}\bar{a}_z)$$

where $E_0 = 200$ kV / m

$$\frac{d^2z}{dt^2} = 0 \quad \longrightarrow \quad z = ct + c_2$$

$$m\frac{d^2x}{dt^2} = -2eE_0 \quad \longrightarrow \quad x = \frac{-2eE_0t^2}{2m} + c_3t + c_4$$

$$m\frac{d^2y}{dt^2} = eE_0 \quad \longrightarrow \quad y = \frac{eE_0t^2}{2m} + c_5t + c_6$$

At $t = 0$, $(x, y, z) = (0,0,0)$ $\quad c_1 = 0 = c_4 = c_6$

Also, $(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}) = (0,0,0)$

At $t = 0 \quad \longrightarrow \quad c_1 = 0 = c_3 = c_5$

Hence, $\quad (x, y) = \dfrac{eE_0t^2}{2m} \quad (-2,1)$

*i.e.* $2\,|\,y\,| = |\,x\,|$

Thus the largest value of is

80 cm = 0.8 m

**P.E. 4.4**
**(a)**

Consider an element of area $dS$ of the disk.

The contribution due to $dS = \rho d\phi d\rho$ is

$$dE = \frac{\rho_s dS}{4\pi\varepsilon_0 r^2} = \frac{\rho_s dS}{4\pi\varepsilon_0(\rho^2 + h^2)}$$

The sum of the contribution along $\rho$ gives zero.

$$E_z = \frac{\rho_s}{4\pi\,\varepsilon_0}\int_{\rho=0}^{a}\int_{\phi=0}^{2\pi}\frac{h\rho\,d\rho\,d\phi}{(\rho^2 + h^2)^{3/2}} = \frac{h\,\rho_s}{2\,\varepsilon_0}\int_{\rho-0}^{a}\frac{\rho\,d\rho}{(\rho^2 + h^2)^{3/2}}$$

$$= \frac{h\,\rho_s}{4\,\varepsilon_0}\int_0^a(\rho^2 + h^2)^{-3/2}d(\rho^2) = \frac{h\rho_s}{4\varepsilon_0}(-2(\rho^2 + h^2)^{-1/2}\Big|_0^a$$

$$= \frac{\rho_s}{2\varepsilon_0}[1 - \frac{h}{(h^2 + a^2)^{1/2}}]$$

**(b)**

As $a \longrightarrow \infty$,

$$\boldsymbol{E} = \frac{\rho_s}{2\varepsilon_0}\,\boldsymbol{a}_z$$

**(c)** Let us recall that if *a/h* <<1 then $(1+a/h)^n$ can be approximated by *(1+na/h)*. Thus the expression for $E_z$ from (a) can be modified for *a<<h* as follows.

$$E_z = \frac{\rho_s}{2\varepsilon_o}\left[1-\frac{1}{\sqrt{1+\frac{a^2}{h^2}}}\right] = \frac{\rho_s}{2\varepsilon_o}\left[1-\left(1+\frac{a^2}{h^2}\right)^{-\frac{1}{2}}\right] \xrightarrow{a\to 0,\ but\ \rho_s\pi a^2=Q} \frac{\rho_s}{2\varepsilon_o}\left[\frac{a^2}{2h^2}\right]$$

$$= \frac{\rho_s}{2\varepsilon_o}\left[\frac{\pi a^2}{2\pi h^2}\right] = \frac{Q}{4\pi\varepsilon_o h^2}$$

This is in keeping with original Coulomb's law.

**P. E. 4.5**

$$Q_S = \int \rho_S dS = \int_{-2}^{2}\int_{-2}^{2} 12|y|\, dx\, dy$$

$$= 12(4)\int_0^2 2\,y\,dy = \underline{\underline{192\ \text{mC}}}$$

$$\boldsymbol{E} = \int \frac{\rho_s dS}{4\pi\varepsilon r^2}\boldsymbol{a}_r = \int \frac{\rho_s dS\,|\boldsymbol{r}-\boldsymbol{r}'|}{4\pi\varepsilon_o\,|\boldsymbol{r}-\boldsymbol{r}'|^3}$$

where $\boldsymbol{r}-\boldsymbol{r}' = (0,0,10)-(x,y,z) = (-x,-y,10)$.

$$\boldsymbol{E} = \int_{x=-2}^{2} \int_{y=-2}^{2} \frac{12\,|y|10^{-3}(-x,-y,10)}{4\pi(\frac{10^{-9}}{36\pi})(x^2+y^2+100)^{3/2}}$$

$$= 108(10^6)[\int_{-2}^{2}|y|\int_{-2}^{2}\frac{-xdx\,dy\;\boldsymbol{a}_x}{(x^2+y^2+100)^{3/2}} + \int_{-2}^{2}\int_{-2}^{2}\frac{-y|y|\;dy\,dx\,\boldsymbol{a}_y}{(x^2+y^2+100)^{3/2}}$$

$$+10\boldsymbol{a}_z\int_{-2}^{2}\int_{-2}^{2}\frac{|y|\,dx\,dy}{(x^2+y^2+100)^{3/2}}]$$

$$\boldsymbol{E} = 108(10^7)\boldsymbol{a}_z\int_{-2}^{2}[2\int_{0}^{2}\frac{\frac{1}{2}d(y^2)}{(x^2+y^2+100)^{3/2}}]dx$$

$$= -216(10^7)\boldsymbol{a}_z\int_{-2}^{2}[\frac{1}{(x^2+104)^{1/2}} - \frac{1}{(x^2+100)^{1/2}}]dx$$

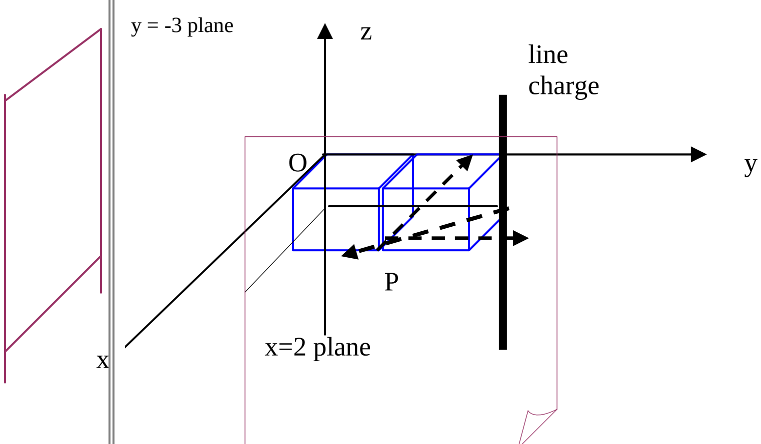
$$= -216(10^7)\boldsymbol{a}_z\;\ln|\frac{x+\sqrt{x^2+104}}{x+\sqrt{x^2+100}}|\;\Big|_{-2}^{2}$$

$$= -216(10^7)\boldsymbol{a}_z(\ln(\frac{2+\sqrt{108}}{2+\sqrt{104}}) - \ln(\frac{-2+\sqrt{108}}{-2+\sqrt{104}}))$$

$$= -216(10^7)\boldsymbol{a}_z(-7.6202\;(10^{-3}))$$

$$\boldsymbol{E} = \underline{\underline{16.46\;\boldsymbol{a}_z\;\text{MV/m}}}$$

**P.E. 4.6**



$\boldsymbol{E_1}$ and $\mathbf{E}_2$ remain the same as in Example 4.6.

$$\boldsymbol{E_3} = \frac{\rho_L}{2\pi\varepsilon_0\rho}\boldsymbol{a}_\rho$$

This expression, which represents the field due to a line charge,
is modified as follows. To get $\boldsymbol{a}_\rho$, consider the $z = -1$ plane. $\rho = \sqrt{2}$

$$\boldsymbol{a}_\rho = \boldsymbol{a}_x \cos 45° - \boldsymbol{a}_y \sin 45°$$

$$= \frac{1}{\sqrt{2}}(\boldsymbol{a}_x - \boldsymbol{a}_y)$$

$$\boldsymbol{E}_3 = \frac{10(10^{-9})}{2\pi(\frac{10^{-9}}{36\pi})}\frac{1}{2}(\boldsymbol{a}_x - \boldsymbol{a}_y)$$

$$= 90\pi(\boldsymbol{a}_x - \boldsymbol{a}_y).$$ Hence,

$$\boldsymbol{E} = \boldsymbol{E}_1 + \boldsymbol{E}_2 + \boldsymbol{E}_3$$
$$= -180\pi\boldsymbol{a}_x + 270\pi\boldsymbol{a}_y + 90\pi\boldsymbol{a}_x - 90\pi\boldsymbol{a}_y$$
$$= \underline{\underline{-282.7\,\boldsymbol{a}_x + 565.5\boldsymbol{a}_y \ \text{V/m}}}$$

**P.E. 4.7**

$$\boldsymbol{D} = \boldsymbol{D}_Q + \boldsymbol{D}_\rho = \frac{Q}{4\pi r^2}\boldsymbol{a}_r + \frac{\rho_s}{2}\boldsymbol{a}_n$$

$$= \frac{30\times10^{-9}}{4\pi(5)^2} \ \frac{[(0,4,3)-(0,0,0)]}{5} + \frac{10\times10^{-9}}{2}\boldsymbol{a}_y$$

$$= \frac{30}{500\pi}(0,4,3) + 5\boldsymbol{a}_y \ \text{nC/m}^2$$

$$= \underline{\underline{5.076\boldsymbol{a}_y + 0.0573\boldsymbol{a}_z \ \text{nC/m}^2}}$$

**P.E. 4.8**

$$(a)\quad \rho_v = \nabla \bullet \boldsymbol{D} = 4x$$

$$\rho_v(-1,0,3) = \underline{\underline{-4\ \text{C/m}^3}}$$

$$(b)\quad \Psi = \oint \boldsymbol{D} \bullet d\boldsymbol{S} = \iint \boldsymbol{D} \bullet d\boldsymbol{S}\Big|_{x=0} + \iint \boldsymbol{D} \bullet d\boldsymbol{S}\Big|_{x=1} + \iint \boldsymbol{D} \bullet d\boldsymbol{S}\Big|_{y=0} + \iint \boldsymbol{D} \bullet d\boldsymbol{S}\Big|_{y=1} + \iint \boldsymbol{D} \bullet d\boldsymbol{S}\Big|_{z=0} + \int$$

$$= -\int_0^1\int_0^1 (2y^2 + z)dydz + \int_0^1\int_0^1 (2y^2 + z)dydz + \int_0^1\int_0^1 4\,\text{x}(1)dxdz - \int_0^1\int_0^1 (\text{x})dxdz + \int_0^1\int_0^1 (\text{x})dxdz$$

$$= 4\int_0^1\int_0^1 \text{x}\,dxdz = 4(1/2)(1) = 2C$$

$$\text{(c)}\quad \Psi = Q = \int \rho_v dv = \int_0^1\int_0^1\int_0^1 4x dxdydz$$

$$= 4(1)(1)(1/2) = \underline{\underline{2\ \text{C}}}$$

$$Q = \Psi = \underline{\underline{2\ \text{C}}}$$

**P.E. 4.9**

$$Q = \int \rho v dv = \psi = \oint \boldsymbol{D} \bullet d\boldsymbol{S}$$

For $0 \le r \le 10$,

$$D_r(4\pi r^2) = \iiint 2r\ (r^2)\sin\theta\, d\theta\, dr\, d\phi$$

$$D_r(4\pi r^2) = 4\pi\left(\frac{2r^4}{4}\Big|_0^r\right) = 2\pi r^4$$

$$D_r = \frac{r^2}{2} \qquad \boldsymbol{E} = \frac{r^2}{2\varepsilon_0}\boldsymbol{a}_r \ \text{ nV/m}$$

$$\boldsymbol{E}(r=2) = \frac{4(10^{-9})}{2(\frac{10^{-9}}{36\pi})}\boldsymbol{a}_r = 72\pi\boldsymbol{a}_r = \underline{\underline{226\ \boldsymbol{a}_r\ \text{ V/m}}}$$

For $r \ge 10$,

$$D_r(4\pi r^2) = 2\pi r_0^4, \qquad r_0 = 10\text{m}$$

$$D_r = \frac{r_0^4}{2r^2} \longrightarrow \boldsymbol{E} = \frac{r_0^4}{2\varepsilon_0 r^2}\boldsymbol{a}_r \ \text{ nV/m}$$

$$\boldsymbol{E}(r=12) = \frac{10^4(10^{-9})}{2(\frac{10^{-9}}{36\pi})(144)}\boldsymbol{a}_r = 1250\pi\boldsymbol{a}_r$$

$$= \underline{\underline{3.927\boldsymbol{a}_r\ \text{ kV/m}}}$$

**P. E. 4.10**

$$V(\mathbf{r}) = \sum_{k=1}^{3} \frac{Q_k}{4\pi\varepsilon_0 |\mathbf{r}-\mathbf{r_k}|} + C$$

$$At \quad V(\infty) = 0, \quad C = 0$$

$$|\mathbf{r}-\mathbf{r}_1| = |(-1,5,2)-(2,-1,3)| = \sqrt{46}$$

$$|\mathbf{r}-\mathbf{r}_2| = |(-1,5,2)-(0,4,-2)| = \sqrt{18}$$

$$|\mathbf{r}-\mathbf{r}_3| = |(-1,5,2)-(0,0,0)| = \sqrt{30}$$

$$V(-1,5,2) = \frac{10^{-6}}{4\pi(\frac{10^{-9}}{36\pi})}\left[\frac{-4}{\sqrt{46}} + \frac{5}{\sqrt{18}} + \frac{3}{\sqrt{30}}\right]$$

$$= \underline{\underline{10.23\ k\text{V}}}$$

**P.E. 4.11**

$$V = \frac{Q}{4\pi\varepsilon_0 r} + C$$

$$\text{If } V(0,6,-8) = V(r=10) = 2;$$

$$2 = \frac{5(10^{-9})}{4\pi(\frac{10^{-9}}{36\pi})(10)} + C \quad \longrightarrow \quad C = -2.5$$

(a)

$$V_A = \frac{5(10^{-9})}{4\pi(\frac{10^{-9}}{36\pi})|(-3,2,6)-(0,0,0)|} - 2.5$$

$$= \underline{\underline{3.929\ V}}$$

(b)

$$V_B = \frac{45}{\sqrt{7^2+1^2+5^2}} - 2.5 = \underline{\underline{2.696\,V}}$$

(c ) $V_{AB} = V_B - V_A = 2.696 - 3.929 = \underline{\underline{-1.233\,V}}$

**P.E. 4.12**

(a)

$$\frac{-W}{Q} = \int \boldsymbol{E} \bullet d\boldsymbol{l} = \int (3x^2 + y)dx + xdy$$

$$= \int_0^2 (3x^2 + y)dx \,\Big|_{y=5} + \int_5^{-1} x\,dy \,\Big|_{x=2}$$

$$= 18 - 12 = 6 \text{ kV}$$

$$W = -6Q = \underline{\underline{12 \text{ mJ}}}$$

**(b)**

$$dy = -3\,dx$$

$$-\frac{W}{Q} = \int \boldsymbol{E} \bullet d\boldsymbol{l} = \int_0^2 (3x^2 + 5 - 3x)dx + x(-3)dx$$

$$= \int_0^2 (3\text{x}^2 - 6x + 5)dx = 8 - 12 + 10 = 6$$

$$W = \underline{\underline{12 \text{ mJ}}}$$

**P.E. 4.13**

(a)

$$(0,0,10) \longrightarrow (r = 10, \theta = 0, \phi = 0)$$

$$V = \frac{100\cos 0}{4\pi\varepsilon_0(10^2)}(10^{-12}) = \frac{10^{-12}}{4\pi(\frac{10^{-9}}{36\pi})} = \underline{\underline{9 \text{ mV}}}$$

$$\boldsymbol{E} = \frac{100(10^{-12})}{4\pi(\frac{10^{-9}}{36\pi})10^3}[2\cos 0\,\boldsymbol{a}_r + \sin 0\,\boldsymbol{a}_\theta]$$

$$= \underline{\underline{1.8\,\boldsymbol{a}_r \text{ mV/m}}}$$

**(b)**

$$At \ (1, \frac{\pi}{3}, \frac{\pi}{2}),$$

$$V = \frac{100\cos\frac{\pi}{3}(10^{-12})}{4\pi(\frac{10^{-9}}{36\pi})(1)^2} = \underline{\underline{0.45\,V}}$$

$$\boldsymbol{E} = \frac{100\,(10^{-12})}{4\pi(\frac{10^{-9}}{36\pi})(1)^2}(2\cos\frac{\pi}{3}\boldsymbol{a}_r + \sin\frac{\pi}{3}\boldsymbol{a}_\theta)$$

$$= \underline{\underline{0.9\boldsymbol{a}_r + 0.7794\boldsymbol{a}_\theta \text{ V/m}}}$$

**P.E. 4.14**

After $Q_1$, $W_1 = 0$

After $Q_2$, $W_2 = Q_2 V_{21} = \dfrac{Q_2\ Q_1}{4\pi\varepsilon_0 |(1,0,0) - (0,0,0)|}$

$$= \frac{1(-2)(10^{-18})}{4\pi(10^{-9})\dfrac{1}{36\pi}} = \underline{\underline{-18\ \text{nJ}}}$$

After $Q_3$,

$W_3 = Q_3(V_{31} + V_{32}) + Q_2 V_{21}$

$$= 3(9)(10^{-9})\left\{\frac{1}{|(0,0,-1)-(0,0,0)|} + \frac{-2}{|(0,0,-1)-(1,0,0)|}\right\} - 18\,\text{nJ}$$

$$= 27\left(1 - \frac{2}{\sqrt{2}}\right) - 18$$

$$= \underline{\underline{-29.18\ \text{nJ}}}$$

After $Q_4$,

$W_4 = Q_4(V_{41} + V_{42} + V_{43}) + Q_3(V_{31} + V_{32}) + Q_2 V_{21}$

$$= -4(9)(10^{-9})\left\{\frac{1}{|(0,0,1)-(0,0,0)|} + \frac{-2}{|(0,0,1)-(1,0,0)} + \frac{3}{|(0,0,1)-(0,0,-1)}-\right\} + W_3$$

$$= -36\left(1 - \frac{2}{\sqrt{2}} + \frac{3}{2}\right) + W_3$$

$$= -39.09 - 29.18\ nJ = \underline{\underline{-68.27\ \text{nJ}}}$$

**P.E. 4.15**

$$\boldsymbol{E} = -\nabla V = -(y+1)\boldsymbol{a}_x + (1-x)\boldsymbol{a}_y - 2\boldsymbol{a}_z$$

At (1,2,3), $\boldsymbol{E} = \underline{\underline{-3\boldsymbol{a}_x - 2\boldsymbol{a}_z\ \text{V/m}}}$

$$W = \frac{1}{2}\varepsilon_o \int \boldsymbol{E} \bullet \boldsymbol{E} dv = \frac{1}{2}\varepsilon_o \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} (x^2 + y^2 - 2x + 2y + 6) dxdydz$$

$$= \frac{1}{2}\varepsilon_o \left[\int_{-1}^{1} x^2 dx \iint dydz + \int_{-1}^{1} y^2 dy \iint dxdz - 2\int_{-1}^{1} xdx \iint dydz + 2\int_{-1}^{1} ydy \iint dxdz + 6(2)(2)(2)\right]$$

$$= \frac{1}{2}\varepsilon_o \left[2\frac{x^3}{3}\bigg|_{-1}^{1}(2)(2) + 0 + 0 + 6(8)\right] = \frac{80\varepsilon_o}{3}$$

$$= \underline{\underline{0.2358\ \text{nJ}}}$$

**Prob. 4.1**

$$\boldsymbol{F}_{Q_1} = \frac{Q_1 Q_2\,(\boldsymbol{r}_{Q_1} - \boldsymbol{r}_{Q_2})}{4\pi\,\varepsilon \left| \boldsymbol{r}_{Q_1} - \boldsymbol{r}_{Q_2} \right|^3} = \frac{-20(10^{-12})[(3,2,1)-(-4,0,6)]}{4\pi \dfrac{10^{-9}}{36\pi}\left|(3,2,1)-(-4,0,6)\right|^3} = -180\frac{(7,2,-5)}{688.88}\times 10^{-3}$$

$$= \underline{\underline{-1.8291\boldsymbol{a}_x - 0.5226\boldsymbol{a}_y + 1.3065\boldsymbol{a}_z \text{ mN}}}$$

**Prob. 4.2** (a)

$$\boldsymbol{E}(5,0,6) = \frac{Q_1}{4\pi\,\varepsilon_0}\frac{[(5,0,6)-(4,0,-3)]}{|(5,0,6)-(4,0,-3)|^3} + \frac{Q_2}{4\pi\,\varepsilon_0}\frac{[(5,0,6)-(2,0,1)]}{|(5,0,6)-(2,0,1)|^3}$$

$$= \frac{Q_1}{4\pi\,\varepsilon_0}\frac{(1,0,9)}{(\sqrt{82})^3} + \frac{Q_2}{4\pi\,\varepsilon_0}\frac{(3,0,5)}{(34)^{3/2}}$$

If $E_z = 0$, then

$$\frac{9Q_1}{4\pi\,\varepsilon_0}\frac{1}{(82)^{3/2}} + \frac{5Q_2}{4\pi\,\varepsilon_0}\frac{1}{(34)^{3/2}} = 0$$

$$Q_1 = -\frac{5}{9}Q_2\,(\frac{82}{34})^{3/2} = -\frac{5}{9}4\,(\frac{82}{34})^{3/2} \text{ nC}$$

$$= \underline{\underline{-8.3232 \text{ nC}}}$$

**(b)**

$$\boldsymbol{F}(5,0,6) = q\boldsymbol{E}(5,0,6)$$

If $F_x = 0$, then

$$\frac{qQ_1}{4\pi\varepsilon_0(82)^{3/2}} + \frac{3qQ_2}{4\pi\varepsilon_0(34)^{3/2}} = 0$$

$$Q_1 = -3Q_2(\frac{82}{34})^{3/2} = -12(\frac{82}{34})^{3/2} \; nC$$

$$Q_1 = \underline{\underline{-44.945 \; nC}}$$

**Prob. 4.3**

(a)

$$\boldsymbol{E} = \sum_{k=1}^{2} \frac{Q(\boldsymbol{r}-\boldsymbol{r}_k^{'})}{4\pi\varepsilon_o |\boldsymbol{r}-\boldsymbol{r}_k^{'}|^3} = \frac{Q\left[(0,0,0)-(a,0,0)\right]}{4\pi\varepsilon_o |(0,0,0)-(a,0,0)|^3} - \frac{Q\left[(0,0,0)-(-a,0,0)\right]}{4\pi\varepsilon_o |(0,0,0)-(-a,0,0)|^3}$$

$$= \frac{Q(-a,0,0)}{4\pi\varepsilon_o a^3} - \frac{Q(a,0,0)}{4\pi\varepsilon_o a^3} = \underline{\underline{\frac{Q}{2\pi\varepsilon_o a^2}\boldsymbol{a}_x}}$$

(b)

$$\boldsymbol{E} = \frac{Q\left[(0,a,0)-(a,0,0)\right]}{4\pi\varepsilon_o |(0,a,0)-(a,0,0)|^3} - \frac{Q\left[(0,a,0)-(-a,0,0)\right]}{4\pi\varepsilon_o |(0,a,0)-(-a,0,0)|^3}$$

$$= \frac{Q(-a,a,0)}{4\pi\varepsilon_o (2a^2)^{3/2}} - \frac{Q(a,a,0)}{4\pi\varepsilon_o (2a^2)^{3/2}} = \underline{\underline{\frac{-Q}{4\sqrt{2}\pi\varepsilon_o a^2}\boldsymbol{a}_x}}$$

(c )

$$\boldsymbol{E} = \frac{Q\left[(a,0,a)-(a,0,0)\right]}{4\pi\varepsilon_o |(a,0,a)-(a,0,0)|^3} - \frac{Q\left[(a,0,a)-(-a,0,0)\right]}{4\pi\varepsilon_o |(a,0,a)-(-a,0,0)|^3}$$

$$= \frac{Q(0,0,a)}{4\pi\varepsilon_o a^3} - \frac{Q(2a,0,a)}{4\pi\varepsilon_o (5a^2)^{3/2}} = \underline{\underline{\frac{-Q}{10\sqrt{5}\pi\varepsilon_o a^2}\boldsymbol{a}_x + \frac{Q}{4\pi\varepsilon_o a^2}\left[1-\frac{1}{5\sqrt{5}}\right]\boldsymbol{a}_z}}$$

**Prob. 4.4**

$$F = qE = mg \quad \longrightarrow \quad E = \frac{mg}{q} = \frac{2\times 9.8}{4\times 10^{-3}} = \underline{\underline{4.9 \text{ kV/m}}}$$

**Prob. 4.5**

**(a)** $$Q = \int \rho_L dl = \int_0^5 12x^2 dx = 4x^3 \Big|_0^5 mC = \underline{\underline{0.5C}}$$

**(b)**

$$Q = \int \rho_S dS = \int_{z=0}^{4} \int_{\phi=0}^{2\pi} \rho z^2 \rho \, d\phi \, dz \Big|_{\rho=3} = 9(2\pi)\frac{z^3}{3}\Big|_0^4 nC$$

$$= \underline{\underline{1.206 \ \mu C}}$$

(c)

$$Q = \int \rho_V dV = \iiint \frac{10}{r\sin\theta} r^2 \sin\theta \, d\theta \, d\phi \, dr$$

$$= 10\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \int_0^4 r dr = 10(2\pi)(\pi)\frac{4^2}{2}$$

$$= \underline{\underline{1579.1 \ C}}$$

**Prob. 4.6**

$$Q=\int_v \rho_v dv=\int_0^a\int_0^a\int_9^a \frac{\rho_o x}{a}dxdydz=(a)(a)\rho_o\left(\frac{x^2}{2a}\Big|_0^a\right)=\underline{\underline{\frac{a^3\rho_o}{2}}}$$

**Prob. 4.7**

$$Q=\int_v \rho_v dv=\int_{\rho=0}^{2}\int_{z=0}^{1}\int_{\phi=\pi/6}^{\pi/2} 5\rho^2 z\rho d\phi d\rho dz \text{ mC}$$

$$=5\frac{\rho^4}{4}\Big|_0^2 \frac{z^2}{2}\Big|_0^1 \phi\Big|_{\pi/6}^{\pi/2}=\frac{5}{8}(16)(1)(\pi/2-\pi/6)=\frac{10\pi}{3}$$

$$Q=\underline{\underline{10.472 \text{ mC}}}$$

**Prob. 4.8**

$$Q=\int\rho_s dS=\int 6xydxdy$$

$$=\int_{x=0}^{2}\int_{y=0}^{x}6xydxdy+\int_{x=2}^{4}\int_{y=2}^{-x+4}6xydxdy$$

$$=6\int_{x=0}^{2}x\frac{y^2}{2}\Big|_0^x dx+\int_{x=2}^{4}6x\frac{y^2}{2}\Big|_0^{-x+4}dx$$

$$=6\int_{x=0}^{2}x(\frac{x^2}{2}-0)dx+3\int_{x=2}^{4}x\left[(4-x)^2-0\right]dx$$

$$=6\int_0^2\frac{x^3}{2}dx+3\int_2^4(16x-8x^2+x^3)dx$$

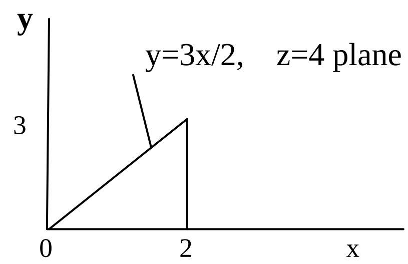
$$=3\frac{x^4}{4}\Big|_0^2+6(8x^2-\frac{8x^3}{3}+\frac{x^4}{4})\Big|_2^4$$

$$=12+3(128-32-\frac{512}{3}+\frac{64}{3}+64-4)$$

$$=12+3(96-\frac{448}{3}+60)$$

$$Q\quad \underline{\underline{=32 \text{ C}}}$$

**Prob. 4.9**




$$Q = \int_S \rho_s dS = \iint 10x^2 yzdxdy \bigg|_{z=4} \text{ mC } = 10(4)\int_{x=0}^{2}\int_{y=0}^{3x/2} x^2 ydydx$$

$$= 40\int_{x=0}^{2} x^2 \left.\frac{y^2}{2}\right|_0^{3x/2} dx = 20\int_{x=0}^{2} x^2 \left(\frac{9}{4}x^2\right)dx = 20(9/4)\left.\frac{x^5}{5}\right|_0^2 = 9(32) = \underline{\underline{288 \text{ mC}}}$$

**Prob. 4.10**

$$Q = \int_v \rho_v dv = \iiint 4\rho^2 z\cos\phi \rho d\rho d\phi dz \qquad \text{nC}$$

$$= 4\int_0^2 \rho^3 d\rho \int_0^1 zdz \int_0^{\pi/4} \cos\phi d\phi = \rho^4 \Big|_0^2 \frac{z^2}{2}\Big|_0^1 (\sin\phi)\Big|_0^{\pi/4}$$

$$= (16)(0.5)(\sin\pi/4) = \underline{\underline{5.657 \text{ nC}}}$$

**Prob. 4.11**

(a) $Q = \int_L \rho_L dl = \int_0^1 12x^2 dx = \left.\frac{12x^3}{3}\right|_0^1 = \underline{\underline{4 \text{ nC}}}$

(b) Method 1:

$$\boldsymbol{E} = \int \frac{\rho_L dl}{4\pi\varepsilon r^3}\boldsymbol{r}$$

$\boldsymbol{r}$ = (0,0,h) - (x,0,0) = (-x,0,h), $r = |\boldsymbol{r}| = \sqrt{h^2 + x^2}$

$$\boldsymbol{E} = \frac{1}{4\pi\varepsilon}\int \frac{12x^2(-x,0,h)dx}{(h^2+x^2)^{3/2}}$$

Since x<<h, we can ignore the x-component.

$$\boldsymbol{E} = \frac{h}{4\pi\varepsilon}\boldsymbol{a}_z \int \frac{12x^2 dx}{h^3} = \frac{Q}{4\pi\varepsilon h^2}\boldsymbol{a}_z$$

Method 2:

The line charge can be regarded as a point charge because it is very far from the observation point.

$$E = \frac{Q}{4\pi\varepsilon r^2}a_r = \frac{4\times10^{-9}}{4\pi\times\frac{10^{-9}}{36\pi}(1000)^2}a_z = \frac{4(9)}{10^6}a_z = \underline{\underline{3.6\times10^{-5}a_z\ V/m}}$$

**Prob. 4.12**

$$E = \int_L \frac{\rho_L dl}{4\pi\varepsilon_o R^2}a_R$$

$$R = -aa_\rho + ha_z, R = |R| = \sqrt{a^2+h^2}, dl = ad\phi$$

$$E = \frac{\rho_L}{4\pi\varepsilon_o}\int \frac{(-aa_\rho + ha_z)}{(a^2+h^2)^{3/2}}ad\phi$$

Due to symmetry, the ρ-component cancels

$$E = \frac{\rho_L}{4\pi\varepsilon_o}\int_0^{2\pi}\frac{haa_z d\phi}{(a^2+h^2)^{3/2}} = \frac{\rho_L haa_z}{4\pi\varepsilon_o(a^2+h^2)^{3/2}}(2\pi)$$

out.

$$F = QE = \frac{4\times10^{-3}\times12\times10^{-6}\times4\times3a_z}{4\pi\times\frac{10^{-9}}{36\pi}\times5^3}(2\pi) = \underline{\underline{260.58a_z\ \text{N}}}$$

**Prob. 4.13**

$$E = \int_S \frac{\rho_s dS}{4\pi\varepsilon_o R^3}R, \quad R = \rho(-a_\rho) + ha_z, dS = \rho d\phi d\rho$$

$$E = \frac{\rho_s}{4\pi\varepsilon_o}\int_S \frac{\rho d\phi d\rho}{(\rho^2+h^2)^{3/2}}(-\rho a_\rho + ha_z)$$

Due to symmetry, the $\rho$-component vanishes.

$$E = \frac{\rho_s h a_z}{4\pi\varepsilon_o}\int_a^b \rho(\rho^2+h^2)^{-3/2}d\rho\int_0^{2\pi}d\phi$$

Let u=$\rho^2 + h^2, du = 2\rho d\rho$

$$E = \frac{\rho_s h a_z}{4\pi\varepsilon_o}(2\pi)\int\frac{1}{2}u^{-3/2}du = \frac{\rho_s h a_z}{2\varepsilon_o}\frac{1/2u^{-1/2}}{-1/2} = \frac{\rho_s h a_z}{2\varepsilon_o}\left(-\frac{1}{\sqrt{\rho^2+h^2}}\Bigg|_a^b\right)$$

$$E = \underline{\underline{\frac{\rho_s h}{4\pi\varepsilon_o}\left[\frac{1}{\sqrt{a^2+h^2}} - \frac{1}{\sqrt{b^2+h^2}}\right]a_z}}$$

**Prob. 4.14**

(a) From eq. (4.26),

$$E = \frac{\rho_s}{2\varepsilon_o}\mathbf{a}_n = \frac{12\times10^{-9}}{2\times\frac{10^{-9}}{36\pi}}\mathbf{a}_n = 18\pi(12)\mathbf{a}_n = \begin{cases} 678.58\mathbf{a}_z \text{ V/m}, & z>0 \\ -678.58\mathbf{a}_z \text{ V/m}, & z<0 \end{cases}$$

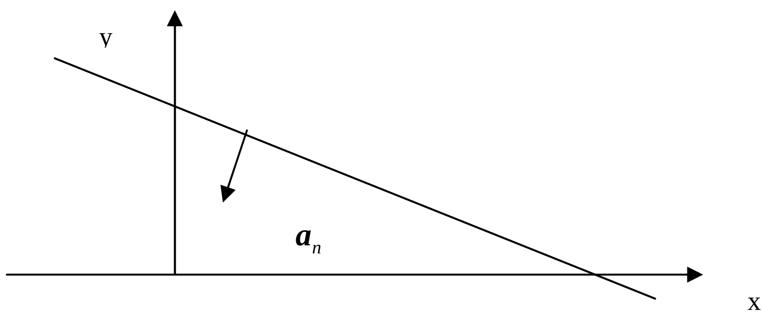
(b) Similarly,

$$E = \frac{\rho_s}{2\varepsilon_o}\mathbf{a}_n = \frac{12\times10^{-9}}{2\times\frac{10^{-9}}{36\pi}}\mathbf{a}_n = 18\pi(12)\mathbf{a}_n = \begin{cases} -678.58\mathbf{a}_z \text{ V/m}, & z>4 \\ 678.58\mathbf{a}_z \text{ V/m}, & z<4 \end{cases}$$

For z > 4 and z < 0, the fields cancel out. For 0 < z < 4, they add up. Thus

$E = 2\times678.58\mathbf{a}_z$ V/m= $1.357\mathbf{a}_z$ kV/m

**Prob. 4.15**




Let $f(x,y) = x + 2y - 5$; $\nabla f = \mathbf{a}_x + 2\mathbf{a}_y$

$$\mathbf{a}_n = \pm\frac{\nabla f}{|\nabla f|} = \pm\frac{(\mathbf{a}_x + 2\mathbf{a}_y)}{\sqrt{5}}$$

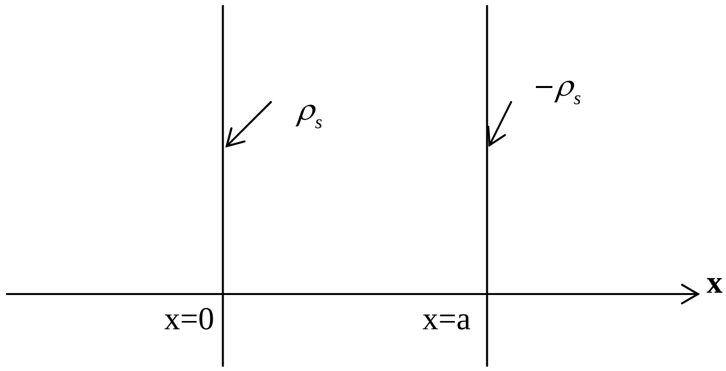
Since point $(-1,0,1)$ is below the plane,

$$\mathbf{a}_n = -\frac{(\mathbf{a}_x + 2\mathbf{a}_y)}{\sqrt{5}}.$$

$$E = \frac{\rho_s}{2\varepsilon_0}\mathbf{a}_n = \frac{6(10^{-9})}{2(10^{-9}/36\pi)}\left(-\frac{(\mathbf{a}_x + 2\mathbf{a}_y)}{\sqrt{5}}\right)$$

$$= -151.7\mathbf{a}_x - 303.5\mathbf{a}_y \text{ V/m}$$

**Prob. 4. 16**




(a) For x < 0,

$$\boldsymbol{E} = \boldsymbol{E}_1 + \boldsymbol{E}_2 = \frac{\rho_s}{2\varepsilon_o}(-\boldsymbol{a}_x) + \frac{(-\rho_s)}{2\varepsilon_o}(-\boldsymbol{a}_x) = \underline{\underline{\boldsymbol{0}}}$$

(b) For 0 < x < a,.

$$\boldsymbol{E} = \frac{\rho_s}{2\varepsilon_o}\boldsymbol{a}_x + \frac{(-\rho_s)}{2\varepsilon_o}(-\boldsymbol{a}_x) = \underline{\underline{\frac{\rho_s}{\varepsilon_o}\boldsymbol{a}_x}}$$

(c ) For x > a,

$$\boldsymbol{E} = \frac{\rho_s}{2\varepsilon_o}\boldsymbol{a}_x + \frac{(-\rho_s)}{2\varepsilon_o}(\boldsymbol{a}_x) = \underline{\underline{\boldsymbol{0}}}$$

**Prob. 4.17**

(a) At P(5,-1,4),

$$\boldsymbol{E} = \sum_{k=1}^{3}\frac{\rho_{sk}}{2\varepsilon_o}\boldsymbol{a}_{nk} = \frac{10\times10^{-6}}{2\times\frac{10^{-9}}{36\pi}}(\boldsymbol{a}_x) + \frac{-20\times10^{-6}}{2\times\frac{10^{-9}}{36\pi}}(\boldsymbol{a}_y) + \frac{30\times10^{-6}}{2\times\frac{10^{-9}}{36\pi}}(-\boldsymbol{a}_z)$$

$$= 36\pi(5,-10,-15)\times10^3 = \underline{\underline{565.5\boldsymbol{a}_x - 1131\boldsymbol{a}_y - 1696.5\boldsymbol{a}_z \text{ kV/m}}}$$

(b) At R(0,-2,1)

$$\boldsymbol{E} = 36\pi\left[5(-\boldsymbol{a}_x) - 10(\boldsymbol{a}_y) + 15(-\boldsymbol{a}_z)\right]\times10^3 = \underline{\underline{-565.5\boldsymbol{a}_x - 1131\boldsymbol{a}_y - 1696.5\boldsymbol{a}_z \text{ kV/m}}}$$

(c ) At Q(3,-4,10),

$$\boldsymbol{E} = 36\pi\left[5\boldsymbol{a}_x - 10(-\boldsymbol{a}_y) + 15\boldsymbol{a}_z\right]\times10^3 = \underline{\underline{565.5\boldsymbol{a}_x + 1131\boldsymbol{a}_y + 1696.5\boldsymbol{a}_z \text{ kV/m}}}$$

**Prob. 4.18**

$$F_e = \frac{e^2}{4\pi\varepsilon_o r^2}\mathbf{a}_r$$

$$\frac{F_e}{F_g} = \frac{e^2}{4\pi\varepsilon_o}\frac{1}{Gm^2} = \frac{1}{4\pi\times 10^{-9}/36\pi \times 6.67\times 10^{-11}}\left(\frac{1.6\times 10^{-19}}{9.1\times 10^{-31}}\right)^2$$

$$= \underline{\underline{4.17\times 10^{42}}}$$

**Prob. 4.19**

Let $Q_1$ be located at the origin. At the spherical surface of radius r,

$$Q_1 = \oint \mathbf{D}\cdot d\mathbf{S} = \varepsilon E_r(4\pi r^2)$$

Or

$$\mathbf{E} = \frac{Q_1}{4\pi\varepsilon r^2}\mathbf{a}_r \quad \text{by Gauss's law}$$

If a second charge $Q_2$ is placed on the spherical surface, $Q_2$ experiences a force

$$\mathbf{F} = Q_2\mathbf{E} = \frac{Q_1 Q_2}{4\pi\varepsilon r^2}\mathbf{a}_r$$

which is Coulomb's law.

**Prob. 4.20**

For a point charge, $\mathbf{D} = \dfrac{Q}{4\pi R^3}\mathbf{R}$

For the given three point charges,

$$\mathbf{D} = \frac{1}{4\pi}\left[\frac{Q\mathbf{R_1}}{R_1^3} + \frac{Q\mathbf{R_2}}{R_2^3} - \frac{2Q\mathbf{R_3}}{R_3^3}\right]$$

$$\mathbf{R}_1 = (0,0) - (-1,0) = (1,0),\ R_1 = 1$$

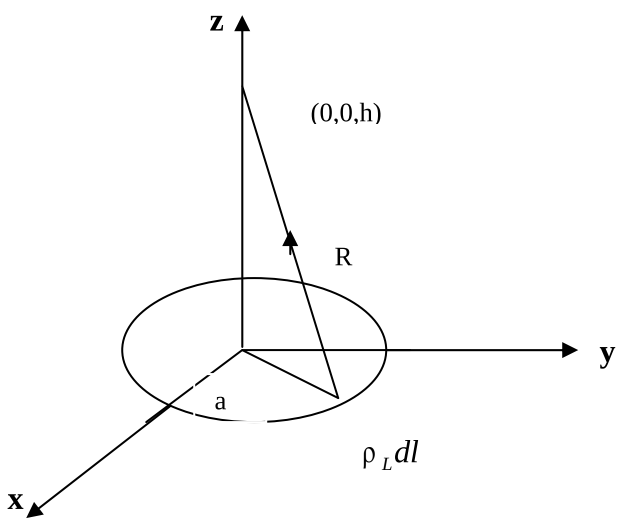
$$\mathbf{R}_2 = (0,0) - (1,0) = (-1,0),\ R_2 = 1$$

$$\mathbf{R}_3 = (0,0) - (0,1) = (0,-1),\ R_3 = 1$$

$$\mathbf{D} = \frac{Q}{4\pi}\left[(1,0) + (-1,0) - 2(0,-1)\right] = \frac{Q}{4\pi}(0,2) = \underline{\underline{\frac{Q}{2\pi}\mathbf{a}_y}}$$

**Prob. 4.21**

(a) Assume for now that the ring is placed on the z=0 plane.




$$\mathbf{D} = \int \frac{\rho_L dl\, \mathbf{R}}{4\pi R^3}, \mathbf{R} = -a\,\mathbf{a}_\rho + h\,\mathbf{a}_z$$

$$\mathbf{D} = \frac{\rho_L}{4\pi} \int_{\phi=0}^{\phi=2\pi} \frac{a d\phi(-a\,\mathbf{a}_\rho + h\,\mathbf{a}_z)}{(a^2+h^2)^{3/2}}$$

Due to symmetry, the $\rho$ component vanishes.

$$\mathbf{D} = \frac{\rho_L\, a(2\pi h)\mathbf{a}_z}{4\pi(a^2+h^2)^{3/2}} = \frac{\rho_L\; a h\,\mathbf{a}_z}{2(a^2+h^2)^{3/2}}$$

$a = 2, \quad h = 3, \quad \rho_L = 5\,\mu\text{C/m}$

Since the ring is actually placed in $x = 0$, $\mathbf{a}_z$ becomes $\mathbf{a}_x$.

$$\mathbf{D} = \frac{(6)(5)\mathbf{a}_x}{2(4+9)^{3/2}} = \underline{\underline{0.32\,\mathbf{a}_x\ \mu\text{C/m}^2}}$$

(b)

$$\boldsymbol{D}_Q = \frac{Q}{4\pi}\frac{[(3,0,0)-(0,-3,0)]}{|(3,0,0)-(0,-3,0)|^3} + \frac{Q}{4\pi}\frac{[(3,0,0)-(0,3,0)]}{|(3,0,0)-(0,3,0)|^3}$$

$$= \frac{Q(3,3,0)}{4\pi(18)^{3/2}} + \frac{Q(3,-3,0)}{4\pi(18)^{3/2}} = \frac{6Q(1,0,0)}{4\pi(18)^{3/2}}$$

$$\boldsymbol{D} = \boldsymbol{D}_R + \boldsymbol{D}_Q = 0$$

$$0.32(10^{-6}) + \frac{6Q}{4\pi(18)^{3/2}} = 0$$

$$\therefore\ Q = -0.32(4\pi)(18^{3/2})10^{-6}\frac{1}{6} = \underline{\underline{-51.182\mu C}}$$

**Prob. 4.22**

(a) $\rho_v = \nabla \bullet \boldsymbol{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = 0 + 2x - 4 = \underline{\underline{2x - 4 \text{ nC/m}^3}}$

(b)

$$\Psi = \int_S \boldsymbol{D} \bullet d\boldsymbol{S} = \iint y^2 dydz \Big|_{x=3} = \int_0^5 dz \int_0^6 y^2 dy = (5)\frac{y^3}{3}\Big|_0^6 = \frac{5}{3}(6)^3 = \underline{\underline{360 \text{ nC}}}$$

**Prob. 4.23**

$$\Psi = \int_S \boldsymbol{D} \bullet d\boldsymbol{S} = \varepsilon_o \int_S \boldsymbol{E} \bullet d\boldsymbol{S}, \qquad d\boldsymbol{S} = d\rho dz \boldsymbol{a}_\phi$$

$$\frac{\Psi}{\varepsilon_o} = \int_S \boldsymbol{E} \bullet d\boldsymbol{S} = -\iint 6\rho z \sin\phi d\rho dz \Big|_{\phi = 90^o}$$

$$= -6\sin 90^o \int_0^2 \rho d\rho \int_0^5 z dz = -6\left(\frac{\rho^2}{2}\Big|_0^2\right)\left(\frac{z^2}{2}\Big|_0^5\right) = -3(4-0)\frac{1}{2}(25-0) = -150$$

$$\Psi = -150\varepsilon_o = -150 \times \frac{10^{-9}}{36\pi} = \underline{\underline{-1.326 \text{ nC}}}$$

**Prob. 4.24**

(a)

$$\rho_v = \nabla \bullet D = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \sin\theta \sin\phi) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\cos\theta\sin\phi) + \frac{1}{r\sin\theta}(-\sin\phi)$$

$$= \frac{2}{r}\sin\theta\sin\phi + \frac{\sin\phi}{\sin\theta}(\cos^2\theta - \sin^2\theta) - \frac{\sin\phi}{r\sin\theta}$$

But $\cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$

$$\rho_v = \frac{2}{r}\sin\theta\sin\phi + \frac{\sin\phi}{\sin\theta} - \frac{2\sin\theta\sin\phi}{r} - \frac{\sin\phi}{\sin\theta} = 0$$

At $(2, 30^o, 60^o)$, $r = 2, \theta = 30^o, \phi = 60^o$

$\rho_v = \underline{\underline{0}}$

$$\Psi = \int_S \boldsymbol{D} \bullet d\boldsymbol{S}, \qquad d\boldsymbol{S} = r^2 \sin\theta d\theta d\phi \boldsymbol{a}_r$$

$$\Psi = \int_{\phi=0}^{60^o}\int_{\theta=0}^{30^o} r^2 \sin^2\theta \sin\phi d\theta d\phi \bigg|_{r=2}$$

(b)

$$= 4\int_0^{60^o} \sin\phi d\phi \int_0^{30^o} \sin^2\theta d\theta$$

But $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$.

$$\Psi = 4\left(-\cos\phi \bigg|_0^{60^o}\right)\frac{1}{2}\int_0^{30^o}(1-\cos 2\theta)d\theta = 2(-\cos 60^o + 1)\left(\theta - \frac{1}{2}\sin 2\theta\right)\bigg|_0^{\pi/6}$$

$$= 2(-1/2+1)\left(\pi/6 - \frac{1}{2}\sin(\pi/3) - 0\right) = 0.5236 - 0.433 = 0.0906 \text{ nC} = \underline{\underline{90.6 \text{ pC}}}$$

**Prob. 4.25**

(a) $$\rho_v = \nabla \cdot \boldsymbol{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \underline{\underline{8y \text{ C/m}^2}}$$

(b) $$\rho_v = \nabla \cdot \boldsymbol{D} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial\phi} + \frac{\partial D_z}{\partial z}$$

$$= 8\sin\phi - 2\sin\phi + 4z$$

$$= \underline{\underline{6\sin\phi + 4z \text{ C/m}^3}}$$

(c) $\rho_v = \nabla \cdot \mathbf{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(D_\theta \sin\theta) + 0$

$$= -\frac{2}{r^4}\cos\theta + \frac{2\cos\theta}{r^4}$$

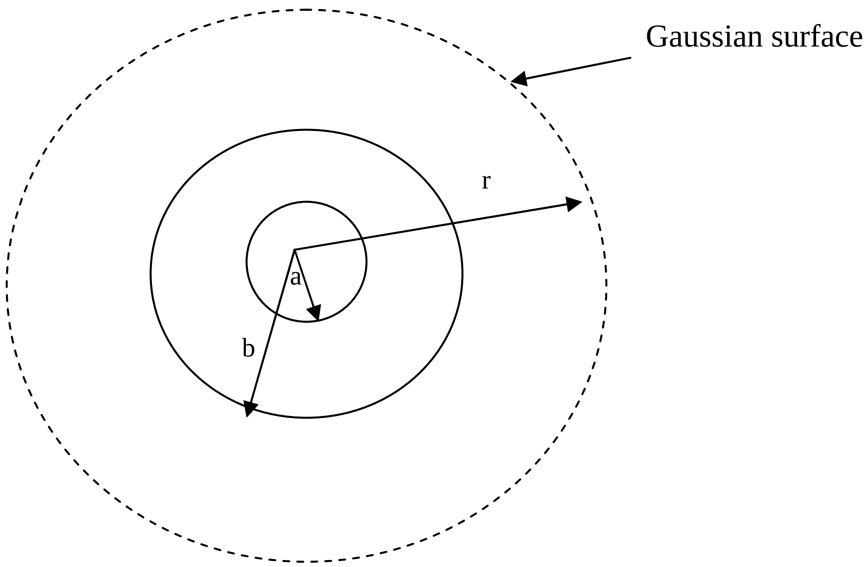
$$= \underline{\underline{0}}$$

**Prob. 4.26**

(a) $Q = \int_v \rho_v dv = \int_0^2\int_0^2\int_0^2 12xyz\,dx\,dy\,dz = 12\left(\frac{x^2}{2}\Big|_0^2\right)\left(\frac{y^2}{2}\Big|_0^2\right)\left(\frac{z^2}{2}\Big|_0^2\right) = 12(2)(2)(2) = \underline{\underline{96 \text{ mC}}}$

(b) $\Psi = Q = \underline{\underline{96 \text{ mC}}}$

**Prob. 4.27**




Apply Gauss's law,

$$\oint \mathbf{D} \cdot d\mathbf{S} = Q_{enc}$$

$$D_r 4\pi r^2 = \rho_{s1} 4\pi a^2 + \rho_{s2} 4\pi b^2 = 8\times10^{-9}\times 4\pi(1)^2 + (-6\times10^{-3})\times 4\pi(2)^2 = -0.3016$$

$$D_r = \frac{-0.3016}{4\pi r^2} = \frac{= 0.3016}{4\pi(3)^2} = -0.0027$$

$$\mathbf{D} = \underline{\underline{-2.7\mathbf{a}_r \text{ mC/m}^2}}$$

**Prob. 4.28**

For r <a.

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q_{enc} = \int_v \rho_v dv$$

$$D_r(4\pi r^2) = \iiint 5r^{1/2} r^2 \sin\theta d\theta d\phi dr = 5\int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\phi \int_0^{r} r^{5/2} dr$$

$$= 5(2)(2\pi)\frac{r^{7/2}}{7/2}\Bigg|_0^r = \frac{40\pi r^{7/2}}{7}$$

$$D_r = \varepsilon_o E_r = \frac{\frac{40\pi r^{7/2}}{7}}{4\pi r^2} = \frac{10}{7} r^{3/2}$$

$$E_r = \frac{10}{7\varepsilon_o} r^{3/2}, 0 < r < a$$

For r > a,

$$D_r(4\pi r^2) = \frac{40}{7}\pi a^{7/2}$$

$$D_r = \varepsilon_o E_r = \frac{\frac{40}{7}\pi a^{7/2}}{4\pi r^2}$$

$$E_r = \frac{10a^{7/2}}{7\varepsilon_o r^2}, r > a$$

Thus,

$$\mathbf{E} = \begin{cases} \dfrac{10}{7\varepsilon_o} r^{3/2} \mathbf{a}_r, 0 < r < a \\ \dfrac{10a^{7/2}}{7\varepsilon_o r^2} \mathbf{a}_r, r > a \end{cases}$$

**Prob. 4.29**

(a) $\rho_v = \nabla \bullet \mathbf{D} = \dfrac{\partial D_x}{\partial x} + \dfrac{\partial D_y}{\partial y} + \dfrac{\partial D_z}{\partial z} = \underline{\underline{2y \text{ C/m}^3}}$

(b) $\Psi = \int \mathbf{D} \cdot d\mathbf{S} = \iint x^2 dx dz \Big|_{y=1} = \int_0^1 x^2 dx \int_0^1 dz = \underline{\underline{\frac{1}{3} \text{ C}}}$

(c ) $Q = \int_v \rho_v dv = \iiint 2y dx dy dz = 2\int_0^1 dx \int_0^1 y dy \int_0^1 dz = \underline{\underline{1 \text{ C}}}$

**Prob. 4.30**

(a)

$$\rho_V = \nabla \bullet \boldsymbol{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z}$$

$$\rho_V = 4(z+1)\cos\phi - (z+1)\cos\phi + 0$$

$$\rho_V = \underline{\underline{3(z+1)\cos\phi \quad \mu\text{C/m}^3}}$$

(b)

$$Q_{enc} = \int \rho_V dv = \iiint 3(z+1)\cos\phi \ \rho d\phi \ d\rho dz$$

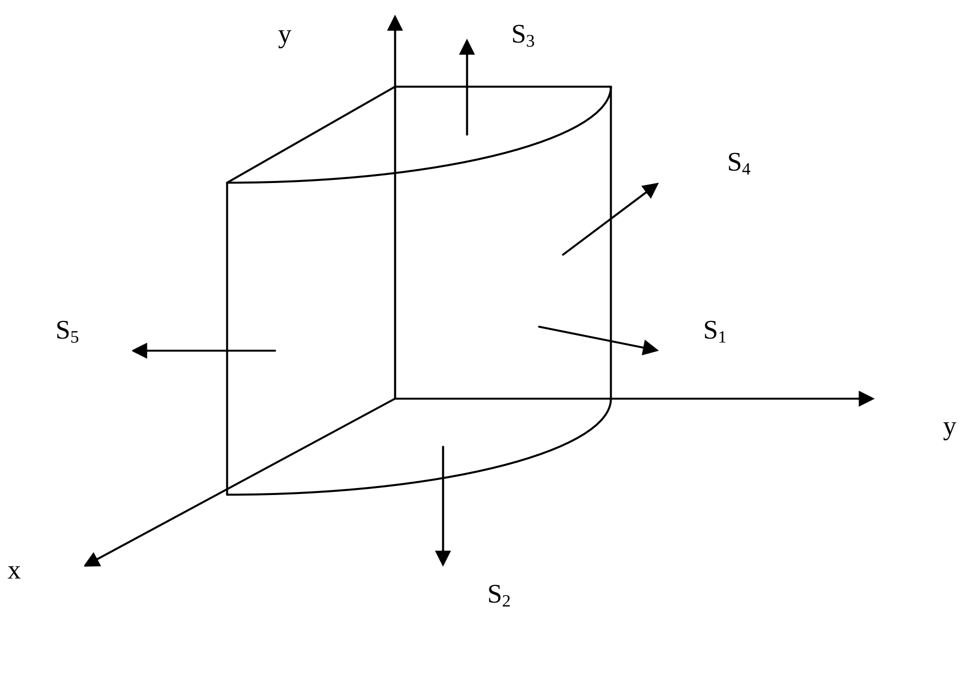
$$= 3\int_0^2 \rho d\rho \int_0^4 (z+1)\,\text{dz} \int_0^{\pi/2} \cos\phi d\phi = 3\left(\frac{1}{2}\rho^2 \Big|_0^2\right)\left(\frac{z^2}{2} + z\right)\Big|_0^4 (\sin\phi\Big|_0^{\pi/2})$$

$$= 3(2)(8+4)(1-0) = \underline{\underline{72\mu\text{C}}}$$

(c)

Let $\psi = \psi_1 + \psi_2 + \psi_3 + \psi_4 + \psi_5 = \oint \boldsymbol{D} \bullet d\boldsymbol{S}$

where $\psi_1, \psi_2, \psi_3, \psi_4, \psi_5$ respectively correspond witn surfaces $S_1, S_2, S_3, S_4, S_4$ (in the figure below) respectively.



For $S_1$ $\rho = 2,\ d\mathbf{S} = \rho d\phi dz \mathbf{a}_\rho$

$$\psi_1 = \iint 2\rho(z+1)\cos\phi dS\Big|_{\rho=2} = 2(2)^2\int_0^4 (z+1)dz \int_0^{\pi/2} \cos\phi d\phi$$

$= 8(12)(1) = 96$

For $S_2$ , $z = 0, d\mathbf{S} = \rho d\phi d\rho(-\mathbf{a}_z)$

$$\psi_2 = -\iint \rho^2 \cos\phi \ \ \rho d\phi \ d\rho = -\int_0^2 \rho^3 d\rho \int_0^{\pi/2} \cos\phi d\phi$$

$$= -\frac{\rho^4}{4}\Big|_0^2 (1) = -4$$

For $S_{3,}$ $z = 4, d\mathbf{S} = \rho d\phi d\rho \mathbf{a}_z$, $\psi_3 = +4$

For $S_4, \phi = \pi/2, d\mathbf{S} = d\rho dz \mathbf{a}_\phi$

$$\psi_4 = -\iint \rho(z+1)\sin\phi \ d\rho dz\Big|_{\phi=\pi/2} = -\left(\sin\frac{\pi}{2}\right)\int_0^2 \rho d\rho \int_0^4 (z+1)dz$$

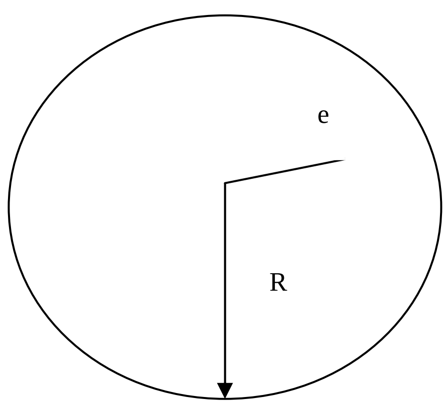
$$= -\frac{\rho^2}{2}\Big|_0^2 (12) = -(2)(12) = -24$$

For $S_5$, $\phi = 0, d\mathbf{S} = d\rho dz(-\mathbf{a}_\phi), \psi_5 = \iint \rho(z+1)\sin\phi \ d\rho dz\Big|_{\phi=0} = 0$

$$\psi = 96 - 4 + 4 - 24 + 0 = \underline{\underline{72\mu\text{C}}}$$

This is exactly the answer obtained in part (b).

**Prob. 4.31**



$F = eE$

$$\rho_0 = \frac{e}{4\pi \frac{R^3}{3}} = \frac{3e}{4\pi R^3}$$

$$\rho_V = \begin{cases} \rho_{0,} & 0 < r < R \\ 0, & \text{elsewhere} \end{cases}$$

$$\oint \boldsymbol{D} \bullet d\boldsymbol{S} = Q_{enc} = \int \rho_V dV = \frac{3e}{4\pi R^3} \frac{4\pi r^3}{3} = D_r (4\pi r^2)$$

$$E_r = \frac{3e\, r}{12\pi\varepsilon_0 R^3}$$

$$F = eE = \frac{e^2 r}{4\pi \varepsilon_0 R^3}$$

**Prob. 4.32**

Using Gauss' law, when $\rho < 0$, $\boldsymbol{E} = \boldsymbol{0}$

For $a < \rho < b$, $Q_{enc} = \oint_S \boldsymbol{D} \bullet d\boldsymbol{S} = \varepsilon_o \oint_S \boldsymbol{E} \bullet d\boldsymbol{S} = \varepsilon_o E_\rho (2\pi\rho L)$

where $L$ is the length of the cable. But

$$Q_{enc} = \int_S \rho_s dS = \int_{\phi=0}^{2\pi} \int_{z=0}^{L} \frac{\rho_o}{\rho} \rho d\phi dz = \rho_o 2\pi L$$

$$\rho_o 2\pi L = \varepsilon_o E_\rho (2\pi\rho L) \quad \rightarrow \quad E_\rho = \frac{\rho_o}{\varepsilon_o \rho}$$

For $\rho > b$, $Q_{enc} = 0$. Thus,

$$\boldsymbol{E} = \begin{cases} \dfrac{\rho_o}{\varepsilon_o \rho} \boldsymbol{a}_\rho, & a < \rho < b \\ 0, & \text{otherwise} \end{cases}$$

**Prob. 4.33**

**(a)**

$\psi = Q_{enc}$ at $r = 2$

$$Q_{enc} = \int \rho_V \, dV = \iiint \frac{10}{r^2} r^2 \sin\theta \, d\theta \, dr \, d\phi$$

$$= 10 \int_{r=1}^{2} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \sin\theta \, d\theta dr d\phi$$

$$= 10(1)\,(2\pi)\,(2) = (40\,\pi) \text{ mC}$$

Thus, $\psi = \underline{\underline{125.7 \text{ mC}}}$

At $r = 6$;

$$Q_{enc.} = 10 \int_{r=1}^{4} dr \int_{\phi=0}^{2\pi} d\phi \int_{\theta=0}^{\pi} \sin\theta \, d\theta$$

$$= 10\,(3)(2\pi)\,(2) = 120\pi \text{ mC}$$

$\psi = \underline{\underline{377 \text{ mC}}}$

**(b)**

$$\psi = Q_{enc}$$

But $\psi = \oint \boldsymbol{D} \bullet d\boldsymbol{S} = D_r \oint dS = D_r (4\pi r^2)$

At $r = 1$,

$Q_{enc} = 0 \longrightarrow \underline{\underline{\boldsymbol{D} = 0}}$

At $r = 5$, $Q_{enc} = 120\,\pi$

$$D_r = \frac{Q_{enc}}{4\pi r^2} = \frac{120\pi}{4\pi(5)^2} = 1.2$$

$\boldsymbol{D} = \underline{\underline{1.2\boldsymbol{a}_r \text{ mC/m}^2}}$

**Prob. 4.34**

$$\rho_v = \frac{Q}{\text{volume}} = \frac{Q}{4\pi a^3/3} = \frac{3Q}{4\pi a^3}$$

For r < a, $\oint \boldsymbol{D} \cdot d\boldsymbol{S} = Q_{enc} = \int \rho_v dv$

$$D_r 4\pi r^2 = \frac{3Q}{4\pi a^3}\frac{4\pi r^3}{3} = \frac{Qr^3}{a^3} \longrightarrow D_r = \frac{Qr}{4\pi a^3}$$

For r > a, $\oint \boldsymbol{D} \cdot d\boldsymbol{S} = Q$

$$D_r 4\pi r^2 = Q \quad \longrightarrow \quad D_r = \frac{Q}{4\pi r^2}$$

Hence,

$$\boldsymbol{D} = \begin{cases} \dfrac{Qr}{4\pi a^3}\boldsymbol{a}_r, & r < a \\ \dfrac{Q}{4\pi r^2}\boldsymbol{a}_r, & r > a \end{cases}$$

**Prob. 4.35**

$$V_P = \frac{Q_1}{4\pi\varepsilon_o r_1} + \frac{Q_2}{4\pi\varepsilon_o r_2} = \frac{10^{-9}}{4\pi \times \dfrac{10^{-9}}{36\pi}}\left[\frac{2}{|(1,-2,3)-(1,0,3)|} - \frac{4}{|(1,-2,3)-(-2,1,5)|}\right]$$

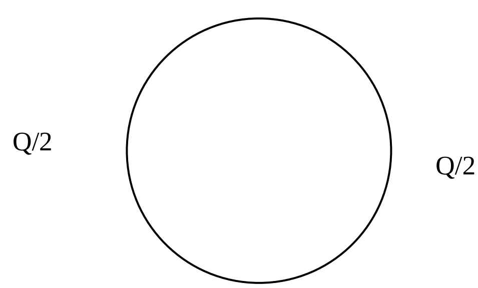
$$= 9\left[\frac{2}{2} - \frac{4}{\sqrt{9+9+4}}\right] = \underline{\underline{1.325 \text{ V}}}$$

**Prob. 4.36**

$$V = 4\frac{Q}{4\pi\varepsilon_o r}, \qquad r = \sqrt{a^2 + a^2 + h^2} = \sqrt{2^2 + 2^2 + 3^2} = \sqrt{17} \text{ cm}$$

$$V = \frac{4 \times 8 \times 10^{-9}}{4\pi \times \dfrac{10^{-9}}{36\pi} \times \sqrt{17} \times 10^{-2}} = \underline{\underline{6.985 \text{ kV}}}$$

**Prob. 4.37 (a)**




$$V = \frac{2\dfrac{Q}{2}}{4\pi\varepsilon_0 r} = \frac{Q}{4\pi\varepsilon_0 r}$$

$$= \frac{60(10^{-6})}{4\pi \times \dfrac{10^{-9}}{36\pi} x4} = \underline{\underline{135 \; k\text{V}}}$$

(b)

$$V = \frac{3\,(\frac{Q}{3})}{4\pi\,\varepsilon_0\, r} = \underline{\underline{135\ k\text{V}}}$$

(c)

$$V = \int \frac{\rho_L dl}{4\pi\,\varepsilon_0\, r} = \frac{\frac{Q}{8\pi}2\pi(4)}{4\pi\,\varepsilon_0\, r}\quad \frac{Q}{4\pi\,\varepsilon_0\, r} = \underline{\underline{135\ k\text{V}}}$$

**Prob. 4.38**

(a)

$$V_P = \sum \frac{Q_k}{4\pi |\boldsymbol{r}_p - \boldsymbol{r}_k|}$$

$$4\pi\,\varepsilon_o V_p = \frac{10^{-3}}{|(-1,1,2)-(0,0,4)|} + \frac{-2(10^{-3})}{|(-1,1,2)-(-2,5,1)|} + \frac{3(10^{-3})}{|(-1,1,2)-(3,-4,6)|}$$

$$4\pi\,\varepsilon_0(10^3)\,V_p = \frac{1}{|(-1,1,-2)|} - \frac{2}{|(1,-4,1)} + \frac{3}{|(-4,5,-4)|} = \frac{1}{\sqrt{6}} - \frac{2}{\sqrt{18}} + \frac{3}{\sqrt{57}}$$

$$4\pi\frac{10^{-9}}{36\pi}(10^3)\,V_p = 0.3542$$

$$\therefore V_p = \underline{\underline{3.008\times10^6\ \text{V}}}$$

(b)

$$V_Q = \sum \frac{Q_k}{4\pi\varepsilon_o |\boldsymbol{r}_p - \boldsymbol{r}_k|}$$

$$4\pi\,\varepsilon_o V_Q = \frac{10^{-3}}{|(1,2,3)-(0,0,4)|} + \frac{-2(10^{-3})}{|(1,2,3)-(-2,5,1)|} + \frac{3(10^{-3})}{|(1,2,3)-(3,-4,6)|}$$

$$4\pi\,\varepsilon_0(10^3)\,V_p = \frac{1}{|(1,2,-1)|} - \frac{2}{|(3,-3,2)|} + \frac{3}{|(-2,6,-3)|} = \frac{1}{\sqrt{6}} - \frac{2}{\sqrt{22}} + \frac{3}{7}$$

$$4\pi\frac{10^{-9}}{36\pi}(10^3)\,V_p = 0.410$$

$$V_Q = \underline{\underline{3.694\ (10^6)\text{V}}}$$

$$\therefore V_{PQ} = V_Q - V_P = 0.686(10^6) = \underline{\underline{686\ k\text{V}}}$$

**Prob. 4.39**

$$-\boldsymbol{E} = \nabla V = \frac{\partial V}{\partial \rho}\boldsymbol{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\boldsymbol{a}_\phi + \frac{\partial V}{\partial z}\boldsymbol{a}_z$$

$$= 2\rho e^{-z}\sin\phi\boldsymbol{a}_\rho + \rho e^{-z}\cos\phi\boldsymbol{a}_\phi - \rho^2 e^{-z}\sin\phi\boldsymbol{a}_z$$

At $(4,\pi/4,-1)$, $\rho=4, \phi=\pi/4, z=-1$

$$-\boldsymbol{E} = 2(4)e^1\sin(\pi/4)\boldsymbol{a}_\rho + 4e^1\cos(\pi/4)\boldsymbol{a}_\phi - 16e^1\sin(\pi/4)\boldsymbol{a}_z$$

$$= 2(4)(2.7183)(0.7071)\boldsymbol{a}_\rho + 4(2.7183)(0,7071)\boldsymbol{a}_\phi - 16(2.7183)(0,7071)\boldsymbol{a}_z$$

$$\underline{\underline{\boldsymbol{E} = -15.38\boldsymbol{a}_\rho - 7.688\boldsymbol{a}_\phi + 30.75\boldsymbol{a}_z \text{ V/m}}}$$

**Prob. 4.40**

(a)

$$\boldsymbol{E} = -\left(\frac{\partial V}{\partial x}\boldsymbol{a}_x + \frac{\partial V}{\partial y}\boldsymbol{a}_y + \frac{\partial V}{\partial z}\boldsymbol{a}_z\right)$$

$$= -2xy(z+3)\boldsymbol{a}_x - x^2(z+3)\boldsymbol{a}_y - x^2 y\boldsymbol{a}_z$$

At $(3,4,-6)$, $x=3$, $y=4$, $z=-6$,

$$\boldsymbol{E} = -2(3)(4)(-3)\boldsymbol{a}_x - 9(-3)\boldsymbol{a}_y - 9(4)\boldsymbol{a}_z$$

$$= \underline{\underline{72\boldsymbol{a}_x + 27\boldsymbol{a}_y - 36\boldsymbol{a}_z \text{ V/m}}}$$

(b)

$$\rho_V = \nabla \bullet \boldsymbol{D} = \varepsilon_0 \nabla \bullet \boldsymbol{E} = -\varepsilon_0(2y)(z+3)$$

$$\psi = Q_{enc} = \int \rho_V dV = -2\varepsilon_0 \iiint y(z+3)dx\,dy\,dz$$

$$= -2\varepsilon_0 \int_0^1 dx \int_0^1 y\,dy \int_0^1 (z+3)dz = -2\varepsilon_0(1)(1/2)\left(\frac{z^2}{2}+3z\right)\Big|_0^1$$

$$= -\varepsilon_0\left(\frac{1}{2}+3\right) = \frac{-7}{2}\left(\frac{10^{-9}}{36\pi}\right)$$

$$Q_{enc} = \underline{\underline{-30.95 \ pC}}$$

**Prob. 4.41**

(a)

$$Q = \int_v \rho_v dv = \iiint \rho_o\left(1-\frac{r^2}{a^2}\right)r^2\sin\theta d\theta d\phi dr$$

$$= \rho_o \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \int_0^a \left(r^2 - \frac{r^4}{a^2}\right)dr = \rho_o(2\pi)(2)\left(\frac{a^3}{3}-\frac{a^3}{5}\right) = \underline{\underline{\frac{8\pi}{15}a^3\rho_o}}$$

(b) Outside the nucleus, r >a,

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q_{enc} \quad \longrightarrow \quad \mathbf{E} = \frac{Q_{enc}}{4\pi\varepsilon_o r^2}\mathbf{a}_r$$

$$\mathbf{E} = \frac{\frac{8\pi a^3\rho_o}{15}}{4\pi\varepsilon_o r^2}\mathbf{a}_r = \underline{\underline{\frac{2a^3\rho_o}{15\varepsilon_o r^2}\mathbf{a}_r}}$$

$$V = -\int \mathbf{E} \cdot d\mathbf{l} = -\int E_r dr = \frac{2a^3\rho_o}{15\varepsilon_o r} + C_1$$

Since $V(\infty) = 0, C_1 = 0.$

$$V = \underline{\underline{\frac{2a^3\rho_o}{15\varepsilon_o r}}}$$

(c ) Inside the nucleus, r <a

$$Q_{enc} = 4\pi\rho_o\left(\frac{r^3}{3} - \frac{r^5}{5a^2}\right)$$

$$\mathbf{E} = \frac{Q_{enc}}{4\pi\varepsilon_o r^2}\mathbf{a}_r = \underline{\underline{\frac{\rho_o}{\varepsilon_o}\left(\frac{r}{3} - \frac{r^3}{5a^2}\right)\mathbf{a}_r}}$$

$$V = -\int E_r dr = -\frac{\rho_o}{\varepsilon_o}\left(\frac{r^2}{6} - \frac{r^4}{20a^2}\right) + C_2$$

$$V(r = a) = \frac{2a^2\rho_o}{15\varepsilon_o} = \frac{\rho_o}{\varepsilon_o}\left(\frac{a^2}{20} - \frac{a^2}{6}\right) + C_2$$

$$C_2 = \frac{2a^2\rho_o}{15\varepsilon_o} + \frac{7a^2\rho_o}{60\varepsilon_o} = \frac{a^2\rho_o}{4\varepsilon_o}$$

$$V = \underline{\underline{\frac{\rho_o}{\varepsilon_o}\left(\frac{r^4}{20a^2} - \frac{r^2}{6}\right) + \frac{a^2\rho_o}{4\varepsilon_o}}}$$

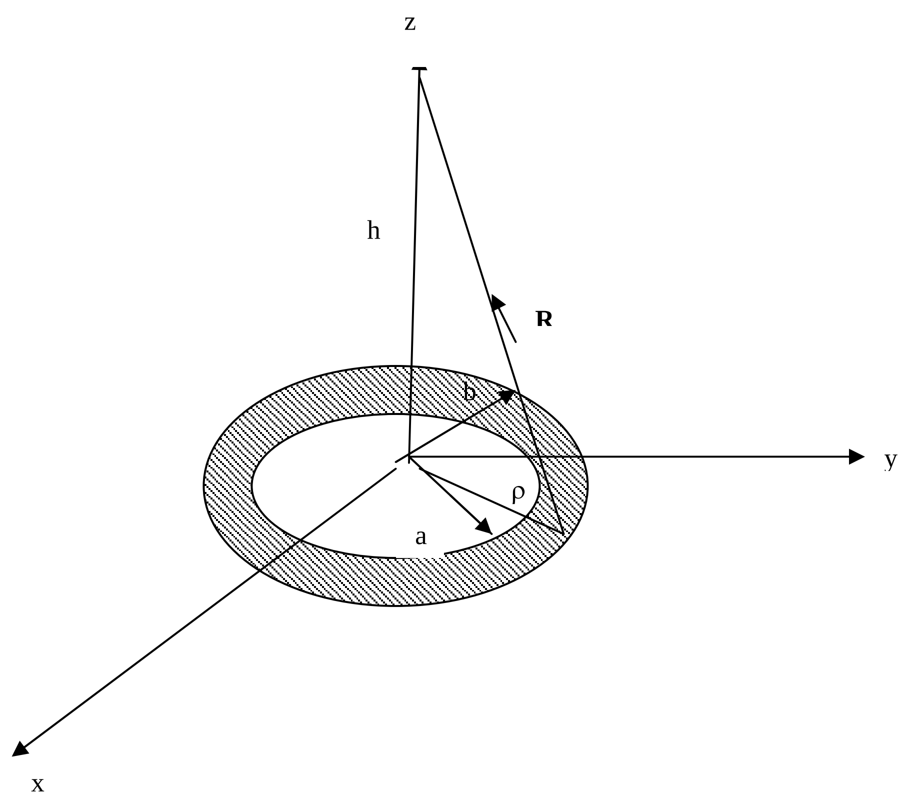
(d) E is maximum when

$$\frac{dE}{dr} = 0 = \frac{\rho_o}{\varepsilon_o}\left(\frac{1}{3} - \frac{3r^2}{5a^2}\right) \quad \longrightarrow \quad 9r^2 = 5a^2$$

$$r = \frac{\sqrt{5}}{3}a = \underline{\underline{0.7454a}}$$

We are able to say maximum because $\frac{d^2E}{dr^2} = -\frac{6r}{5a^2} < 0.$

**Prob. 4.42**




$$\mathbf{D} = \int \frac{\rho_s dS\mathbf{R}}{4\pi R^3}, \qquad \mathbf{R} = -\rho \mathbf{a}_\rho + h\mathbf{a}_z\,, \qquad R = |\,\mathbf{R}\,| = \sqrt{\rho^2 + h^2}, dS = \rho d\phi d\rho$$

$$\rho_s = \frac{Q}{S} = \frac{Q}{\pi(b^2 - a^2)}$$

$$\mathbf{D} = \frac{\rho_s}{4\pi}\int \frac{\rho d\phi d\rho(-\rho \mathbf{a}_\rho + h\mathbf{a}_z)}{(\rho^2 + h^2)^{3/2}}$$

Due to symmetry, the component along $\mathbf{a}_\rho$ vanishes.

$$D_z = \frac{\rho_s h}{4\pi}\int_{\rho=a}^{b}\int_{\phi=0}^{2\pi}\frac{\rho d\phi d\rho}{(\rho^2+h^2)^{3/2}} = \frac{\rho_s h}{4\pi}(2\pi)\int_a^b(\rho^2+h^2)^{-3/2}\rho d\rho$$

$$= \frac{\rho_s h}{2}\left[\frac{-1}{\sqrt{\rho^2+h^2}}\right]_a^b = \frac{\rho_s h}{2}\left[\frac{1}{\sqrt{a^2+h^2}} - \frac{1}{\sqrt{b^2+h^2}}\right]$$

$$\boldsymbol{D} = \frac{Qh}{2\pi(b^2-a^2)}\left[\frac{1}{\sqrt{a^2+h^2}} - \frac{1}{\sqrt{b^2+h^2}}\right]\boldsymbol{a}_z$$

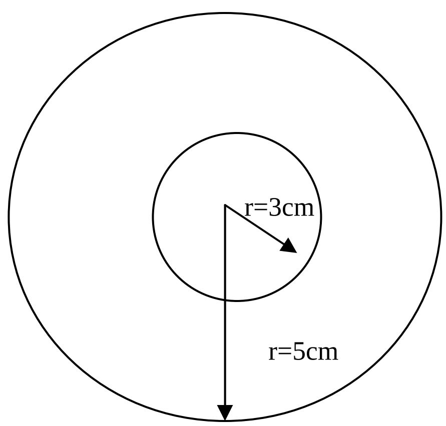
**Prob. 4.43**

$$\rho_v = \nabla \cdot D = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = 4 - 20y + 2z$$

At P(1,2,3), x=1, y= 2, z=3

$$\rho_v = 4 - 20(2) + 2(3) = \underline{\underline{-30 \text{ C/m}^3}}$$

**Prob. 4.44**




For r < 3cm, $Q_{enc} = 0 \longrightarrow \boldsymbol{D} = 0$

For 3 < r < 5cm,

$$\oint \boldsymbol{D} \cdot d\boldsymbol{S} = Q_{enc} = 10 \text{ nC}$$

$$D_r 4\pi r^2 = 10 \text{ nC} \longrightarrow D_r = \frac{10}{4\pi r^2} \text{ nC/m}^2$$

For r > 5 cm,

$$\oint \boldsymbol{D} \cdot d\boldsymbol{S} = Q_{enc} = 10 - 5 = 5\text{nC} \longrightarrow D_r = \frac{5}{4\pi r^2} \text{ nC/m}^2$$

Thus,

$$\boldsymbol{D} = \begin{cases} 0, & r < 3 \text{ cm} \\ \dfrac{10}{4\pi r^2}\boldsymbol{a}_r \ \text{ nC/m}^2, & 3 < r < 5 \text{ cm} \\ \dfrac{5}{4\pi r^2}\boldsymbol{a}_r \ \text{nC/m}^2, & r > 5 \text{ cm} \end{cases}$$

**Prob. 4.45**

$$\rho_v = \nabla \cdot \boldsymbol{D} = \nabla \cdot \varepsilon_o \boldsymbol{E} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\frac{\varepsilon_o E_o \rho^2}{a}\right) \quad = \frac{2\varepsilon_o E_o}{a}, 0 < \rho < a$$

**Prob. 4.46**

Let us choose the following path of two segments.

$(2,1,-1) \rightarrow (5,1,-1) \rightarrow (5,1,2)$

$$W = -q\int \boldsymbol{E} \bullet d\boldsymbol{l}$$

$$-\frac{W}{q} = \int \boldsymbol{E} \bullet d\boldsymbol{l} = \int_{x=2}^{5} 2xyzdx \Bigg|_{z=-1,\, y=1} + \int_{z=-1}^{2} x^2 ydz \Bigg|_{x=5,\, y=1}$$

$$= 2(1)(-1)\frac{x^2}{2}\Bigg|_2^5 + (5)^2(1)z\Bigg|_{-1}^{2} = -21 + 75 = 54$$

$$W = -54q = -108 \ \mu\text{J}$$

**Prob. 4.47**

(a)

$$\rho_v = \nabla \bullet \boldsymbol{D} = \nabla \bullet \varepsilon \boldsymbol{E}$$

$$\frac{\rho_v}{\varepsilon} = \nabla \bullet \boldsymbol{E} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(12\rho^2 z\cos\phi) + \frac{1}{\rho}(-6\rho z\cos\phi) + 0$$

$$= 24z\cos\phi - 6z\cos\phi = 18z\cos\phi$$

At A$(2,180^o,-1)$, $\rho = 2, \phi = 180^o, z = -1$

$$\rho_v = 18\varepsilon z\cos\phi = 18(-1)\cos(180^o)\times\frac{10^{-9}}{36\pi} = \frac{10^{-9}}{2\pi} = 0.1592 \text{ nC/m}^3$$

(b)

$$W = -Q\int_L \boldsymbol{E} \bullet d\boldsymbol{l}, \qquad d\boldsymbol{l} = \rho d\phi \boldsymbol{a}_\phi$$

$$W = Q\int_{\phi=180^o}^{0^o} 6\rho z\sin\phi\ \rho d\phi \Big|_{\rho=2,z=-1} = Q6(2)^2(-1)\left(-\cos\phi \Big|_{180^o}^{0^o}\right) = -24Q(-1-1)$$

$$= 48Q = 48\times10\times10^{-6} = \underline{\underline{480\ \mu\text{J}}}$$

**Prob. 4.48**

(a)

From $A$ to $B$, $d\boldsymbol{l} = rd\theta\, \boldsymbol{a}_\theta$,

$$W_{AB} = -Q\int_{\theta=30°}^{90°} 10\, r\cos\theta\, r\, d\theta \Big|_{r=5} = -Q(10)(5)^2(\sin\theta)\Big|_{30^o}^{90^o}\ \underline{\underline{-1250\ \text{nJ}}}$$

(b)

From $A$ to $C$, $d\boldsymbol{l} = dr\, \boldsymbol{a}_r$,

$$W_{AC} = -Q\int_{r=5}^{10} 20\, r\sin\theta\, dr \Big|_{\theta=30°} = -Q(20)(\sin 30^o)\left(\frac{1}{2}r^2\right)\Big|_5^{10} = \underline{\underline{-3750\ \text{nJ}}}$$

(c )

From $A$ to $D$, $d\boldsymbol{l} = r\sin\theta\, d\phi\ \boldsymbol{a}_\phi$,

$$W_{AD} = -Q\int 0(r\sin\theta)\, d\phi = \underline{\underline{0\ \text{J}}}$$

(d)

$$W_{AE} = W_{AD} + W_{DF} + W_{FE}$$

where $F$ is $(10, 30^o, 60^o)$. Hence,

$$W_{AE} = -Q\left\{\int_{r=5}^{10} 20\, r\sin\theta\, dr \Big|_{\theta=30°} + \int_{\theta=30^o}^{90^o} 10\, r\cos\theta\, r\, d\theta \Big|_{r=10}\right\}$$

$$= -100[\frac{75}{2} + \frac{100}{2}]\ \text{nJ} = \underline{\underline{-8750\ \text{nJ}}}$$

**Prob. 4.49**

$$V_{AB} = -\int_A^B \boldsymbol{E}\cdot d\boldsymbol{l} = -\int_1^5 \frac{10}{r^2}dr$$

$$= \frac{10}{r}\Big|_1^5 = 10(\frac{1}{5} - 1) = \underline{\underline{-8V}}$$

**Prob. 4.50**

*Method 1:*

$$W = -Q\int_L \mathbf{E}\cdot d\mathbf{l}, \qquad d\mathbf{l} = \rho d\phi \mathbf{a}_\phi$$

$$\begin{bmatrix} E_\rho \\ E_\phi \\ E_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

$$E_\phi = -E_x \sin\phi + E_y \cos\phi = -20x\sin\phi + 40y\cos\phi$$

$$x = \rho\cos\phi, y = \rho\sin\phi$$

$$E_\phi = -20\rho\cos\phi\sin\phi + 40\rho\sin\phi\cos\phi = 20\rho\cos\phi\sin\phi$$

$$W = -Q\int_L \mathbf{E}\cdot d\mathbf{l} = -2\times10^{-3}\int 20\rho\cos\phi\sin\phi\rho d\phi\Big|_{\rho=2}$$

$$= -2(20)(2)^2\int_0^{\pi/2}\sin\phi d(\sin\phi)\text{ mJ} = 160\frac{\sin^2\phi}{2}\Big|_0^{\pi/2} = \underline{\underline{-80\text{ mJ}}}$$

*Method 2:*

$$-\frac{W}{Q} = \int_L \mathbf{E}\cdot d\mathbf{l} = \int 20\text{xdx} + 40\text{ydy}$$

$$y = 2 - x, dy = -dx$$

$$-\frac{W}{Q} = \int 20\text{xdx} + 40(2-\text{x})(-\text{dx}) = \int_{\text{x}=2}^{0}(60\text{x} - 80)\text{dx}$$

$$= \frac{60x^2}{2} - 80x\Big|_2^0 = 40$$

$$W = -40Q = \underline{\underline{-80\text{ mJ}}}$$

*Method 3:*

$$\nabla\times\mathbf{E} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 20x & 40y & -10z \end{vmatrix} = 0$$

$$V = -\int_L \mathbf{E}\cdot d\mathbf{l} = -10\text{x}^2 - 20y^2 + 5\text{z}^2 + C$$

$$W = Q(V_2 - V_1) = Q(-20\times4 + 10\times4) = -40Q$$

$$W = -40Q = \underline{\underline{-80\text{ mJ}}}$$

**Prob. 4.51**

$$W = -Q\int_L \boldsymbol{E} \cdot d\boldsymbol{l}$$

$$\boldsymbol{E} = \frac{\rho_s}{2\varepsilon_o}\boldsymbol{a}_n = \frac{\rho_s}{2\varepsilon_o}\boldsymbol{a}_x, \qquad d\boldsymbol{l} = dx\boldsymbol{a}_x$$

$$W = -\frac{Q\rho_s}{2\varepsilon_o}\int_3^1 dx = -\frac{Q\rho_s}{2\varepsilon_o}(-2) = \frac{Q\rho_s}{\varepsilon_o}$$

$$= \frac{10\times10^{-6}\times 40\times10^{-9}}{\frac{10^{-9}}{36\pi}} = 400\times36\pi\times10^{-6} = \underline{\underline{45.24 \text{ mJ}}}$$

**Prob. 4.52**

(a) $$\boldsymbol{E} = -\nabla V = -\frac{\partial V}{\partial x}\boldsymbol{a}_x - \frac{\partial V}{\partial y}\boldsymbol{a}_y - \frac{\partial V}{\partial z}\boldsymbol{a}_y = \underline{\underline{-4x\boldsymbol{a}_x - 8y\boldsymbol{a}_y}}$$

$$\rho_v = \nabla \cdot D = \varepsilon_o \nabla \cdot E = \varepsilon_o(-4-8) = -12\varepsilon_o = \underline{\underline{-106.25 \ \text{ pC/m}^3}}$$

(b) $$\boldsymbol{E} = -\nabla V = -\frac{\partial V}{\partial \rho}\boldsymbol{a}_\rho - \frac{1}{\rho}\frac{\partial V}{\partial \phi}\boldsymbol{a}_\phi - \frac{\partial V}{\partial z}\boldsymbol{a}_z$$

$$= \underline{\underline{-(20\rho\sin\phi + 6z)\boldsymbol{a}_\rho - 10\rho\cos\phi\boldsymbol{a}_\phi - 6\rho\boldsymbol{a}_z}}$$

$$\rho_v = \nabla \cdot D = \varepsilon_o \nabla \cdot E = \varepsilon_o\left(-\frac{1}{\rho}\left[40\rho\sin\phi + 6z\right] + 10\sin\phi\right)$$

$$= \underline{\underline{-\left(30\sin\phi + \frac{6z}{\rho}\right)\varepsilon_o \ \text{ C/m}^3}}$$

(c) $$\boldsymbol{E} = -\nabla V = -\frac{\partial V}{\partial r}\boldsymbol{a}_r - \frac{1}{r}\frac{\partial V}{\partial \theta}\boldsymbol{a}_\theta - \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\boldsymbol{a}_\phi$$

$$= \underline{\underline{-10r\cos\theta\sin\phi\boldsymbol{a}_r + 5r\sin\theta\sin\phi\boldsymbol{a}_\theta - 5r\cot\theta\cos\phi\boldsymbol{a}_\phi}}$$

$$\rho_v = \nabla \cdot D = \varepsilon_o \nabla \cdot E$$

$$\frac{\rho_v}{\varepsilon_o} = \frac{1}{r^2}(-30r^2\cos\theta\sin\phi) + \frac{5r\sin\phi}{r\sin\theta}2\sin\theta\cos\theta + \frac{5r\cot\theta\sin\phi}{r\sin\theta}$$

$$\rho_v = \underline{\underline{\varepsilon_o(5\sin\phi\csc^2\theta\cos\theta - 20\cos\theta\sin\phi) \ \text{ C/m}^3}}$$

**Prob. 4.53**

(a)

$$-\boldsymbol{E} = \nabla V = \frac{\partial V}{\partial \rho}\boldsymbol{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\boldsymbol{a}_\phi + \frac{\partial V}{\partial z}\boldsymbol{a}_z = e^{-z}\sin\phi\boldsymbol{a}_\rho + e^{-z}\cos\phi\boldsymbol{a}_\phi - \rho e^{-z}\sin\phi\boldsymbol{a}_z$$

$$\boldsymbol{E} = -e^{-z}\sin\phi\boldsymbol{a}_\rho - e^{-z}\cos\phi\boldsymbol{a}_\phi + \rho e^{-z}\sin\phi\boldsymbol{a}_z$$

(b)

$$\nabla\times\boldsymbol{E} = \left[\frac{1}{\rho}\frac{\partial E_z}{\partial \phi} - \frac{\partial E_\phi}{\partial z}\right]\boldsymbol{a}_\rho + \left[\frac{\partial E_\rho}{\partial z} - \frac{\partial E_z}{\partial \rho}\right]\boldsymbol{a}_\phi + \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho E_\phi) - \frac{\partial E_\rho}{\partial \phi}\right]\boldsymbol{a}_z$$

$$= \left[\frac{1}{\rho}\rho e^{-z}\cos\phi - e^{-z}\cos\phi\right]\boldsymbol{a}_\rho + \left[e^{-z}\sin\phi - e^{-z}\sin\phi\right]\boldsymbol{a}_\phi + \frac{1}{\rho}\left[-e^{-z}\cos\phi + e^{-z}\cos\phi\right]\boldsymbol{a}_z = \boldsymbol{0}$$

since each component is zero. Alternatively,

$\nabla\times\boldsymbol{E} = \nabla\times\nabla V = 0.$

**Prob. 4.54**

$$V = -r^{-3}\sin\theta\cos\phi$$

$$-\boldsymbol{E} = \nabla V = \frac{\partial V}{\partial r}\boldsymbol{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\boldsymbol{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\boldsymbol{a}_\phi$$

$$= -3r^{-4}\sin\theta\cos\phi\boldsymbol{a}_r + r^{-4}\cos\theta\cos\phi\boldsymbol{a}_\theta + \frac{r^{-4}\sin\theta}{\sin\theta}(-\sin\phi)\boldsymbol{a}_\phi$$

$$\boldsymbol{E} = \frac{3}{r^4}\sin\theta\cos\phi\boldsymbol{a}_r - \frac{1}{r^4}\cos\theta\cos\phi\boldsymbol{a}_\theta + \frac{\sin\phi}{r^4}\boldsymbol{a}_\phi$$

At $(1, 30^o, 60^o), r = 1, \theta = 30^o, \phi = 60^o$

$$\boldsymbol{E} = 3\sin 30^o\cos 60^o\boldsymbol{a}_r - \cos 30^o\cos 60^o\boldsymbol{a}_\theta + \sin 60^o\boldsymbol{a}_\phi$$

$$= 0.75\boldsymbol{a}_r - 0.433\boldsymbol{a}_\theta + 0.866\boldsymbol{a}_\phi$$

$$\boldsymbol{D} = \varepsilon_o\boldsymbol{E} = \frac{10^{-9}}{36\pi}(0.75\boldsymbol{a}_r - 0.433\boldsymbol{a}_\theta + 0.866\boldsymbol{a}_\phi)$$

$$= 6.635\boldsymbol{a}_r - 3.829\boldsymbol{a}_\theta + 7.657\boldsymbol{a}_\phi \text{ pC/m}^2$$

**Prob. 4.55**

For $a < r < b$, we apply Gauss's law.

$$\oint_S D \cdot dS = Q_{enc} = Q$$

$$D_r(4\pi r^2) = Q \longrightarrow E_r = \frac{D_r}{\varepsilon_o} = \frac{Q}{4\pi\varepsilon_o r^2}$$

$$V_{ab} = -\int_a^b \boldsymbol{E} \cdot d\boldsymbol{l} = -\frac{Q}{4\pi\varepsilon_o}\int_a^b \frac{1}{r^2}dr = \frac{Q}{4\pi\varepsilon_o}\frac{1}{r}\Big|_a^b = -\frac{Q}{4\pi\varepsilon_o}\left(\frac{1}{a} - \frac{1}{b}\right)$$

**Prob. 4.56**

$$V = \int_S \frac{\rho_s dS}{4\pi\varepsilon_o r} = \frac{\rho_s}{4\pi\varepsilon_o}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2} \frac{r^2 \sin\theta d\theta d\phi}{r} = \frac{\rho_s}{4\pi\varepsilon_o r}(2\pi)\int_0^{\pi/2} \sin\theta d\theta$$

$$= \frac{\rho_s}{2\varepsilon_o r}\left(-\cos\theta\Big|_0^{\pi/2}\right)\Big|_{r=a}$$

$$V = \frac{\rho_s}{2\varepsilon_o a}$$

**Prob. 4.57**

$$\nabla \times \boldsymbol{E} = 0 \longrightarrow \nabla \times \boldsymbol{D} = 0$$

$$\nabla \times \boldsymbol{D} = \left[\frac{1}{\rho}\frac{\partial D_z}{\partial \phi} - \frac{\partial D_\phi}{\partial z}\right]\boldsymbol{a}_\rho + \left[\frac{\partial D_\rho}{\partial z} - \frac{\partial D_z}{\partial \rho}\right]\boldsymbol{a}_\phi + \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho D_\phi) - \frac{\partial D_\rho}{\partial \phi}\right]\boldsymbol{a}_z$$

$$= 0\boldsymbol{a}_\rho - 0\boldsymbol{a}_\phi - \frac{1}{\rho}2\rho\cos\phi\boldsymbol{a}_z \neq 0$$

Hence ***D* is not** a genuine EM field.

$$\psi = \int_S \boldsymbol{D} \cdot d\boldsymbol{S} = \int_{\phi=0}^{\pi/4}\int_{z=0}^{1} 2\rho\sin\phi\rho d\phi dz = 2\int_0^{\pi/4}\sin\phi d\phi\int_0^1 dz \rho^2 \Big|_{\rho=1}$$

$$= -2\cos\phi\Big|_0^{\pi/4}(1)(1)^2 = -2(\cos\pi/4 - 1) = \underline{\underline{0.5858 \text{ C}}}$$

**Prob. 4.58**

(a)

$m\dfrac{d^2y}{dt^2} = eE$; divide by $m$, and integrate once, one obtains:

$$u = \frac{dy}{dt} = \frac{eEt}{m} + c_0$$

$$y = \frac{eEt^2}{2m} + c_0t + c_1 \qquad (1)$$

"From rest" implies $c_1 = 0 = c_0$

At $t = t_0$, $y = d$, $E = \dfrac{V}{d}$ or $V = Ed$.

Substituting this in (1) yields:

$$t^2 = \frac{2md}{eE}$$

Hence:

$$u = \frac{eE}{m}\sqrt{\frac{2md}{eE}} = \sqrt{\frac{2eEd}{m}} = \sqrt{\frac{2eV}{m}}$$

that is, $u \ \alpha \ \sqrt{V}$

or $u = k\sqrt{V}$

(b)

$$k = \sqrt{\frac{2e}{m}} = \sqrt{\frac{2\,(1.603)\,10^{-19}}{9.1066\ \ (10^{-31})}}$$

$$= \underline{\underline{5.933\times10^{5}}}$$

(c )

$$V = \frac{u^2\ m}{2e} = \frac{9(10^{16})\ \frac{1}{100}}{2(1.76)\ (10^{11})} = \underline{\underline{2.557\ \ k\ \text{V}}}$$

**Prob. 4.59**

(a)

This is similar to Example 4.3.

$$u_y = \frac{eEt}{m}, \quad u_x = u_0$$

$$y = \frac{eEt^2}{2m}, \quad x = u_0 t$$

$$t = \frac{x}{u_0} = \frac{10(10^{-2})}{10^7} = 10 \text{ ns}$$

Since $x = 10\,\text{cm}$ when $y = 1\text{cm}$,

$$E = \frac{2my}{et^2} = \frac{2(10^{-2})}{1.76(10^{11})(10^{-16})} = 1.136 \ k\text{V/m}$$

$$\boldsymbol{E} = -1.136\, \boldsymbol{a}_y \ \text{kV/m}$$

(b)

$$u_x = u_0 = 10^7,$$

$$u_y = \frac{2000}{1.76}(1.76)10^{11}(10^{-8}) = 2(10^6)$$

$$\boldsymbol{u} = (\boldsymbol{a}_x + 0.2\boldsymbol{a}_y)(10^7) \ \text{m/s}$$

**Prob. 4.60**

$$V = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2} = \frac{k\cos\theta}{r^2}$$

$At$ (0,1 nm), $\theta = 0$, $r = 1$ nm, $V = 9$;

that is, $9 = \frac{k(1)}{1(10^{-18})}$, $\therefore\ k = 9(10^{-18})$

$$V = 9(10^{-18})\frac{\cos\theta}{r^2}$$

At (1,1) nm, $r = \sqrt{2}$ nm, $\theta = 45°$,

$$V = \frac{9(10^{-18})\cos 45°}{10^{-18}(\sqrt{2})^2} = \frac{9}{2\sqrt{2}} = 3.182 \ \text{V}$$

**Prob. 4.61**

The dipole is oriented along $y-$axis.

$$V = \frac{\boldsymbol{p}\bullet\boldsymbol{r}}{4\pi\varepsilon_0 r^2};\ \boldsymbol{p}\bullet\boldsymbol{r} = Q\,d\ \boldsymbol{a}_y\bullet\boldsymbol{a}_r\ = Qd\sin\theta\sin\phi$$

$$V = \frac{Qd\sin\theta\sin\phi}{4\pi\,\varepsilon_0\,r^2}$$

$$\boldsymbol{E} = -\nabla V = -\frac{\partial V}{\partial r}\boldsymbol{a}_r - \frac{1}{r}\frac{\partial V}{\partial\theta}\boldsymbol{a}_\theta - \frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi}\boldsymbol{a}_\phi$$

$$= \frac{Q\,d}{4\pi\,\varepsilon_0}\left\{\frac{2\sin\theta\sin\phi}{r^3}\boldsymbol{a}_r - \frac{\cos\theta\sin\phi}{r^3}\boldsymbol{a}_\theta - \frac{\cos\phi}{r^3}\boldsymbol{a}_\phi\right\}$$

$$\boldsymbol{E} = \frac{Q\,d}{4\pi\,\varepsilon_0 r^3}(2\sin\theta\sin\phi\,\boldsymbol{a}_r - \cos\theta\sin\phi\,\boldsymbol{a}_\theta - \cos\phi\,\boldsymbol{a}_\phi)$$

**Prob. 4.62**

Using eq. (4.81),

$$V = \frac{\boldsymbol{p}\bullet(\boldsymbol{r}-\boldsymbol{r}')}{4\pi\varepsilon_o\,|\boldsymbol{r}-\boldsymbol{r}'|^3}$$

$$\boldsymbol{r}-\boldsymbol{r}' = (4,0,1)-(2,3,-1) = (2,-3,2)$$

$$|\boldsymbol{r}-\boldsymbol{r}'| = \sqrt{4+9+4} = \sqrt{17}$$

$$\boldsymbol{p}\bullet(\boldsymbol{r}-\boldsymbol{r}') = (2,6,-4)\bullet(2,-3,2) = 4-18-8 = -22$$

$$V = \frac{-22\times10^{-6}}{4\pi\times\frac{10^{-9}}{36\pi}\times17^{3/2}} = -\frac{198}{70.093}\ \text{kV} = \underline{\underline{-2.825\ \text{kV}}}$$

**Prob. 4.63**

$$\boldsymbol{E} = k(2\cos\theta\boldsymbol{a}_r + \sin\theta\boldsymbol{a}_\theta)$$

$$\begin{bmatrix}E_\rho\\E_\phi\\E_z\end{bmatrix} = \begin{bmatrix}\sin\theta & \cos\theta & 0\\0 & 0 & 1\\\cos\theta & -\sin\theta & 0\end{bmatrix}\begin{bmatrix}2k\cos\theta\\k\sin\theta\\0\end{bmatrix}$$

$$E_z = 2k\cos^2\theta - k\sin^2\theta = 2k\cos^2\theta - k(1-\cos^2\theta) = 3k\cos^2\theta - k$$

Setting this to zero gives

$$3\cos^2\theta = 1 \quad\rightarrow\quad \cos\theta = \pm\frac{1}{\sqrt{3}} = \pm0.5773 \quad\rightarrow\quad \underline{\underline{\theta = 54.74^o,\ 125.26^o}}$$

**Prob. 4.64**

$$W = W_1 + W_2 = 0 + Q_2V_{21} = Q_2 \frac{Q_1}{4\pi\varepsilon_o \,|\,(2,0,0)-(0,0,1)\,|}$$

$$= \frac{40\times10^{-9}\times(-50)\times10^{-9}}{4\pi\times\frac{10^{-9}}{36\pi}\,|\,(2,0,-1)\,|} = \frac{40\times9\times(-50)\times10^{-9}}{\sqrt{4+1}}$$

$$= \underline{\underline{-8.05\ \mu\text{J}}}$$

**Prob. 4.65**

$$\boldsymbol{E} = -\nabla V = -\frac{\partial V}{\partial x}\boldsymbol{a}_x - \frac{\partial V}{\partial y}\boldsymbol{a}_y = -4x\boldsymbol{a}_x - 12y\boldsymbol{a}_y \quad \text{V/m}$$

$$W = \frac{1}{2}\varepsilon_o\int_v |\boldsymbol{E}|^2\,dv = \frac{1}{2}\varepsilon_o\int_{z=-1}^{1}\int_{y=-1}^{1}\int_{x=-1}^{1}(16x^2+144y^2)\,dxdydz$$

$$= \frac{1}{2}\varepsilon_o\left[16(4)\frac{x^3}{3}\Bigg|_{-1}^{1} + 144(4)\frac{y^3}{3}\Bigg|_{-1}^{1}\right] = \frac{1}{2}\frac{10^{-9}}{36\pi}(160)(4)\frac{1}{3}(1+1)$$

$$= \underline{\underline{1.886\ \text{nJ}}}$$

**Prob. 4.66**

Given that $\boldsymbol{E} = 2r\sin\theta\cos\phi\boldsymbol{a}_r + r\cos\theta\cos\phi\boldsymbol{a}_\theta - r\sin\phi\boldsymbol{a}_\phi$

$$E^2 = 4r^2\sin^2\theta\cos^2\phi + r^2\cos^2\theta\cos^2\phi + r^2\sin^2\phi$$

$$= r^2\cos^2\phi\left(4\sin^2\theta + \cos^2\theta\right) + r^2\sin^2\phi$$

$$= r^2\cos^2\phi + 3r^2\cos^2\phi\sin^2\theta + r^2\sin^2\phi$$

$$= r^2(1 + 3\cos^2\phi\sin^2\theta)$$

$$W = \frac{\varepsilon}{2}\iiint E^2 r^2\sin\theta\,drd\theta d\phi$$

$$= \frac{\varepsilon}{2}\int_0^2 r^4 dr\int_{\theta=0}^{\pi}\int_{\phi}^{\pi}(1+3\cos^2\phi\sin^2\theta)\sin\theta\,d\theta d\phi$$

$$= \frac{16\varepsilon}{5}\int_0^{\pi}(\pi\sin\theta + \frac{3\pi}{2}\sin^2\theta)\,d\theta$$

$$= \frac{16}{5}x\frac{10^{-9}}{36\pi}(4\pi) = \frac{16}{45}\ \text{nJ} = \underline{\underline{0.36\ \text{nJ}}}$$

**Prob. 4.67**

*Method 1:*

$$W = \frac{1}{2}\int_S \rho_s V dS = \frac{V}{2}\int_S \rho_s dS = \frac{1}{2}QV$$

But $\quad V = \frac{Q}{4\pi\varepsilon_o a}$

$$W = \frac{Q^2}{8\pi\varepsilon_o a}$$

*Method 2:*

$$W = \frac{1}{2}\int_v \mathbf{D}\cdot\mathbf{E}dv = \frac{1}{2}\varepsilon_o\int_v E^2 dv$$

$$= \frac{1}{2}\varepsilon_o\iiint\left(\frac{Q}{4\pi\varepsilon_o r^2}\right)^2 r^2 \sin\theta d\theta dr d\phi$$

$$= \frac{1}{2}\varepsilon_o\int_0^{2\pi} d\phi\int_0^{\pi}\sin\theta d\theta\int_{r=a}^{\infty}\frac{Q^2}{16\pi^2\varepsilon_o^2 r^2}dr = \frac{\varepsilon_o}{2}(2\pi)(2)\frac{Q^2}{16\pi^2\varepsilon_o^2 a}$$

$$W = \frac{Q^2}{8\pi\varepsilon_o a}$$

**Prob. 68**

$$W = \frac{1}{2}\int_v \mathbf{D}\bullet\mathbf{E}dv = \frac{1}{2}\varepsilon_o\int_v |\mathbf{E}|^2\, dv = \frac{1}{2}\varepsilon_o\int_v\left(y^4 + 4x^2y^2 + 16z^2\right)dxdydz$$

$$\frac{2W}{\varepsilon_o} = \int_0^2 dx\int_0^4 dz\int_{-1}^1 y^4 dy + 4\int_0^2 x^2 dx\int_0^4 dz\int_{-1}^1 y^2 dy + 16\int_0^2 dx\int_0^4 z^2 dz\int_{-1}^1 dy$$

$$= 2(4)\left(2\frac{y^5}{5}\Big|_0^1\right) + 4(4)\left(\frac{x^3}{3}\Big|_0^2\right)\left(2\frac{y^3}{3}\Big|_0^1\right) + 16(2)(2)\left(\frac{z^3}{3}\Big|_0^4\right)$$

$$= \frac{16}{5} + \frac{256}{9} + \frac{64\times 64}{3} = 1396.98$$

$$W = \frac{1}{2}\times\frac{10^{-9}}{36\pi}\times 1396.98 = \underline{\underline{6.176 \text{ nJ}}}$$

**Prob. 4.69**

(a)
$$\boldsymbol{E} = -\nabla V = -\left( \frac{\partial V}{\partial \rho}\boldsymbol{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\boldsymbol{a}_\phi + \frac{\partial V}{\partial z}\boldsymbol{a}_z \right)$$

$$= \underline{\underline{-e^{-z}\sin\phi\boldsymbol{a}_\rho - e^{-z}\cos\phi\boldsymbol{a}_\phi + \rho e^{-z}\sin\phi\boldsymbol{a}_z}}$$

$$\boldsymbol{E}\cdot\boldsymbol{E} = |\boldsymbol{E}|^2 = e^{-2z}\sin^2\phi + e^{-2z}\cos^2\phi + \rho^2 e^{-2z}\sin^2\phi = e^{-2z} + \rho^2 e^{-2z}\sin^2\phi$$

$$W = \frac{1}{2}\varepsilon_o \int_v |\boldsymbol{E}|^2 dv$$

$$\frac{2W}{\varepsilon_o} = \int_v |\boldsymbol{E}|^2 dv = \iiint \left( e^{-2z} + \rho^2 e^{-2z}\sin^2\phi \right)\rho d\rho d\phi dz$$

(b)
$$= \int_0^1 \rho d\rho \int_0^{2\pi} d\phi \int_0^2 e^{-2z}dz + \int_0^1 \rho^3 d\rho \int_0^{2\pi} \sin^2\phi d\phi \int_0^2 e^{-2z}dz$$

$$= \frac{\rho^2}{2}\Big|_0^1 (2\pi)\left( \frac{e^{-2z}}{-2}\Big|_0^2 \right) + \frac{\rho^4}{4}\Big|_0^1 \left( \frac{\phi}{2} - \frac{\sin 2\phi}{4} \right)\Big|_0^{2\pi} \left( \frac{e^{-2z}}{-2}\Big|_0^2 \right)$$

$$= \pi(-1/2)(e^{-4}-1) + \frac{1}{4}(\pi - 0)(-1/2)(e^{-4}-1) = \frac{5\pi}{8}(1-e^{-4}) = 1.9275$$

$$W = \frac{1}{2}\varepsilon_o(1.9275) = \frac{1}{2}\times\frac{10^{-9}}{36\pi}(1.9275) = \underline{\underline{8.512 \text{ pJ}}}$$

# CHAPTER 5

**P. E. 5.1** $d\mathbf{S} = \rho d\phi dz \mathbf{a}_\rho$

$$I = \int J \bullet dS = \int_{\phi=0}^{2\pi} \int_{z=1}^{5} 10z \sin^2 \phi \rho dz d\phi|_{\rho=2} =$$

$$10(2)\frac{z^2}{2}\Big|_1^5 \int_0^{2\pi} \frac{1}{2}(1-\cos 2\phi)d\phi = 10(5^2-1^2)\frac{1}{2}\left(\phi - \frac{1}{2}\sin 2\phi\right)\Big|_0^{2\pi} = 10(24)\pi = 240\pi$$

I = 754 A

**P. E. 5.2**

$$I = \rho_s wu = 0.5\times10^{-6}\times0.1\times10 = 0.5\mu \text{ A}$$

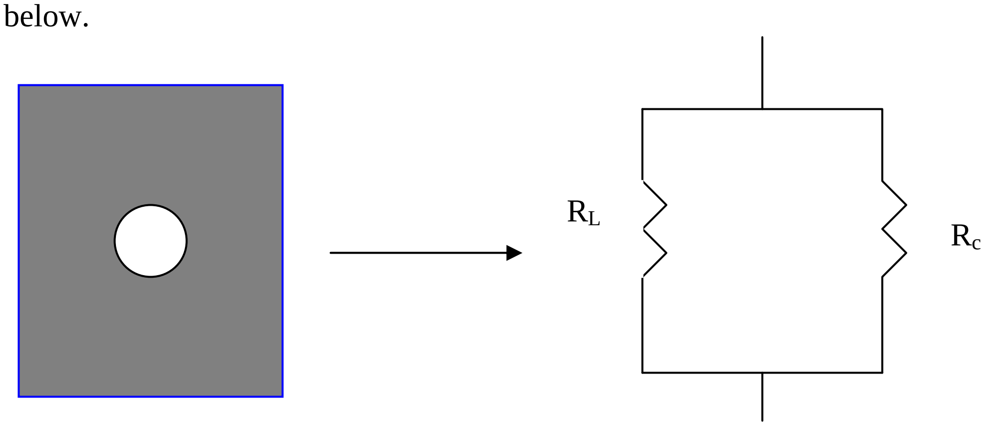
$$V = IR = 0.5\times10^{-6}\times10^{14} = \underline{50 \text{ MV}}$$

**P. E. 5.3** $\sigma = 5.8\times10^7$ S/m

$$J = \sigma E \longrightarrow E = \frac{J}{\sigma} = \frac{8\times10^6}{5.8\times10^7} = \underline{0.138 \text{ V/m}}$$

$$J = \rho_v u \longrightarrow u = \frac{J}{\rho_v} = \frac{8\times10^6}{1.81\times10^{10}} = \underline{4.42\times10^{-4} \text{ m/s}}$$

**P. E. 5.4** The composite bar can be modeled as a parallel combination of resistors as shown below.



$$R_L = \frac{l}{\sigma_L S_L}, \quad l = 4\text{m}, S_L = d^2 - \pi r^2 = 9 - \frac{\pi}{4}\ \text{cm}^2,\ \sigma_L = 5\times10^6\ \text{S/m}$$

For the lead,

$$\text{R}_L = \frac{4}{(5\times10^6\ )\left(9-\frac{\pi}{4}\right)\times10^{-4}} = 973.8\mu\Omega$$

For copper, $R_c = \dfrac{l}{\sigma_c S_c}, \quad S_c = \pi r^2 = \dfrac{\pi}{4}, \sigma_c = 5.8\times10^7\ \text{S/m}\ \ \text{cm}^2$

$$R_c = \frac{4}{5.8\times10^7\times\frac{\pi}{4}\times10^{-4}} = 878.5\ \mu\Omega$$

$$R = \frac{R_L R_c}{R_L + R_c} = \frac{973.8\times878.5}{973.8+878.5} = \underline{\underline{461.8\mu\Omega}}$$

**P. E. 5.5** $\rho_{Ps} = \boldsymbol{P} \bullet \boldsymbol{a}_s = ax^2 + b$

$$\rho_{ps}\Big|_{x=0} = \boldsymbol{P}\bullet(-\boldsymbol{a}_x)\Big|_{x=0} = -(ax^2+b)\Big|_{x=0} = b$$

$$\rho_{ps}\Big|_{x=L} = \boldsymbol{P}\bullet\boldsymbol{a}_x\ \Big|_{x=L} = (ax^2+b)\Big|_{x=L} = \underline{\underline{aL^2+b}}$$

$$Q_s = \int \rho_{ps} dS = -bA + (aL^2+b)A = AaL^2$$

$$\rho_{pv} = -\nabla\bullet\boldsymbol{P} = -\frac{d}{dx}(ax^2+b) = -2ax$$

$$\rho_{pv}\Big|_{x=0} = \underline{\underline{0}}, \qquad \rho_{pv}\Big|_{x=L} = \underline{\underline{-2aL}}$$

$$Q_v = \int \rho_{pv} dv = \int_0^L (-2ax)A dx = -AaL^2$$

Hence,

$$Q_T = Q_v + Q_s = -AaL^2 + AaL^2 = 0$$

**P. E. 5.6**

$$\mathbf{E} = \frac{V}{d}\mathbf{a}_x = \frac{10^3}{2x10^{-3}}\mathbf{a}_x = 500\mathbf{a}_x \text{ kV/m}$$

$$\mathbf{P} = \chi_e\varepsilon_o\mathbf{E} = (2.55-1)x\frac{10^{-9}}{36\pi}x0.5x10^6\,\mathbf{a}_x = \underline{\underline{6.853\mathbf{a}_x\mu C/\text{m}^2}}$$

$$\rho_{ps} = \mathbf{P}\bullet\mathbf{a}_x = \underline{\underline{6.853\mu\text{C}/\text{m}^2}}$$

**P. E. 5.7** (a) Since $P = \varepsilon_o\chi_e E, \quad P_x = \varepsilon_o\chi_e E_x$

$$\chi_e = \frac{P_x}{\varepsilon_o E_x} = \frac{3x10^{-9}}{10\pi}\frac{1}{5}x36\pi x10^9 = \underline{\underline{2.16}}$$

(b) $E = \dfrac{P}{\chi_e\varepsilon_o} = \dfrac{36\pi\times10^9}{2.16}\dfrac{1}{10\pi}(3,-1,4)10^{-9} = \underline{\underline{5a_x - 1.67a_y + 6.67a_z}}$ V/m

(c )

$$D = \varepsilon_o\varepsilon_r E = \frac{\varepsilon_r P}{\chi_e} = \frac{3.16}{2.16}\left(\frac{1}{10\pi}\right)(3,-1,4) \quad \text{nC/m}^2 = \underline{\underline{139.7a_x}}\ \underline{\underline{-46.6a_y + 186.3a_z}} \quad \text{pC/m}^2$$

**P. E. 5.8** From Example 5.8,

$$F = \frac{\rho_s^2 S}{2\varepsilon_o} \longrightarrow \rho_s^2 = \frac{2\varepsilon_o F}{S}$$

But $\rho_s = \varepsilon_o E = \varepsilon_o\dfrac{V_d}{d}$. Hence

$$\rho_s^2 = \frac{2\varepsilon_o F}{S} = \frac{\varepsilon_o^2 V_d^2}{d^2} \longrightarrow V_d^2 = \frac{2Fd^2}{\varepsilon_o S}$$

i.e.

$$V_d = V_1 - V_2 = \sqrt{\frac{2Fd^2}{\varepsilon_o S}}$$

as required.

**P. E. 5.9** (a) Since $a_n = a_x$,

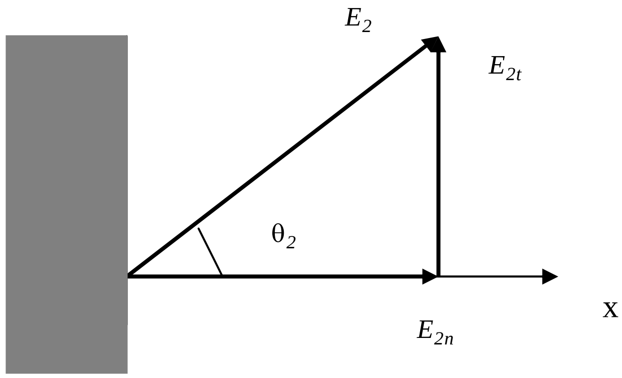
$$D_{1n} = 12a_x, \quad D_{1t} = -10a_x + 4a_z, \quad D_{2n} = D_{1n} = 12a_x$$

$$E_{2t} = E_{1t} \longrightarrow D_{2t} = \frac{\varepsilon_2 D_{1t}}{\varepsilon_1} = \frac{1}{2.5}(-10a_y + 4a_z) = -4a_y + 1.6a_z$$

$$D_2 = D_{2n} + D_{2t} = \underline{\underline{12a_x - 4a_y + 1.6a_z}} \text{ nC/m}^2.$$

$$\tan\theta_2 = \frac{D_{2t}}{D_{2n}} = \frac{\sqrt{(-4)^2 + (1.6)^2}}{12} = 0.359 \quad \longrightarrow \quad \underline{\underline{\theta_2 = 19.75^o}}$$

(b ) $E_{1t} = E_{2t} = E_2 \sin\theta_2 = 12\sin 60^o = 10.392$




$$E_{1n} = \frac{\varepsilon_{r2}}{\varepsilon_{r1}} E_{2n} = \frac{1}{2.5} 12\cos 60^o = 2.4$$

$$E_1 = \sqrt{E_{1t}^{\ 2} + E_{1n}^{\ 2}} = \underline{\underline{10.67}}$$

$$\tan\theta_1 = \frac{E_{1t}}{E_{1n}} = \frac{E_{2t}}{(\varepsilon_{r1}/\varepsilon_{r2})E_{2n}} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}}\frac{E_{2t}}{E_{2n}} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}}\tan\theta_2 = \frac{2.5}{1}\tan 60^o = 4.33 \quad \longrightarrow \quad \underline{\underline{\theta_1 = 77^o}}$$

Note that $\theta_1 > \theta_2$.

**P. E. 5.10**

$$D = \varepsilon_o E = \frac{10^{-9}}{36\pi}(60,20,-30)x10^{-3} = \underline{\underline{0.531a_x + 0.177a_y - 0.265a_z}} \text{ pC/m}^2$$

$$\rho_s = D_n = |D| = \frac{10^{-9}}{36\pi}(10)\sqrt{36+4+9}(10^{-3}) = \underline{\underline{0.619}} \text{ pC/m}^2$$

**Prob. 5.1**

$$I = \int \boldsymbol{J} \bullet d\boldsymbol{S}, \qquad d\boldsymbol{S} = dydz\boldsymbol{a}_x$$

$$I = \iint e^{-x}\cos(4y)dydz \Big|_{x=2} = e^{-2}\int_0^{\pi/3}\cos(4y)dy\int_0^4 dz$$

$$= 4e^{-2}\left(\frac{\sin 4y}{4}\Big|_0^{\pi/3}\right) = e^{-2}\left(\sin(\frac{4\pi}{3}) - 0\right) = \underline{\underline{-0.1172 \text{ A}}}$$

**Prob. 5.2**

*Method 1:*

$$I = \int \boldsymbol{J} \bullet d\boldsymbol{S} = \iint \frac{10}{r}e^{-10^3 t}r^2 \sin d\theta d\phi \Big|_{t = 2ms, r = 4m}$$

$$= 10(4)e^{-10^3 \times 2\times 10^{-3}}\int_{\theta=0}^{\pi}\sin\theta d\theta\int_{\phi=0}^{2\pi}d\phi = 40e^{-2}(2)(2\pi) = 160\pi e^{-2}$$

$$= \underline{\underline{68.03 \text{ A}}}$$

*Method 2:*

$$I = \int \boldsymbol{J} \bullet d\boldsymbol{S} = \iint \frac{10}{r}e^{-10^3 t}dS = \frac{10}{r}e^{-10^3 t}(4\pi r^2)$$

since r is constant on the surface.

$$\text{I}=40\pi r e^{-2} = 160\pi e^{-2} \quad = \underline{\underline{68.03 \text{ A}}}$$

**Prob. 5.3**

$$I = \int \boldsymbol{J}\cdot d\boldsymbol{S} = \iint \frac{10}{\rho}\sin\phi\rho d\phi dz = 10\int_0^5 dz\int_0^{\pi}\sin\phi d\phi$$

$$= 10(5)(-\cos\phi)\Big|_0^{\pi} = \underline{\underline{100 \text{ A}}}$$

**Prob. 5.4**

$$I = \int \boldsymbol{J} \bullet d\boldsymbol{S} \qquad = 5\int_{\rho=0}^{a}\int_{\phi=0}^{2\pi}e^{-10\rho}\rho d\phi d\rho = 5\int_0^{2\pi}d\phi\int_{\rho=0}^{a}\rho e^{-10\rho}d\rho$$

$$= 5(2\pi)\left(\frac{e^{-10\rho}}{100}(-10\rho - 1)\right)\Big|_0^a = \frac{10\pi}{100}\left[e^{-10a}(-10a-1) - 1(-0-1)\right], \qquad a = 0.004$$

$$= \frac{\pi}{10}\left[e^{-0.04}(-0.04-1)+1\right] = \frac{\pi}{10}(0.00078) = \underline{\underline{244.7\ \mu\text{A}}}$$

**Prob. 5.5**

$$I=\int_S \mathbf{J}\bullet d\mathbf{S},\qquad d\mathbf{S}=r^2\sin\theta d\theta d\phi \mathbf{a}_r$$

$$I=\int_{\phi=0}^{2\pi}\int_{\theta=\pi/4}^{\pi/2}\frac{20\cos\theta}{r+3}r^2\sin\theta d\theta d\phi\Bigg|_{r=3}=\frac{20(9)}{6}\int_0^{2\pi}d\phi\int_{\pi/4}^{\pi/2}\cos\theta\sin\theta d\theta$$

$$=30(2\pi)\int_{\pi/4}^{\pi/2}\cos\theta d(-\cos\theta)=-60\pi\frac{\cos^2\theta}{2}\Bigg|_{\pi/4}^{\pi/2}=-60\pi(0-\cos^2(\pi/4))=-60\pi(-1/2)=30\pi$$

$\underline{\underline{I=94.2\text{ A}}}$

**Prob. 5.6**

$$R=\frac{l}{\sigma S}\quad\longrightarrow\quad \sigma=\frac{l}{RS}=\frac{2\times10^{-2}}{10^6(\pi)(4\times10^{-3})^2}=\underline{\underline{3.978\times10^{-4}\text{ S/m}}}$$

**Prob. 5.7** (a) $R=\frac{l}{\sigma S}=\frac{8\times10^{-2}}{3\times10^4\pi(25)10^{-6}}=\frac{8}{75\pi}=\underline{\underline{33.95\text{m}\Omega}}$

(b) $I=V/R=9\times\frac{75\pi}{8}=\underline{\underline{265.1}}\text{ A}$

(c ) P = IV = $\underline{\underline{2.386\text{ kW}}}$

**Prob. 5.8**

(a) $E=\frac{V}{l}=\frac{9}{100}=\underline{\underline{90\text{ mV/m}}}$

(b) $R=\frac{V}{I}=\frac{9}{0.3}=30\ \Omega$

$$R=\frac{l}{\sigma S}\quad\longrightarrow\quad \sigma=\frac{l}{RS}=\frac{100}{30(\pi\times2^2)10^{-6}}=\underline{\underline{2.653\times10^5\text{ S/m}}}$$

**Prob. 5.9**

If R and S are the same,

$$R_1 = \frac{\ell_1}{\sigma_1 S} = R_2 = \frac{\ell_2}{\sigma_2 S} \longrightarrow \ell_1 = \ell_2 \frac{\sigma_1}{\sigma_2}$$

If 1 corresponds to copper and 2 to silver,

$\sigma_1 = 5.8\times10^7$ S/m, $\sigma_2 = 6.1\times10^7$ S/m

$$\ell_1 = \ell_2 \frac{5.8}{6.1} = 0.951\ell_2$$

That is, the copper wire is shorter than silver wire or the silver wire is longer.

**Prob. 5.10**

$$R = \frac{\ell}{\sigma S} = \frac{V}{I} \quad \to \quad \sigma = \frac{I\ell}{SV}, \quad S = \pi r^2$$

$$\sigma = \frac{2(5)}{\pi(2^2\times10^{-6})(12)} = \frac{10^7}{48\pi} = \underline{\underline{6.635\times10^4 \text{ S/m}}}$$

**Prob. 5.11** (a) $S_i = \pi r_i^2 = \pi(1.5)^2 x10^{-4} = 7.068x10^{-4}$

$$S_o = \pi(r_o^2 - r_i^2) = \pi(4-2.25)\times10^{-4} = 5.498\times10^{-4}$$

$$R_I = \frac{\rho_I l}{S_I} = \frac{11.8\times10^{-8}\times10}{7.068X10^{-4}} = 16.69\times10^{-4}$$

$$R_o = \frac{\rho_o l}{S_o} = \frac{1.77\times10^{-8}\times10}{5.498\times10^{-4}} = 3.219\times10^{-4}$$

$$R = R_i // R_o = \frac{R_i R_o}{R_i + R_o} = \frac{16.69\times3.219\times10^{-4}}{16.69+3.219} = \underline{\underline{0.27\text{m}\Omega}}$$

(b) $V = I_i R_i = I_o R_o \longrightarrow \frac{I_i}{I_o} = \frac{R_o}{R_i} = \frac{0.3219}{1.669} = 0.1929$

$$I_i + I_o = 1.1929 I_o = 60 \text{ A}$$

$I_o = \underline{\underline{50.3 \text{ A}}}$ (copper), $I_i = \underline{\underline{9.7 \text{ A}}}$ (steel)

Alternatively, using the principle of current division,

$$I_o = 60\frac{R_i}{R_i + R_o} = 50.3 \text{ A}$$

$$I_i = 60\frac{R_o}{R_i + R_o} = 9.7 \text{ A}$$

(c) $$R = \frac{10\times 1.77\times 10^{-8}}{\pi(2^2)\times 10^{-4}} = \underline{\underline{0.141\text{m}\Omega}}$$

**Prob. 5.12**

From eq. (5.16),

$$R_1 = \frac{\rho_1 \ell}{S_1} = \frac{\rho_1 \ell}{ab}, \qquad R_2 = \frac{\rho_2 \ell}{S_2} = \frac{\rho_2 \ell}{ac}$$

$$R = R_1 \| R_2 = \frac{R_1 R_2}{R_1 + R_2} = \frac{\frac{\rho_1 \ell}{ab}\frac{\rho_2 \ell}{ac}}{\frac{\rho_1 \ell}{ab} + \frac{\rho_2 \ell}{ac}} = \frac{\rho_1 \rho_2 \ell^2}{ac\rho_1 \ell + ab\rho_2 \ell}$$

$$R = \underline{\underline{\frac{\rho_1 \rho_2 \ell}{a(c\rho_1 + b\rho_2)}}}$$

**Prob. 5.13**

$$R = \frac{\ell}{\sigma S} = \frac{\ell}{\pi r^2 \sigma} = \frac{V}{I}$$

$$I = V\frac{\pi r^2 \sigma}{\ell} = \frac{12\pi(0.84\times 10^{-3})^2 \times 6.1\times 10^7}{12.4} = \underline{\underline{130.86 \text{ A}}}$$

**Prob. 5.14**

$$|\boldsymbol{P}| = n|\boldsymbol{p}| = nQd = 2ned = \chi_e \varepsilon_o E \qquad (Q = 2e)$$

$$\chi_e = \frac{2ned}{\varepsilon_o E} = \frac{2\times 5\times 10^{25}\times 1.602\times 10^{-19}\times 10^{-18}}{\frac{10^{-9}}{36\pi}\times 10^4} = 0.000182$$

$$\varepsilon_r = 1 + \chi_e = \underline{\underline{1.000182}}$$

**Prob. 5.15**

$$\boldsymbol{P} = \frac{\sum_{i=1}^{N} q_i \boldsymbol{d}_i}{v} = \frac{\sum_{i=1}^{N} \boldsymbol{p}_i}{v}$$

$$|\,\boldsymbol{P}\,| = \frac{N}{v}|\,\boldsymbol{p}\,| = 2\times10^{19}\times1.8\times10^{-27} = 3.6\times10^{-8}$$

$$\boldsymbol{P} = |\,\boldsymbol{P}\,|\,\boldsymbol{a}_x = \underline{\underline{3.6\times10^{-8}\,\boldsymbol{a}_x\ \text{C/m}^2}}$$

But $P = \chi_e \varepsilon_o E$ or $\chi_e = \frac{P}{\varepsilon_o E} = \frac{3.6\times10^{-8}}{(10^{-9}/36\pi)10^5} = 0.0407$

$$\varepsilon_r = 1 + \chi_e = \underline{\underline{1.0407}}$$

**Prob. 5.16**

$$\boldsymbol{E} = \frac{Q}{4\pi\varepsilon_o\varepsilon_r r^2}\boldsymbol{a}_r$$

$$\boldsymbol{P} = \chi_e\varepsilon_o\boldsymbol{E} = \frac{\chi_e Q}{4\pi\varepsilon_r r^2}\boldsymbol{a}_r = \frac{3(10)10^{-3}}{4\pi(4)1^2}\boldsymbol{a}_r = \underline{\underline{596.8\boldsymbol{a}_r\ \ \mu\text{C/m}^2}}$$

**Prob. 5.17**

$$\boldsymbol{P} = \chi_e\varepsilon_o\boldsymbol{E} \longrightarrow \boldsymbol{E} = \frac{\boldsymbol{P}}{\chi_e\varepsilon_o} = \frac{100\times10^{-9}}{2.5\frac{10^{-9}}{36\pi}(2)}\boldsymbol{a}_\rho = \underline{\underline{2.261\boldsymbol{a}_\rho\ \text{kV/m}}}$$

$$\boldsymbol{D} = \varepsilon_o\varepsilon_r\boldsymbol{E} = 3.5\times\frac{10^{-9}}{36\pi}2.261\times10^3\boldsymbol{a}_\rho = \underline{\underline{70\boldsymbol{a}_\rho\ \text{nC/m}^2}}$$

**Prob. 5.18**

$$\rho_{pv} = -\nabla\cdot\boldsymbol{P} = -\frac{1}{\rho}\frac{\partial}{\partial\rho}(p_o\rho^2) = \underline{\underline{-2p_o}}$$

The surface polarization charge is

$$\rho_{ps} = \boldsymbol{P}\cdot\boldsymbol{a}_\rho\Big|_{\rho=a} = \underline{\underline{p_o a}}$$

**Prob. 5.19**

(a)

$$Q_{s1} = \int_S \boldsymbol{P} \cdot d\boldsymbol{S}, d\boldsymbol{S} = r^2 \sin d\theta d\phi(-\boldsymbol{a}_r)$$

$$= -\iint 4r\, r^2 \sin\theta d\theta d\phi \Big|_{r=1.2cm}$$

$$= -4(1.2)^3 (10^{-6}) \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta (10^{-12})$$

$$= -6.912(2\pi)(2) \times 10^{-18}$$

$$= \underline{\underline{-86.86 \times 10^{-18} \text{ C}}}$$

(b)

$$Q_{s2} = \int_S \boldsymbol{P} \cdot d\boldsymbol{S}, \quad d\boldsymbol{S} = r^2 \sin d\theta d\phi(\boldsymbol{a}_r)$$

$$= -\iint 4r\, r^2 \sin\theta d\theta d\phi \Big|_{r=2.6cm}$$

$$= -4(2.6)^3 (10^{-6}) \int_0^{2\pi} d\phi \int \sin\theta d\theta (10^{-12})$$

$$= -4(2.6)^3 (2\pi)(2) \times 10^{-18} = \underline{\underline{883.5}} \underline{\underline{\times 10^{-18} \text{ C}}}$$

(c )

$$\rho_{pv} = -\nabla \cdot \boldsymbol{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}(4r^3) \text{ pC/m}^3 = -12\text{pC/m}^3$$

$$Q_v = \int_v \rho_{pv} dv = -12\iiint dv = -12\int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\phi \int_{1.2}^{2.6} r^2 dr (10^{-18})$$

$$= -12(2)(2\pi)\frac{r^3}{3}\Big|_{1.2}^{2.6}(10^{-18}) = -16\pi(2.6^3 - 1.2^3)(10^{-18})$$

$$= \underline{\underline{-796.61 \times 10^{-18} \text{ C}}}$$

**Prob. 5.20**

$$\boldsymbol{D} = \varepsilon_o \varepsilon_r \boldsymbol{E} = 2.1x\frac{10^{-9}}{36\pi}(6,12,-20) = \underline{\underline{0.1114\boldsymbol{a}_x + 0.2228\boldsymbol{a}_y - 0.3714\boldsymbol{a}_z \text{ nC/m}^2}}$$

$$\boldsymbol{P} = \chi_e \varepsilon_o \boldsymbol{E} = 1.1x\frac{10^{-9}}{36\pi}(6,12,-20) = \underline{\underline{0.0584\boldsymbol{a}_x + 0.1167\boldsymbol{a}_y - 0.1945\boldsymbol{a}_z \text{ nC/m}^2}}$$

**Prob. 5.21**

At $P(-2,5,3)$, $\quad x = -2, y = 5, z = 3$

$V = 4(4)(5)(27) = \underline{\underline{2.16 \text{ kV}}}$

$$\boldsymbol{E} = -\nabla V = -\left(\frac{\partial V}{\partial x}\boldsymbol{a}_x + \frac{\partial V}{\partial y}\boldsymbol{a}_y + \frac{\partial V}{\partial z}\boldsymbol{a}_z\right) = -\left(8xyz^3\boldsymbol{a}_x + 4x^2z^3\boldsymbol{a}_y + 12x^2yz^2\boldsymbol{a}_z\right)$$

At P,

$$\boldsymbol{E} = -(-16)(5)(27)\boldsymbol{a}_x - 4(4)(27)\boldsymbol{a}_y - 12(4)(5)(9)\boldsymbol{a}_z = \underline{\underline{2.16\boldsymbol{a}_x - 0.432\boldsymbol{a}_y - 2.16\boldsymbol{a}_z \text{ kV/m}}}$$

$$\boldsymbol{P} = \chi_e\varepsilon_o\boldsymbol{E} = \frac{7\times10^{-9}}{36\pi}(2160, -432, -2160) = \underline{\underline{133.69\boldsymbol{a}_x - 26.74\boldsymbol{a}_y - 133.69\boldsymbol{a}_z \text{ nC/m}^2}}$$

**Prob. 5.22**

**(a)** $\boldsymbol{E} = -\nabla V = -\left(\frac{\partial V}{\partial x}\boldsymbol{a}_x + \frac{\partial V}{\partial y}\boldsymbol{a}_y + \frac{\partial V}{\partial z}\boldsymbol{a}_z\right) = \underline{\underline{-20xyz\boldsymbol{a}_x - 10x^2z\boldsymbol{a}_y - 10(x^2y - z)\boldsymbol{a}_z \text{V/m}}}$

**(b)** $\boldsymbol{D} = \varepsilon\boldsymbol{E} = 5\varepsilon_o\boldsymbol{E} = \underline{\underline{-0.8842xyz\boldsymbol{a}_x - 0.4421x^2z\boldsymbol{a}_y - 0.4421(x^2y - z)\boldsymbol{a}_z \text{ nC/m}^2}}$

**(c )** $\boldsymbol{P} = \chi_e\varepsilon_o\boldsymbol{E} = 4\varepsilon_o\boldsymbol{E} = \underline{\underline{-0.7073xyz\boldsymbol{a}_x - 0.3537x^2z\boldsymbol{a}_y - 0.3537(x^2y - z)\boldsymbol{a}_z \text{ nC/m}^2}}$

**(d)** $\rho_v = -\varepsilon\nabla^2 V$

$$\nabla^2 V = \frac{\partial}{\partial x}(20xyz) + \frac{\partial}{\partial y}(10x^2z) + \frac{\partial}{\partial z}(10x^2y - 10z) = 20yz - 10$$

$$\rho_v = -5\varepsilon_o 10(2yz - 1) = \underline{\underline{-0.8854yz + 0.4427 \text{ nC/m}^3}}$$

**Prob. 5.23**
Using Gauss' law,

$$Q_{enc} = \oint \mathbf{D} \bullet d\mathbf{S}$$

For r < a, $Q_{enc} = 0 \quad \to \quad \mathbf{E} = \mathbf{0} = \mathbf{D} = \mathbf{P}$

For a < r <b, $Q_{enc} = 4C$

$$4 = D_r(4\pi r^2) \quad \to \quad \mathbf{D} = \frac{4}{4\pi r^2}\mathbf{a}_r = \frac{\mathbf{a}_r}{\pi r^2}$$

$$\mathbf{E} = \frac{\mathbf{D}}{\varepsilon} = \frac{\mathbf{D}}{2\varepsilon_o} = \frac{\mathbf{a}_r}{2\varepsilon_o \pi r^2}$$

$$\mathbf{P} = \chi_e \varepsilon_o \mathbf{E} = (1)\varepsilon_o \mathbf{E} = \frac{\mathbf{a}_r}{2\pi r^2}$$

For b < r < c, $Q_{enc} = 4 - 6 = -2C$

$$-2 = D_r(4\pi r^2) \quad \to \quad \mathbf{D} = \frac{-2}{4\pi r^2}\mathbf{a}_r = -\frac{\mathbf{a}_r}{2\pi r^2}$$

$$\mathbf{E} = \frac{\mathbf{D}}{\varepsilon} = \frac{\mathbf{D}}{5\varepsilon_o} = -\frac{\mathbf{a}_r}{10\varepsilon_o \pi r^2}$$

$$\mathbf{P} = \chi_e \varepsilon_o \mathbf{E} = (4)\varepsilon_o \mathbf{E} = -\frac{4\mathbf{a}_r}{10\pi r^2} = -\frac{2\mathbf{a}_r}{5\pi r^2}$$

For r >c, $Q_{enc} = 4 - 6 + 10 = 8C$

$$8 = D_r(4\pi r^2) \quad \to \quad \mathbf{D} = \frac{8}{4\pi r^2}\mathbf{a}_r = \frac{2\mathbf{a}_r}{\pi r^2}$$

$$\mathbf{E} = \frac{\mathbf{D}}{\varepsilon_o} = \frac{2\mathbf{a}_r}{\varepsilon_o \pi r^2}$$

$$\mathbf{P} = \chi_e \varepsilon_o \mathbf{E} = (0)\varepsilon_o \mathbf{E} = \mathbf{0}$$

**Prob. 5.24** (a) Applying Coulomb's law, we obtain the electric field intensity due to a point charge as

$$E_r = \begin{cases} \dfrac{D_r}{\varepsilon_o} = \dfrac{Q}{4\pi\varepsilon_o r^2}, & r > b \\ \dfrac{D_r}{\varepsilon} = \dfrac{Q}{4\pi\varepsilon r^2}, & a < r < b \end{cases}$$

$$P = \frac{\varepsilon_r - 1}{\varepsilon_r} D \qquad (= D - \varepsilon_O E)$$

Hence

$$P_r = \frac{\varepsilon_r - 1}{\varepsilon_r} \cdot \frac{Q}{4\pi r^2}, \qquad a < r < b$$

(b) $$\rho_{pv} = -\nabla \bullet \mathbf{P} = -\frac{1}{r^2}\frac{d}{dr}(r^2 P_r) = 0$$

(c )

$$\rho_{ps} = \mathbf{P} \bullet (-\mathbf{a}_r) = -\frac{Q}{4\pi a^2}(\frac{\varepsilon_r - 1}{\varepsilon_r}), \qquad r = a$$

$$\rho_{ps} = \mathbf{P} \bullet (\mathbf{a}_r) = -\frac{Q}{4\pi b^2}(\frac{\varepsilon_r - 1}{\varepsilon_r}), \qquad r = b$$

**Prob. 5.25**

$$F_1 = \frac{Q_1 Q_2}{4\pi\varepsilon_o d^2} = 2.6 \text{ nN}, \qquad F_2 = \frac{Q_1 Q_2}{4\pi\varepsilon_o \varepsilon_r d^2} = 1.5 \text{ nN}$$

$$\frac{F_1}{F_2} = \frac{2.6}{1.5} = \varepsilon_r = 1.733$$

**Prob. 5.26**

(a) By Gauss's law,

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q_{enc} \longrightarrow D_r = \frac{Q}{4\pi r^2}$$

$$E_r = \frac{D_r}{\varepsilon} = \frac{Q}{4\pi\varepsilon r^2}$$

$$W = \int_v \frac{1}{2}\varepsilon |\mathbf{E}|^2 \, dv, \qquad dv = r^2 \sin\theta dr d\phi d\theta$$

$$W = \frac{1}{2}\varepsilon \int_{\phi=0}^{2\pi}\int_{\theta}^{\pi}\int_{r=a}^{\infty} \frac{Q^2}{16\pi^2\varepsilon^2 r^4} r^2 \sin\theta dr d\phi d\theta = \frac{Q^2}{8\pi\varepsilon a}$$

(b) $D_r$ remains the same but

$$E_r = \frac{D_r}{\varepsilon} = \frac{Q}{4\pi r^2 \varepsilon_o \left(1+\frac{a}{r}\right)^2} = \frac{Q}{4\pi\varepsilon_o (r+a)^2}$$

$$W = \int_v \frac{1}{2}\varepsilon |E|^2\, dv = \frac{1}{2}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=a}^{\infty} \frac{Q^2 r^2 \sin\theta dr d\theta d\phi}{16\pi^2\varepsilon^2 (r+a)^4}\varepsilon_o \left(\frac{r+a}{r}\right)^2$$

$$= \frac{Q^2}{32\pi^2\varepsilon_o}(4\pi)\int_a^{\infty}\frac{dr}{(r+a)^2} = \frac{Q^2}{8\pi\varepsilon_o}\left(-\frac{1}{r+a}\Bigg|_a^{\infty}\right) = \frac{Q^2}{8\pi\varepsilon_o}\frac{1}{2a}$$

$$W = \frac{Q^2}{16a\pi\varepsilon_o}$$

**Prob. 5.27**

(a)

$$\rho_v = \begin{cases} \rho_o, & 0 < r < a \\ 0, & r > a \end{cases}$$

For $r < a$, $\varepsilon E_r (4\pi r^2) = \rho_o \frac{4\pi r^3}{3} \longrightarrow E_r = \frac{\rho_o r}{3\varepsilon}$

$$V = -\int \mathbf{E} \bullet d\mathbf{l} = -\frac{\rho_o r^2}{6\varepsilon} + c_1$$

For $r > a$, $\varepsilon_o E_r (4\pi r^2) = \rho_o \frac{4\pi a^3}{3} \longrightarrow E_r = \frac{\rho_o a^3}{3\varepsilon_o r^2}$

$$V = -\int E \bullet dl = \frac{\rho_o a^3}{3\varepsilon_o r} + c_2$$

As $r \longrightarrow \infty$, $V = 0$ and $c_2 = 0$

At $r = a$, $V(a^+) = V(a^-)$

$$-\frac{\rho_o a^2}{6\varepsilon_o\varepsilon_r} + c_1 = \frac{\rho_o a^2}{3\varepsilon_o} \longrightarrow c_1 = \frac{\rho_o a^2}{6\varepsilon_o\varepsilon_r}(2\varepsilon_r + 1)$$

$$V(r=0) = c_1 = \frac{\rho_o a^2 (2\varepsilon_r + 1)}{6\varepsilon_o\varepsilon_r}$$

(b) $V(r = a) = \frac{\rho_o a^2}{3\varepsilon_o}$

**Prob. 5.28**

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \varepsilon_o E_o \begin{bmatrix} 4 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$D_x = \varepsilon_o E_o (4+1-1) = 4\varepsilon_o E_o$$
$$D_y = \varepsilon_o E_o (1+3-1) = 3\varepsilon_o E_o$$
$$D_z = \varepsilon_o E_o (1+1-2) = 0$$
$$\boldsymbol{D} = \underline{\underline{\varepsilon_o E_o (4\boldsymbol{a}_x + 3\boldsymbol{a}_y) \text{ C/m}^2}}$$

**Prob. 5.29**

Since $\dfrac{\partial \rho_v}{\partial t} = 0, \qquad \nabla \bullet \boldsymbol{J} = 0$ must hold.

(a) $\nabla \bullet \boldsymbol{J} = 6x^2 y + 0 - 6x^2 y = 0 \longrightarrow$ This is possible.

(b) $\nabla \bullet \boldsymbol{J} = y + (z+1) \neq 0 \longrightarrow$ This is not possible.

(c) $\nabla \bullet \boldsymbol{J} = \dfrac{1}{\rho}\dfrac{\partial}{\partial \rho}(z^2) + \cos\phi \neq 0 \longrightarrow$ This is not possible.

(d) $\nabla \bullet \boldsymbol{J} = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}(\sin\theta) = 0 \longrightarrow$ This is possible.

**Prob. 5.30**

$$\nabla \cdot J = \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} = 2e^{-2y}\cos 2x - 2e^{-2y}\cos 2x + 1 = 1 = -\frac{\partial \rho_v}{\partial t}$$

Hence, $\dfrac{\partial \rho_v}{\partial t} = \underline{\underline{-1 \text{ C/m}^3 s}}$

**Prob. 5.31**

(a) $\nabla \bullet \boldsymbol{J} = \dfrac{1}{\rho}\dfrac{\partial}{\partial \rho}\left(\dfrac{100}{\rho}\right) = -\dfrac{100}{\rho^3}$

$$-\frac{\partial \rho_v}{\partial t} = \nabla \bullet \boldsymbol{J} = -\frac{100}{\rho^3} \longrightarrow \frac{\partial \rho_v}{\partial t} = \underline{\underline{\frac{100}{\rho^3} \text{ C/m}^3.s}}$$

(b) $I = \int \mathbf{J} \bullet d\mathbf{S} = \iint \frac{100}{\rho^2} \rho d\phi \, dz \Big|_{\rho=2} = \frac{100}{2} \int_0^{2\pi} d\phi \int_0^1 dz = 100\pi = \underline{\underline{314.16 \text{ A}}}$

**Prob. 5.32**

$\rho = \rho_{vo} e^{-t/T_r}$

where $\rho_{vo}$ is the initial value (t=0). When $t = 80\mu s$, $\rho = \frac{1}{2}\rho_{vo}$. Then,

$$\frac{1}{2}\rho_{vo} = \rho_{vo} e^{-(80\times10^{-6})/T_r} \quad \rightarrow \quad (80\times10^{-6})/T_r = \ln 2$$

$$T_r = \frac{80\times10^{6}}{\ln 2} = \underline{\underline{115.42\mu s}}$$

$$T_r = \frac{\varepsilon}{\sigma} \quad \rightarrow \quad \sigma = \frac{\varepsilon}{T_r} = \frac{7.5\times\frac{10^{-9}}{36\pi}\ln 2}{80\times10^{-6}} = \underline{\underline{5.746\times10^{-7}\text{ S/m}}}$$

**Prob. 5.33**

From the continuity equation,

$$\nabla \bullet \mathbf{J} = -\frac{\partial \rho_v}{\partial t} \qquad (1)$$

But $\mathbf{J} = \rho_v \mathbf{u}$

Applying the vector identity

$$\nabla \bullet (V\mathbf{A}) = V\nabla \bullet \mathbf{A} + \mathbf{A} \bullet \nabla V$$

$$\nabla \bullet \mathbf{J} = \rho_v (\nabla \bullet \mathbf{u}) + \mathbf{u} \bullet \nabla \rho_v \qquad (2)$$

Substituting (2) into (1) gives

$$(\mathbf{u} \bullet \nabla)\rho_v + \rho_v(\nabla \bullet \mathbf{u}) + \frac{\partial \rho_v}{\partial t} = 0$$

as required.

**Prob. 5.34**

$$-\frac{\partial \rho_v}{\partial t} = \nabla \bullet \mathbf{J} = \frac{\partial J_x}{\partial x} = 0.5\pi \cos \pi x$$

At P(2,4,-3), x= 2

$$\frac{\partial \rho_v}{\partial t} = -0.5\pi \cos(2\pi) = -0.5\pi = \underline{\underline{-1.571 \text{ C/m}^2 s}}$$

**Prob. 5.35**

(a)

$$\frac{\varepsilon}{\sigma} = \frac{3.1 \times \frac{10^{-9}}{36\pi}}{10^{-15}} = \underline{\underline{2.741 \times 10^4 \text{ s}}}$$

(b) $$\frac{\varepsilon}{\sigma} = \frac{6 \times \frac{10^{-9}}{36\pi}}{10^{-15}} = \underline{\underline{5.305 \times 10^4 \text{ s}}}$$

(c) $$\frac{\varepsilon}{\sigma} = \frac{80 \times \frac{10^{-9}}{36\pi}}{10^{-4}} = \underline{\underline{7.07\ \mu\text{s}}}$$

**Prob. 5.36**

$$T_r = \frac{\varepsilon}{\sigma} = \frac{2.5 \times 10^{-9}}{5 \times 10^{-6} \times 36\pi} = 4.42\ \mu\text{s}$$

$$\rho_{vo} = \frac{Q}{V} = \frac{1}{\frac{4\pi}{3} \times 10^{-6} \times 8} = \underline{\underline{29.84 \text{ kC/m}^3}}$$

$$\rho_v = \rho_{vo} e^{-t/T_r} = 29.84 e^{-2/4.42} = \underline{\underline{18.98 \text{ kC/m}^3}}$$

**Prob. 5.37**

The normal component of a solenoidal vector field is continuous across an interface.

Hence, since $\nabla \bullet \boldsymbol{J} = 0$,

$$J_{1n} = J_{2n}$$

Similarly, the tangential component of a curl-free vector field is continuous.

Hence, since $\nabla \times (\boldsymbol{J}/\sigma) = 0$,

$$\frac{J_{1t}}{J_{2t}} = \frac{\sigma_1}{\sigma_2}$$

Alternatively,

$$E_{1n} = E_{2n} \qquad \rightarrow \qquad \frac{J_{1t}}{\sigma_1} = \frac{J_{2t}}{\sigma_2} \qquad \text{or} \qquad \frac{J_{1t}}{J_{2t}} = \frac{\sigma_1}{\sigma_2}$$

as required.

**Prob. 5.38**

(a)

$$\boldsymbol{P}_1 = \chi_{e1}\varepsilon_o \boldsymbol{E}_1 = 3\times\frac{10^{-9}}{36\pi}(60,-100,40) = \underline{\underline{1.591\boldsymbol{a}_x - 2.6526\boldsymbol{a}_y + 1.061\boldsymbol{a}_z \text{ nC/m}^2}}$$

(b)

$$\boldsymbol{E}_{2t} = \boldsymbol{E}_{1t} = 60\boldsymbol{a}_x - 100\boldsymbol{a}_y$$

$$\boldsymbol{D}_{2n} = \boldsymbol{D}_{1n} \quad \rightarrow \quad \varepsilon_2 \boldsymbol{E}_{2n} = \varepsilon_1 \boldsymbol{E}_{1n}$$

$$\boldsymbol{E}_{2n} = \frac{\varepsilon_1}{\varepsilon_2}\boldsymbol{E}_{1n} = \frac{4}{7.5}(40\boldsymbol{a}_z) = 21.33\boldsymbol{a}_z$$

$$\boldsymbol{E}_2 = 60\boldsymbol{a}_x - 100\boldsymbol{a}_y + 21.33\boldsymbol{a}_z$$

$$\boldsymbol{D}_2 = \varepsilon_o\varepsilon_{r2}\boldsymbol{E}_2 = 7.5\times\frac{10^{-9}}{36\pi}(60,-100,21.33) = \underline{\underline{3.979\boldsymbol{a}_x - 6.631\boldsymbol{a}_y + 1.414\boldsymbol{a}_z \text{ nC/m}^2}}$$

**Prob. 5.39**

(a)

$$\boldsymbol{E}_{2t} = \boldsymbol{E}_{1t} = -10\boldsymbol{a}_y + 8\boldsymbol{a}_z$$

$$\boldsymbol{D}_{2n} = \boldsymbol{D}_{1n} \quad \rightarrow \quad \varepsilon_2 \boldsymbol{E}_{2n} = \varepsilon_1 \boldsymbol{E}_{1n}$$

$$\boldsymbol{E}_{1n} = \frac{\varepsilon_2}{\varepsilon_1}\boldsymbol{E}_{2n} = \frac{2\varepsilon_o}{4\varepsilon_o}(6\boldsymbol{a}_x) = 3\boldsymbol{a}_x$$

$$\boldsymbol{E}_1 = 3\boldsymbol{a}_x - 10\boldsymbol{a}_y + 8\boldsymbol{a}_z$$

$$\boldsymbol{P}_1 = \chi_{e1}\varepsilon_o \boldsymbol{E}_1 = 3\times\frac{10^{-9}}{36\pi}(3,-10,8) = \underline{\underline{79.6\boldsymbol{a}_x - 265.3\boldsymbol{a}_y + 212.2\boldsymbol{a}_z \text{ nC/m}^2}}$$

$$\boldsymbol{P}_2 = \chi_{e2}\varepsilon_o \boldsymbol{E}_2 = 1\times\frac{10^{-9}}{36\pi}(6,-10,8) = \underline{\underline{53.05\boldsymbol{a}_x - 88.42\boldsymbol{a}_y + 70.74\boldsymbol{a}_z \text{ nC/m}^2}}$$

(b)

$$w_1 = \frac{1}{2}\varepsilon_1 E_1^2 = \frac{1}{2}(4\varepsilon_o)(9+100+64) = 346\varepsilon_o = \underline{\underline{3.0593 \text{ nJ/m}^2}}$$

$$w_2 = \frac{1}{2}\varepsilon_2 E_2^2 = \frac{1}{2}(2\varepsilon_o)(36+100+64) = 200\varepsilon_o = \underline{\underline{1.7684 \text{ nJ/m}^2}}$$

**Prob. 5.40**

f(x,y)= 4x +3y –10=0

$$\nabla f = 4\boldsymbol{a}_x + 3\boldsymbol{a}_y \quad \longrightarrow \quad \boldsymbol{a}_n = -\frac{\nabla f}{|\nabla f|} = \frac{-(4\boldsymbol{a}_x + 3\boldsymbol{a}_y)}{5} = -0.8\boldsymbol{a}_x - 0.6\boldsymbol{a}_y$$

The minus sign is chosen for $\mathbf{a}_n$ because it is directed toward the origin.

$$\boldsymbol{D}_{1n} = (\boldsymbol{D}_1 \cdot \boldsymbol{a}_n)\boldsymbol{a}_n = (1.6 - 2.4)\boldsymbol{a}_n = -0.64\boldsymbol{a}_x - 0.48\boldsymbol{a}_y$$

$$\boldsymbol{D}_{1t} = \boldsymbol{D}_1 - \boldsymbol{D}_{1n} = 2.64\boldsymbol{a}_x - 3.52\boldsymbol{a}_y + 6.5\boldsymbol{a}_z$$

$$\boldsymbol{D}_{2n} = \boldsymbol{D}_{1n} = -0.64\boldsymbol{a}_x - 0.48\boldsymbol{a}_y$$

$$\boldsymbol{E}_{2t} = \boldsymbol{E}_{1t} \quad \longrightarrow \quad \frac{\boldsymbol{D}_{2t}}{\varepsilon_2} = \frac{\boldsymbol{D}_{1t}}{\varepsilon_2}$$

$$D_{2t} = \frac{\varepsilon_2}{\varepsilon_1} D_{1t} = \frac{2.5}{1}(2.64, -3.52, 6.5) = (6.6, -8.8, 16.25)$$

$$D_2 = D_{2n} + D_{2t} = \underline{\underline{5.96a_x - 9.28a_y + 16.25a_z \text{ nC/m}^2}}$$

$$\theta_2 = \cos^{-1}\frac{D_2 \cdot a_n}{|D_2|} = \underline{\underline{87.66^o}}$$

**Prob. 5.41**

(a)

Let $f(x, y, z) = x + 2y + z - 1 = 0$

$$\nabla f = a_x + 2a_y + a_z \quad \rightarrow \quad a_n = \frac{\nabla f}{|\nabla f|} = \frac{1}{\sqrt{6}}(a_x + 2a_y + a_z)$$

$$E_{1n} = (E_1 \bullet a_n)a_n = \frac{1}{\sqrt{6}}(20 - 20 + 40)\frac{1}{\sqrt{6}}(a_x + 2a_y + a_z) = \underline{\underline{6.667a_x + 13.33a_y + 6.667a_z \text{ V/m}}}$$

$$E_{1t} = E_1 - E_{1n} = \underline{\underline{13.3a_x - 23.3a_y + 33.3a_z \text{ V/m}}}$$

(b)

$$E_{2t} = E_{1t} = 13.3a_x - 23.3a_y + 33.3a_z$$

$$D_{2n} = D_{1n} \quad \rightarrow \quad \varepsilon_2 E_{2n} = \varepsilon_1 E_{1n}$$

$$E_{2n} = \frac{\varepsilon_1}{\varepsilon_2} E_{1n} = \frac{2}{5}(6.667, 13.33, 6.667) = 2.7a_x + 5.3a_y + 33.3a_z$$

$$E_2 = E_{2t} + E_{2n} = \underline{\underline{16a_x - 18a_y + 36a_z \text{ V/m}}}$$

**Prob. 5.42**

(a) $P_1 = \varepsilon_o \chi_{e1} E_1 = 2 \times \frac{10^{-9}}{36\pi}(10, -6, 12) = \underline{\underline{0.1768a_x - 0.1061a_y + 0.2122a_z \text{ nC/m}^2}}$

(b) $E_{1n} = -6a_y, \quad E_{2t} = E_{1t} = 10a_x + 12a_z$

$$D_{2n} = D_{1n} \longrightarrow \varepsilon_2 E_{2n} = \varepsilon_1 E_{1n}$$

or $E_{2n} = \frac{\varepsilon_1}{\varepsilon_2} E_{1n} = \frac{3\varepsilon_o}{4.5\varepsilon_o}(-6a_z) = -4a_y$

$$E_2 = \underline{\underline{10a_x - 4a_y + 12a_z \text{ V/m}}}$$

$$\tan\theta_2 = \frac{E_{2t}}{E_{2n}} = \frac{\sqrt{10^2 + 12^2}}{4} = 3.905 \longrightarrow \underline{\underline{\theta_2 = 75.64^o}}$$

(c ) $w_E = \frac{1}{2}\boldsymbol{D} \bullet \boldsymbol{E} = \frac{1}{2}\varepsilon |\boldsymbol{E}|^2$

$$w_{E1} = \frac{1}{2}\varepsilon_1 |\boldsymbol{E}_1|^2 = \frac{1}{2}x3x\frac{10^{-9}}{36\pi}(10^2 + 6^2 + 12^2) = \underline{\underline{3.7136 \ \text{nJ/m}^3}}$$

$$w_{E2} = \frac{1}{2}\varepsilon_2 |\boldsymbol{E}_2|^2 = \frac{1}{2}x4.5x\frac{10^{-9}}{36\pi}(10^2 + 4^2 + 12^2) = \underline{\underline{5.1725 \ \text{nJ/m}^3}}$$

**Prob. 5.43** (a) $\boldsymbol{D}_{2n} = 12\boldsymbol{a}_\rho = \boldsymbol{D}_{1n}, \qquad \boldsymbol{D}_{2t} = -6\boldsymbol{a}_\phi + 9\boldsymbol{a}_z$

$$\boldsymbol{E}_{2t} = \boldsymbol{E}_{2t} \qquad \longrightarrow \qquad \frac{\boldsymbol{D}_{1t}}{\varepsilon_1} = \frac{\boldsymbol{D}_{2t}}{\varepsilon_2}$$

$$\boldsymbol{D}_{1t} = \frac{\varepsilon_1}{\varepsilon_2}\boldsymbol{D}_{2t} = \frac{3.5\varepsilon_o}{1.5\varepsilon_o}(-6\boldsymbol{a}_\phi + 9\boldsymbol{a}_z) = -14\boldsymbol{a}_\phi + 21\boldsymbol{a}_z$$

$$\underline{\underline{\boldsymbol{D}_1 = 12\boldsymbol{a}_\rho - 14\boldsymbol{a}_\phi + 21\boldsymbol{a}_z \ \ \text{nC/m}^2}}$$

$$\boldsymbol{E}_1 = \boldsymbol{D}_1 / \varepsilon_1 = \frac{(12, -14, 21) \times 10^{-9}}{3.5 \times \frac{10^{-9}}{36\pi}} = 387.8\boldsymbol{a}_\rho - 452.4\boldsymbol{a}_\phi + 678.6\boldsymbol{a}_z$$

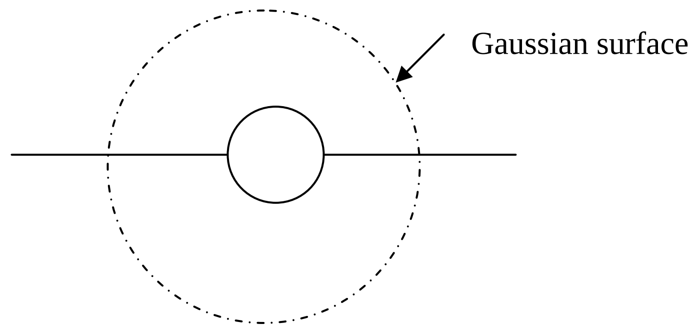
(b) $\boldsymbol{P}_2 = \varepsilon_o \chi_{e2}\boldsymbol{E}_2 = 0.5\varepsilon_o \frac{\boldsymbol{D}_2}{\varepsilon_2} = \frac{0.5\varepsilon_o}{1.5\varepsilon_o}(12, -6, 9) = \underline{\underline{4\boldsymbol{a}_\rho - 2\boldsymbol{a}_\phi + 3\boldsymbol{a}_z \ \ \text{nC/m}^2}}$

$$\rho_{v2} = \nabla \bullet \boldsymbol{P}_2 = 0$$

(c) $w_{E1} = \frac{1}{2}\boldsymbol{D}_1 \bullet \boldsymbol{E}_1 = \frac{1}{2}\frac{\boldsymbol{D}_1 \bullet \boldsymbol{D}_1}{\varepsilon_o \varepsilon_{r1}} = \frac{1}{2}\frac{(12^2 + 14^2 + 21^2)x10^{-18}}{3.5x\frac{10^{-9}}{36\pi}} = \underline{\underline{12.62 \ \ \mu\text{J/m}^2}}$

$$w_{E2} = \frac{1}{2}\frac{\boldsymbol{D}_2 \bullet \boldsymbol{D}_2}{\varepsilon_o \varepsilon_{r2}} = \frac{1}{2}\frac{(12^2 + 6^2 + 9^2)x10^{-18}}{1.5x\frac{10^{-9}}{36\pi}} = \underline{\underline{9.839 \ \ \mu\text{J/m}^2}}$$

**Prob. 5.44**




$$Q = \int \boldsymbol{D}.dS = \varepsilon_1 E_r \frac{4\pi r^2}{2} + \varepsilon_2 E_r \frac{4\pi r^2}{2} = 2\pi r^2 (\varepsilon_1 + \varepsilon_2) E_r$$

$$E_r = \begin{cases} \dfrac{Q}{2\pi(\varepsilon_1 + \varepsilon_2) r^2}, & r > a \\ 0, & r < a \end{cases}$$

**Prob. 5.45**

$$\boldsymbol{E}_{1t} = \boldsymbol{E}_{2t} = 5\cos\theta \boldsymbol{a}_\theta$$

$$\boldsymbol{D}_{1n} = \boldsymbol{D}_{2n} \quad \rightarrow \quad \varepsilon_1 \boldsymbol{E}_{1n} = \varepsilon_2 \boldsymbol{E}_{2n}$$

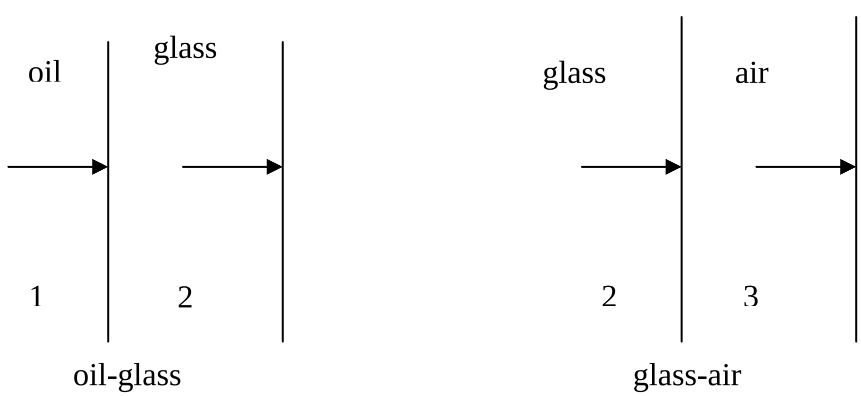
$$\boldsymbol{E}_{1n} = \frac{\varepsilon_2}{\varepsilon_1} \boldsymbol{E}_{2n} = \frac{6\varepsilon_o}{2\varepsilon_o}(10\sin\theta)\mathbf{a}_r = 30\sin\theta \boldsymbol{a}_r$$

$$\boldsymbol{E}_{1t} = \boldsymbol{E}_{1t} + \boldsymbol{E}_{1n} = \underline{\underline{30\sin\theta \boldsymbol{a}_r + 5\cos\theta \boldsymbol{a}_\theta}}$$

$$\boldsymbol{D}_1 = \varepsilon_1 \boldsymbol{E}_1 = 2\varepsilon_o \boldsymbol{E}_1 = \underline{\underline{\varepsilon_o(60\sin\theta \boldsymbol{a}_r + 10\cos\theta \boldsymbol{a}_\theta)}}$$

**Prob. 5.46**

(a) The two interfaces are shown below



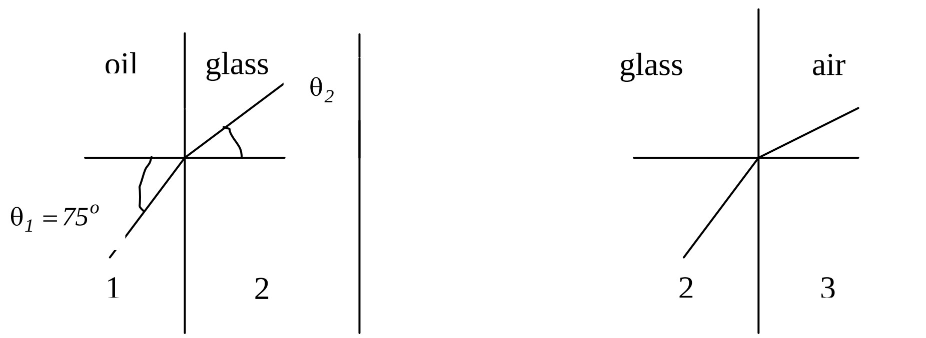
$E_{1n} = 2000, \quad E_{1t} = 0 = E_{2t} = E_{3t}$

$D_{1n} = D_{2n} = D_{3n} \longrightarrow \varepsilon_1 E_{1n} = \varepsilon_2 E_{2n} = \varepsilon_3 E_{3n}$

$$E_{2n} = \frac{\varepsilon_1}{\varepsilon_2} E_{1n} = \frac{3.0}{8.5}(2000) = \underline{\underline{705.9 \text{ V/m}, \quad \theta_2 = 0^o}}$$

$$E_{3n} = \frac{\varepsilon_1}{\varepsilon_3} E_{1n} = \frac{3.0}{1.0}(2000) = \underline{\underline{6000 \text{ V/m}, \quad \theta_3 = 0^o}}$$

(b)




$E_{1n} = 2000\cos 75^o = 517.63, \quad E_{1t} = 2000\sin 75^o = E_{2t} = E_{3t} = 1931.85$

$E_{2n} = \frac{\varepsilon_1}{\varepsilon_2} E_{1n} = \frac{3}{8.5}(517.63) = 182.7, \quad E_{3n} = \frac{\varepsilon_1}{\varepsilon_3} E_{1n} = \frac{3}{1}(517.63) = 1552.89$

$$E_2 = \sqrt{E_{2n}^2 + E_{2t}^2} = 1940.5, \quad \theta_2 = \tan^{-1}\frac{E_{2t}}{E_{2n}} = \underline{\underline{84.6^o}},$$

$$E_3 = \sqrt{E_{3n}^2 + E_{3t}^2} = 2478.6, \quad \theta_3 = \tan^{-1}\frac{E_{3t}}{E_{3n}} = \underline{\underline{51.2^o}}$$

**Prob. 5.47**

$$\rho_s = D_n = \varepsilon_o E = \frac{10^{-9}}{36\pi}\sqrt{30^2 + 40^2 + 20^2} \times 10^{-3} = \frac{\sqrt{2900}}{36\pi} \text{ pC/m}^2$$
$$= \underline{\underline{0.476 \text{ pC/m}^2}}$$

**Prob. 5.48** (a) $\rho_s = D_n = \varepsilon_o E_n = \frac{10^{-9}}{36\pi}\sqrt{15^2 + 8^2} = \underline{\underline{0.1503 \text{ nC/m}^2}}$

(b) $D_n = \rho_s = -20 \text{ nC/m}^2$

$$D = D_n a_n = (-20 \text{ nC})(-a_y) = \underline{\underline{20a_y \text{ nC/m}^2}}$$

**Prob. 5.49**

At the interface between $\varepsilon_o$ and $2\varepsilon_o$,

$$E_{1n} = E_o \cos 30^o, \quad E_{1t} = E_o \sin 30^o$$

$$E_{2t} = E_{1t} = 0.5E_o$$

$$D_{2n} = D_{1n} \longrightarrow \varepsilon_1 E_{1n} = \varepsilon_2 E_{2n}$$

$$E_{2n} = \frac{\varepsilon_1}{\varepsilon_2} E_{1n} = \frac{\varepsilon_o}{2\varepsilon_o}(0.866E_o) = 0.433E_o$$

The angle **E** makes with the z-axis is

$$\theta_1 = \tan^{-1}\frac{E_{2t}}{E_{2n}} = \tan^{-1}\frac{0.5}{0.433} = \underline{\underline{49.11^o}}$$

At the interface between $2\varepsilon_o$ and $3\varepsilon_o$,

$$E_{3t} = E_{2t} = 0.5E_o$$

$$D_{3n} = D_{2n} \longrightarrow E_{3n} = \frac{\varepsilon_2}{\varepsilon_3} E_{2n} = \frac{2\varepsilon_o}{3\varepsilon_o}(0.433E_o) = 0.2887E_o$$

The angle **E** makes with the z-axis is

$$\theta_2 = \tan^{-1}\frac{E_{3t}}{E_{3n}} = \tan^{-1}\frac{0.5}{0.2887} = \underline{\underline{60^o}}$$

At the interface between $3\varepsilon_o$ and $\varepsilon_o$,

$$E_{4t} = E_{3t} = 0.5E_o$$

$$D_{4n} = D_{3n} \longrightarrow E_{4n} = \frac{\varepsilon_3}{\varepsilon_4} E_{3n} = \frac{3\varepsilon_o}{\varepsilon_o}(0.2887E_o) = 0.866E_o$$

The angle **E** makes with the z-axis is

$$\theta_3 = \tan^{-1}\frac{E_{4t}}{E_{4n}} = \tan^{-1}\frac{0.5}{0.866} = \underline{\underline{30^o}}$$

# CHAPTER 6

**P. E. 6.1**

$$\nabla^2 V = -\frac{\rho}{\varepsilon} \longrightarrow \frac{d^2V}{dx^2} = -\frac{\rho_o x}{\varepsilon a}$$

$$V = -\frac{\rho_o x^3}{6\varepsilon a} + Ax + B$$

$$\boldsymbol{E} = -\frac{dV}{dx}\boldsymbol{a}_x = \left(\frac{\rho_o x^2}{2\varepsilon a} - A\right)\boldsymbol{a}_x$$

If **E** = 0 at x =0, then

$$0 = 0 - A \longrightarrow A = 0$$

If V= 0 at x =a, then

$$0 = -\frac{\rho_o a^3}{6\varepsilon a} + B \longrightarrow B = \frac{\rho_o a^2}{6\varepsilon}$$

Thus

$$\underline{\underline{V = \frac{\rho_o}{6\varepsilon a}(a^3 - x^3)}}, \qquad \boldsymbol{E} = \underline{\underline{\frac{\rho_o x^2}{2\varepsilon a}\boldsymbol{a}_x}}$$

**P. E. 6.2** $\quad V_1 = A_1 x + B_1, \quad V_2 = A_2 x + B_2$

$$V_1(x = d) = V_o = A_1 d + B_1 \longrightarrow B_1 = V_o - A_1 d$$

$$V_1(x = 0) = 0 = 0 + B_2 \longrightarrow B_2 = 0$$

$$V_1(x = a) = V_2(x = a) \longrightarrow aA_1 + B_1 = A_2 a$$

$$D_{1n} = D_{2n} \longrightarrow \varepsilon_1 A_1 = \varepsilon_2 A_2 \longrightarrow A_2 = \frac{\varepsilon_1}{\varepsilon_2} A_1$$

$$A_1 a + V_o - A_1 d = \frac{\varepsilon_1}{\varepsilon_2} a A_1 \longrightarrow V_o = A_1\left(-a + d + \frac{\varepsilon_1}{\varepsilon_2} a\right)$$

or

$$A_1 = \frac{V_o}{d - a + \varepsilon_1 a / \varepsilon_2}, \qquad A_2 = \frac{\varepsilon_1}{\varepsilon_2} A_1 \frac{\varepsilon_1 V_o}{\varepsilon_2 d - \varepsilon_2 a + \varepsilon_1 a}$$

Hence

$$\boldsymbol{E}_1 = -A_1 \boldsymbol{a}_x = \underline{\underline{\frac{-V_o a_x}{d - a + \varepsilon_1 a / \varepsilon_2}}}, \qquad \boldsymbol{E}_2 = -A_2 \boldsymbol{a}_x = \underline{\underline{\frac{-V_o \boldsymbol{a}_x}{a + \varepsilon_2 d / \varepsilon_1 - \varepsilon_2 a / \varepsilon_1}}}$$

**P. E. 6.3** From Example 6.3,

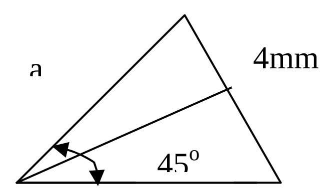
$$\mathbf{E} = -\frac{V_o}{\rho\phi_o}\mathbf{a}_\phi, \qquad D = \varepsilon_o E$$

$$\rho_s = D_n(\phi = 0) = -\frac{V_o \varepsilon}{\rho \phi_o}$$

The charge on the plate $\phi = 0$ is

$$Q = \int \rho_s dS = -\frac{V_o \varepsilon}{\phi_o}\int_{z=0}^{L}\int_{\rho=a}^{b}\frac{1}{\rho}dz d\rho = -\frac{V_o\varepsilon}{\phi_o}L\ln(b/a)$$

$$C = \frac{|Q|}{V_o} = \frac{\varepsilon L}{\phi_o}\ln\frac{b}{a}$$




$$a\sin\frac{45^o}{2} = 2 \longrightarrow a = \frac{2}{\sin 22.5^o} = 5.226 \text{ mm}$$

$$C = \frac{1.5 \times \frac{10^{-9}}{36\pi}}{\frac{\pi}{4}} 5\ln\frac{1000}{5.226} = 444 \text{ pF}$$

$$Q = CV_o = 444 \times 10^{-12} \times 50 \text{ C} = \underline{\underline{22.2 \text{ nC}}}$$

**P. E. 6.4** From Example 6.4,

$$V_o = 50, \quad \theta_2 = 45^o, \quad \theta_1 = 90^o, \quad r = \sqrt{3^2 + 4^2 + 2^2} = \sqrt{29}, \quad \theta = \tan^{-1}\frac{\rho}{z} =$$

$$\tan^{-1}\frac{5}{2} \longrightarrow \theta = 68.2^o; \qquad \tan 45^o = 1$$

$$V = \frac{50\ln(\tan 34.1^o)}{\ln(\tan 22.5^o)} = \underline{\underline{22.125 \text{ V}}},$$

$$\mathbf{E} = \frac{50\mathbf{a}_\theta}{\sqrt{29}\sin 68.2^o \ln(\tan 22.5^o)} = \underline{\underline{11.35\mathbf{a}_\theta \text{ V/m}}}$$

**P. E. 6.5**

From Example 6.5,

$$V(x,y) = \frac{4V_o}{\pi}\sum_{n=\text{odd}}^{\infty}\frac{\sin(n\pi x/b)\sinh[n\pi y/b]}{n\sinh(n\pi a/b)}$$

$$\boldsymbol{E} = -\nabla V = -\frac{\partial V}{\partial x}\boldsymbol{a}_x - \frac{\partial V}{\partial y}\boldsymbol{a}_y$$

$$= -\frac{4V_o}{b}\sum_{n=\text{odd}}^{\infty}\frac{1}{\sinh n\pi a/b}\left[\cos(n\pi x/b)\sinh(n\pi y/b)\,\mathbf{a}_x + \sin(n\pi x/b)\cosh(n\pi y/b)\mathbf{a}_y\right]$$

(a) At (x,y) = (a, a/2),

$$V = \frac{400}{\pi}(0.3775 - 0.0313 + 0.00394 - 0.000585 + \ldots) = \underline{\underline{44.51\text{ V}}}$$

$$\boldsymbol{E} = 0\boldsymbol{a}_x + (-115.12 + 19.127 - 3.9411 + 0.8192 - 0.1703 + 0.035 - 0.0074 + \ldots)\boldsymbol{a}_y$$
$$= \underline{\underline{-99.25\boldsymbol{a}_y\ \text{V/m}}}$$

(b) At (x,y) = (3a/2, a/4),

$$V = \frac{400}{\pi}(0.1238 + 0.006226 - 0.00383 + 0.0000264 + \ldots) = \underline{\underline{16.50\text{ V}}}$$

$$\boldsymbol{E} = (24.757 - 3.7358 - 0.3834 + 0.0369 + 0.00351 - 0.00033 + \ldots)\boldsymbol{a}_x$$
$$+(-66.25 - 4.518 + 0.3988 + 0.03722 - 0.00352 - 0.000333 + \ldots)\boldsymbol{a}_y$$
$$= \underline{\underline{20.68\boldsymbol{a}_x - 70.34\boldsymbol{a}_y\ \text{V/m}}}$$

**P. E. 6.6**

$$V(y=a) = V_o\sin(7\pi x/b) = \sum_{n=1}^{\infty} c_n \sin(n\pi x/b)\sinh(n\pi a/b)$$

By equating coefficients, we notice that $c_n = 0$ for $n \neq 7$. For n=7,

$$V_o\sin(7\pi x/b) = c_7\sin(7\pi x/b)\sinh(7\pi a/b) \longrightarrow c_7 = \frac{V_o}{\sinh(7\pi a/b)}$$

Hence

$$V(x,y) = \underline{\underline{\frac{V_o}{\sinh(7\pi a/b)}\sin(7\pi x/b)\sinh(7\pi y/b)}}$$

**P. E. 6.7** Let $V(r,\theta,\phi) = R(r)F(\theta)\Phi(\phi)$.
Substituting this in Laplace's equation in spherical coordinates gives

$$\frac{\Phi F}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{R\Phi}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dF}{d\theta}\right)+\frac{RF}{r^2\sin^2\theta}\frac{d^2\Phi}{d\phi^2}=0$$

Dividing by $RF\Phi\,/\,r^2\sin^2\theta$ gives

$$\frac{\sin^2\theta}{R}\frac{d}{dr}\left(r^2R'\right)+\frac{\sin\theta}{F}\frac{d}{d\theta}(\sin\theta F')=-\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}=\lambda^2$$

$$\underline{\underline{\Phi''+\lambda^2\Phi=0}}$$

$$\frac{1}{R}\frac{d}{dr}\left(r^2R'\right)+\frac{1}{F\sin\theta}\frac{d}{d\theta}(\sin\theta F')=\lambda^2/\sin^2\theta$$

$$\frac{1}{R}\frac{d}{dr}\left(r^2R'\right)=\frac{\lambda^2}{\sin^2\theta}-\frac{1}{F\sin\theta}\frac{d}{d\theta}(\sin\theta F')=\mu^2$$

$$2rR'+r^2R''=\mu^2R$$

or

$$\underline{\underline{R''+\frac{2}{r}R'-\frac{\mu^2}{r^2}R=0}}$$

$$\frac{\sin\theta}{F}\frac{d}{d\theta}(\sin\theta F')-\lambda^2+\mu^2\sin^2\theta=0$$

or

$$\underline{\underline{F''+\cos\theta F'+(\mu^2\sin\theta-\lambda^2\csc\theta)F=0}}$$

**P. E. 6.8** (a) This is similar to Example 6.8(a) except that here $0<\phi<2\pi$ instead of $0<\phi<\pi/2$. Hence

$$I=\frac{2\pi tV_o\sigma}{\ln(b/a)} \quad \text{and} \quad R=\frac{V_o}{I}=\underline{\underline{\frac{\ln\frac{b}{a}}{2\pi t\sigma}}}$$

(b) This to similar to Example 6.8(b) except that here $0<\phi<2\pi$. Hence

$$I=\frac{V_o\sigma}{t}\int_a^b\int_0^{2\pi}\rho\,d\rho\,d\phi=\frac{V_o\sigma\pi(b^2-a^2)}{t}$$

and $R = \dfrac{V_o}{I} = \dfrac{t}{\sigma\pi(b^2 - a^2)}$

**P. E. 6.9** From Example 6.9,

$$J_1 = \frac{\sigma_1 V_o}{\rho \ln\frac{b}{a}}, \qquad J_2 = \frac{\sigma_2 V_o}{\rho \ln\frac{b}{a}}$$

$$I = \int J \bullet dS = \int_{z=0}^{L}\left[\int_{\phi=0}^{\pi} J_1 \rho\, d\phi + \int_{\phi=\pi}^{2\pi} J_2 \rho\, d\phi\right] dz = \frac{V_o l}{\ln\frac{b}{a}}\left[\pi\sigma_1 + \pi\sigma_2\right]$$

$$R = \frac{V_o}{I} = \frac{\ln\frac{b}{a}}{\pi l\left[\sigma_1 + \sigma_2\right]}$$

**P. E. 6.10** (a) $C = \dfrac{4\pi\varepsilon}{\frac{1}{a} - \frac{1}{b}}$, $C_1$ and $C_2$ are in series.

$$C_1 = 4\pi x \frac{10^{-9}}{36\pi}\left(\frac{2.5}{\frac{10^3}{2} - \frac{10^3}{3}}\right) = 5/3 \text{ pF}, \qquad C_2 = 4\pi x \frac{10^{-9}}{36\pi}\left(\frac{3.5}{\frac{10^3}{1} - \frac{10^3}{2}}\right) = 7/9 \text{ pF}$$

$$C = \frac{C_1 C_2}{C_1 + C_2} = \frac{(5/3)(7/9)}{(5/3) + (7/9)} = 0.53 \text{ pF}$$

(b) $C = \dfrac{2\pi\varepsilon}{\frac{1}{a} - \frac{1}{b}}$, $C_1$ and $C_2$ are in parallel.

$$C_1 = 2\pi \times \frac{10^{-9}}{36\pi}\left(\frac{2.5}{\frac{10^3}{1} - \frac{10^3}{3}}\right) = 5/24 \text{ pF}, \qquad C_2 = 2\pi \times \frac{10^{-9}}{36\pi}\left(\frac{3.5}{\frac{10^3}{1} - \frac{10^3}{3}}\right) = 7/24 \text{ pF}$$

$$C = C_1 + C_2 = 0.5 \text{ pF}$$

**P. E. 6.11** As in Example 6.8, the solution of Laplace's equation yields
$V(\rho) = A \ln \rho + B$

Using the boundary conditions $V(\rho = a) = 0, \quad V(\rho = b) = V_o$,
0 = Alna + B
$V_o$ = Aln b + B
Solving this yields

$$V = V_o \frac{\ln \rho / a}{\ln b / a}, \qquad \boldsymbol{E} = -\nabla V = -\frac{V_o}{\rho \ln b / a} \boldsymbol{a}_\rho$$

$$Q = \int \varepsilon E \bullet dS = \frac{V_o \varepsilon}{\ln b / a} \int_{z=0}^{L} \int_{\phi=0}^{2\pi} \frac{1}{\rho} dz \rho d\phi = \frac{V_o 2\pi\varepsilon L}{\ln b / a}$$

$$C = \frac{Q}{V_o} = \underline{\underline{\frac{2\pi\varepsilon L}{\ln b / a}}}$$

**P. E. 6.12**

(a) Let $C_1$ and $C_2$ be capacitances per unit length of each section and $C_T$ be the total capacitance of 10m length. $C_1$ and $C_2$ are in series.

$$C_1 = \frac{2\pi\varepsilon_{r1}\varepsilon_o}{\ln b / c} = \frac{2\pi x 2.5}{\ln 3/2} \frac{10^{-9}}{36\pi} = 342.54 \text{ pF/m},$$

$$C_2 = \frac{2\pi\varepsilon_{r2}\varepsilon_o}{\ln c / a} = \frac{2\pi x 3.5}{\ln 2} \frac{10^{-9}}{36\pi} = 280.52 \text{ pF/m}$$

$$C = \frac{C_1 C_2}{C_1 + C_2} = \frac{342.54 x 280.52}{342.54 + 280.52} = 154.22 \text{ pF}$$

$$C_T = Cl = \underline{\underline{1.54}} \text{ nF}$$

(b) $C_1$ and $C_2$ are in parallel.

$$C = C_1 + C_2 = \frac{\pi\varepsilon_{r1}\varepsilon_o}{\ln b / a} + \frac{\pi\varepsilon_{r2}\varepsilon_o}{\ln b / a} = \frac{\pi(\varepsilon_{r1} + \varepsilon_{r2})\varepsilon_o}{\ln b / a} = \frac{6\pi}{\ln 3} \frac{10^{-9}}{36\pi} = 151.7 \text{ pF/m}$$

$$C_T = Cl = \underline{\underline{1.52}} \text{ nF}$$

**P. E. 6.13** Instead of Eq. (6.31), we now have

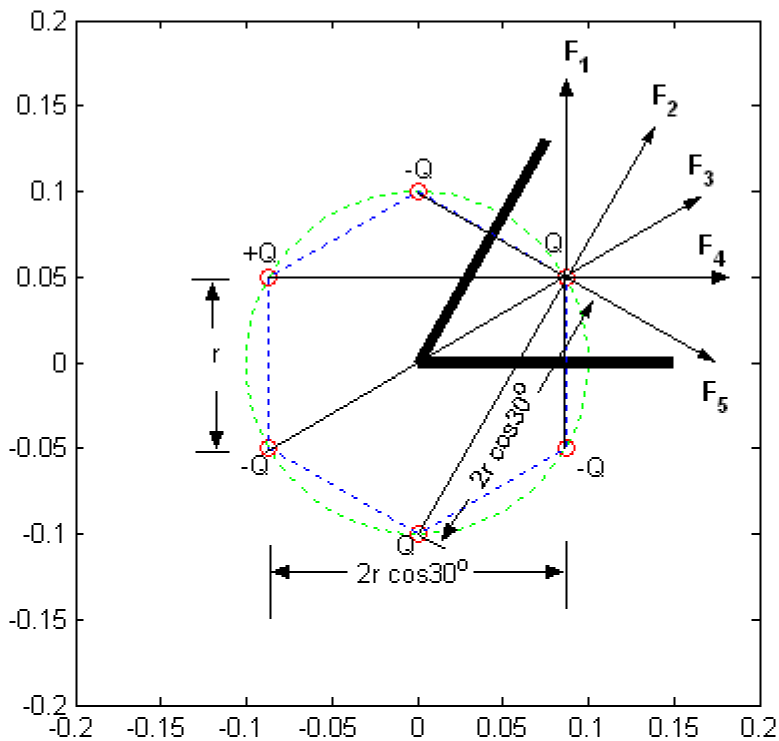
$$V = -\int_b^a \frac{Qdr}{4\pi\varepsilon r^2} = -\int_b^a \frac{Qdr}{4\pi \frac{10\varepsilon_o}{r} r^2} = -\frac{Q}{40\pi\varepsilon_o} \ln b / a$$

$$C = \frac{Q}{|V|} = \frac{40\pi}{\ln 4/1.5}\frac{10^{-9}}{36\pi} = \underline{\underline{1.13 \text{ nF}}}$$

**P. E. 6.14** Let

$$F = F_1 + F_2 + F_3 + F_4 + F_5$$

where $F_i$, $i = 1,2,\ldots,5$ are shown on in the figure below.




$$\boldsymbol{F} = -\frac{Q^2}{4\pi\varepsilon_o r^2}\boldsymbol{a}_y + \frac{Q^2(\boldsymbol{a}_x \sin 30^o + \boldsymbol{a}_y \cos 30^o)}{4\pi\varepsilon_o(2r\cos 30^o)^2} - \frac{Q^2(\boldsymbol{a}_x \cos 30^o + \boldsymbol{a}_y \sin 30^o)}{4\pi\varepsilon_o(2r)^2}$$

$$+ \frac{Q^2\boldsymbol{a}_x}{4\pi\varepsilon_o(2r\cos 30^o)^2} - \frac{Q^2(\boldsymbol{a}_x \cos 30^o - \boldsymbol{a}_y \sin 30^o)}{4\pi\varepsilon_o r^2}$$

$$= \frac{Q^2}{4\pi\varepsilon_o r^2}\left[-\boldsymbol{a}_y + \frac{1}{3}\left(\frac{\boldsymbol{a}_x}{2} + \frac{\sqrt{3}\boldsymbol{a}_y}{2}\right) - \frac{1}{4}\left(\frac{\sqrt{3}\boldsymbol{a}_x}{2} + \frac{\boldsymbol{a}_y}{2}\right) + \frac{1}{3}a_x - \frac{\sqrt{3}\boldsymbol{a}_x}{2} + \frac{\boldsymbol{a}_y}{2}\right]$$

$$= 9x10^{-5}\left[\boldsymbol{a}_x\left(\frac{1}{2}-\frac{5\sqrt{3}}{8}\right)+\boldsymbol{a}_y\left(\frac{-5}{8}+\frac{\sqrt{3}}{6}\right)\right] = -52.4279\boldsymbol{a}_x - 30.27\boldsymbol{a}_y \quad \mu\text{N}$$

$$|\boldsymbol{F}| = 60.54 \ \mu\text{N}$$

Note that the force tends to pull Q toward the origin.

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**Prob. 6.1**

(a)

$$\boldsymbol{E} = -\nabla V = -\left(\frac{\partial V}{\partial x}\boldsymbol{a}_x + \frac{\partial V}{\partial z}\boldsymbol{a}_y + \frac{\partial V}{\partial z}\boldsymbol{a}_z\right)$$

$$= -(15x^2y^2z\boldsymbol{a}_x + 10x^3yz\boldsymbol{a}_y + 5x^3y^2\boldsymbol{a}_z)$$

At P, x=-3, y=1, z=2,

$$\boldsymbol{E} = -15(9)(1)(2)\boldsymbol{a}_x + 10(-27)(1)(2)\boldsymbol{a}_y - 5(-27)(1)\boldsymbol{a}_z = -270\boldsymbol{a}_x + 540\boldsymbol{a}_y + 135\boldsymbol{a}_z \quad \text{V/m}$$

(b) $\rho_v = \nabla \bullet \boldsymbol{D}$ or $\rho_v = -\varepsilon\nabla^2 V$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial y^2} = \frac{\partial}{\partial x}(15x^2y^2z) + \frac{\partial}{\partial y}(10x^3yz) + \frac{\partial}{\partial z}(5x^3y^2)$$

$$= 30xy^2z + 10x^3z$$

At P,

$$\rho_v = -\varepsilon\nabla^2 V = -2.25\times\frac{10^{-9}}{36\pi}\left[30(-3)(1)(2) + 10(-27)(2)\right] = 14.324 \text{ nC/m}^3$$

**Prob. 6.2**

(a)

$$-\boldsymbol{E} = \nabla V = -\frac{20}{r^3}\cos\theta\sin\phi\boldsymbol{a}_r - \frac{10}{r^3}\sin\theta\sin\phi\boldsymbol{a}_\theta + \frac{1}{\text{r}\sin\theta}\frac{10}{r^2}\cos\theta\cos\phi\boldsymbol{a}_\phi$$

At $P(1,60^o,30^o), r = 1, \theta = 60^o, \phi = 30^o$

$$\boldsymbol{E} = \frac{20}{1^3}\cos 60^o\sin 30^o\boldsymbol{a}_r + \frac{10}{1^3}\sin 60^o\sin 30^o\boldsymbol{a}_\theta - \frac{1}{\sin 60^o}\frac{10\cos 60^o\cos 30^o}{1^3}\boldsymbol{a}_\phi$$

$$= 5\boldsymbol{a}_r + 4.33\boldsymbol{a}_\theta - 5\boldsymbol{a}_\phi \quad \text{V/m}$$

(b)

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{-20\cos\theta\sin\phi}{r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{-10\sin^2\theta\sin\phi}{r^2}\right)$$

$$-\frac{1}{r^2\sin^2\theta}\frac{10\cos\theta\sin\phi}{r^2}$$

$$= \frac{20\cos\theta\sin\phi}{r^4} - \frac{20\sin\theta\cos\theta\sin\phi}{r^4\sin\theta} - \frac{10\cos\theta\sin\phi}{r^4\sin^2\theta}$$

$$= -\frac{10\cos\theta\sin\phi}{r^4\sin^2\theta}$$

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon} \longrightarrow \rho_v = -\varepsilon\nabla^2 V = \frac{10\varepsilon_o\cos\theta\sin\phi}{r^4\sin^2\theta}$$

At P, r =1, $\theta$=60$^o$, $\phi$=30$^o$

$$\rho_v = 10\times\frac{10^{-9}}{36\pi}\frac{\cos 60^o\sin 30^o}{1^4\sin^2 60^o} = \underline{\underline{29.47\text{ pC/m}^3}}$$

**Prob. 6.3**

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon}$$

$$\frac{d^2V}{dy^2} = -\frac{y}{4\pi}\cdot\frac{10^{-9}}{4\varepsilon_o} = -\frac{y}{4\pi}\cdot\frac{10^{-9}}{4\frac{10^{-9}}{36\pi}} = -2.25y$$

$$\frac{dV}{dy} = -2.25\frac{y^2}{2} + B$$

$$V = -0.375y^3 + By + C$$

$$V(1) = 0 = -0.375 + B + C \qquad (1)$$

$$V(3) = 50 = -10.125 + 3B + C \qquad (2)$$

From (1) and (2), B=29.875 and C=-29.5

$$V = -0.375y^3 + 29.875y - 29.5$$

$$V(2) = \underline{\underline{27.25\text{ V}}}$$

**Prob. 6.4**

$$\boldsymbol{E} = -\nabla V = -\frac{dV}{d\rho}\boldsymbol{a}_\rho = -(0.8)(10)\rho^{-0.2}\boldsymbol{a}_\rho = -8(0.6)^{-0.2}\boldsymbol{a}_\rho = \underline{\underline{-8.861\boldsymbol{a}_\rho}}$$

$$\nabla^2 V = \frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dV}{d\rho}\right) = -\frac{\rho_v}{\varepsilon_o} = \frac{1}{\rho}\frac{d}{d\rho}\left(8\rho^{0.8}\right) = \frac{1}{\rho}(8)0.8\rho^{-0.2} = 6.4\rho^{-1.2}$$

$$\rho_v = -\varepsilon_o\nabla^2 V = -6.4\times\frac{10^{-9}}{36\pi}\rho^{-1.2} = -\frac{6.4(0.6)^{-1.2}}{36\pi}\text{ nC/m}^3 = \underline{\underline{-0.1044\text{ nC/m}^3}}$$

**Prob. 6.5**

$$\nabla^2 V = \frac{d^2V}{dx^2} = -\frac{\rho_v}{\varepsilon} = -\frac{50(1-y^2)x10^{-6}}{\varepsilon} = -k(1-y^2)$$

where $k = \dfrac{50\times10^{-6}}{3\times\dfrac{10^{-9}}{36\pi}} = 600\pi\times10^3$

$$\frac{dV}{dy} = -k(y - y^3/3) + A$$

$$V = -k\left(\frac{y^2}{2} - \frac{y^4}{12}\right) + Ay + B = 50\pi.10^3 y^4 - 300\pi.10^3 y^2 + Ay + B$$

When y=2cm, V=30X10$^3$,

$$30\times10^3 = 50\pi\times10^3\times16\times10^{-6} - 300\pi\times10^3\times4\times10^{-4} + Ay + B$$

or

$$30,376.77 = 0.02A + B \qquad (1)$$

When y=-2cm, V=30x10$^3$,

$$30,376.77 = -0.02A + B \qquad (2)$$

From (1) and (2), A=0, B=30,376.77. Thus,

$$\underline{\underline{V = 157.08y^4 - 942.5y^2 + 30.377\text{ kV}}}$$

**Prob. 6.6**

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon} \longrightarrow \frac{d^2V}{dz^2} = -\frac{\rho_o z}{\varepsilon d}$$

$$\frac{dV}{dz} = -\frac{\rho_o z^2}{2\varepsilon d} + A$$

$$V = -\frac{\rho_o z^3}{6\varepsilon d} + Az + B$$

$$z = 0, V = 0 \longrightarrow 0 = 0 + B, \quad \text{i.e. } B = 0$$

$$z = d, V = V_o \longrightarrow V_o = -\frac{\rho_o d^2}{6\varepsilon} + Ad$$

$$A = \frac{V_o}{d} + \frac{\rho_o d}{6\varepsilon}$$

Hence,

$$V = \underline{\underline{-\frac{\rho_o z^3}{6\varepsilon d} + \left(\frac{V_o}{d} + \frac{\rho_o d}{6\varepsilon}\right) z}}$$

**Prob. 6.7**

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon} = \frac{-\frac{10}{\rho}\times 10^{-12}}{3.6\times\frac{10^{-9}}{36\pi}} = -\frac{0.1\pi}{\rho}$$

Let $\alpha = 0.1\pi$.

$$\nabla^2 V = -\frac{\alpha}{\rho} = \frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dV}{d\rho}\right)$$

$$-\alpha = \frac{d}{d\rho}\left(\rho\frac{dV}{d\rho}\right)$$

$$\rho\frac{dV}{d\rho} = -\alpha\rho + A$$

$$\frac{dV}{d\rho} = -\alpha + \frac{A}{\rho}$$

$$V = -\alpha\rho + A\ln\rho + B$$

At $\rho$=2, V=0 $\longrightarrow$ $0 = -2\alpha + A\ln 2 + B$ (1)

At $\rho$=5, V=60 $\longrightarrow$ $60 = -5\alpha + A\ln 5 + B$ (2)

Subtracting (1) from (2),

$$60 = -3\alpha + A\ln 5/2 \longrightarrow A = \frac{60+3\alpha}{\ln 2.5} = 66.51$$

From (1),

$$B = 2\alpha - A\ln 2 = -45.473$$

$$\boldsymbol{E} = -\frac{dV}{d\rho}\boldsymbol{a}_\rho = (\alpha - \frac{A}{\rho})\boldsymbol{a}_\rho = \underline{\underline{(0.3142 - \frac{66.51}{\rho})\boldsymbol{a}_\rho}}$$

**Prob. 6.8**

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dV}{d\rho}\right)=-\frac{\rho_o}{\varepsilon}$$

$$\frac{d}{d\rho}\left(\rho\frac{dV}{d\rho}\right)=-\frac{\rho_o\rho}{\varepsilon}$$

Integrating gives

$$\rho\frac{dV}{d\rho}=-\frac{\rho_o}{\varepsilon}\frac{\rho^2}{2}+A \quad\rightarrow\quad \frac{dV}{d\rho}=-\frac{\rho_o}{2\varepsilon}\rho+\frac{A}{\rho}$$

Integrating again,

$$\underline{\underline{V=-\frac{\rho_o\rho^2}{4\varepsilon}+A\ln\rho+B}}$$

where A and B are integration constants.

**Prob. 6.9**

$$\nabla^2V=\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dV}{dr}\right)=-\frac{\rho_v}{\varepsilon}=-\frac{10\times10^{-9}}{6r\times\frac{10^{-9}}{36\pi}}=-\frac{60\pi}{r}$$

$$\frac{d}{dr}\left(r^2\frac{dV}{dr}\right)=-60\pi r \quad\rightarrow\quad r^2\frac{dV}{dr}=-30\pi r^2+A$$

$$\frac{dV}{dr}=-30\pi+\frac{A}{r^2} \quad\rightarrow\quad V=-30\pi r-\frac{A}{r}+B$$

$$V(r=1)=0 \quad\rightarrow\quad 0=-30\pi-A+B \qquad (1)$$

$$V(r=4)=50 \quad\rightarrow\quad 50=-120\pi-A/4+B \qquad (2)$$

Solving (1) and (2) yields $A = 443.66$, $B = 537.91$

Thus,

$$V=-30\pi r-\frac{443.66}{r}+537.91$$

$$V(r=2)=-60\pi-\frac{443.66}{2}+537.91=\underline{\underline{127.58\text{ V}}}$$

**Prob. 6.10**

(a)

$$\nabla^2V_1=\frac{\partial^2V_1}{\partial x^2}+\frac{\partial^2V_1}{\partial y^2}+\frac{\partial^2V_1}{\partial z^2}=0+0-2\neq0$$

It <u>does not</u> satisfy Laplace's equation.

(b)

$$\nabla^2 V_2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho(-\rho^{-2}10\sin\phi)\right] + \frac{1}{\rho^2}\left(-\frac{10\sin\phi}{\rho}\right) = \frac{10}{\rho}(\rho^{-2})\sin\phi - \frac{10\sin\phi}{\rho^3} = 0$$

It does satisfy Laplace's equation.

( c)

$$\nabla^2 V_3 = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2(-r^{-2}5\sin\theta)\right] + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{(5\cos\theta)}{r^2}\right]$$

$$= 0 + \frac{5}{r^3\sin\theta}(1 - 2\sin^2\theta) \neq 0$$

It does not satisfy Laplace's equation.

**Prob. 6.11**

$$\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 6xy + 0 + 2c = 0$$

$c = -3xy$

**Prob. 6.12**

(a) $\frac{\partial V}{\partial x} = 4xyz, \quad \frac{\partial^2 V}{\partial x^2} = 4yz$

$$\frac{\partial V}{\partial y} = 2x^2 z - 3y^2 z, \quad \frac{\partial^2 V}{\partial y^2} = -6yz$$

$$\frac{\partial V}{\partial z} = 2x^2 y - y^3, \quad \frac{\partial^2 V}{\partial z^2} = 0$$

$$\nabla^2 V = 4yz - 6yz + 0 = -2yz$$

$\nabla^2 V \neq 0$, V does not satisfy Laplace's equation.

(b)

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon} = -2yz \longrightarrow \rho_v = 2yz\varepsilon$$

$$Q = \int \rho_v dv = \int_0^1\int_0^1\int_0^1 (2yz\varepsilon)dxdydz = 2\varepsilon(1)\frac{y^2}{2}\Big|_0^1 \frac{z^2}{2}\Big|_0^1 = \varepsilon/2 = 2\varepsilon_o/2 = \varepsilon_o$$

Q = 8.854 pC

**Prob. 6.13**

$$\nabla^2 V = \frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dV}{d\rho}\right) = 0 \quad \rightarrow \quad V = A\ln\rho + B$$

Let a = 1 cm, b = 1.5 cm, $V_o = 50V$

$$V(\rho = b) = 0 \quad \rightarrow \quad 0 = A\ln b + B \text{ or } B = -A\ln b$$

$$V(\rho = a) = V_o \quad \rightarrow \quad V_o = A\ln a - A\ln b = A\ln\frac{a}{b} \text{ or } A = \frac{V_o}{\ln\frac{a}{b}}$$

$$V = A\ln\rho - A\ln b = A\ln\frac{\rho}{b}$$

$$\boldsymbol{E} = -\nabla V = -\frac{dV}{d\rho}\boldsymbol{a}_\rho = -\frac{A}{\rho}\boldsymbol{a}_\rho = -\frac{V_o}{\rho\ln\frac{a}{b}}\boldsymbol{a}_\rho$$

$$\rho_s = D_n = \varepsilon E_n = \frac{\varepsilon_o\varepsilon_r V_o}{a\ln\frac{b}{a}} = \frac{50(4)\frac{10^{-9}}{36\pi}}{10^{-2}\ln 1.5} = \frac{400(50)}{36\pi\ln 1.5} \text{ nC/m}^2 = \underline{\underline{436.14 \text{ nC/m}^2}}$$

**Prob. 6.14**

$$\nabla^2 V = \frac{d^2V}{dz^2} = 0 \quad \longrightarrow \quad V = Az + B$$

When z=0, V = 0 $\longrightarrow$ B=0

When z=d, V = $V_o$ $\longrightarrow$ $V_o$=Ad or A = $V_o$/d

Hence,

$$V = \frac{V_o z}{d}$$

$$\boldsymbol{E} = -\nabla V = -\frac{dV}{dz}\boldsymbol{a}_z = -\frac{V_o}{d}\boldsymbol{a}_z$$

$$\boldsymbol{D} = \varepsilon\boldsymbol{E} = -\varepsilon_o\varepsilon_r\frac{V_o}{d}\boldsymbol{a}_z$$

Since $V_o$ = 50 V and d = 2mm,

$\underline{\underline{V = 25z \text{ kV},}}$ $\underline{\underline{\mathbf{E} = -25\mathbf{a}_z \text{ kV/m}}}$

$$\boldsymbol{D} = -\frac{10^{-9}}{36\pi}(1.5)25\times10^3\boldsymbol{a}_z = \underline{\underline{-332\boldsymbol{a}_z \text{ nC/m}^2}}$$

$$\rho_s = D_n = \underline{\underline{\pm 332 \text{ nC/m}^2}}$$

The surface charge density is positive on the plate at z=d and negative on the plate at z=0.

**Prob. 6.15** From Example 6.8, solving $\nabla^2 V = 0$ when $V = V(\rho)$ leads to

$$V = \frac{V_o \ln \rho / a}{\ln b / a} = V_o \frac{\ln(a/\rho)}{\ln(a/b)}$$

$$\mathbf{E} = -\nabla V = -\frac{V_o}{\rho \ln b/a}\mathbf{a}_\rho = \frac{V_o}{\rho \ln a/b}\mathbf{a}_\rho, \qquad \mathbf{D} = \varepsilon \mathbf{E} = -\frac{\varepsilon_o \varepsilon_r V_o}{\rho \ln b/a}\mathbf{a}_\rho$$

$$\rho_s = D_n = \pm \left.\frac{\varepsilon_o \varepsilon_r V_o}{\rho \ln b/a}\right|_{\rho = a,b}$$

In this case, $V_o$=100 V, b=5mm, a=15mm, $\varepsilon_r = 2$. Hence at $\rho = 10$mm,

$$V = \frac{100 \ln(10/15)}{\ln(5/15)} = \underline{\underline{36.91 \text{ V}}}$$

$$\mathbf{E} = \frac{100}{10x10^{-3} \ln 3}\mathbf{a}_\rho = \underline{\underline{9.102\mathbf{a}_\rho \text{ kV/m}}}$$

$$\mathbf{D} = 9.102x10^3 x \frac{10^{-9}}{36\pi} 2\mathbf{a}_\rho = \underline{\underline{161\mathbf{a}_\rho \text{ nC/m}^2}}$$

$$\rho_s(\rho = 5\text{mm}) = \frac{10^{-9}}{36\pi}(2)\frac{10^5}{5 \ln 3} = \underline{\underline{322 \text{ nC/m}^2}}$$

$$\rho_s(\rho = 15\text{mm}) = -\frac{10^{-9}}{36\pi}(2)\frac{10^5}{15 \ln 3} = \underline{\underline{-107.3 \text{ nC/m}^2}}$$

**Prob. 6.16**

$$\frac{1}{\rho}\frac{d^2V}{d\phi^2} = 0 \longrightarrow \frac{d^2V}{d\phi^2} = 0 \longrightarrow \frac{dV}{d\phi} = A$$

$$V = A\phi + B$$

$$0 = 0 + B \longrightarrow B = 0$$

$$50 = A\pi/2 \longrightarrow A = \frac{100}{\pi}$$

$$\mathbf{E} = -\nabla V = -\frac{1}{\rho}\frac{dV}{d\phi}\mathbf{a}_\phi = -\frac{A}{\rho}\mathbf{a}_\phi = \underline{\underline{-\frac{100}{\pi\rho}\mathbf{a}_\phi}}$$

**Prob. 6.17**

(a)

$$\frac{\partial V}{\partial \rho} = V_o(1+\frac{a^2}{\rho^2})\sin\phi$$

$$\rho\frac{\partial V}{\partial \rho} = V_o(\rho+\frac{a^2}{\rho})\sin\phi$$

$$\frac{\partial}{\partial \rho}\left(\rho\frac{\partial V}{\partial \rho}\right) = V_o(1-\frac{a^2}{\rho^2})\sin\phi$$

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial V}{\partial \rho}\right) = V_o(\frac{1}{\rho}-\frac{a^2}{\rho^3})\sin\phi$$

$$\frac{\partial^2 V}{\partial \phi^2} = -V_o(\rho-\frac{a^2}{\rho})\sin\phi$$

$$\frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} = -V_o(\frac{1}{\rho}-\frac{a^2}{\rho^3})\sin\phi$$

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial V}{\partial \rho}\right) + \frac{\partial^2 V}{\partial \phi^2} = 0$$

(b)

If $\rho^2 \gg a^2$, then $a^2/\rho^2 \ll 1$ and $V \approx V_o\rho\sin\phi$

$$\boldsymbol{E} = -\nabla V = -\frac{\partial V}{\partial \rho}\boldsymbol{a}_\rho - \frac{1}{\rho}\frac{\partial V}{\partial \phi}\boldsymbol{a}_\phi = \underline{\underline{-V_o\sin\phi\boldsymbol{a}_\rho - V_o\cos\phi\boldsymbol{a}_\phi}}$$

**Prob. 6.18**

$$\nabla^2 V = \frac{d^2V}{dx^2} = 0 \quad \longrightarrow \quad V = Ax + B$$

At $x = 20$ mm $= 0.02$ m, $V = 0$

$$0 = 0.02A + B \quad \longrightarrow (1)$$

$$\boldsymbol{E} = -100\boldsymbol{a}_x - \frac{dV}{dx}\boldsymbol{a}_x \quad \longrightarrow A = 110 \longrightarrow \quad (2)$$

From (1) $\quad B = -0.02A = -2.2$

Then $\quad V = 110x - 2.2$

At $x = 0 \quad \underline{\underline{V = -2.2V}}$

At x = 50 mm = 0.05 m,

$$V = 110\times 0.05 - 2.2 = \underline{\underline{3.3V}}$$

**Prob. 6.19**

$$\nabla^2 V = 0 \longrightarrow V = -A/r + B$$

At r=0.5, V=-50 $\longrightarrow$ -50 = -A/0.5 + B

Or

-50 = -2A + B (1)

At r = 1, V =50 $\longrightarrow$ 50 = -A + B (2)

From (1) and (2), A = 100, B = 150, and

$$V = -\frac{100}{r} + 150$$

$$\mathbf{E} = -\nabla V = -\frac{A}{r^2}\mathbf{a}_r = -\frac{100}{r^2}\mathbf{a}_r \quad \text{V/m}$$

**Prob. 6.20** From Example 6.4,

$$V = \frac{V_o \ln\left(\dfrac{\tan\theta/2}{\tan\theta_1/2}\right)}{\ln\left(\dfrac{\tan\theta_2/2}{\tan\theta_1/2}\right)}$$

$V_o = 100$, $\theta_1 = 30^o$, $\theta_2 = 120^o$, $r = \sqrt{3^2 + 0^2 + 4^2} = 5$, $\theta = \tan^{-1}\rho/z = \tan^{-1} 3/4 = 36.87^o$

$$V = 100\frac{\ln\left(\dfrac{\tan 18.435^o}{\tan 15^o}\right)}{\ln\left(\dfrac{\tan 60^o}{\tan 15^o}\right)} = \underline{\underline{117\ \text{V}}}$$

$$\mathbf{E} = -\frac{1}{r}\frac{\partial V}{\partial\theta}\mathbf{a}_\theta = \frac{-V_o\mathbf{a}_\theta}{r\ln\left(\dfrac{\tan\theta_2}{\tan\theta_1}\right)}\frac{\dfrac{\sec^2\theta/2}{\tan\theta_1/2}}{\dfrac{\tan\theta/2}{\tan\theta_1/2}} = \frac{-V_o\mathbf{a}_\theta}{r\ln\left(\dfrac{\tan\theta_2}{\tan\theta_1}\right)2\sin(\theta/2)\cos(\theta/2)}$$

$$\boldsymbol{E} = \frac{-V_o \boldsymbol{a}_\theta}{r\sin\theta \ln\left(\dfrac{\tan\theta_2/2}{\tan\theta_1/2}\right)} = \frac{-100\boldsymbol{a}_\theta}{5\sin 36.87^o \ln 6.464} = \underline{\underline{-17.86\boldsymbol{a}_\theta \ \ \text{V/m}}}$$

**Prob. 6.21**

(a)

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) = 0 \quad \longrightarrow \quad V = A\ln\rho + B$$

$$V(\rho = b) = 0 \quad \longrightarrow \quad 0 = A\ln b + B \quad \longrightarrow \quad B = -A\ln b$$

$$V(\rho = a) = V_o \quad \longrightarrow \quad V_o = A\ln a/b \quad \longrightarrow \quad A = -\frac{V_o}{\ln b/a}$$

$$V = -\frac{V_o}{\ln b/a}\ln\rho/b = \frac{V_o \ln b/\rho}{\ln b/a}$$

$$V(\rho = 15\text{mm}) = 70\frac{\ln 2}{\ln 50} = \underline{\underline{12.4 \text{ V}}}$$

(b) As the electron decelerates, potential energy gained = K.E. loss

$$e[70 - 12.4] = \frac{1}{2}m[(10^7)^2 - u^2] \quad \longrightarrow \quad 10^{14} - u^2 = \frac{2e}{m}\times 57.6$$

$$u^2 = 10^{14} - \frac{2\times 1.6\times 10^{-19}}{9.1\times 10^{-31}}\times 57.6 = 10^{12}(100 - 20.25)$$

$$\underline{\underline{u = 8.93\times 10^6 \ \ \text{m/s}}}$$

**Prob. 6.22** This is similar to case 1 of Example 6.5.

$$X = c_1 x + c_2, \quad Y = c_3 y + c_4$$

But $X(0) = 0 \quad \longrightarrow \quad 0 = c_2, \quad Y(0) = 0 \quad \longrightarrow \quad 0 = c_4$

Hence,

$$V(x,y) = XY = a_o xy, \quad a_o = c_1 c_3$$

Also, $V(xy = 4) = 20 \quad \longrightarrow \quad 20 = 4a_o \quad \longrightarrow \quad a_o = 5$

Thus,

$$V(x,y) = 5xy \ \text{ and } \ \boldsymbol{E} = -\nabla V = -5y\boldsymbol{a}_x - 5x\boldsymbol{a}_y$$

At (x,y) = (1,2),

$$\underline{\underline{V = 10 \text{ V}, \quad \mathbf{E} = -10\mathbf{a}_x - 5\mathbf{a}_y \text{ V/m}}}$$

**Prob. 6.23** (a) As in Example 6.5, $X(x) = A\sin(n\pi x/b)$

For Y,

$$Y(y) = c_1\cosh(n\pi y/b) + c_2\sinh(n\pi y/b)$$

$$Y(a) = 0 \longrightarrow 0 = c_1\cosh(n\pi a/b) + c_2\sinh(n\pi a/b) \longrightarrow c_1 = -c_2\tanh(n\pi a/b)$$

$$V = \sum_{n=1}^{\infty} a_n \sin(n\pi x/b)\left[\sinh(n\pi y/b) - \tanh(n\pi a/b)\cosh(n\pi y/b)\right]$$

$$V(x, y=0) = V_o = -\sum_{n=1}^{\infty} a_n \tanh(n\pi a/b)\sin(n\pi x/b)$$

$$-a_n\tanh(n\pi a/b) = \frac{2}{b}\int_0^b V_o\sin(n\pi x/b)dx = \begin{cases} \dfrac{4V_o}{n\pi}, & n = \text{odd} \\ 0, & n = \text{even} \end{cases}$$

Hence,

$$V = -\frac{4V_o}{\pi}\sum_{n=\text{odd}}^{\infty}\sin(n\pi x/b)\left[\frac{\sinh(n\pi y/b)}{n\tanh(n\pi a/b)} - \frac{\cosh(n\pi y/b)}{n}\right]$$

$$= -\frac{4V_o}{\pi}\sum_{n=\text{odd}}^{\infty}\frac{\sin(n\pi x/b)}{n\sinh(n\pi a/b)}\left[\sinh(n\pi y/b)\cosh(n\pi a/b) - \cosh(n\pi y/b)\sinh(n\pi a/b)\right]$$

$$\underline{\underline{= \frac{4V_o}{\pi}\sum_{n=\text{odd}}^{\infty}\frac{\sin(n\pi x/b)\sinh[n\pi(a-y)/b]}{n\sinh(n\pi a/b)}}}$$

Alternatively, for Y

$$Y(y) = c_1\sinh n\pi(y - c_2)/b$$

$$Y(a) = 0 \longrightarrow 0 = c_1\sinh[n\pi(a - c_2)/b] \longrightarrow c_2 = a$$

$$V = \sum_{n=1}^{\infty} b_n \sin(n\pi x/b)\sinh[n\pi(y-a)/b]$$

where

$$b_n = \begin{cases} -\dfrac{4V_o}{n\pi \sinh(n\pi a/b)}, & n = \text{odd} \\ 0, & n = \text{even} \end{cases}$$

(b) This is the same as Example 6.5 except that we exchange y and x. Hence

$$V(x,y) = \frac{4V_o}{\pi} \sum_{n=\text{odd}}^{\infty} \frac{\sin(n\pi y/a)\sinh(n\pi x/a)]}{n\sinh(n\pi b/a)}$$

(c) This is the same as part (a) except that we must exchange x and y. Hence

$$V(x,y) = \frac{4V_o}{\pi} \sum_{n=\text{odd}}^{\infty} \frac{\sin(n\pi y/a)\sinh[n\pi(b-x)/a]}{n\sinh(n\pi b/a)}$$

**Prob. 6.24** (a) X(x) is the same as in Example 6.5. Hence

$$V(x,y) = \sum_{n=1}^{\infty} \sin(n\pi x/b)\left[a_n \sinh(n\pi y/b) + b_n \cosh(n\pi y/b)\right]$$

At y=0, V = $V_1$

$$V_1 = \sum_{n=1}^{\infty} b_n \sin(n\pi x/b) \longrightarrow b_n = \begin{cases} \dfrac{4V_1}{n\pi}, & n = \text{odd} \\ 0, & \text{n = even} \end{cases}$$

At y=a, V = $V_2$

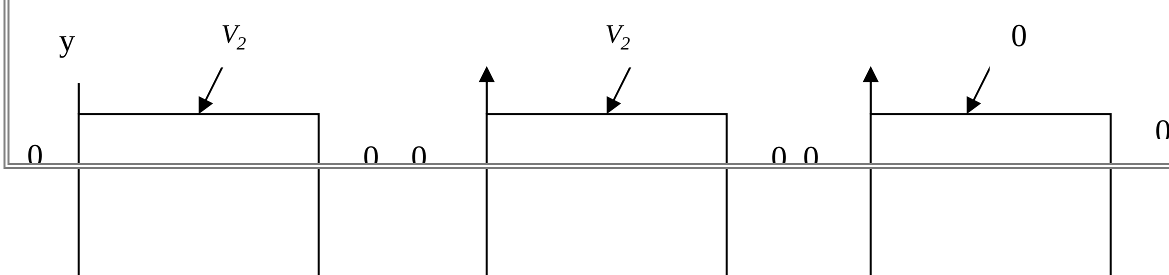
$$V_2 = \sum_{n=1}^{\infty} \sin(n\pi x/b)\left[a_n \sinh(n\pi a/b) + b_n \cosh(n\pi a/b)\right]$$

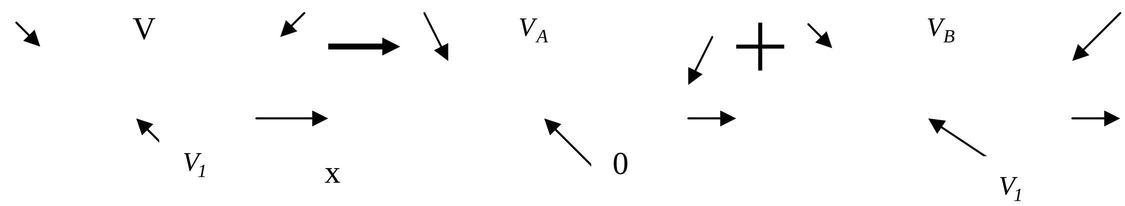
$$a_n \sinh(n\pi a/b) + b_n \cosh(n\pi a/b) = \begin{cases} \dfrac{4V_2}{n\pi}, & n = \text{odd} \\ 0, & \text{n = even} \end{cases}$$

or

$$a_n = \begin{cases} \dfrac{4V_2}{n\pi \sinh(n\pi a/b)}\left(V_2 - V_1 \cosh(n\pi a/b)\right), & n = \text{odd} \\ 0, & \text{n = even} \end{cases}$$

Alternatively, we may apply superposition principle.






i.e. $V = V_A + V_B$

$V_A$ is exactly the same as Example 6.5 with $V_o = V_2$, while $V_B$ is exactly the same **as** Prob. 6.19(a). Hence

$$V = \frac{4}{\pi}\sum_{n=\text{odd}}^{\infty} \frac{\sin(n\pi x/b)}{n\sinh(n\pi a/b)}\left[V_1 \sinh[n\pi(a-y)/b] + V_2 \sinh(n\pi y/b)\right]$$

(b)

$$V(x,y) = (a_1 e^{-\alpha x} + a_2 e^{+\alpha x})(a_3 \sin\alpha y + a_4 \cos\alpha y)$$

$$\lim_{x \longrightarrow \infty} V(x,y) = 0 \quad \longrightarrow \quad a_2 = 0$$

$$V(x, y=0) = 0 \quad \longrightarrow \quad a_4 = 0$$

$$V(x, y=a) = 0 \quad \longrightarrow \quad \alpha = n\pi/a, \quad n = 1,2,3,\ldots$$

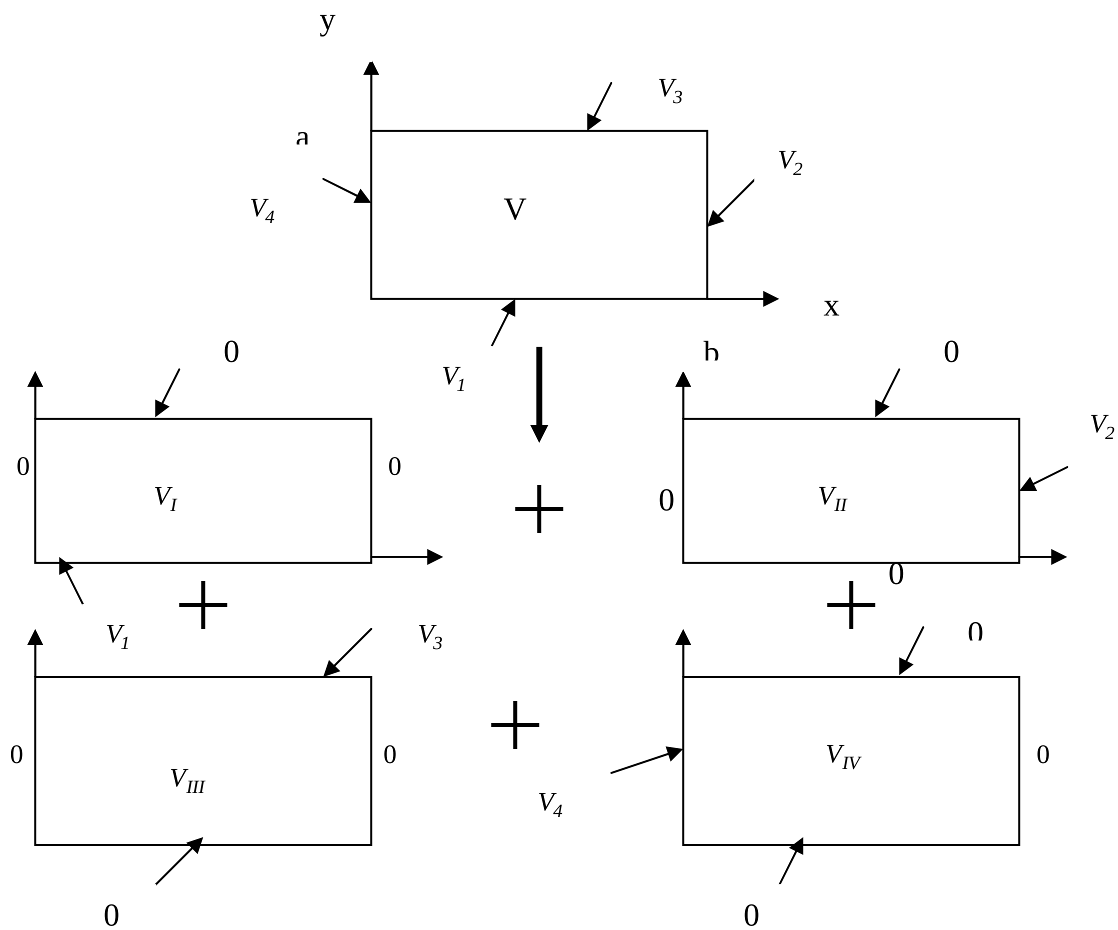
Hence,

$$V(x,y) = \sum_{n=1}^{\infty} a_n e^{-n\pi x/a} \sin(n\pi y/a)$$

$$V(x=0,y) = V_o = \sum_{n=1}^{\infty} a_n \sin(n\pi y/a) \quad \longrightarrow \quad a_n = \begin{cases} \frac{4V_o}{n\pi}, & n = \text{odd} \\ 0, & n = \text{even} \end{cases}$$

$$V(x,y) = \frac{4V_o}{\pi}\sum_{n=\text{odd}}^{\infty} \frac{\sin(n\pi y/a)}{n} \exp(-n\pi x/a)$$

(c ) The problem is easily solved using superposition theorem, as illustrated below.




Therefore,

$$V = V_I + V_{II} + V_{III} + V_{IV}$$

$$= \frac{4}{\pi}\sum_{n=odd}^{\infty}\frac{1}{n}\left\{\begin{aligned}&\frac{\sin(n\pi x/b)}{\sinh(n\pi a/b)}\left[V_1\sinh(n\pi(a-y)/b)+V_3\sinh(n\pi y/b)\right]\\&+\frac{\sin(n\pi x/a)}{\sinh(n\pi b/a)}\left[V_2\sinh(n\pi y/a)+V_4\sinh(n\pi(b-x)/a)\right]\end{aligned}\right\}$$

where

$$V_I = \frac{4V_1}{\pi}\sum_{n=\text{odd}}^{\infty}\frac{\sin(n\pi x/b)\sinh[n\pi(a-y)/b]}{n\sinh(n\pi a/b)}$$

$$V_{II} = \frac{4V_2}{\pi}\sum_{n=\text{odd}}^{\infty}\frac{\sin(n\pi x/a)\sinh(n\pi y/a)}{n\sinh(n\pi b/a)}$$

$$V_{III} = \frac{4V_3}{\pi}\sum_{n=\text{odd}}^{\infty}\frac{\sin(n\pi x/b)\sinh(n\pi y/b)}{n\sinh(n\pi a/b)}$$

$$V_{IV} = \frac{4V_4}{\pi}\sum_{n=\text{odd}}^{\infty}\frac{\sin(n\pi y/a)\sinh[n\pi(b-x)/a]}{n\sinh(n\pi b/a)}$$

**Prob. 6.25**

$$\boldsymbol{E} = -\nabla V = -\frac{\partial V}{\partial x}\boldsymbol{a}_x - \frac{\partial V}{\partial y}\boldsymbol{a}_y$$

$$E_x = \frac{4V_o}{\pi}\sum_{n=odd}^{\infty}\frac{n\pi}{a}\frac{\sin(n\pi y/a)}{n}\exp(-n\pi x/a)$$

$$E_y = -\frac{4V_o}{\pi}\sum_{n=odd}^{\infty}\frac{n\pi}{a}\frac{\cos(n\pi y/a)}{n}\exp(-n\pi x/a)$$

$$\boldsymbol{E} = \frac{4V_o}{a}\sum_{n=odd}^{\infty}\exp(-n\pi x/a)\left[\sin(n\pi y/a)\boldsymbol{a}_x - \cos(n\pi y/a)\boldsymbol{a}_y\right]$$

**Prob. 6.26**

This is similar to Example 6.5 except that we must exchange x and y. Going through the same arguments, we have

$$V(x,y) = \sum c_n \sinh\left(\frac{n\pi x}{b}\right)\sin\left(\frac{n\pi y}{b}\right)$$

Applying the condition at x=a, we get

$$V_o \sin\left(\frac{\pi y}{b}\right) = \sum c_n \sinh\left(\frac{n\pi a}{b}\right)\sin\left(\frac{n\pi y}{b}\right)$$

This yields

$$c_n \sinh\left(\frac{n\pi a}{b}\right) = \begin{cases} V_o, & n=1 \\ 0, & n \neq 1 \end{cases}$$

Hence,

$$V(x,y) = V_o \frac{\sinh\left(\frac{\pi x}{b}\right)\sin\left(\frac{\pi y}{b}\right)}{\sinh\left(\frac{\pi a}{b}\right)}$$

**Prob. 6.27**

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial\phi^2} = 0$$

If we let $V(\rho,\phi) = R(\rho)\Phi(\phi)$,

$$\frac{\Phi}{\rho}\frac{\partial}{\partial\rho}(\rho R') + \frac{1}{\rho^2} R\Phi'' = 0$$

or

$$\frac{\rho}{R}\frac{\partial}{\partial\rho}(\rho R') = -\frac{\Phi''}{\Phi} = \lambda$$

Hence

$$\Phi'' + \lambda\Phi = 0$$

and

$$\frac{\partial}{\partial\rho}(\rho R') - \frac{\lambda R}{\rho} = 0$$

or

$$R'' + \frac{R'}{\rho} - \frac{\lambda R}{\rho^2} = 0$$

**Prob. 6.28**

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial V}{\partial\theta}) = 0$$

If $V(r,\theta) = R(r)F(\theta), \quad r \neq 0,$

$$F\frac{d}{dr}(r^2 R') + \frac{R}{\sin\theta}\frac{d}{d\theta}(\sin\theta F') = 0$$

Dividing through by RF gives

$$\frac{1}{R}\frac{d}{dr}(r^2 R') = -\frac{1}{F\sin\theta}\frac{d}{d\theta}(\sin\theta F') = \lambda$$

Hence,

$$\sin\theta F'' + \cos\theta F' + \lambda F\sin\theta = 0$$

or

$$F'' + \cot\theta F' + \lambda F = 0$$

Also,

$$\frac{d}{dr}(r^2 R') - \lambda R = 0$$

or

$$R'' + \frac{2R'}{r} - \frac{\lambda}{r^2} R = 0$$

**Prob. 6.29** If the centers at $\phi = 0$ and $\phi = \pi/2$ are maintained at a potential difference of $V_o$, from Example 6.3,

$$E_\phi = \frac{2V_o}{\pi\rho}, \qquad J = \sigma E$$

Hence,

$$I = \int J \bullet dS = \frac{2V_o\sigma}{\pi} \int_{\rho=a}^{b} \int_{z=0}^{t} \frac{1}{\rho} d\rho dz = \frac{2V_o\sigma t}{\pi} \ln(b/a)$$

and

$$R = \frac{V_o}{I} = \frac{\pi}{2\sigma t \ln(b/a)}$$

**Prob. 6.30** If $V(r=a) = 0, \quad V(r=b) = V_o$, from Example 6.9,

$$E = \frac{V_o}{r^2(1/a - 1/b)}, \qquad J = \sigma E$$

Hence,

$$I = \int J \bullet dS = \frac{V_o\sigma}{1/a - 1/b} \int_{\theta=0}^{\alpha} \int_{\phi=0}^{2\pi} \frac{1}{r^2} r^2 \sin\theta d\theta d\phi = \frac{2\pi V_o\sigma}{1/a - 1/b} (-\cos\theta)\Big|_0^\alpha$$

$$R = \frac{V_o}{I} = \frac{\frac{1}{a} - \frac{1}{b}}{2\pi\sigma(1 - \cos\alpha)}$$

**Prob. 6.31**

This is the same as Problem 6.30 except that $\alpha = \pi$. Hence,

$$R = \frac{1}{2\pi\sigma(1 - \cos\pi)}\left(\frac{1}{a} - \frac{1}{b}\right) = \frac{1}{4\pi\sigma}\left(\frac{1}{a} - \frac{1}{b}\right)$$

**Prob. 6.32** For a spherical capacitor, from Eq. (6.38),

$$R = \frac{\frac{1}{a} - \frac{1}{b}}{4\pi\sigma}$$

For the hemisphere, $R' = 2R$ since the sphere consists of two hemispheres in parallel. As $b \longrightarrow \infty$,

$$R' = \lim_{b \longrightarrow \infty} \frac{2\left[\frac{1}{a} - \frac{1}{b}\right]}{4\pi\sigma} = \frac{1}{2\pi a\sigma}$$

$$G = 1/R' = 2\pi a\sigma$$

Alternatively, for an isolated sphere, $C = 4\pi\varepsilon a.$ But

$$RC = \frac{\varepsilon}{\sigma} \longrightarrow R = \frac{1}{4\pi a\sigma}$$

$$R' = 2R = \frac{1}{2\pi a\sigma} \quad \text{or} \quad G = 2\pi a\sigma$$

**Prob. 6.33**

(a) For the parallel-plate capacitor,

$$\mathbf{E} = -\frac{V_o}{d}\mathbf{a}_x$$

From Example 6.11,

$$C = \frac{1}{V_o^2}\int \varepsilon |E|^2 dv = \frac{1}{V_o^2}\int \varepsilon \frac{V_o^2}{d^2} dv = \frac{\varepsilon}{d^2} Sd = \frac{\varepsilon S}{d}$$

(b) For the cylindrical capacitor,

$$\mathbf{E} = -\frac{V_o}{\rho \ln b/a}\mathbf{a}_\rho$$

From Example 6.8,

$$C = \frac{1}{V_o^2}\iiint \frac{\varepsilon V_o^2}{(\rho \ln b/a)^2}\rho d\rho d\phi dz = \frac{2\pi\varepsilon L}{(\ln b/a)^2}\int_a^b \frac{d\rho}{\rho} = \frac{2\pi\varepsilon L}{\ln b/a}$$

(c )For the spherical capacitor,

$$\mathbf{E} = \frac{V_o}{r^2(1/a - 1/b)}\mathbf{a}_r$$

From Example 6.10,

$$C = \frac{1}{V_o^2}\iiint \frac{\varepsilon V_o^2}{r^4(1/a - 1/b)^2} r^2 \sin\theta d\theta dr d\phi = \frac{\varepsilon}{(1/a - 1/b)^2} 4\pi \int_a^b \frac{dr}{r^2} = \frac{4\pi\varepsilon}{\frac{1}{a} - \frac{1}{b}}$$

**Prob. 6.34**

Assume V($\rho$=a) = 0 and V($\rho$=b)=$V_o$. Following Example 6.8,

$$V = A\ln\rho + B = \frac{V_o}{\ln\frac{b}{a}}\ln\frac{\rho}{a}$$

$$\boldsymbol{J} = \sigma\boldsymbol{E} = -\sigma\nabla V, \qquad d\boldsymbol{S} = \rho d\phi dz\boldsymbol{a}_\rho$$

$$I = \int_S \boldsymbol{J}\bullet d\boldsymbol{S} = \int_{\phi=0}^{2\pi}\int_{z=0}^{L}\frac{\sigma V_o}{\rho\ln\frac{b}{a}}\rho d\phi dz = \frac{\sigma V_o}{\ln\frac{b}{a}}(2\pi L)$$

$$R = \frac{V_o}{I} = \frac{\ln\frac{b}{a}}{2\pi\sigma L}$$

$$G = \frac{I}{V_o} = \frac{2\pi\sigma L}{\ln\frac{b}{a}}$$

The conductance per unit length is

$$\text{G}' = \frac{G}{L} = \frac{2\pi\sigma}{\ln\frac{b}{a}}$$

**Prob. 6.35**

From eq. (6.37) or from previous problem

$$R = \frac{\ln\frac{b}{a}}{2\pi\sigma L}$$

$$P = VI = \frac{V^2}{R} = \frac{2\pi\sigma LV^2}{\ln\frac{b}{a}}$$

The power loss per unit length is

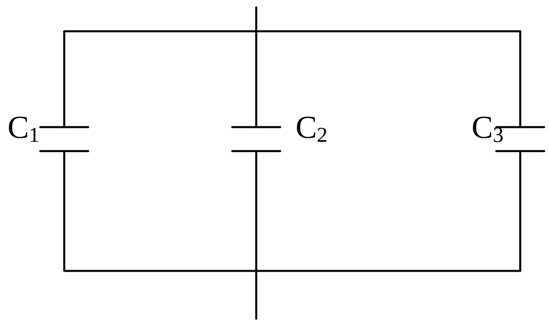
$$P' = \frac{P}{L} = \frac{2\pi\sigma V^2}{\ln\frac{b}{a}}$$

**Prob. 6.36**

$$C = \frac{\varepsilon S}{d} \quad \longrightarrow \quad S = \frac{Cd}{\varepsilon_o \varepsilon_r} = \frac{2\times10^{-9}\times10^{-6}}{4\times10^{-9}/36\pi}\ \text{m}^2 = \underline{\underline{0.5655\ \text{cm}^2}}$$

**Prob. 6.37**

This can be regarded as three capacitors in parallel.




$$C = C_1 + C_2 + C_3 = \sum \frac{\varepsilon_o \varepsilon_{rk} S_k}{d_k}$$

$$= \frac{\varepsilon_o}{2\times10^{-3}}\left[3\times15\times10^{-2}\times20\times10^{-2} + 5\times15\times10^{-2}\times20\times10^{-2} + 8\times15\times10^{-2}\times20\times10^{-2}\right]$$

$$= \frac{10^{-9}}{36\pi}\times\frac{15\times10^{-2}\times20\times10^{-2}}{2\times10^{-3}}[3+5+8] = \underline{\underline{2.122\ \text{nF}}}$$

**Prob. 6.38**

The structure may be treated as consisting of three capacitors in series.

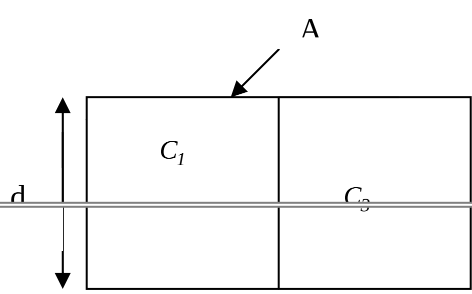
$$C_1 = \frac{\varepsilon_o A}{a}, \qquad C_2 = \frac{\varepsilon_r \varepsilon_o A}{a}, \qquad C_3 = \frac{\varepsilon_o A}{a}$$

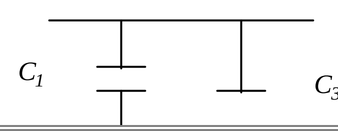
$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} = \frac{a}{\varepsilon_o A} + \frac{a}{\varepsilon_o \varepsilon_r A} + \frac{a}{\varepsilon_o A}$$

$$\frac{A}{aC} = \frac{2}{\varepsilon_o} + \frac{1}{\varepsilon_o \varepsilon_r} = \frac{2\varepsilon_r + 1}{\varepsilon_o \varepsilon_r}$$

$$C = \underline{\underline{\frac{\varepsilon_o \varepsilon_r A}{a(1+2\varepsilon_r)}}}$$

**Prob. 6.39**




From the figure above,

$$C = \frac{C_1 C_2}{C_1 + C_2} + C_3$$

here

$$C_1 = \frac{\varepsilon_o A/2}{d/2} = \frac{\varepsilon_o A}{d}, \quad C_2 = \frac{\varepsilon_o \varepsilon_r A}{d}, \quad C_3 = \frac{\varepsilon_o A}{2d}$$

$$C = \frac{\varepsilon_o^{\,2} \varepsilon_r A^2 / d^2}{\varepsilon_o(\varepsilon_r + 1)A/d} + \frac{\varepsilon_o A}{2d} = \frac{\varepsilon_o A}{d}\left(\frac{1}{2} + \frac{\varepsilon_r}{\varepsilon_r + 1}\right) = \frac{10^{-9}}{36\pi}\frac{10\times 10^{-4}}{2\times 10^{-3}}\left(\frac{1}{2} + \frac{6}{7}\right) \cong \underline{\underline{6 \text{ pF}}}$$

**Prob. 6.40**

$$C = \frac{\varepsilon_o S}{d} \longrightarrow S = \frac{Cd}{\varepsilon_o}$$

$$S = \frac{1\times 1\times 10^{-3}}{10^{-9}/36\pi} = 36\pi\times 10^{6}$$

$$\underline{\underline{S = 1.131\times 10^{8} \text{ m}^2}}$$

**Prob. 6.41**

$$F dx = dW_E \quad \longrightarrow \quad F = \frac{dW_E}{dx}$$

$$W_E = \int \frac{1}{2}\varepsilon |E|^2 \, dv = \frac{1}{2}\varepsilon_o \varepsilon_r E^2 xad + \frac{1}{2}\varepsilon_o E^2 da(1 - x)$$

where $E = V_o / d$ .

$$\frac{dW_E}{dx} = \frac{1}{2}\varepsilon_o \frac{V_o^{\,2}}{d^2}(\varepsilon_r - 1)da \quad \longrightarrow \quad F = \frac{\varepsilon_o(\varepsilon_r - 1)V_o^{\,2} a}{2d}$$

Alternatively, $W_E = \frac{1}{2} C V_o^{\,2}$, where

$$C = C_1 + C_2 = \frac{\varepsilon_o \varepsilon_r ax}{d} + \frac{\varepsilon_o \varepsilon_r (L - x)}{d}$$

$$\frac{dW_E}{dx} = \frac{1}{2}\varepsilon_o \frac{V_o^{\,2} a}{d}(\varepsilon_r - 1)$$

$$F = \frac{\varepsilon_o(\varepsilon_r - 1)V_o^2 a}{2d}$$

**Prob. 6.42**

(a)

$$C = \frac{\varepsilon_o S}{d} = \frac{10^{-9}}{36\pi}\frac{200\times10^{-4}}{3\times10^{-3}} = \underline{\underline{59 \text{ pF}}}$$

(b) $\rho_s = D_n = 10^{-6} \text{ nC/m}^2$. But

$$D_n = \varepsilon E_n = \frac{\varepsilon_o V_o}{d} = \rho_s$$

or

$$V_o = \frac{\rho_s d}{\varepsilon_o} = 10^{-6}\times3\times10^{-3}\times36\pi\times10^{9} = \underline{\underline{339.3 \text{ V}}}$$

(c )

$$F = \frac{Q^2}{2S\varepsilon_o} = \frac{\rho_s^{\,2} S}{2\varepsilon_o} = \frac{10^{-12}\times200\times10^{-4}\times36\pi\times10^{9}}{2} = \underline{\underline{1.131 \text{ mN}}}$$

**Prob. 6.43**

$$C_1 = \frac{\varepsilon_o\varepsilon_r S}{d}, \qquad C_2 = \frac{\varepsilon_o S}{d}$$

$$\frac{C_1}{C_2} = \varepsilon_r \qquad \longrightarrow \qquad \varepsilon_r = \frac{56\mu\text{F}}{32\mu\text{F}} = \underline{\underline{1.75}}$$

**Prob. 6.44**

(a) $$C = \frac{\varepsilon S}{d} = \frac{6.8\times\dfrac{10^{-9}}{36\pi}\times0.5}{4\times10^{-3}} = \underline{\underline{7.515 \text{ nF}}}$$

(b)

$$\rho_s = \frac{Q}{S}, \qquad C = \frac{Q}{V} \qquad \longrightarrow \qquad Q = CV$$

$$\rho_s = \pm\frac{CV}{S} = \pm\frac{7.515\times10^{-9}\times9}{0.5} = \underline{\underline{\pm135.27 \text{ nC/m}^2}}$$

**Prob. 6.45**

$$C_o = \frac{\varepsilon S}{d}, \qquad C = \frac{\varepsilon S}{3d} = \underline{\underline{C_o / 3}}$$

$$C_o = \frac{Q_o}{V} \quad \rightarrow \quad Q_o = C_o V$$

$$Q = CV = (C_o/3)V = \underline{\underline{Q_o / 3}}$$

$$E_o = \frac{V}{d}, \qquad E = \frac{V}{3d} = \underline{\underline{\frac{E_o}{3}}}$$

$$W = \frac{1}{2}CV^2 = \frac{1}{2}(C_o/3)V^2 = \underline{\underline{W_o / 3}}$$

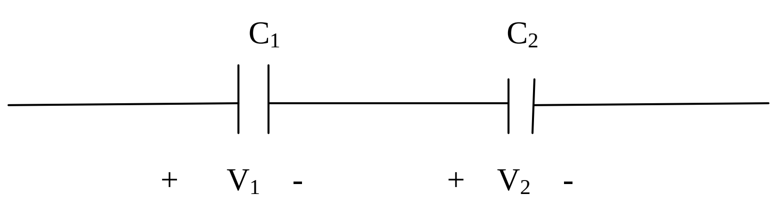
This indicates that two-thirds of the energy stored is lost in the connecting wires and source resistance.

**Prob. 6.46**

$$(a) \qquad C_1 = \frac{\varepsilon_o \varepsilon_{r1} S}{d}, \qquad C_2 = \frac{\varepsilon_o \varepsilon_{r2} S}{d}$$

$$C = \frac{C_1 C_2}{C_1 + C_2} = \frac{\frac{\varepsilon_o \varepsilon_{r1} S}{d} \frac{\varepsilon_o \varepsilon_{r2} S}{d}}{\frac{\varepsilon_o \varepsilon_{r1} S}{d} + \frac{\varepsilon_o \varepsilon_{r2} S}{d}} = \frac{\frac{\varepsilon_o \varepsilon_{r1} \varepsilon_{r2} S}{d}}{\varepsilon_{r1} + \varepsilon_{r2}} = \frac{10^{-9}}{36\pi} \frac{40 \times 10^{-4}}{2 \times 10^{-3}} \frac{4 \times 6}{4+6} = \frac{4.8}{36\pi} \text{ nF} = \underline{\underline{42.44 \text{ pF}}}$$

$$(b) \qquad C = \frac{Q}{V} \quad \rightarrow \quad Q = CV = \underline{\underline{509.3 \text{ pC}}}$$



$(c)$ $C_1$ and $C_2$ are in series as shown above

$Q = C_1V_1 + C_2V_2, \quad V_1 + V_2 = V = 12$

Solving these gives

$$V_1 = \frac{C_2}{C_1 + C_2}V = \frac{6}{10}(12) = 7.2$$

$$V_2 = \frac{C_1}{C_1 + C_2}V = \frac{4}{10}(12) = 4.8$$

$$E_1 = \frac{V_1}{d} = \underline{\underline{3.6 \text{ kV/m}}}$$

$$E_2 = \frac{V_2}{d} = \underline{\underline{2.4 \text{ kV/m}}}$$

$$D = D_1 = D_2 = \varepsilon_1 E_1 = \underline{\underline{1.2732 \times 10^{-7} \text{ C/m}^2}}$$

$$P_1 = \chi_{e_1}\varepsilon_o E_1 = \underline{\underline{9.549 \times 10^{-8} \text{ C/m}^2}}$$

$$P_2 = \chi_{e2}\varepsilon_o E_2 = \underline{\underline{1.061 \times 10^{-8} \text{ C/m}^2}}$$

**Prob.6.47**

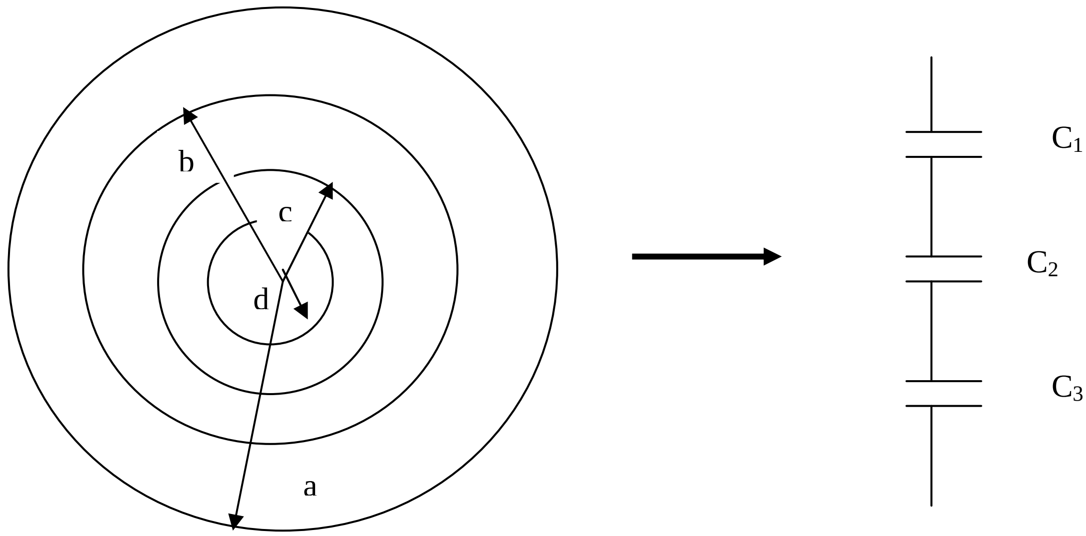
(a)

$$C = \frac{4\pi\varepsilon}{\frac{1}{a} - \frac{1}{b}} = \frac{4\pi \times 2.25 \times \frac{10^{-9}}{36\pi}}{\frac{1}{5x10^{-2}} - \frac{1}{10x10^{-2}}} = \underline{\underline{25 \text{ pF}}}$$

(b) Q = C $V_o$= 25x80 pC

$$\rho_s = \frac{Q}{4\pi r^2} = \frac{25 \times 80}{4\pi \times 25 \times 10^{-4}} \text{ pC/m}^2 = \underline{\underline{63.66 \text{ nC/m}^2}}$$

**Prob. 6.48**




$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$$

where $C_1 = \dfrac{4\pi\varepsilon_3}{\dfrac{1}{b} - \dfrac{1}{a}}, \qquad C_2 = \dfrac{4\pi\varepsilon_2}{\dfrac{1}{c} - \dfrac{1}{b}}, \qquad C_3 = \dfrac{4\pi\varepsilon_1}{\dfrac{1}{d} - \dfrac{1}{c}},$

$$\frac{4\pi}{C} = \frac{1/b - 1/a}{\varepsilon_3} + \frac{1/c - 1/b}{\varepsilon_2} + \frac{1/d - 1/c}{\varepsilon_1}$$

$$C = \frac{4\pi}{\dfrac{\varepsilon_1}{\dfrac{1}{d} - \dfrac{1}{c}} + \dfrac{\varepsilon_2}{\dfrac{1}{c} - \dfrac{1}{b}} + \dfrac{\varepsilon_3}{\dfrac{1}{b} - \dfrac{1}{a}}}$$

**Prob. 6.49**

We may place a charge Q on the inner conductor. The negative charge –Q is on the outer surface of the shell. Within the shell, **E** = 0, i.e. between r=c and r=b. Otherwise,

$$\boldsymbol{E} = \frac{Q}{4\pi\varepsilon_o r^2}\boldsymbol{a}_r$$

The potential at r=a is

$$V_a = -\int_{-\infty}^{a} \boldsymbol{E}\cdot d\boldsymbol{l} = -\int_{-\infty}^{c} E_r dr - \int_{c}^{b} E_r dr - \int_{b}^{a} E_r dr$$

$$= -\frac{Q}{4\pi\varepsilon_o}\int_{-\infty}^{c}\frac{dr}{r^2} - 0 - \frac{Q}{4\pi\varepsilon_o}\int_{b}^{a}\frac{dr}{r^2} = \frac{Q}{4\pi\varepsilon_o c} + \frac{Q}{4\pi\varepsilon_o}\left(\frac{1}{a}-\frac{1}{b}\right)$$

$$C = \frac{Q}{V_a} = \underline{\underline{\frac{1}{\frac{1}{4\pi\varepsilon_o c} + \frac{1}{4\pi\varepsilon_o}\left(\frac{1}{a}-\frac{1}{b}\right)}}}$$

**Prob. 6.50**

We can regard this as having two cylindrical capacitors in series.

$$C_1 = \frac{2\pi\varepsilon_o\varepsilon_{r1}L}{\ln\frac{c}{a}}, \qquad C_2 = \frac{2\pi\varepsilon_o\varepsilon_{r2}L}{\ln\frac{b}{c}}$$

$$C = \frac{C_1C_2}{C_1+C_2} = \frac{\frac{2\pi\varepsilon_o\varepsilon_{r1}L}{\ln\frac{c}{a}}\frac{2\pi\varepsilon_o\varepsilon_{r2}L}{\ln\frac{b}{c}}}{\frac{2\pi\varepsilon_o\varepsilon_{r1}L}{\ln\frac{c}{a}} + \frac{2\pi\varepsilon_o\varepsilon_{r2}L}{\ln\frac{b}{c}}} = \underline{\underline{\frac{2\pi\varepsilon_o\varepsilon_{r1}\varepsilon_{r2}L}{\varepsilon_{r1}\ln\frac{b}{c} + \varepsilon_{r2}\ln\frac{c}{a}}}}$$

**Prob. 6.51**

$$C = \frac{2\pi\varepsilon L}{\ln(b/a)} = \frac{2\pi\times 2.5\times\frac{10^{-9}}{36\pi}\times 3\times 10^3}{\ln(8/5)} = \underline{\underline{0.8665\ \mu\text{F}}}$$

**Prob. 6.52**

Let the plate at $\phi=0$ be 0, i.e. V(0)=0 and let the plate at $\phi=\pi/4$ be $V_o$, i.e. $V(\pi/4)=V_o$.

$$\nabla^2 V = \frac{1}{\rho^2}\frac{d^2V}{d\phi^2} = 0 \quad \longrightarrow \quad \frac{dV}{d\phi} = A \quad \longrightarrow \quad V = A\phi + B$$

$$V(0) = 0 \quad \longrightarrow \quad 0 = 0 + B \quad \longrightarrow \quad B = 0$$

$$V(\pi/4) = V_o \quad \longrightarrow \quad V_o = A\pi/4 \quad \longrightarrow \quad A = \frac{4V_o}{\pi}$$

$$\boldsymbol{E} = -\nabla V = -\frac{1}{\rho}\frac{dV}{d\phi}\boldsymbol{a}_\phi = -\frac{A}{\rho}\boldsymbol{a}_\phi = -\frac{4V_o}{\pi\rho}\boldsymbol{a}_\phi$$

$$\boldsymbol{D} = \varepsilon\boldsymbol{E} = -\frac{4\varepsilon V_o}{\pi\rho}\boldsymbol{a}_\phi$$

$$\rho_s = D_n = -\frac{4\varepsilon V_o}{\pi\rho}$$

$$Q = \int \rho_s dS = -\int_{\rho=a}^{b}\int_{z=0}^{L}\frac{4\varepsilon V_o}{\pi\rho}d\rho dz = -\frac{4\varepsilon V_o}{\pi}L\ln(b/a)$$

$$C = \frac{|Q|}{V_o} = \underline{\underline{\frac{4\varepsilon L}{\pi}\ln(b/a)}}$$

**Prob. 6.53**

Since $V = V(\phi)$,

$$\nabla^2 V = 0 \quad \rightarrow \quad \frac{1}{\rho^2}\frac{d^2V}{d\phi^2} = 0$$

But $\rho \neq 0$, $\frac{d^2V}{d\phi^2} = 0 \quad \rightarrow \quad \frac{dV}{d\phi} = A \quad \rightarrow \quad V = A\phi + B$

For $\phi$=0, V=0 $\quad \rightarrow \quad$ 0=0+$B$ $\quad \rightarrow \quad B = 0$

For $\phi=\alpha$, V=$V_o$ $\quad \rightarrow \quad V_o = A\alpha \quad \rightarrow \quad A = \frac{V_o}{\alpha}$

$$\boldsymbol{E} = -\nabla V = -\frac{1}{\rho}\frac{dV}{d\phi}\boldsymbol{a}_\rho = -\frac{A}{\rho}\boldsymbol{a}_\rho$$

$$\rho_s = D_n = \varepsilon E_\phi = \frac{A}{\rho}\varepsilon, \qquad dS = d\rho dz$$

$$Q = \int_S \rho_s dS = \int_{\rho=\rho_1}^{\rho_2}\int_{z=0}^{L}\frac{A}{\rho}\varepsilon d\rho dz = A\varepsilon L\ln\frac{\rho_2}{\rho_1} = \frac{V_o}{\alpha}\varepsilon L\ln\frac{\rho_2}{\rho_1}$$

$$C = \frac{Q}{V_o} = \underline{\underline{\frac{\varepsilon L}{\alpha}\ln\frac{\rho_2}{\rho_1}}}$$

**Prob. 6.54**

$$C = \frac{2\pi\varepsilon_o L}{\ln(b/a)} = \frac{2\pi \times \frac{10^{-9}}{36\pi} \times 100 \times 10^{-6}}{\ln(600/20)} = 1.633 \times 10^{-15} \text{ F}$$

$$V = Q/C = \frac{50 \times 10^{-15}}{1.633 \times 10^{-15}} = \underline{\underline{30.62 \text{ V}}}$$

**Prob. 6.55**

$$C_1 = \frac{2\pi\varepsilon_1}{\ln(b/a)}, \qquad C_2 = \frac{2\pi\varepsilon_2}{\ln(c/b)}$$

Since the capacitance are in series, the total capacitance per unit length is

$$C = \frac{C_1 C_2}{C_1 + C_2} = \underline{\underline{\frac{2\pi\varepsilon_1\varepsilon_2}{\varepsilon_2 \ln(b/a) + \varepsilon_1 \ln(c/b)}}}$$

**Prob. 6.56**

(a) This is similar to Example 6.10.

$$\nabla^2 V = 0 \qquad \rightarrow \qquad V = \frac{V_o\left(\frac{1}{r} - \frac{1}{b}\right)}{\frac{1}{a} - \frac{1}{b}}$$

$$\boldsymbol{E} = -\nabla V = -\frac{dV}{dr}\boldsymbol{a}_r = \frac{-V_o}{r^2\left(\frac{1}{a} - \frac{1}{b}\right)}\boldsymbol{a}_r$$

At r=a, $\rho_s = D_n = \varepsilon E_r = \dfrac{\varepsilon_o \varepsilon_r V_o}{a^2\left(\frac{1}{a} - \frac{1}{b}\right)}$

$$\varepsilon_r = \frac{\rho_s a^2\left(\frac{1}{a} - \frac{1}{b}\right)}{\varepsilon_o V} = \frac{400 \times 10^{-9}(4 \times 10^{-4})\left(\frac{1}{2} - \frac{1}{4}\right)\frac{1}{10^{-2}}}{\frac{10^{-9}}{36\pi}(100)} = 16(36\pi)10^{-2}(1/4) = \underline{\underline{4.524}}$$

(b)

$$C = \frac{4\pi\varepsilon_o\varepsilon_r}{\frac{1}{a} - \frac{1}{b}} = \frac{4\pi \times \frac{10^{-9}}{36\pi} 4.524}{\left(\frac{1}{2} - \frac{1}{4}\right)\frac{1}{10^{-2}}} = \frac{452.4}{9(1/4)} = \underline{\underline{201.1 \text{ nF}}}$$

**Prob. 6.57**
Each half has capacitance given by

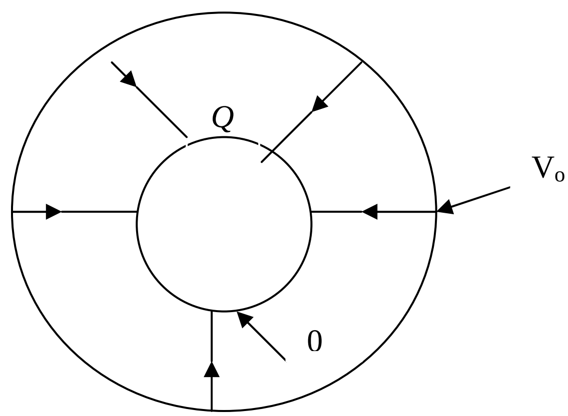
$$C = \frac{2\pi\varepsilon}{\frac{1}{a} - \frac{1}{b}} = \frac{2\pi\varepsilon ab}{b-a}$$

The two halves may be regarded as capacitors in parallel. Hence,

$$C = C_1 + C_2 = \frac{2\pi\varepsilon_1 ab}{b-a} + \frac{2\pi\varepsilon_2 ab}{b-a} = \underline{\underline{\frac{2\pi(\varepsilon_1+\varepsilon_2)ab}{b-a}}}$$

**Prob. 6.58**

$$\boldsymbol{E} = \frac{Q}{4\pi\varepsilon r^2}\boldsymbol{a}_r$$




$$W = \frac{1}{2}\int \varepsilon\,|\boldsymbol{E}|^2\, dv = \iiint \frac{Q^2}{32\pi^2\varepsilon^2 r^4}\varepsilon r^2 \sin\theta d\theta d\phi dr$$

$$= \frac{Q^2}{32\pi^2\varepsilon}(2\pi)(2)\int_c^b \frac{dr}{r^2} = \frac{Q^2}{8\pi\varepsilon}\left(\frac{1}{c} - \frac{1}{b}\right)$$

$$W = \frac{Q^2(b-c)}{8\pi\varepsilon bc}$$

**Prob. 6.59**

(a) Method 1: $\boldsymbol{E} = \frac{\rho_s}{\varepsilon}(-\boldsymbol{a}_x)$, where $\rho_s$ is to be determined.

$$V_o = -\int \boldsymbol{E}\bullet dl = -\int \frac{-\rho_s}{\varepsilon}dx = \rho_s\int_0^d \frac{1}{\varepsilon_o}\frac{d}{d+x}dx = \frac{\rho_s}{\varepsilon}d\ln(x+d)\Big|_0^d$$

$$V_o = \rho_s d \ln\frac{2d}{d} \quad \longrightarrow \quad \rho_s = \frac{V_o \varepsilon_o}{d \ln 2}$$

$$\mathbf{E} = -\frac{\rho_s}{\varepsilon}\mathbf{a}_x = -\frac{V_o}{(x+d)\ln 2}\mathbf{a}_x$$

<u>Method 2</u>: We solve Laplace's equation

$$\nabla \bullet (\varepsilon \nabla V) = \frac{d}{dx}(\varepsilon \frac{dV}{dx}) = 0 \quad \longrightarrow \quad \varepsilon \frac{dV}{dx} = A$$

$$\frac{dV}{dx} = \frac{A}{\varepsilon} = \frac{Ad}{\varepsilon_o (x+d)} = \frac{c_1}{x+d}$$

$$V = c_1 \ln(x+d) + c_2$$

$$V(x=0) = 0 \quad \longrightarrow \quad 0 = c_1 \ln d + c_2 \quad \longrightarrow \quad c_2 = -c_1 \ln d$$

$$V(x=d) = V_o \quad \longrightarrow \quad V_o = c_1 \ln 2d - c_1 \ln d = c_1 \ln 2$$

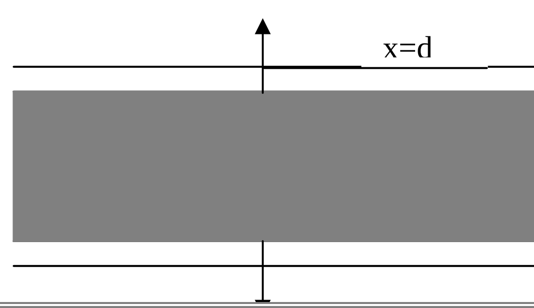
$$c_1 = \frac{V_o}{\ln 2}$$

$$V = c_1 \ln\frac{x+d}{d} = \frac{V_o}{\ln 2}\ln\frac{x+d}{d}$$

$$\mathbf{E} = -\frac{dV}{dx}\mathbf{a}_x = -\frac{V_o}{(x+d)\ln 2}\mathbf{a}_x$$

(b) $\mathbf{P} = (\varepsilon_r - 1)\varepsilon_o \mathbf{E} = -\left(\frac{x+d}{d} - 1\right)\frac{\varepsilon_o V_o}{(x+d)\ln 2}\mathbf{a}_x = -\frac{\varepsilon_o x V_o}{d(x+d)\ln 2}\mathbf{a}_x$

(c )



x=0

$$\rho_{ps}\,|_{x=0} = \mathbf{P} \bullet (-\mathbf{a}_x)\,|_{x=0} = \underline{\underline{0}}$$

$$\rho_{ps}\,|_{x=d} = \mathbf{P} \bullet \mathbf{a}_x\,|_{x=d} = \underline{\underline{-\frac{\varepsilon_o V_o}{2d\ln 2}}}$$

(d) $$E = \frac{\rho_s}{\varepsilon}\mathbf{a}_x = \frac{Q}{\varepsilon S}\mathbf{a}_x = \frac{Q}{\varepsilon_o(1+\frac{x}{d})S}\mathbf{a}_x$$

$$V = -\int \mathbf{E}\cdot d\mathbf{l} = -\frac{Q}{\varepsilon_o S}\int_a^d \frac{dx}{(1+\frac{x}{d})} = \frac{Q}{\varepsilon_o S} d\ln 2$$

$$C = \frac{Q}{V} = \underline{\underline{\frac{\varepsilon_o S}{d\ln 2}}}$$

**Prob. 6.60**

We solve Laplace’s equation for an inhomogeneous medium.

$$\nabla \cdot (\varepsilon \nabla V) = \frac{d}{dx}\left(\varepsilon \frac{dV}{dx}\right) = 0 \quad \longrightarrow \quad \varepsilon \frac{dV}{dx} = A$$

$$\frac{dV}{dx} = \frac{A}{\varepsilon} = \frac{A}{2\varepsilon_o}\left[1 + \left(\frac{x}{d}\right)^2\right]$$

$$V = \frac{A}{2\varepsilon_o}(x + \frac{x^3}{3d^2}) + B$$

When x=d, V=$V_o$,

$$V_o = \frac{A}{2\varepsilon_o}(d + \frac{d}{3}) + B \quad \longrightarrow \quad V_o = \frac{2Ad}{3\varepsilon_o} + B \qquad (1)$$

When x = -d, V=0,

$$0 = \frac{A}{2\varepsilon_o}(-d - \frac{d}{3}) + B \quad \longrightarrow \quad 0 = -\frac{2Ad}{3\varepsilon_o} + B \qquad (2)$$

Adding (1) and (2), $V_o = 2B \quad \longrightarrow \quad B = V_o / 2$

From (2),

$$B = \frac{2Ad}{3\varepsilon_o} = \frac{V_o}{2} \quad \longrightarrow \quad A = \frac{3\varepsilon_o V_o}{4d}$$

$$\boldsymbol{E} = -\nabla V = -\frac{dV}{dx}\boldsymbol{a}_x = -\frac{A}{\varepsilon}\boldsymbol{a}_x = -\frac{3\varepsilon_o V_o}{4d}\frac{\left[1 + \left(\frac{x}{d}\right)^2\right]}{2\varepsilon_o}\boldsymbol{a}_x = -\frac{-3V_o}{8d}\left[1 + \left(\frac{x}{d}\right)^2\right]\boldsymbol{a}_x$$

$$\rho_s = D \cdot a_n = \varepsilon E \cdot a_x \Big|_{x=d} = -A = -\frac{3\varepsilon_o V_o}{4d}$$

$$Q = \int_S \rho_s dS = \rho_s S = -\frac{3S\varepsilon_o V_o}{4d}$$

$$C = \frac{|Q|}{V_o} = \underline{\underline{\frac{3\varepsilon_o S}{4d}}}$$

**Prob. 6.61**

Method 1: Using Gauss's law,

$$Q = \int \boldsymbol{D} \bullet dS = 4\pi r^2 D_r \quad \longrightarrow \quad \boldsymbol{D} = \frac{Q}{4\pi r^2}\boldsymbol{a}_r, \quad \varepsilon = \frac{\varepsilon_o k}{r^2}$$

$$\boldsymbol{E} = \boldsymbol{D} / \varepsilon = \frac{Q}{4\pi\varepsilon_o k}\boldsymbol{a}_r$$

$$V = -\int \mathbf{E} \bullet d\mathbf{l} = -\frac{Q}{4\pi\varepsilon_o k}\int_b^a dr = -\frac{Q}{4\pi\varepsilon_o k}(b-a)$$

$$C = \frac{Q}{|V|} = \underline{\underline{\frac{4\pi\varepsilon_o k}{b-a}}}$$

Method 2: Using the inhomogeneous Laplace's equation,

$$\nabla \bullet (\varepsilon \nabla V) = 0 \quad \longrightarrow \quad \frac{1}{r^2}\frac{d}{dr}\left(\frac{\varepsilon_o k}{r^2} r^2 \frac{dV}{dr}\right) = 0$$

$$\varepsilon_o k \frac{dV}{dr} = A' \quad \longrightarrow \quad \frac{dV}{dr} = A \text{ or } V = Ar + B$$

$$V(r=a) = 0 \quad \longrightarrow \quad 0 = Aa + B \quad \longrightarrow \quad B = -Aa$$

$$V(r=b) = V_o \quad \longrightarrow \quad V_o = Ab + B = A(b-a) \quad \longrightarrow \quad A = \frac{V_o}{b-a}$$

$$\mathbf{E} = -\frac{dV}{dr}\mathbf{a}_r = -A\mathbf{a}_r = -\frac{V_o}{b-a}\mathbf{a}_r$$

$$\rho_s = D_n = -\frac{V_o}{b-a}\frac{\varepsilon_o k}{r^2}\Big|_{r=a,b}$$

$$Q = \int \rho_s dS = -\frac{V_o \varepsilon_o k}{b-a}\iint \frac{1}{r^2} r^2 \sin\theta \, d\theta \, d\phi = -\frac{V_o \varepsilon_o k}{b-a} 4\pi$$

$$C = \frac{|Q|}{V_o} = \underline{\underline{\frac{4\pi\varepsilon_o k}{b-a}}}$$

**Prob. 6.62**

$$C = 4\pi\varepsilon_o a = 4\pi \times \frac{10^{-9}}{36\pi} \times 6.37 \times 10^6 = \underline{\underline{0.708 \text{ mF}}}$$

**Prob.6.63**

$$C = \frac{Q}{V}$$

$$\boldsymbol{D} = \frac{Q}{2\pi\rho L}\boldsymbol{a}_\rho$$

$$\boldsymbol{E} = \frac{D}{\varepsilon} = \frac{Q}{2\pi\rho L\varepsilon_o(3)(1+\rho)}$$

$$V = -\int \boldsymbol{E}\cdot dl = \frac{Q}{6\pi L\varepsilon_o}\int_a^b \frac{d\rho}{\rho(1+\rho)}$$

Let $\frac{1}{\rho(1+\rho)} = \frac{A}{\rho} + \frac{B}{1+\rho}$

Using partial fractions

A=1, B= -1

$$V = \frac{Q}{6\pi\varepsilon_0 L}\left[\int_a^b \frac{d\rho}{\rho} - \int_a^b \frac{d\rho}{1+\rho}\right]$$

$$= \frac{Q}{6\pi\varepsilon_0 L}\left[\ln\rho - \ln(1+\rho)\right]\Big|_a^b$$

$$= \frac{Q}{6\pi\varepsilon_0 L}\left[\ln\frac{b}{1+b} - \ln\frac{a}{1+a}\right]$$

If a=1 mm, and b=5 mm

$$C = \frac{Q}{|V|}$$

$$= \frac{6\pi\varepsilon_o}{\ln\frac{b}{1+b} - \ln\frac{a}{1+a}}$$

$$= \frac{6\pi\times\frac{10^{-9}}{36\pi}}{\ln\frac{5}{6} - \ln\frac{1}{2}}$$

$$= \frac{\frac{1}{6}\times 10^{-9}}{\ln 0.8333 - \ln 0.5} = \frac{1}{6}\times 1.9591 \text{ nF}$$

$$C = \underline{\underline{0.326 \text{ nF}}}$$

**Prob. 6.64**

$$\frac{D-a}{a} = \frac{D}{a} - 1 = 11$$

$$C \simeq \frac{\pi \times \frac{10^{-9}}{36\pi}(4)}{\ln 11} = \frac{4}{36 \ln 11} \text{ nF/m} = \underline{\underline{46.34 \text{ pF/m}}}$$

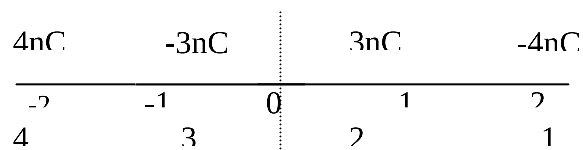
**Prob. 6.65**

(a) From eq. (6.46),

$$\rho_s = \frac{-Qh}{2\pi[x^2 + y^2 + h^2]^{3/2}} = \frac{-10 \times 10^{-9}(10)}{2\pi[4 + 16 + 100]^{3/2}} = \frac{-10^{-7}}{2\pi(120)^{3/2}} = \underline{\underline{-12.107 \text{ pF/m}^2}}$$

(b) $Q_{in} = -Q = \underline{\underline{-10 \text{ nC}}}$

**Prob. 6.66**




(a) $Q_i$ = -(3nC – 4nC) = $\underline{\underline{\text{1nC}}}$

(b) The force of attraction between the charges and the plates is

$$\boldsymbol{F} = \boldsymbol{F}_{13} + \boldsymbol{F}_{14} + \boldsymbol{F}_{23} + \boldsymbol{F}_{24}$$

$$|F| = \frac{10^{-18}}{4\pi \times 10^{-9}/36\pi}\left[\frac{9}{2^2} - \frac{2(12)}{3^2} + \frac{16}{4^2}\right] = \underline{\underline{5.25 \text{ nN}}}$$

**Prob. 6.67**

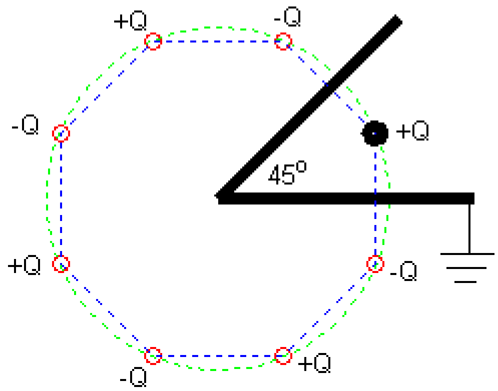
We have 7 images as follows: -Q at (-1,1,1), -Q at (1,-1,1), -Q at (1,1,-1),

-Q at (-1,-1,-1), Q at (1,-1,-1), Q at (-1,-1,1), and Q at (-1,1,-1). Hence,

$$\boldsymbol{F} = \frac{Q^2}{4\pi\varepsilon_o}\left[\begin{array}{l} -\frac{2}{2^3}\boldsymbol{a}_x - \frac{2}{2^3}\boldsymbol{a}_y - \frac{2}{2^3}\boldsymbol{a}_z - \frac{(2\boldsymbol{a}_x + 2\boldsymbol{a}_y + 2\boldsymbol{a}_z)}{12^{3/2}} + \frac{(2\boldsymbol{a}_y + 2\boldsymbol{a}_z)}{8^{3/2}} \\ + \frac{(2\boldsymbol{a}_x + 2\boldsymbol{a}_y)}{8^{3/2}} + \frac{(2\boldsymbol{a}_x + 2\boldsymbol{a}_z)}{8^{3/2}} \end{array}\right]$$

$$= 0.9(\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z)\left(-\frac{1}{4} - \frac{1}{12\sqrt{3}} + \frac{1}{4\sqrt{2}}\right) = \underline{\underline{-0.1092(\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z)\ \text{N}}}$$

**Prob. 6.68**




$$N = \left(\frac{360^o}{45^o} - 1\right) = \underline{\underline{7}}$$

**Prob. 6.69**

(a)

$$\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_- = \frac{\rho_L}{2\pi\varepsilon_o}\left(\frac{\mathbf{a}_{\rho1}}{\rho_1} - \frac{\mathbf{a}_{\rho2}}{\rho_2}\right) = \frac{16\times10^{-9}}{2\pi\times\dfrac{10^{-9}}{36\pi}}\left[\frac{(2,-2,3)-(3,-2,4)}{|(2,-2,3)-(3,-2,4)|^2} - \frac{(2,-2,3)-(3,-2,-4)}{|(2,-2,3)-(3,-2,-4)|^2}\right]$$

$$= 18x16\left[\frac{(-1,0,-1)}{2} - \frac{(-1,0,7)}{50}\right] = \underline{\underline{-138.2\mathbf{a}_x - 184.3\mathbf{a}_y\ \text{V/m}}}$$

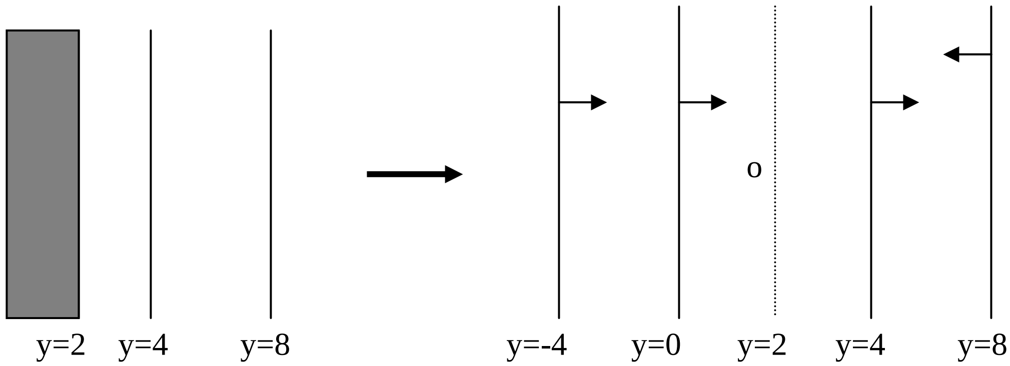
(b) $\rho_s = D_n$

$$\mathbf{D} = \mathbf{D}_+ + \mathbf{D}_- = \frac{\rho_L}{2\pi}\left(\frac{\mathbf{a}_{\rho1}}{\rho_1} - \frac{\mathbf{a}_{\rho2}}{\rho_2}\right) = \frac{16x10^{-9}}{2\pi}\left[\frac{(5,-6,0)-(3,-6,4)}{|(5,-6,0)-(3,-6,4)|^2} - \frac{(5,-6,0)-(3,-6,-4)}{|(5,-6,0)-(3,-6,-4)|^2}\right]$$

$$= \frac{8}{\pi}\left[\frac{(2,0,-4)}{20} - \frac{(2,0,4)}{20}\right]\ \text{nC/m}^2 = -1.018\mathbf{a}_z\ \text{nC/m}^2$$

$\rho_s = -1.018 \text{ nC/m}^2$

**Prob. 6.70**




At P(0,0,0), **E**=0 since **E** does not exist for y<2.

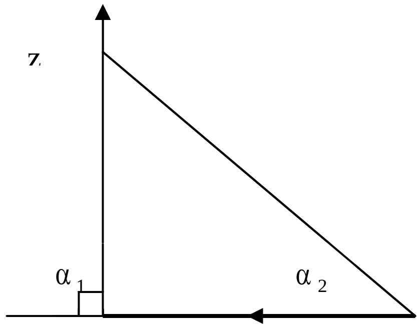
At Q(-4,6,2), y=6 and

$$\mathbf{E} = \sum \frac{\rho_s}{2\varepsilon_o}\mathbf{a}_n = \frac{10^{-9}}{2x10^{-9}/36\pi}(-30\mathbf{a}_y + 20\mathbf{a}_y - 20\mathbf{a}_y - 30\mathbf{a}_y) = 18\pi(-60)\mathbf{a}_y$$

$$= -3.4\mathbf{a}_y \text{ kV/m}$$

# CHAPTER 7

**P.E. 7.1**




$$\rho = 5, \ \cos\alpha_1 = 0, \quad \cos\alpha_2 = \sqrt{\frac{2}{27}}$$

$$\boldsymbol{a}_\phi = \boldsymbol{a}_l \times \boldsymbol{a}_\rho = \left(\frac{-\boldsymbol{a}_x - \boldsymbol{a}_y}{\sqrt{2}}\right) \times \boldsymbol{a}_z = \frac{-\boldsymbol{a}_x + \boldsymbol{a}_y}{\sqrt{2}}$$

$$\boldsymbol{H}_3 = \frac{10}{4\pi(5)}\left(\sqrt{\frac{2}{27}} - 0\right)\left(\frac{-\boldsymbol{a}_x + \boldsymbol{a}_y}{\sqrt{2}}\right) = \underline{\underline{-30.63\boldsymbol{a}_x + 30.63\boldsymbol{a}_y}} \quad \text{mA/m}$$

**P.E. 7.2**

(a) $\boldsymbol{H} = \frac{2}{4\pi(2)}\left(1 + \frac{3}{\sqrt{13}}\right)\boldsymbol{a}_z = \underline{\underline{0.1458\boldsymbol{a}_z}}$ A/m

(b) $\rho = \sqrt{3^2 + 4^2} = 5, \alpha_2 = 0, \cos\alpha_1 = -\frac{12}{13},$

$$\boldsymbol{a}_\phi = -\boldsymbol{a}_y \times \left(\frac{3\boldsymbol{a}_x - 4\boldsymbol{a}_z}{5}\right) = \frac{4\boldsymbol{a}_x + 3\boldsymbol{a}_z}{5}$$

$$\boldsymbol{H} = \frac{2}{4\pi(5)}\left(1 + \frac{12}{13}\right)\left(\frac{4\boldsymbol{a}_x + 3\boldsymbol{a}_z}{5}\right) = \frac{1}{26\pi}\left(4\boldsymbol{a}_x + 3\boldsymbol{a}_z\right)$$

$$= \underline{\underline{48.97\boldsymbol{a}_x + 36.73\boldsymbol{a}_z}} \quad \text{mA/m}$$

**P.E. 7.3**

(a) From Example 7.3,

$$\boldsymbol{H} = \frac{Ia^2}{2(a^2 + z^2)^{3/2}}\boldsymbol{a}_z$$

At (0,0,-1cm), z = 2cm,

$$\boldsymbol{H} = \frac{50\times10^{-3} \times 25\times10^{-4}}{2(5^2 + 2^2)^{3/2} \times 10^{-6}}\boldsymbol{a}_z = \underline{\underline{400.2\boldsymbol{a}_z \text{ mA/m}}}$$

(b) At (0,0,10cm), z = 9cm,

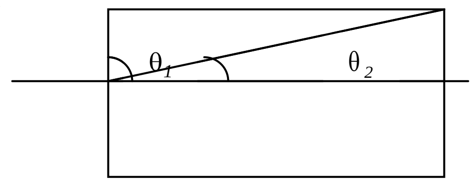
$$H = \frac{50\times10^{-3}\times25\times10^{-4}}{2(5^2+9^2)^{3/2}\times10^{-6}}a_z = \underline{\underline{57.3a_z}} \text{ mA/m}$$

**P.E. 7.4**

$$H = \frac{NI}{2L}\left(\cos\theta_2 - \cos\theta_1\right)a_z = \frac{2\times10^3\times50\times10^{-3}(\cos\theta_2-\cos\theta_1)a_z}{2\times0.75}$$

$$= \frac{100}{1.5}\left(\cos\theta_2 - \cos\theta_1\right)a_z$$

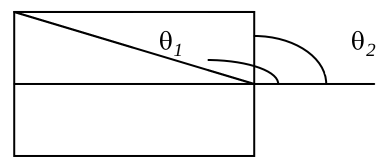
(a) At (0,0,0), $\theta = 90^o$, $\cos\theta_2 = \dfrac{0.75}{\sqrt{0.75^2+0.05^2}}$ = 0.9978




$$H = \frac{100}{1.5}\left(0.9978-0\right)a_z$$

= $\underline{\underline{66.52\ a_z}}$ A/m

(b) At (0,0,0.75), $\theta_2 = 90^o, \cos\theta_1 = -0.9978$

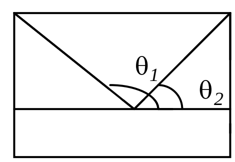



$$H = \frac{100}{1.5}\left(0+0.9978\right)a_z$$

= $\underline{\underline{66.52a_z}}$ A/m

(c) At (0,0,0.5), $\cos\theta_1 = \dfrac{-0.5}{\sqrt{0.5^2+0.05^2}} = -0.995$

$$\cos\theta_1 = \frac{0.25}{\sqrt{0.25^2+0.05^2}} = 0.9806$$




$$H = \frac{100}{1.5}\left(0.9806+0.995\right)a_z$$

= $\underline{\underline{131.7a_z}}$ A/m

**P.E. 7.5**

$$H = \frac{1}{2}K\times a_n$$

(a) $H(0,0,0) = \frac{1}{2}50a_z\times(-a_y) = \underline{\underline{25a_x}}$ mA/m

(b) $H(1,5,-3) = \frac{1}{2}50a_z\times a_y = \underline{\underline{-25a_x}}$ mA/m

**P.E. 7.6**

$$\left|H\right| = \begin{cases} \dfrac{NI}{2\pi\rho}, \rho - a < \rho < \rho + a, & 9<\rho<11 \\ 0, & \text{otherwise} \end{cases}$$

(a) At (3,-4,0), $\rho = \sqrt{3^2 + 4^2}$ =5cm ‹ 9cm

$$|\boldsymbol{H}| = \underline{\underline{0}}$$

(b) At (6,9,0), $\rho = \sqrt{6^2 + 9^2} = \sqrt{117}$ ‹ 11

$$|\boldsymbol{H}| = \frac{10^3 \times 100 \times 10^{-3}}{2\pi\sqrt{117} \times 10^2} = \underline{\underline{147.1}} \text{ A/m}$$

**P.E. 7.7**

(a) $\boldsymbol{B} = \nabla \times \boldsymbol{A} = (-4xz - 0)\boldsymbol{a}_x + (0 + 4yz)\boldsymbol{a}_y + (y^2 - x^2)\boldsymbol{a}_z$

$$\boldsymbol{B}(-1,2,5) = \underline{\underline{20\boldsymbol{a}_x + 40\boldsymbol{a}_y + 3\boldsymbol{a}_z}} \text{ Wb/m}^2$$

(b) $\psi = \int \boldsymbol{B}.d\boldsymbol{S} = \int_{y=-1}^{4}\int_{x=0}^{1} (y^2 - x^2)dxdy = \int_{-1}^{4} y^2 dy - 5\int_{0}^{1} x^2 dx$

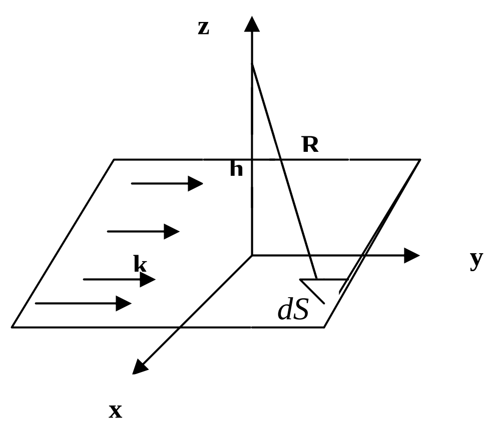
$$= \frac{1}{3}(64+1) - \frac{5}{3} = \underline{\underline{20 \text{ Wb}}}$$

Alternatively,

$$\psi = \int \boldsymbol{A}.d\boldsymbol{l} = \int_0^1 x^2(-1)dx + \int_{-1}^{4} y^2(1)dy + \int_1^0 x^2(4)dx + 0$$

$$= -\frac{5}{3} + \frac{65}{3} = \underline{\underline{20 \text{ Wb}}}$$

**P.E. 7.8**




$$\boldsymbol{H} = \int \frac{\boldsymbol{k}dS \times \boldsymbol{R}}{4\pi R^3},$$

$dS = dxdy, \boldsymbol{k} = k_y \boldsymbol{a}_{y,}$

$R = (-x, -y, h),$

$$\boldsymbol{k}\times\boldsymbol{R}=(h\boldsymbol{a}_x+x\boldsymbol{a}_z)k_y,$$

$$\boldsymbol{H}=\int\frac{k_y(h\boldsymbol{a}_x+x\boldsymbol{a}_z)dxdy}{4\pi(x^2+y^2+h^2)^{3/2}}$$

$$=\frac{k_y h\boldsymbol{a}_x}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dxdy}{(x^2+y^2+h^2)^{3/2}}+\frac{k_y\boldsymbol{a}_z}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{xdxdy}{(x^2+y^2+h^2)^{3/2}}$$

The integrand in the last term is zero because it is an odd function of x.

$$\boldsymbol{H}=\frac{k_y h\boldsymbol{a}_x}{4\pi}\int_{\phi=0}^{2\pi}\int_{\rho=0}^{\infty}\frac{\rho d\phi d\rho}{(\rho^2+h^2)^{3/2}}=\frac{k_y h2\pi\boldsymbol{a}_x}{4\pi}\int_0^{\infty}(\rho^2+h^2)^{-3/2}\frac{d(\rho^2)}{2}$$

$$=\frac{k_y h}{2}\boldsymbol{a}_x\left(\frac{-1}{(\rho^2+h^2)^{1/2}}\right)\Bigg|_0^{\infty}=\frac{k_y}{2}\boldsymbol{a}_x$$

Similarly, for point (0,0,-h), **H** = $-\frac{1}{2}k_y\boldsymbol{a}_x$

Hence,

$$\boldsymbol{H}=\begin{cases}\frac{1}{2}k_y\boldsymbol{a}_x, & z>0\\ \frac{1}{2}k_y\boldsymbol{a}_x, & z<0\end{cases}$$

---

**Prob. 7.1**

(a) See text

(b) Let **H = H**$_y$ + **H**$_z$

For $\boldsymbol{H}_z=\frac{I}{2\pi\rho}\boldsymbol{a}_\phi \qquad \rho=\sqrt{(-3)^2+4^2}=5$

$$\mathbf{a}_\phi=-\mathbf{a_z}\times\frac{(-3\mathbf{a_x}+4\mathbf{a_y})}{5}=\frac{(3\mathbf{a_y}-4\mathbf{a_x})}{5}$$

$$\mathbf{H_z}=\frac{20}{2\pi(25)}(4\mathbf{a_x}+3\mathbf{a_y})=0.5093\mathbf{a_x}+0.382\mathbf{a_y}$$

For $\mathbf{H_y}=\frac{\mathbf{I}}{2\pi\rho}\mathbf{a}_\phi$, $\rho=\sqrt{(-3)^2+5^2}=\sqrt{34}$

$$\boldsymbol{a}_\phi=\boldsymbol{a}_y\times\frac{(-3\boldsymbol{a}_x+5\boldsymbol{a}_z)}{\sqrt{34}}=\frac{3\boldsymbol{a}_z+5\boldsymbol{a}_x}{\sqrt{34}}$$

$$\boldsymbol{H}_y = \frac{10}{2\pi(34)}(5\boldsymbol{a}_x + 3\boldsymbol{a}_z) = 0.234\boldsymbol{a}_x + 0.1404\boldsymbol{a}_z$$

**H = H**$_y$ + **H**$_z$

$= 0.7433\mathbf{a_x} + 0.382\mathbf{a_y} + 0.1404\mathbf{a_z}$ A/m

**Prob. 7.2**

$$H = \frac{I}{2\pi\rho} \quad \rightarrow \quad \rho = \frac{I}{2\pi H} = \frac{2}{2\pi(10\times10^{-3})} = \frac{100}{\pi} = \underline{\underline{31.83 \text{ m}}}$$

**Prob. 7.3**

Let $\boldsymbol{H} = \boldsymbol{H}_1 + \boldsymbol{H}_2$

where $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$ are respectively due to the lines located at (0,0) and (0,5).

$$\boldsymbol{H}_1 = \frac{I}{2\pi\rho}\boldsymbol{a}_\phi, \quad \rho = 5, \quad \boldsymbol{a}_\phi = \boldsymbol{a}_\ell \times \boldsymbol{a}_\rho = \boldsymbol{a}_z \times \boldsymbol{a}_x = \boldsymbol{a}_y$$

$$\boldsymbol{H}_1 = \frac{10}{2\pi(5)}\boldsymbol{a}_y = \frac{\boldsymbol{a}_y}{\pi}$$

$$\boldsymbol{H}_2 = \frac{I}{2\pi\rho}\boldsymbol{a}_\phi, \quad \rho = 5\sqrt{2}, \quad \boldsymbol{a}_\phi = \boldsymbol{a}_\ell \times \boldsymbol{a}_\rho, \boldsymbol{a}_\ell = -\boldsymbol{a}_z$$

$$\boldsymbol{a}_\rho = \frac{5\boldsymbol{a}_x - 5\boldsymbol{a}_y}{5\sqrt{2}} = \frac{\boldsymbol{a}_x - \boldsymbol{a}_y}{\sqrt{2}}$$

$$\boldsymbol{a}_\phi = -\boldsymbol{a}_z \times \left(\frac{\boldsymbol{a}_x - \boldsymbol{a}_y}{\sqrt{2}}\right) = \frac{-\boldsymbol{a}_x - \boldsymbol{a}_y}{\sqrt{2}}$$

$$\boldsymbol{H}_2 = \frac{10}{2\pi 5\sqrt{2}}\left(\frac{-\boldsymbol{a}_x - \boldsymbol{a}_y}{\sqrt{2}}\right) = \frac{1}{2\pi}(-\boldsymbol{a}_x - \boldsymbol{a}_y)$$

$$\boldsymbol{H} = \boldsymbol{H}_1 + \boldsymbol{H}_2 = \frac{\boldsymbol{a}_y}{\pi} + \frac{1}{2\pi}(-\boldsymbol{a}_x - \boldsymbol{a}_y) = \underline{\underline{-0.1592\boldsymbol{a}_x + 0.1592\boldsymbol{a}_y}}$$

**Prob. 7.4**

$$\boldsymbol{H} = d\boldsymbol{H} = \frac{Id\boldsymbol{l} \times \boldsymbol{R}}{4\pi R^3} = \boldsymbol{H}_1 + \boldsymbol{H}_2$$

For $\boldsymbol{H}_1$, $\quad \boldsymbol{R} = (3,1,-2) - (0,0,0) = (3,1,-2), \quad R = \sqrt{9+1+4} = \sqrt{14}$

$$Id\boldsymbol{l} \times \boldsymbol{R} = 4\times 10^{-5}\begin{vmatrix} 1 & 0 & 0 \\ 3 & 1 & -2 \end{vmatrix} = 4\times 10^{-5}(0,2,1)$$

$$\boldsymbol{H}_1 = \frac{4\times 10^{-5}(0,2,1)}{4\pi(14)^{3/2}} = (0, 0.01215, 0.006076)10^{-5}$$

For $\boldsymbol{H}_2$, $\quad \boldsymbol{R} = (3,1,-2) - (0,0,1) = (3,1,-3), \quad R = \sqrt{9+1+9} = \sqrt{19}$

$$Id\boldsymbol{l} \times \boldsymbol{R} = 6\times 10^{-5}\begin{vmatrix} 0 & 1 & 0 \\ 3 & 1 & -3 \end{vmatrix} = 6\times 10^{-5}(-3,0,-3)$$

$$\boldsymbol{H}_1 = \frac{6\times 10^{-5}(-3,0,-3)}{4\pi(19)^{3/2}} = (-0.017, 0, -0.017)10^{-5}$$

$$\boldsymbol{H} = \boldsymbol{H}_1 + \boldsymbol{H}_2 = (-0.0173, 0.01215, -0.01122)10^{-5} = \underline{\underline{(-0.173\boldsymbol{a}_x + 1.215\boldsymbol{a}_y - 0.1122\boldsymbol{a}_z)\ \mu\text{A/m}}}$$

**Prob. 7.5**

Let $\boldsymbol{H} = \boldsymbol{H}_y + \boldsymbol{H}_z$

For $\boldsymbol{H}_z$, $\quad \boldsymbol{H}_z = \dfrac{I}{2\pi\rho}\boldsymbol{a}_\phi$

where I = 20, $\boldsymbol{\rho} = -3\boldsymbol{a}_x + 4\boldsymbol{a}_y$, $\rho = \sqrt{3^2+4^2} = 5$

$$\boldsymbol{a}_\phi = \boldsymbol{a}_\ell \times \boldsymbol{a}_\rho = -\boldsymbol{a}_z \times \left(\frac{-3\boldsymbol{a}_x + 4\boldsymbol{a}_y}{5}\right) = \begin{vmatrix} 0 & 0 & -1 \\ -0.6 & 0.8 & 0 \end{vmatrix} = 0.8\boldsymbol{a}_x + 0.6\boldsymbol{a}_y$$

$$\boldsymbol{H}_z = \frac{20}{2\pi(5)}(0.8\boldsymbol{a}_x + 0.6\boldsymbol{a}_y) = 0.5093\boldsymbol{a}_x + 0.382\boldsymbol{a}_y$$

For $\boldsymbol{H}_y$, $\quad \boldsymbol{H}_y = \dfrac{I}{2\pi\rho}\boldsymbol{a}_\phi$

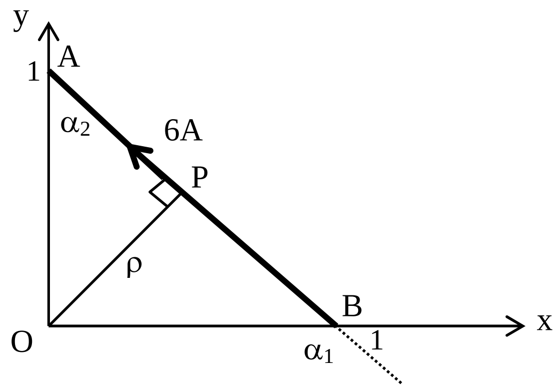
where I = 10, $\boldsymbol{\rho} = -3\boldsymbol{a}_x + 5\boldsymbol{a}_z$, $\rho = \sqrt{3^2+5^2} = \sqrt{34}$

$$\boldsymbol{a}_\phi = \boldsymbol{a}_\ell \times \boldsymbol{a}_\rho = \frac{1}{\sqrt{34}}\begin{vmatrix} 0 & 1 & 0 \\ -3 & 0 & 5 \end{vmatrix} = \frac{1}{\sqrt{34}}(5\boldsymbol{a}_x + 3\boldsymbol{a}_z)$$

$$\boldsymbol{H}_y = \frac{10}{2\pi(34)}(5\boldsymbol{a}_x + 3\boldsymbol{a}_z) = 0.234\boldsymbol{a}_x + 0.1404\boldsymbol{a}_z$$

$$\boldsymbol{H} = \boldsymbol{H}_y + \boldsymbol{H}_z = \underline{\underline{0.7432\boldsymbol{a}_x + 0.382\boldsymbol{a}_y + 0.1404\boldsymbol{a}_z\ \text{A/m}}}$$

**Prob. 7.6**




$$\boldsymbol{H} = \frac{I}{4\pi\rho}\left(\cos\alpha_2 - \cos\alpha_1\right)\boldsymbol{a}_\phi$$

$$\alpha_1 = 135^o, \quad \alpha_2 = 45^o, \quad \rho = \frac{1}{2}\sqrt{2} = \frac{\sqrt{2}}{2}$$

$$\boldsymbol{a}_\phi = \boldsymbol{a}_l \times \boldsymbol{a}_\rho = \left(\frac{-\boldsymbol{a}_x + \boldsymbol{a}_y}{\sqrt{2}}\right) \times \left(\frac{-\boldsymbol{a}_x - \boldsymbol{a}_y}{\sqrt{2}}\right) = \frac{1}{2}\begin{vmatrix} -1 & 1 & 0 \\ -1 & -1 & 0 \end{vmatrix} = \boldsymbol{a}_z$$

$$H = \frac{6}{4\pi\frac{\sqrt{2}}{2}}\left(\cos 45^o - \cos 135^o\right)\boldsymbol{a}_z = \frac{3}{\pi}\boldsymbol{a}_z$$

$$\boldsymbol{H}\left(0, 0, 0\right) = \underline{\underline{0.954\boldsymbol{a}_z \text{ A/m}}}$$

**Prob. 7.7**

(a) At (5,0,0), $\rho = 5$, $\boldsymbol{a}_\phi = \boldsymbol{a}_y$, $\cos\alpha_1 = 0$, $\cos\alpha_2 = \frac{10}{\sqrt{125}}$

$$\boldsymbol{H} = \frac{2}{4\pi(5)}\left(\frac{10}{\sqrt{125}}\right)\boldsymbol{a}_y = \underline{\underline{28.471\boldsymbol{a}_y \text{ mA/m}}}$$

(b) At (5,5,0), $\rho = 5\sqrt{2}$, $\cos\alpha_1 = 0$, $\cos\alpha_2 = \frac{10}{\sqrt{150}}$

$$\boldsymbol{a}_\phi = \boldsymbol{a}_z \times \left(\frac{\boldsymbol{a}_x + \boldsymbol{a}_y}{\sqrt{2}}\right) = \frac{-\boldsymbol{a}_x + \boldsymbol{a}_y}{\sqrt{2}}$$

$$\boldsymbol{H} = \frac{2}{4\pi(5\sqrt{2})}\left(\frac{10}{\sqrt{150}}\right)\left(\frac{-\boldsymbol{a}_x + \boldsymbol{a}_y}{\sqrt{2}}\right) = \underline{\underline{13(-\boldsymbol{a}_x + \boldsymbol{a}_y) \text{ mA/m}}}$$

(c ) At (5,15,0), $\rho = \sqrt{250} = 5\sqrt{10}$, $\cos\alpha_1 = 0$, $\cos\alpha_2 = \frac{10}{\sqrt{350}}$

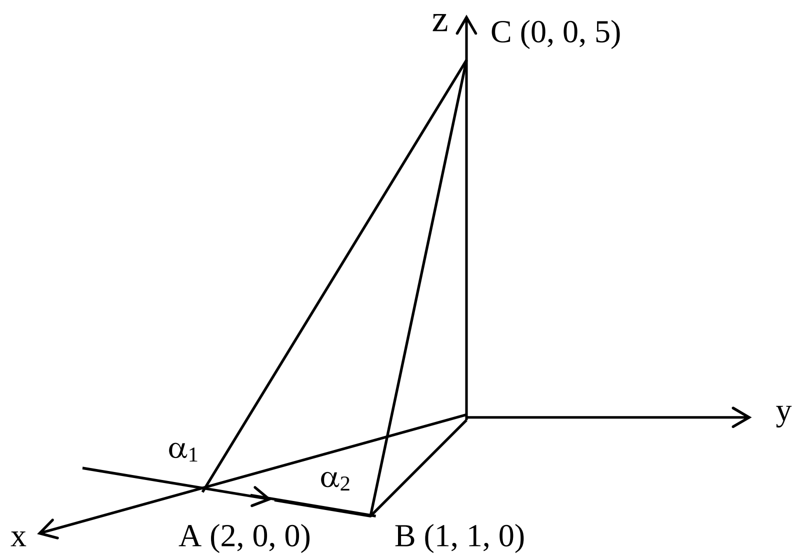
$$\mathbf{a}_\phi = \mathbf{a}_z \times \left( \frac{5\mathbf{a}_x + 15\mathbf{a}_y}{5\sqrt{10}} \right) = \frac{5\mathbf{a}_y - 15\mathbf{a}_x}{5\sqrt{10}}$$

$$\mathbf{H} = \frac{2}{4\pi(5\sqrt{10})}\left(\frac{10}{\sqrt{350}}\right)\left( \frac{-15\mathbf{a}_x + 5\mathbf{a}_y}{5\sqrt{10}} \right) = \underline{\underline{-5.1\mathbf{a}_x + 1.7\mathbf{a}_y \text{ mA/m}}}$$

d) At (5,-15,0), by symmetry,

$$\mathbf{H} = \underline{\underline{5.1\mathbf{a}_x + 1.7\mathbf{a}_y \text{ mA/m}}}$$

**Prob. 7.8**



($a$) Consider the figure above.

$$AB = (1,1,0) - (2,0,0) = (-1,1,0)$$

$$AC = (0,0,5) - (2,0,0) = (-2,0,5)$$

$AB \cdot AC = 2$, i.e AB and AC are not perpendicular.

$$\cos(180^o - \alpha_1) = \frac{AB \cdot AC}{|AB||AC|} = \frac{2}{\sqrt{2}\sqrt{29}} \rightarrow \cos\alpha_1 = -\sqrt{\frac{2}{29}}$$

$$BC = (0,0,5) - (1,1,0) = (-1,-1,5)$$

$$BA = (1,-1,0)$$

$$\cos\alpha_2 = \frac{\bar{B}C \cdot B\bar{A}}{|BC||BA|} = \frac{-1+1}{|BC||BA|} = 0$$

i.e. $BC = \boldsymbol{\rho} = (-1,-1,5), \quad \rho = \sqrt{27}$

$$\boldsymbol{a}_\phi = \boldsymbol{a}_l \times \boldsymbol{a}_\rho = \frac{(-1,1,0)}{\sqrt{2}} \times \frac{(-1,-1,5)}{\sqrt{27}} = \frac{(5,5,2)}{\sqrt{54}}$$

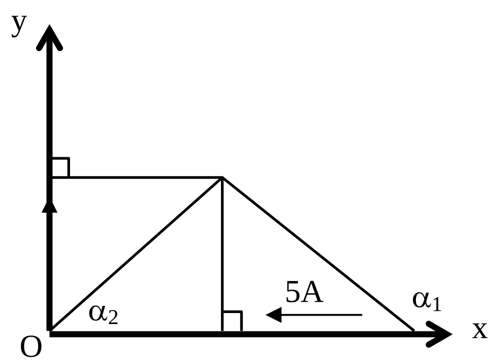
$$\boldsymbol{H}_2 = \frac{10}{4\pi\sqrt{27}}\left(0 + \sqrt{\frac{2}{29}}\right)\frac{(5,5,2)}{\sqrt{2}\sqrt{27}} = \frac{5}{2\pi\sqrt{29}} \cdot \frac{(5,5,2)}{27} \text{ A/m}$$

$$= 27.37\,\boldsymbol{a}_x + 27.37\boldsymbol{a}_y + 10.95\,\boldsymbol{a}_z \text{ mA/m}$$

(b) $\boldsymbol{H} = \boldsymbol{H}_1 + \boldsymbol{H}_2 + \boldsymbol{H}_3 = (0, -59.1, 0) + (27.37, 27.37, 10.95) + (-30.63, 30.63, 0)$

$$= -3.26\,\boldsymbol{a}_x - 1.1\,\boldsymbol{a}_y + 10.95\boldsymbol{a}_z \text{ mA/m}$$

**Prob. 7.9**




($a$) Let $\boldsymbol{H} = \boldsymbol{H}_x + \boldsymbol{H}_y = 2\boldsymbol{H}_x$

$$\boldsymbol{H}_x = \frac{I}{4\pi\rho}(\cos\alpha_2 - \cos\alpha_1)\boldsymbol{a}_\phi$$

where $\boldsymbol{a}_\phi = -\boldsymbol{a}_x \times \boldsymbol{a}_y = -\boldsymbol{a}_z, \ \alpha_1 = 180^o, \ \alpha_2 = 45^o$

$$\boldsymbol{H}_x = \frac{5}{4\pi(2)}\left(\cos\ 45^o - \cos\ 180^o\right)\ (-\boldsymbol{a}_z) = -0.3397\boldsymbol{a}_z$$

$$\boldsymbol{H} = 2\boldsymbol{H}_x = \underline{\underline{-0.6792\ \boldsymbol{a}_z\ \text{A/m}}}$$

(*b*) $\boldsymbol{H} = \boldsymbol{H}_x + \boldsymbol{H}_y$

where $\boldsymbol{H}_x = \frac{5}{4\pi(2)}(1-0)\ \boldsymbol{a}_\phi, \quad \boldsymbol{a}_\phi = -\boldsymbol{a}_x \times -\boldsymbol{a}_y = \boldsymbol{a}_z$

$$= 198.9\boldsymbol{a}_z \ \text{mA/m}$$

$\boldsymbol{H}_y = 0$ since $\alpha_1 = \alpha_2 = 0$

$$\boldsymbol{H} = \underline{\underline{0.1989\ \boldsymbol{a}_z\ \text{A/m}}}$$

(*c*) $\boldsymbol{H} = \boldsymbol{H}_x + \boldsymbol{H}_y$

where $\boldsymbol{H}_x = \frac{5}{4\pi(2)}(1-0)\ (-\boldsymbol{a}_x \times \boldsymbol{a}_z) = 198.9\ \boldsymbol{a}_y\ \text{mA/m}$

$$\boldsymbol{H}_y = \frac{5}{4\pi(2)}(1-0)\ (\boldsymbol{a}_y \times \boldsymbol{a}_z) = 198.9\ \boldsymbol{a}_x\ \text{mA/m}$$

$$\boldsymbol{H} = \underline{\underline{0.1989\ \boldsymbol{a}_x + 0.1989\ \boldsymbol{a}_y\ \text{A/m.}}}$$

**Prob. 7.10**

For the side of the loop along y-axis,

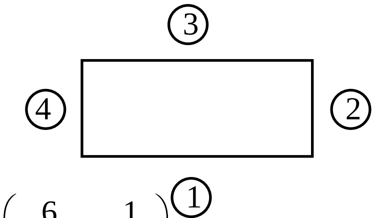
$$\boldsymbol{H}_1 = \frac{I}{4\pi\rho}\left(\cos\alpha_2 - \cos\alpha_1\right)\ \boldsymbol{a}_\phi$$

where $\boldsymbol{a}_\phi = -\boldsymbol{a}_x, \ \rho = 2\tan 30^o = \frac{2}{\sqrt{3}}, \ \alpha_2 = 30^o, \ \alpha_1 = 150^o$

$$\boldsymbol{H}_1 = \frac{5}{4\pi}\frac{\sqrt{3}}{2}\left(\cos\ 30^o - \cos\ 150^o\right)(-\boldsymbol{a}_x) = -\frac{15}{8\pi}\boldsymbol{a}_x$$

$$\boldsymbol{H} = 3\boldsymbol{H}_1 = -1.79\boldsymbol{a}_x\ \text{A/m}$$

**Prob. 7.11**




Let $\boldsymbol{H} = \boldsymbol{H}_1 + \boldsymbol{H}_2 + \boldsymbol{H}_3 + \boldsymbol{H}_4$

where $\bar{H}_n$ is the contribution by side n.

(a) $\boldsymbol{H} = 2\boldsymbol{H}_1 + \boldsymbol{H}_2 + \boldsymbol{H}_4$ since $\boldsymbol{H}_1 = \boldsymbol{H}_3$

$$\boldsymbol{H}_1 = \frac{I}{4\pi\rho}\left(\cos\alpha_2 - \cos\alpha_1\right)\boldsymbol{a}_\phi = \frac{10}{4\pi(2)}\left(\frac{6}{\sqrt{40}} + \frac{1}{\sqrt{2}}\right)\boldsymbol{a}_z$$

$$\boldsymbol{H}_2 = \frac{10}{4\pi(6)}\left(2\times\frac{2}{\sqrt{40}}\right)\boldsymbol{a}_z, \quad \boldsymbol{H}_4 = \frac{10}{4\pi(2)}\left(2\cdot\frac{1}{\sqrt{2}}\right)\boldsymbol{a}_z$$

$$\boldsymbol{H} = \left[\frac{5}{2\pi}\left(\frac{3}{\sqrt{10}} + \frac{1}{\sqrt{2}}\right) + \frac{5}{6\pi\sqrt{10}} + \frac{5}{2\pi\sqrt{2}}\right]\boldsymbol{a}_z = \underline{\underline{1.964\boldsymbol{a}_z \text{ A/m}}}$$

($b$) At $(4, 2, 0)$, $\boldsymbol{H} = 2\left(\boldsymbol{H}_1 + \boldsymbol{H}_4\right)$

$$\boldsymbol{H}_1 = \frac{10}{4\pi(2)}\ \frac{8}{\sqrt{20}}\ \boldsymbol{a}_z, \quad \boldsymbol{H}_4 = \frac{10}{4\pi(4)}\ \frac{4}{\sqrt{20}}\ \boldsymbol{a}_z$$

$$\boldsymbol{H} = \frac{2\sqrt{5}}{\pi}\left(1 + \frac{1}{4}\right)\boldsymbol{a}_z = \underline{\underline{1.78\boldsymbol{a}_z \text{ A/m}}}$$

($c$) At $(4, 8, 0)$, $\boldsymbol{H} = \boldsymbol{H}_1 + 2\boldsymbol{H}_2 + \boldsymbol{H}_3$

$$\boldsymbol{H}_1 = \frac{10}{4\pi(8)}\left(2\cdot\frac{4}{4\sqrt{5}}\right)\boldsymbol{a}_z, \quad \boldsymbol{H}_2 = \frac{10}{4\pi(4)}\left(\frac{8}{4\sqrt{5}} - \frac{1}{\sqrt{2}}\right)\boldsymbol{a}_z$$

$$\boldsymbol{H}_3 = \frac{10}{4\pi(4)}\left(\frac{2}{\sqrt{2}}\right)\left(-\boldsymbol{a}_z\right)$$

$$\boldsymbol{H} = \frac{5}{8\pi}\left(\boldsymbol{a}_z\right)\left(\frac{1}{\sqrt{5}} + \frac{4}{\sqrt{5}} - \frac{4}{\sqrt{2}}\right) = \underline{\underline{-0.1178\boldsymbol{a}_z \text{ A/m}}}$$

($d$) At $(0, 0, 2)$,

$$\boldsymbol{H}_1 = \frac{10}{4\pi(2)}\left(\frac{8}{\sqrt{68}} - 0\right)\left(\boldsymbol{a}_x \times \boldsymbol{a}_z\right) = -\frac{10}{\pi\sqrt{68}}\ \boldsymbol{a}_y$$

$$\boldsymbol{H}_2 = \frac{10}{4\pi\sqrt{68}}\left(\frac{4}{\sqrt{84}} - 0\right)\boldsymbol{a}_y \times \left(\frac{2\boldsymbol{a}_z - 8\boldsymbol{a}_x}{\sqrt{68}}\right) = \frac{5\left(\boldsymbol{a}_x + 4\boldsymbol{a}_z\right)}{17\pi\sqrt{84}}$$

$$H_3 = \frac{10}{4\pi\sqrt{20}}\left(-\frac{8}{\sqrt{84}}-0\right)\mathbf{a}_x \times \left(\frac{2\mathbf{a}_x - 4\mathbf{a}_y}{\sqrt{20}}\right) = \frac{\mathbf{a}_y + 2\mathbf{a}_z}{\pi\sqrt{21}}$$

$$H_4 = \frac{10}{4\pi\sqrt{2}}\left(0+\frac{4}{\sqrt{20}}\right)\left(-\mathbf{a}_y \times \mathbf{a}_z\right) = \frac{-5\mathbf{a}_x}{\pi\sqrt{20}}$$

$$H = \left(\frac{5}{34\pi\sqrt{21}}-\frac{5}{\pi\sqrt{20}}\right)\mathbf{a}_x + \left(\frac{1}{\pi\sqrt{21}}-\frac{10}{\pi\sqrt{68}}\right)\mathbf{a}_y + \left(\frac{20}{34\pi\sqrt{21}}+\frac{2}{\pi\sqrt{21}}\right)\mathbf{a}_z$$

$$= \underline{\underline{-0.3457\,\bar{a}_x - 0.3165\,\bar{a}_y + 0.1798\,\bar{a}_z \text{ A/m}}}$$

**Prob. 7.12**

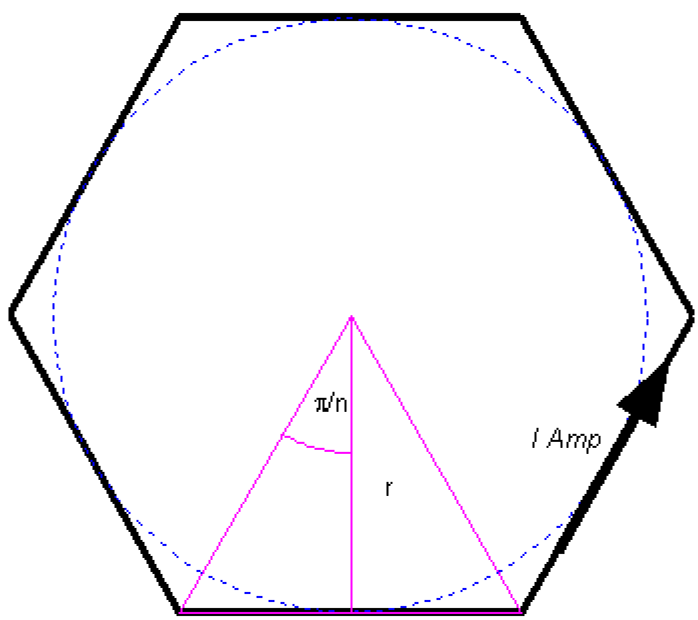
$$\mathbf{H} = 4\mathbf{H}_1 = 4\frac{I}{4\pi\rho}(\cos\alpha_2 - \cos\alpha_1)\mathbf{a}_\phi$$

$$\rho = a = 2cm, I = 5mA, \alpha_2 = 45^o, \alpha_1 = 90^o + 45^o = 135^o$$

$$\mathbf{a}_\phi = \mathbf{a}_\ell \times \mathbf{a}_\rho = \mathbf{a}_y \times (-\mathbf{a}_x) = \mathbf{a}_z$$

$$\mathbf{H} = \frac{I}{\pi a}\left(\frac{1}{\sqrt{2}} - -\frac{1}{\sqrt{2}}\right)\mathbf{a}_z = \frac{\sqrt{2}I}{\pi a}\mathbf{a}_z = \frac{\sqrt{2}\times 5\times 10^{-3}}{\pi \times 2\times 10^{-2}}\mathbf{a}_z = \underline{\underline{0.1125\mathbf{a}_z}}$$

**Prob. 7.13**



(a) Consider one side of the polygon as shown. The angle subtended by the Side At the center of the circle

$$\frac{360^o}{n} = \frac{2\pi}{n}$$

The field due to this side is

$$H_1 = \frac{I}{4\pi\rho}(\cos\alpha_2 - \cos\alpha_1)$$

where $\rho = r,\quad \cos\alpha_2 = \cos(90 - \frac{\pi}{n}) = \sin\frac{\pi}{n}$

$$\cos\alpha_1 = -\sin\frac{\pi}{n}$$

$$H_1 = \frac{I}{4\pi r}\, 2\sin\frac{\pi}{n}$$

$$H = nH_1 = \frac{nI}{2\pi r}\sin\frac{\pi}{n}$$

(*b*) For $n = 3,\quad H = \frac{3I}{2\pi r}\sin\frac{\pi}{3}$

$$r\cot 30^o = 2 \rightarrow r = \frac{2}{\sqrt{3}}$$

$$H = \frac{3\times 5}{2\pi\, 2/\sqrt{3}}\cdot\frac{\sqrt{3}}{2} = \frac{45}{8\pi} = 1.79 \text{ A/m.}$$

For $n = 4,\quad H = \frac{4I}{2\pi r}\sin\frac{\pi}{4} = \frac{4\times 5}{2\pi(2)}\cdot\frac{1}{\sqrt{2}}$

$= 1.128$ A/m.

(c) As $n \rightarrow \infty$,

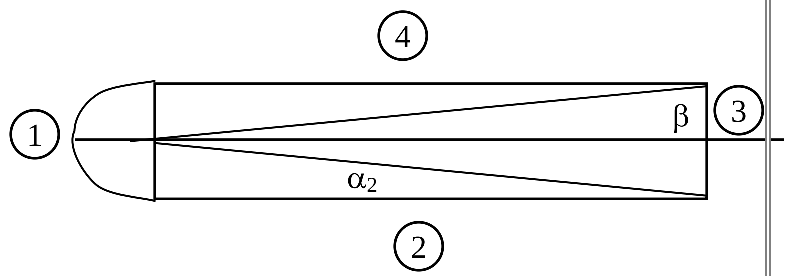
$$H = \lim_{n\rightarrow\infty}\frac{nI}{2\pi r}\sin\frac{\pi}{n} = \frac{nI}{2\pi r}\cdot\frac{\pi}{n} = \frac{I}{2r}$$

From Example 7.3, when $h = 0$,

$$H = \frac{I}{2r}$$

which agrees.

**Prob. 7.14**




Let $\boldsymbol{H} = \boldsymbol{H}_1 + \mathbf{H}_2 + \mathbf{H}_3 + \mathbf{H}_4$

$$\mathbf{H}_1 = \frac{I}{4a}\,\mathbf{a}_z = \frac{10}{4\times 4\times 10^{-2}}\,\mathbf{a}_z = 62.5\,\mathbf{a}_z$$

$$\boldsymbol{H}_2 = \boldsymbol{H}_4 = \frac{I}{4\pi\times 4\times 10^{-2}}\left(\cos\ \alpha_2 - \cos\ 90^o\right)\mathbf{a}_z\ ,\quad \alpha_2 = \tan^{-1}\frac{4}{100} = 2.29^o$$

$$= 19.88\,\mathbf{a}_z$$

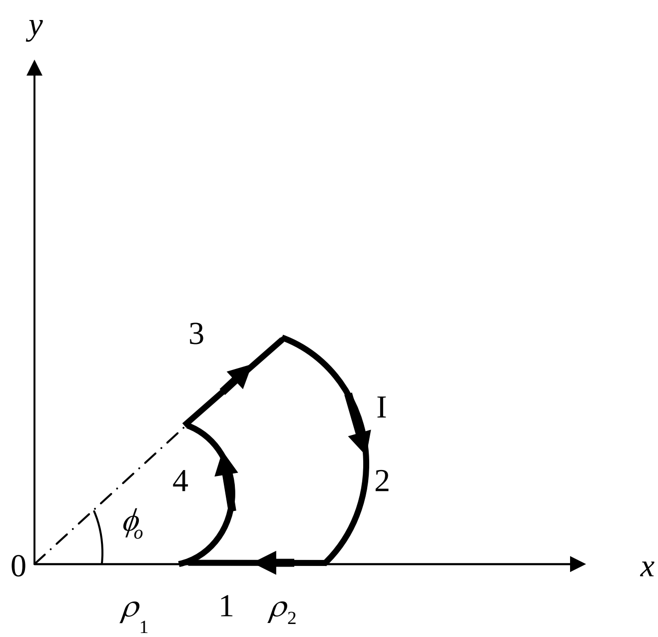
$$\boldsymbol{H}_3 = \frac{I}{4\pi(1)}\,2\ \cos\ \beta\mathbf{a}_z\ ,\quad \beta = \tan^{-1}\frac{100}{4} = 87.7^o$$

$$= \frac{10}{4\pi}\,2\ \cos\ 87.7^o\mathbf{a}_z = 0.06361\,\mathbf{a}_z$$

$$\boldsymbol{H} = (62.5 + 2\times 19.88 + 0.06361)\,\mathbf{a}_z$$

$$= 102.32\,\mathbf{a}_z\ \text{A/m.}$$

**Prob. 7.15**



Let $\boldsymbol{H} = \boldsymbol{H}_1 + \boldsymbol{H}_2 + \boldsymbol{H}_3 + \boldsymbol{H}_4$

which correspond with sides 1, 2, 3, and 4 as shown in the figure above.

$\boldsymbol{H}_1$ and $\boldsymbol{H}_3$ can be found using eq. (7.12). It can be shown that

$\boldsymbol{H}_3 = -\boldsymbol{H}_1$

For $\boldsymbol{H}_4$, $\quad d\boldsymbol{H}_4 = \dfrac{Id\boldsymbol{l} \times \boldsymbol{R}}{4\pi R^3}, \quad d\boldsymbol{l} = \rho_1 d\phi \boldsymbol{a}_\phi, \boldsymbol{R} = -\rho_1 \boldsymbol{a}_\rho$

$$d\boldsymbol{H}_4 = \frac{I\rho_1^2 d\phi \boldsymbol{a}_\phi \times (-\boldsymbol{a}_\rho)}{4\pi\rho_1^3} = \frac{Id\phi(\boldsymbol{a}_z)}{4\pi\rho_1}$$

$$\boldsymbol{H}_4 = \frac{I(\boldsymbol{a}_z)}{4\pi\rho_1}\int_0^{\phi_o} d\phi = \frac{I\phi_o(\boldsymbol{a}_z)}{4\pi\rho_1}$$

Similarly, for $\boldsymbol{H}_2$,

$$\boldsymbol{H}_2 = \frac{I\phi_o(-\boldsymbol{a}_z)}{4\pi\rho_2}$$

$$\boldsymbol{H} = \boldsymbol{H}_1 + \boldsymbol{H}_2 + \boldsymbol{H}_3 + \boldsymbol{H}_4 = \underline{\underline{\frac{I\phi_o}{4\pi}\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)\boldsymbol{a}_z}}$$

**Prob. 7.16**

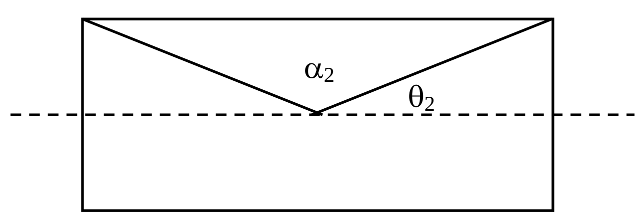
From Example 7.3,

$$\boldsymbol{H}_1 = \frac{I}{2a}\boldsymbol{a}_z, \qquad \boldsymbol{H}_2 = \frac{Ia^2}{2[a^2 + d^2]^{3/2}}\boldsymbol{a}_z$$

$$\boldsymbol{H} = \boldsymbol{H}_1 + \boldsymbol{H}_2 = \frac{I}{2}\left[\frac{1}{a} + \frac{a^2}{[a^2 + d^2]^{3/2}}\right]\boldsymbol{a}_z = \frac{10}{2}\left[\frac{1}{3\times 10^{-2}} + \frac{3^2}{[3^2 + 4^2]^{3/2}(10^{-2})}\right]\boldsymbol{a}_z$$

$$= 500\left[\frac{1}{3} + \frac{9}{125}\right]\boldsymbol{a}_z = \underline{\underline{202.67\boldsymbol{a}_z \text{ A/m}}}$$

**Prob. 7.17**



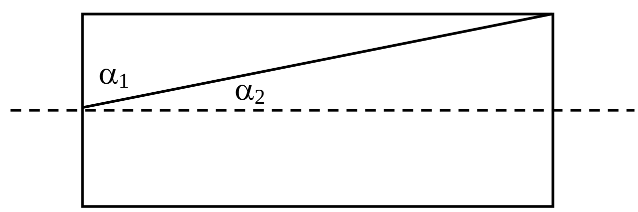

 

$$|\boldsymbol{H}| = \frac{nI}{2}(\cos\theta_2 - \cos\theta_1)$$

$$\cos\theta_2 = -\cos\theta_1 = \frac{\ell/2}{\left(a^2 + \ell^2/4\right)^{1/2}}$$

$$|\boldsymbol{H}| = \frac{nI\ell}{2\left(a^2 + \ell^2/4\right)^{1/2}} = \frac{0.5\times150\times2\times10^{-2}}{2\times10^{-3}\times\sqrt{4^2+10^2}} = \underline{\underline{69.63\ \text{A/m}}}$$

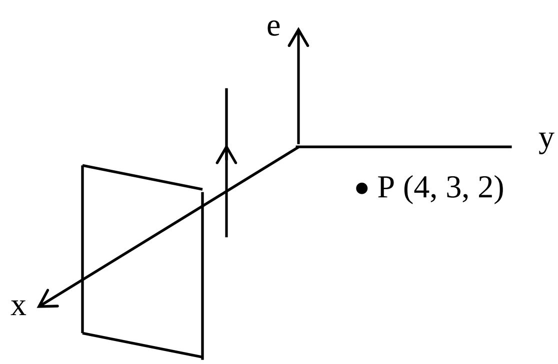
(b)




$$\alpha_1 = 90^o,\quad \tan\theta_2 = \frac{a}{b} = \frac{4}{20} = 0.2 \rightarrow \theta_2 = 11.31^o$$

$$|\boldsymbol{H}| = \frac{nI}{2}\cos\theta_2 = \frac{150\times0.5}{2}\cos 11.31^o = \underline{\underline{36.77\ \text{A/m}}}$$

**Prob. 7.18**



Let $\boldsymbol{H} = \boldsymbol{H}_l + \boldsymbol{H}_p$

$$\boldsymbol{H}_l = \frac{1}{2\pi\rho}\mathbf{a}_\phi$$

$$\boldsymbol{\rho} = (4, 3, 2) - (1, -2, 2) = (3, 5, 0), \quad \rho = |\boldsymbol{\rho}| = \sqrt{34}$$

$$\mathbf{a}_\rho = \frac{3\mathbf{a}_x + 5\mathbf{a}_y}{\sqrt{34}}, \quad \mathbf{a}_l = \mathbf{a}_z$$

$$\mathbf{a}_\phi = \mathbf{a}_l \times \mathbf{a}_\rho = \mathbf{a}_z \times \left(\frac{3\mathbf{a}_x + 5\mathbf{a}_y}{\sqrt{34}}\right) = \frac{3\mathbf{a}_y - 5\mathbf{a}_x}{\sqrt{34}}$$

$$\boldsymbol{H}_1 = \frac{20\pi}{2\pi}\left(\frac{-5\mathbf{a}_x + 3\mathbf{a}_y}{34}\right)\text{x}10^{-3} = (-1.47\mathbf{a}_y + 0.88\mathbf{a}_y) \text{ mA/m}$$

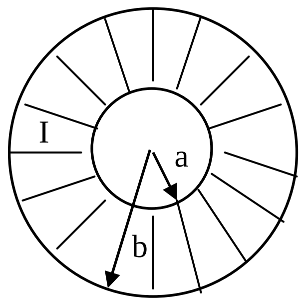
$$\boldsymbol{H}_p = \frac{1}{2}\mathbf{K} \times \mathbf{a}_n = \frac{1}{2}\left(100\times10^{-3}\right) \mathbf{a}_z \times (-\mathbf{a}_x) = -0.05\mathbf{a}_y \text{ A/m}$$

$$\boldsymbol{H} = \boldsymbol{H}_1 + \mathbf{H}_p = \underline{\underline{-1.47\mathbf{a}_x - 49.12\,\mathbf{a}_y \text{ mA/m}}}$$

**Prob. 7.19**

(a) See text

(b)




*For* $\rho < a$, $\oint \boldsymbol{H} \cdot d\boldsymbol{l} = I_{enc} = 0 \rightarrow H = 0$

*For* $a < \rho < b$, $H_\phi \cdot 2\pi\rho = \frac{I\pi(\rho^2 - a^2)}{\pi(b^2 - a^2)}$

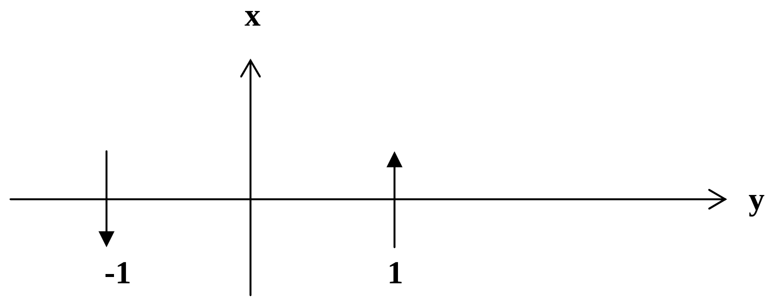
$$H_\phi = \frac{I}{2\pi\rho}\left(\frac{\rho^2 - a^2}{b^2 - a^2}\right)$$

*For* $\rho > b$, $H_\phi \cdot 2\pi\rho = I \rightarrow H_\phi = \frac{I}{2\pi\rho}$

Thus,

$$H_\phi = \begin{cases} 0, & \rho < a \\ \frac{I}{2\pi\rho}\left(\frac{\rho^2 - a^2}{b^2 - a^2}\right), & a < \rho < b \\ \frac{I}{2\pi\rho}, & \rho > b \end{cases}$$

**Prob.7.20**




$$\boldsymbol{H} = \sum \frac{1}{2}\boldsymbol{K} \times \boldsymbol{a}_n$$

$$= \frac{1}{2}(20\boldsymbol{a}_x) \times (-\boldsymbol{a}_y) + \frac{1}{2}(-20\boldsymbol{a}_x) \times \boldsymbol{a}_y$$

$$= 10(-\boldsymbol{a}_z) - 10(\boldsymbol{a}_z)$$

$$\underline{\underline{= -20\boldsymbol{a}_z \text{ A/m}}}$$

**Prob. 7.21**

$$\boldsymbol{H}_{\boldsymbol{P}} = \frac{1}{2}\boldsymbol{k} \times \boldsymbol{a}_n = \frac{1}{2}10\boldsymbol{a}_x \times \boldsymbol{a}_z = -5\boldsymbol{a}_y$$

$$\boldsymbol{H}_L = \frac{I}{2\pi\rho}\boldsymbol{a}_\phi = \frac{I}{2\pi(3)}(\boldsymbol{a}_x \times -\boldsymbol{a}_z) = \frac{I}{6\pi}\boldsymbol{a}_y$$

$$\boldsymbol{H}_P + \boldsymbol{H}_L = -5\boldsymbol{a}_y + \frac{I}{6\pi}\boldsymbol{a}_y = 0 \quad \longrightarrow \quad I = 30\pi = \underline{\underline{94.25 \text{ A}}}$$

**Prob. 7.22**

(a)

From eq. (7.29),

$$\underline{\underline{\boldsymbol{H} = \begin{cases} \dfrac{I\rho}{2\pi a^2}\boldsymbol{a}_\phi, & 0 < \rho < a \\ \dfrac{I}{2\pi\rho}\boldsymbol{a}_\phi, & \rho > a \end{cases}}}$$

(b)

$$\boldsymbol{J} = \nabla \times \boldsymbol{H} = \frac{1}{\rho}\frac{d}{d\rho}(\rho H_\phi)\boldsymbol{a}_z = \underline{\underline{\begin{cases} \dfrac{I}{\pi a^2}\boldsymbol{a}_z, & \rho < a \\ 0\boldsymbol{a}_z, & \rho > a \end{cases}}}$$

**Prob. 7.23**

For $0 < \rho < a$

$$\oint_L \boldsymbol{H} \cdot d\boldsymbol{l} = I_{enc} = \int \boldsymbol{J} \cdot d\boldsymbol{S}$$

$$H_\phi 2\pi\rho = \int_{\phi=0}^{2\pi} \int_{\rho=0}^{\rho} \frac{J_o}{\rho} \rho d\phi d\rho$$

$$= J_o 2\pi\rho$$

$$H_\phi = J_o$$

For $\rho > a$

$$\int \mathbf{H} \cdot d\boldsymbol{l} = \int \boldsymbol{J} \cdot d\boldsymbol{S} = \int_{\phi=0}^{2\pi} \int_{\rho=0}^{a} \frac{J_o}{\rho} \rho d\phi d\rho$$

$$H_\rho 2\pi\rho = J_o 2\pi a$$

$$H_\phi = \frac{J_o a}{\rho}$$

$$\text{Hence } H_\phi = \begin{cases} J_o, & 0<\rho<a \\ \dfrac{J_o a}{\rho}, & \rho>a \end{cases}$$

**Prob. 7.24**

(a) $$\boldsymbol{J} = \nabla \times \boldsymbol{H} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & 0 \end{vmatrix} = \underline{\underline{2(x-y)\boldsymbol{a}_z}}$$

(b) $d\boldsymbol{S} = dxdy\boldsymbol{a}_z$ ,

$$I = \int_S \boldsymbol{J} \bullet d\boldsymbol{S} = \iint (2x-2y)dxdy = 2\int_1^5 dy \int_0^2 xdx - 2\int_0^2 dx \int_1^5 ydy = 2(4)\left(\frac{x^2}{2}\bigg|_0^2\right) - 2(2)\left(\frac{y^2}{2}\bigg|_1^5\right)$$

$$= 4(4-0) - 2(25-1) = 16 - 48 = \underline{\underline{-32 \text{ A}}}$$

**Prob. 7.25**

(a) $$\boldsymbol{J} = \nabla \times \boldsymbol{H} = \frac{1}{\rho}\frac{d}{d\rho}(\rho H_\phi)\boldsymbol{a}_z = \frac{1}{\rho}\frac{d}{d\rho}(k_o \frac{\rho^2}{a})\boldsymbol{a}_z = \underline{\underline{\frac{2k_o}{a}\boldsymbol{a}_z}}$$

(b)For ρ>a,

$$\oint \boldsymbol{H} \cdot d\boldsymbol{l} = I_{enc} = \int \boldsymbol{J} \cdot dS = \int_{\rho=0}^{a} \int_{\phi=0}^{2\pi} \frac{2k_o}{a} \rho d\rho d\phi = \frac{2k_o}{a}(2\pi)\frac{\rho^2}{2}\bigg|_0^a$$

$$H_\phi 2\pi\rho = 2\pi k_o a \quad \longrightarrow \quad H_\phi = \frac{k_o a}{\rho}$$

$$\boldsymbol{H} = k_o\left(\frac{a}{\rho}\right)\boldsymbol{a}_\phi, \qquad \rho > a$$

**Prob. 7.26**

$$\boldsymbol{J} = \nabla \times \boldsymbol{H} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & 0 \end{vmatrix} = (2x - 2y)\boldsymbol{a}_z$$

At (1,-4,7), x =1, y = -4, z=7,

$$\boldsymbol{J} = [2(1) - 2(-4)]\boldsymbol{a}_z = 10\boldsymbol{a}_z \text{ A/m}^2$$

**Prob. 7.27**

(a)

$$\boldsymbol{J} = \nabla \times \boldsymbol{H} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho H_\phi)\boldsymbol{a}_z = \frac{1}{\rho}\frac{\partial}{\partial\rho}(10^3\rho^3)\boldsymbol{a}_z$$

$$= 3\rho \times 10^3 \boldsymbol{a}_z \text{ A/m}^2$$

(b)

*Method 1:*

$$I = \int_S \boldsymbol{J} \cdot d\boldsymbol{S} = \iint 3\rho \; \rho d\phi d\rho 10^3 = 3\times 10^3 \int_0^2 \rho^2 d\rho \int_0^{2\pi} d\phi$$

$$= 3\times 10^3 (2\pi)\frac{\rho^3}{3}\bigg|_2^2 = 16\pi \times 10^3 A = 50.265 \text{ kA}$$

*Method 2:*

$$I = \oint_L \boldsymbol{H} \cdot d\boldsymbol{l} = 10^3 \int_0^{2\pi} \rho^2 \; \rho d\phi = 10^3(8)(2\pi) = 50.265 \text{ kA}$$

**Prob. 7.28**

$$\boldsymbol{J} = \nabla \times \boldsymbol{H} = \frac{1}{\rho}\frac{d}{d\rho}(\rho H_\phi)\boldsymbol{a}_z = \frac{1}{\rho}\frac{d}{d\rho}(4\rho^2)\boldsymbol{a}_z = 8\boldsymbol{a}_z$$

$$I = \int_S \boldsymbol{J} \bullet d\boldsymbol{S} = JS = 8(\pi a^2) = 8\pi \times 10^{-4} = 2.513 \text{ mA}$$

**Prob. 7.29**

(a) $\boldsymbol{B} = \dfrac{\mu_o I}{2\pi\rho}\boldsymbol{a}_\phi$

At (-3,4,5), ρ=5.

$$B = \frac{4\pi \times 10^{-7} \times 2}{2\pi(5)} a_\phi = \underline{\underline{80 a_\phi \text{ nWb/m}^2}}$$

(b) $$\Psi = \int \boldsymbol{B} \bullet dS = \frac{\mu_o I}{2\pi} \iint \frac{d\rho dz}{\rho} = \frac{4\pi \times 10^{-7} \times 2}{2\pi} \ln \rho \Big|_2^6 z \Big|_0^4$$

$$= 16 \times 10^{-7} \ln 3 = \underline{\underline{1.756\ \mu\text{Wb}}}$$

**Prob. 7.30**

Let $\boldsymbol{H} = \boldsymbol{H}_1 + \boldsymbol{H}_2$

where $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$ are due to the wires centered at x = 0 and x = 10cm respectively.

(*a*) For $\boldsymbol{H}_1$, $\rho = 50$ cm, $\boldsymbol{a}_\phi = \boldsymbol{a}_l \times \boldsymbol{a}_\rho = \boldsymbol{a}_z \times \boldsymbol{a}_x = \boldsymbol{a}_y$

$$\boldsymbol{H}_1 = \frac{5}{2\pi\left(5 \times 10^{-2}\right)} \boldsymbol{a}_y = \frac{50}{\pi} \boldsymbol{a}_y$$

For $\boldsymbol{H}_2$, $\rho = 5$ cm, $\boldsymbol{a}_\phi = -\boldsymbol{a}_z \times -\boldsymbol{a}_x = \boldsymbol{a}_y$, $\boldsymbol{H}_2 = \boldsymbol{H}_1$

$$\mathbf{H} = 2\boldsymbol{H}_1 = \frac{100}{\pi} \boldsymbol{a}_y$$

$$= \underline{\underline{31.83\, \boldsymbol{a}_y \text{ A/m}}}$$

(*b*) For $\boldsymbol{H}_1$, $\boldsymbol{a}_\phi = \boldsymbol{a}_z \times \left(\dfrac{2\boldsymbol{a}_x + \boldsymbol{a}_y}{\sqrt{5}}\right) = \dfrac{2\boldsymbol{a}_y - \boldsymbol{a}_x}{\sqrt{5}}$

$$\boldsymbol{H}_1 = \frac{5}{2\pi 5\sqrt{5} \times 10^{-2}} \left(\frac{-\boldsymbol{a}_x + 2\boldsymbol{a}_y}{\sqrt{5}}\right) = -3.183\boldsymbol{a}_x + 6.366\boldsymbol{a}_y$$

For $\boldsymbol{H}_2$, $\boldsymbol{a}_\rho = -\boldsymbol{a}_z \times \boldsymbol{a}_y = \boldsymbol{a}_x$

$$\boldsymbol{H}_2 = \frac{5}{2\pi(5)} \boldsymbol{a}_x = 15.915\boldsymbol{a}_x$$

$$\boldsymbol{H} = \boldsymbol{H}_1 + \boldsymbol{H}_2$$

$$= \underline{\underline{12.3\, \boldsymbol{a}_x + 6.366\boldsymbol{a}_y \text{ A/m}}}$$

**Prob. 7.31**

(a) $I = \int \boldsymbol{J} \cdot d\boldsymbol{S}$

$$= \int_{\phi=0}^{2\pi} \int_{\rho=0}^{a} J_o (1 - \frac{\rho^2}{a^2}) \rho d\rho d\phi \qquad = J_o \int_0^{2\pi} d\phi \int_0^a (\rho - \frac{\rho^3}{a^2}) d\rho$$

$$= 2\pi J_o \left( \frac{\rho^2}{2} - \frac{\rho^4}{4a^2} \right) \Big|_0^a \qquad = \frac{2\pi}{2} J_o \left( a^2 - \frac{a^2}{2} \right)$$

$$= \frac{1}{2} \pi a^2 J_o$$

(b) $\oint \boldsymbol{H} \cdot d\boldsymbol{l} = I_{enc} = \int \boldsymbol{J} \cdot d\boldsymbol{S}$

For $\rho$ < a,

$$H_\phi 2\pi\rho = \int \boldsymbol{J} \cdot d\boldsymbol{S}$$

$$= 2\pi J_o \left( \frac{\rho^2}{2} - \frac{\rho^4}{4a^2} \right)$$

$$H_\rho \cdot 2\pi\rho = 2\pi J_o \frac{\rho^2}{4} \left( 2 - \frac{\rho^2}{a^2} \right)$$

$$H_\rho = \frac{J_o \rho}{4} \left( 2 - \frac{\rho^2}{a^2} \right)$$

For $\rho$ > a,

$$\oint \mathbf{H} \cdot dl = \int J_o d\boldsymbol{S} = I$$

$$H_\phi \cdot 2\pi\rho = \frac{1}{2} \pi a^2 J_o$$

$$H_\phi = \frac{a^2 J_o}{4\rho}$$

Hence $H_\phi = \begin{cases} \frac{J_o \rho}{4} \left( 2 - \frac{\rho^2}{a^2} \right), \rho < a \\ \frac{aJ_o}{4\rho}, \quad \rho > a \end{cases}$

**Prob. 7.32**

$$\boldsymbol{B} = \frac{\mu_0 I}{2\pi\rho}\boldsymbol{a}_\phi$$

$$\psi = \int \boldsymbol{B}\cdot d\boldsymbol{S} = \int_{\rho=d}^{d+a}\int_{z=0}^{b}\frac{\mu_0 I}{2\pi\rho}\,d\rho dz$$

$$= \underline{\underline{\frac{\mu_0 Ib}{2\pi}\ln\frac{d+a}{d}}}$$

**Prob.7.33**

For a whole circular loop of radius $a$, Example 7.3 gives

$$\boldsymbol{H} = \frac{Ia^2\boldsymbol{a}_z}{2\left[a^2+h^2\right]^{3/2}}$$

Let h $\longrightarrow 0$

$$\boldsymbol{H} = \frac{I}{2a}\boldsymbol{a}_z$$

For a semicircular loop, H is halfed

$$\boldsymbol{H} = \frac{I}{4a}\boldsymbol{a}_z$$

$$\boldsymbol{B} = \mu_o\boldsymbol{H} = \underline{\underline{\frac{\mu_o I}{4a}\boldsymbol{a}_z}}$$

**Prob. 7.34**

(a) $\nabla\bullet\boldsymbol{B} = \frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z} = 0$

showing that **B** satisfies Maxwell's equation.

(b) $d\boldsymbol{S} = dydz\boldsymbol{a}_x$

$$\Psi = \int\boldsymbol{B}\bullet d\boldsymbol{S} = \int_{z=1}^{4}\int_{y=0}^{1} y^2 dydz = \frac{y^3}{3}\Big|_0^1 (z)\Big|_1^4 = \underline{\underline{1\text{ Wb}}}$$

(c ) $\nabla \times \mathbf{H} = \mathbf{J} \quad \longrightarrow \quad \boldsymbol{J} = \nabla \times \frac{\boldsymbol{B}}{\mu_o}$

$$\nabla \times \boldsymbol{B} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & z^2 & x^2 \end{vmatrix} = -2z\boldsymbol{a}_x - 2x\boldsymbol{a}_y - 2y\boldsymbol{a}_z$$

$$\boldsymbol{J} = -\frac{2}{\mu_o}(z\boldsymbol{a}_x + x\boldsymbol{a}_y + y\boldsymbol{a}_z) \text{ A/m}^2$$

(d)
Since $\nabla \bullet \boldsymbol{B}=0$,

$$\Psi = \oint_S \boldsymbol{B} \bullet d\boldsymbol{S} = \int_v \nabla \bullet \boldsymbol{B} dv = 0$$

**Prob. 7.35**

On the slant side of the ring, $z = \frac{h}{6}(\rho - a)$

where $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$ are due to the wires centered at x = 0 and x = 10cm respectively.

$$\psi = \int \boldsymbol{B}.\mathrm{d}\boldsymbol{S} = \int \frac{\mu_o I}{2\pi\rho} \,\mathrm{d}\rho \,\mathrm{dz}$$

$$= \frac{\mu_o I}{2\pi} \int_{\rho=a}^{a+b} \int_{z=0}^{\frac{h}{b}(\rho-a)} \frac{dz\, \mathrm{d}\rho}{\rho} = \frac{\mu_o Ih}{2\pi b} \int_{\rho=a}^{a+b} \left(1 - \frac{a}{\rho}\right) \mathrm{d}\rho$$

$$= \frac{\mu_o Ih}{2\pi b}\left(b - a \ln \frac{a+b}{a}\right) \text{ as required.}$$

If a = 30 cm, b = 10 cm, h = 5 cm, I = 10 A,

$$\psi = \frac{4\pi \times 10^{-7} \times 10 \times 0.05}{2\pi\left(10 \times 10^{-2}\right)} \left(0.1 - 0.3 \ln \frac{4}{3}\right)$$

$$= 1.37 \times 10^{-8} \text{ Wb}$$

**Prob. 7.36**

$$\psi = \int \mathbf{B} \cdot \mathrm{d}\boldsymbol{S} = \mu_o \int_{z=0}^{0.2} \int_{\phi=0}^{50^o} \frac{10^6}{\rho} \sin 2\phi \; \rho \; \mathrm{d}\phi \; \mathrm{dz}$$

$$\psi = 4\pi \times 10^{-7} \times 10^6 (0.2) \left(-\frac{\cos 2\phi}{2}\right) \Bigg|_0^{50^o}$$

$$= 0.04\pi \left(1 - \cos 100^o\right)$$

$$= 0.1475 \text{ Wb}$$

**Prob. 7.37**

$$\psi = \int_S \mathbf{B} \cdot d\mathbf{S} = \int_{\phi=0}^{\pi/4} \int_{\rho=1}^{2} \frac{20}{\rho} \sin^2\phi \; \rho d\rho d\phi = 20 \int_1^2 d\rho \int_0^{\pi/4} \sin^2\phi d\phi$$

$$= 20(1) \int_0^{\pi/4} \frac{1}{2}(1-\cos 2\phi) d\phi = 10(\phi - \frac{1}{2}\sin 2\phi) \Big|_0^{\pi/4}$$

$$= 10(\frac{\pi}{4} - \frac{1}{2}) = \underline{\underline{2.854 \text{ Wb}}}$$

**Prob. 7.38**

$$\psi = \int_S \mathbf{B} \cdot d\mathbf{S}, \quad d\mathbf{S} = r^2 \sin\theta \, d\theta \, d\phi \, \mathbf{a}_r$$

$$\psi = \iint \frac{2}{r^3} \cos\theta \; r^2 \sin\theta d\theta d\phi \Big|_{r=1} = 2 \int_0^{2\pi} d\phi \int_0^{\pi/3} \cos\theta \sin\theta d\theta$$

$$= 2(2\pi) \int_0^{\pi/3} \sin\theta d(\sin\theta) = 4\pi \frac{\sin^2\theta}{2} \Big|_0^{\pi/3} = 2\pi \sin^2(\pi/3)$$

$$= \underline{\underline{4.7123 \text{ Wb}}}$$

**Prob. 7.39**

$$\mathbf{B} = \mu_o \mathbf{H} = \frac{\mu_o}{4\pi} \int_v \frac{\mathbf{J} \times \mathbf{R}}{R^3} dv$$

Since current is the flow of charge, we can express this in terms of a charge moving with velocity **u**. **J**dv = dq**u**.

$$\mathbf{B} = \frac{\mu_o}{4\pi} \left[ \frac{q\mathbf{u} \times \mathbf{R}}{R^3} \right]$$

In our case, $\mathbf{u}$ and $\mathbf{R}$ are perpendicular. Hence,

$$B = \frac{\mu_o}{4\pi} \frac{qu}{R^2} = \frac{4\pi \times 10^{-7}}{4\pi} \times \frac{1.6 \times 10^{-19} \times 2.2 \times 10^6}{(5.3 \times 10^{-11})^2} = \frac{1.6 \times 10^{-20}}{(5.3)^2 \times 10^{-22}}$$

$$= \underline{\underline{12.53 \text{ Wb/m}^2}}$$

**Prob. 7.40**

(a) $\nabla\cdot A = -ya \sin ax \neq 0$

$$\nabla\times A = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y\cos ax & 0 & y+e^{-x} \end{vmatrix}$$

$$= \boldsymbol{a}_x + e^{-x}\boldsymbol{a}_y - \cos ax\,\boldsymbol{a}_z \neq \boldsymbol{0}$$

***A*** is neither electrostatic nor magnetostatic field

(b) $\nabla\cdot\boldsymbol{B} = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho B_\rho\right) = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(20\right) = 0$

$\nabla\times\boldsymbol{B} = \boldsymbol{0}$

**B** can be **E**-field in a charge-free region.

(c) $\nabla\cdot\mathbf{C} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(r^2\sin\theta) = 0$

$\nabla\times\boldsymbol{C} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(r^2\sin^2\theta\right)\boldsymbol{a}_r - \frac{1}{r}\frac{\partial}{\partial r}(r^3\sin\theta)\boldsymbol{a}_\theta \neq \boldsymbol{0}$

***C*** is possibly ***H*** field.

**Prob. 7.41**

(a) $\nabla\cdot\boldsymbol{D} = 0$

$$\nabla\times\boldsymbol{D} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 z & 2(x+1)yz & -(x+1)z^2 \end{vmatrix}$$

$$= 2(x+1)y\boldsymbol{a}_x + \ldots \neq \boldsymbol{0}$$

***D*** is possibly a magnetostatic field.

(b) $\nabla\cdot\mathbf{E} = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left((z+1)\cos\phi\right) + \frac{\partial}{\partial z}\left(\frac{\sin\phi}{\rho}\right) = 0$

$\nabla\times\boldsymbol{E} = \frac{1}{\rho^2}\cos\theta\,\boldsymbol{a}_\rho + \ldots \neq \boldsymbol{0}$

***E*** could be a magnetostatic field.

(c) $\nabla\cdot\boldsymbol{F} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(2\cos\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{r^2}\right) \neq 0$

$\nabla\times\boldsymbol{F} = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r^{-1}\sin\theta\right) + \frac{2\sin\theta}{r^2}\right]\boldsymbol{a}_\theta \neq \boldsymbol{0}$

***F*** can be neither electrostatic nor magnetostatic field.

**Prob. 7.42**

$$\boldsymbol{A} = \int \frac{\mu_o I d\boldsymbol{l}}{4\pi r} = \frac{\mu_o I L \boldsymbol{a_z}}{4\pi r}$$

This requires no integration since L << r.

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} = \frac{1}{\rho}\frac{\partial A_z}{\partial \phi}\boldsymbol{a}_\rho - \frac{\partial A_z}{\partial \rho}\boldsymbol{a}_\phi$$

But r $= \sqrt{\rho^2 + z^2}$

$$\boldsymbol{A} = \frac{\mu_o I L \boldsymbol{a}_z}{4\pi(\rho^2 + z^2)^{1/2}}$$

$$\frac{\partial A_z}{\partial \rho} = \frac{\mu_o I L}{4\pi}\frac{\partial}{\partial \rho}(\rho^2 + z^2)^{1/2} = \frac{\mu_o I L}{4\pi}(-\frac{1}{2})(\rho^2 + z^2)^{-3/2}(2\rho)$$

$$\boldsymbol{B} = \frac{\mu_o I L \rho \mathbf{a}_\phi}{4\pi(\rho^2 + z^2)^{3/2}} = \frac{\mu_o I L \rho \mathbf{a}_\phi}{4\pi r^3}$$

**Prob. 7.43**

$$\boldsymbol{B} = \mu_o \boldsymbol{H} = \nabla \times \boldsymbol{A} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 10\sin \pi y & 0 & 4 + \cos \pi x \end{vmatrix} = \pi \sin \pi x \boldsymbol{a}_y - 10\pi \cos \pi y \boldsymbol{a}_z$$

$$\boldsymbol{H} = \frac{\pi}{\mu_o}\left(\sin \pi x \boldsymbol{a}_y - 10\cos \pi y \boldsymbol{a}_z\right)$$

$$\boldsymbol{J} = \nabla \times \boldsymbol{H} = \frac{\pi}{\mu_o}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & \sin \pi x & -10\cos \pi y \end{vmatrix} = \frac{\pi}{\mu_o}\left(10\pi \sin \pi y \boldsymbol{a}_x + \pi \cos \pi x \boldsymbol{a}_z\right)$$

$$\boldsymbol{J} = \frac{\pi^2}{\mu_o}\left(10\sin \pi y \boldsymbol{a}_x + \cos \pi x \boldsymbol{a}_z\right)$$

**Prob. 7.44**

(a)

$$\nabla \bullet A = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + 0 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{2\cos\theta}{r}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\sin^2\theta}{r^3}\right)$$

$$= -\frac{1}{r^4}2\cos\theta + \frac{1}{r^4\sin\theta}2\sin\theta\cos\theta = 0$$

(b)

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(A_\phi \sin\theta) - \frac{\partial A_\theta}{\partial\phi}\right]\boldsymbol{a}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(rA_\phi)\right]\boldsymbol{a}_\theta$$

$$+\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]\boldsymbol{a}_\phi$$

$$= 0\,\boldsymbol{a}_r + 0\boldsymbol{a}_\theta + \frac{1}{r}\left[\frac{\partial}{\partial r}\left(\frac{\sin\theta}{r^2}\right) + \frac{2\sin\theta}{r^3}\right]\boldsymbol{a}_\phi = \frac{1}{r}\left[-\frac{2\sin\theta}{r^3} + \frac{2\sin\theta}{r^3}\right]\boldsymbol{a}_\phi = \underline{\underline{\boldsymbol{0}}}$$

**Prob. 7.45**

$$(a)\quad \boldsymbol{B} = \nabla \times \mathbf{A} = \begin{vmatrix} \frac{\partial}{\partial \mathrm{x}} & \frac{\partial}{\partial \mathrm{y}} & \frac{\partial}{\partial \mathrm{z}} \\ 2\mathrm{x}^2\mathrm{y} + \mathrm{yz} & \mathrm{xy}^2 - \mathrm{xz}^3 & -6xy + 2\mathrm{z}^2\mathrm{y}^2 \end{vmatrix}$$

$$\boldsymbol{B} = \underline{\underline{(-6\mathrm{xz} + 4x^2y + 3xz^2)\boldsymbol{a}_x + \left(\mathrm{y} + 6\mathrm{yz}\text{-}4\mathrm{xy}^2\right)\boldsymbol{a}_y + \left(\mathrm{y}^2 - z^3 - 2x^2 - z\right)\boldsymbol{a}_z}}\ \mathrm{Wb/m}^2$$

(b)

$$\psi = \int_{z=0}^{2}\int_{y=0}^{2}\left(-6xz + 4x^2y + 3xz^2\right)\,dy\,dz\,\Big|_{x=1}$$

$$= \int\int_0 \left(-6xz\right)\,dy\,dz + 4\int\int_0 \mathrm{x}^2\mathrm{y}\,dy\,dz + 3\int\int_0 \mathrm{xz}^2\,dy\,dz$$

$$= -6\int_0^2 \mathrm{dz}\int_0^2 dy + 4\int_0^2 dz\int_0^2 \mathrm{y}\,dy + 3\int_0^2 dy\int_0^2 z^2 dz$$

$$= -6(2)(2) + 4(2)\left(\frac{\mathrm{y}^2}{2}\Big|_0^2\right) + 3(2)\left(\frac{z^3}{3}\Big|_0^2\right) = -24+16+16$$

$$\psi = \underline{\underline{8\ \mathrm{Wb}}}$$

$$(c)\quad \nabla\cdot\boldsymbol{A} = \frac{\partial \mathrm{A_x}}{\partial \mathrm{x}} + \frac{\partial \mathrm{A_y}}{\partial \mathrm{y}} + \frac{\partial \mathrm{A_z}}{\partial \mathrm{z}} = 4\mathrm{xy} + 2\mathrm{xy} - 6xy = 0$$

$$\nabla\cdot\boldsymbol{B} = -6z + 8xy + 3z^3 + 6z - 8xy + 1 - 3z^3 - 1 = 0$$

As a matter of mathematical necessity,

$$\nabla \bullet \boldsymbol{B} = \nabla \bullet (\nabla \times \boldsymbol{A}) = 0$$

**Prob. 7.46**

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(A_\phi \sin\theta) - \frac{\partial A_\theta}{\partial\phi}\right]\boldsymbol{a}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(rA_\phi)\right]\boldsymbol{a}_\theta$$

$$= \frac{1}{r\sin\theta}\frac{k}{r^2}\frac{\partial}{\partial\theta}(\sin^2\theta)\boldsymbol{a}_r - \frac{1}{r}\frac{\partial}{\partial r}(kr^{-1})\sin\theta\boldsymbol{a}_\theta$$

$$= \frac{k}{r^3\sin\theta}2\sin\theta\cos\theta\,\boldsymbol{a}_r + \frac{k\sin\theta}{r^3}\boldsymbol{a}_\theta = \underline{\underline{\frac{2k\cos\theta}{r^3}\boldsymbol{a}_r + \frac{k\sin\theta}{r^3}\boldsymbol{a}_\theta}}$$

**Prob. 7.47**

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} = \frac{1}{\rho}\frac{\partial A_z}{\partial\phi}\boldsymbol{a}_\rho - \frac{\partial A_z}{\partial\rho}\boldsymbol{a}_\phi$$

$$= \frac{15}{\rho}\mathrm{e}^{-\rho}\cos\phi\,\boldsymbol{a}_\rho + 15\,\mathrm{e}^{-\rho}\sin\phi\,\boldsymbol{a}_\phi$$

$$\boldsymbol{B}\left(3, \frac{\pi}{4}, -10\right) = 5\,\mathrm{e}^{-3}\frac{1}{\sqrt{2}}\boldsymbol{a}_\rho + 15\,\mathrm{e}^{-3}\frac{1}{\sqrt{2}}\boldsymbol{a}_\phi$$

$$\mathbf{H} = \frac{\boldsymbol{B}}{\mu_o} = \frac{10^7}{4\pi}\frac{15}{\sqrt{2}}\mathrm{e}^{-3}\left(\frac{1}{3}\boldsymbol{a}_\rho + \boldsymbol{a}_\phi\right)$$

$$\mathbf{H} = \left(14\,\boldsymbol{a}_\rho + 42\,\boldsymbol{a}_\phi\right)\cdot 10^4 \text{ A/m}$$

$$\psi = \int \boldsymbol{B}\cdot \mathrm{d}\boldsymbol{S} = \iint \frac{15}{\rho}\mathrm{e}^{-\rho}\cos\phi\,\rho\,\mathrm{d}\phi\,\mathrm{dz}$$

$$= 15\,\mathrm{z}\Big|_0^{10}\left(\sin\phi\right)\Big|_0^{\pi/2}\mathrm{e}^{-5} = 150\,\mathrm{e}^{-5} \quad\Rightarrow\quad \psi = \underline{\underline{1.011\text{ Wb}}}$$

**Prob. 7.48**

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(A_\phi \sin\theta) - \frac{\partial A_\theta}{\partial\phi}\right]\boldsymbol{a}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(rA_\phi)\right]\boldsymbol{a}_\theta$$

$$+\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]\boldsymbol{a}_\phi$$

$$= \frac{1}{r\sin\theta}\frac{10}{r}2\sin\theta\cos\theta\boldsymbol{a}_r - \frac{1}{r}\frac{\partial}{\partial r}(10)\sin\theta\boldsymbol{a}_\theta + 0\boldsymbol{a}_\phi$$

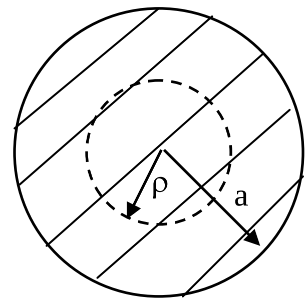
$$\boldsymbol{B} = \frac{20}{r^2}\cos\theta\boldsymbol{a}_r$$

At $(4, 60^o, 30^o)$, r = 4, $\theta$=60$^o$

$$\boldsymbol{H} = \frac{\boldsymbol{B}}{\mu_o} = \frac{1}{4\pi\times 10^{-7}}\left[\frac{20}{4^2}\cos 60^o\boldsymbol{a}_r\right] = \underline{\underline{4.974\times 10^5\boldsymbol{a}_r \text{ A/m}}}$$

**Prob. 7.49**

Applying Ampere's law gives




$$H_\phi \cdot 2\pi\rho \quad = \quad J_o \cdot \pi\rho^2$$

$$H_\phi \quad = \quad \frac{J_o}{2}\rho$$

$$B_\phi \quad = \quad \mu_o\, H_\phi \quad = \quad \mu_o \frac{J_o\rho}{2}$$

But $\boldsymbol{B} \quad = \quad \nabla\times\mathbf{A} \quad = \quad -\frac{\partial A_z}{\partial\rho}\bar{a}_\varphi \quad + \quad . \; . \; .$

$$-\frac{\partial A_z}{\partial\rho} \quad = \quad \frac{1}{2}\mu J_o\rho \quad \longrightarrow \quad A_z \quad = \quad -\mu_o\frac{J_o\rho^2}{4}$$

$$\text{or} \quad \mathbf{A} \; = \; -\frac{1}{4}\mu_o J_o\rho^2\, \boldsymbol{a}_z$$

**Prob. 7.50**

$$\boldsymbol{B} = \nabla\times\boldsymbol{A} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & A_z(x,y) \end{vmatrix} = \frac{\partial A_z}{\partial y}\boldsymbol{a}_x - \frac{\partial A_z}{\partial x}\boldsymbol{a}_y$$

$$= -\frac{\pi}{2}\sin\frac{\pi x}{2}\sin\frac{\pi y}{2}\boldsymbol{a}_x - \frac{\pi}{2}\cos\frac{\pi x}{2}\cos\frac{\pi y}{2}\boldsymbol{a}_y$$

**Prob. 7.51**

$$\boldsymbol{B} = \nabla\times\boldsymbol{A} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\phi\sin\theta)\boldsymbol{a}_r - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\boldsymbol{a}_\theta$$

$$= \frac{1}{r\sin\theta}\frac{A_o}{r^2}(2\sin\theta\cos\theta)\boldsymbol{a}_r - \frac{A_o}{r}\sin\theta(-r^{-2})\boldsymbol{a}_\theta$$

$$= \frac{A_o}{r^3}(2\cos\theta\boldsymbol{a}_r + \sin\theta\boldsymbol{a}_\theta)$$

**Prob. 7.52**

(a) $$\boldsymbol{J} = \nabla \times \boldsymbol{H} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & x^2 z & -y^2 z \end{vmatrix} = (-2yz - x^2)\boldsymbol{a}_x + (2xz - 2xy)\boldsymbol{a}_z$$

At (2,-1,3), x=2, y=-1, z=3.

$$\boldsymbol{J} = \underline{\underline{2\boldsymbol{a}_x + 16\boldsymbol{a}_z \text{ A/m}^2}}$$

(b) $$-\frac{\partial \rho_v}{\partial t} = \nabla \bullet \boldsymbol{J} = 0 - 2x + 2x = 0$$

At (2,-1,3),

$$\frac{\partial \rho_v}{\partial t} = \underline{\underline{0 \text{ C/m}^3\text{s}}}$$

**Prob. 7.53**

(a) $$\mathbf{B} = \nabla \times \boldsymbol{A}$$

$$= \left[\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right]\boldsymbol{a}_\rho + \left[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right]\boldsymbol{a}_\phi + \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho A_\phi) - \frac{\partial A_\rho}{\partial \phi}\right]\boldsymbol{a}_z$$

$$= -\frac{\partial A_z}{\partial \rho}\boldsymbol{a}_\phi = 20\rho \boldsymbol{a}_\phi \ \mu\text{Wb/m}^2$$

$$\boldsymbol{H} = \frac{\boldsymbol{B}}{\mu_o} = \underline{\underline{\frac{-20\rho}{\mu_o}\boldsymbol{a}_\phi \ \mu\text{A/m}}}$$

$$\boldsymbol{J} = \nabla \times \boldsymbol{H} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\phi)\boldsymbol{a}_z$$

$$= \frac{1}{\mu_o \rho}(-40\rho)\boldsymbol{a}_z = \underline{\underline{\frac{-40}{\mu_o}\boldsymbol{a}_z \ \mu\text{A/m}^2}}$$

(b) $$I = \int \boldsymbol{J} \cdot d\boldsymbol{S} = \frac{-40}{\mu_o}\int_{\rho=0}^{2}\int_{\phi=0}^{2\pi} \rho \, d\phi \, d\rho, \quad d\boldsymbol{S} = \rho \, d\phi \, d\rho \, \boldsymbol{a}_z$$

$$= \frac{-40}{\mu_o}\int_0^2 \rho \, d\rho \int_0^{2\pi} d\phi = \frac{-40}{\mu_o}\frac{\rho^2}{2}\Big|_0^2 (2\pi)$$

$$= \frac{-80\pi \times 2 \times 10^{-6}}{4\pi \times 10^{-7}} = \underline{\underline{-400 \text{ A}}}$$

**Prob. 7.54**
$$\mathbf{H} = -\nabla V_m \quad \rightarrow \quad V_m = -\int \boldsymbol{H}\cdot d\boldsymbol{l} = -\text{mmf}$$

From Example 7.3, $\boldsymbol{H} = \dfrac{Ia^2}{2\left(z^2+a^2\right)^{3/2}}\boldsymbol{a}_z$

$$V_m = -\frac{Ia^2}{2}\int \left(z^2+a^2\right)^{-3/2} dz = \frac{-Iz}{2\left(z^2+a^2\right)^{1/2}} + c$$

As $z \rightarrow \infty$, $V_m = 0$, i.e.

$$0 = -\frac{I}{2} + c \quad \rightarrow \quad c = \frac{I}{2}$$

Hence,

$$V_m = \frac{I}{2}\left[1 - \frac{z}{\sqrt{z^2+a^2}}\right]$$

**Prob. 7.55**

$$\boldsymbol{H} = \frac{I}{2\pi\rho}\,\boldsymbol{a}_\phi$$

*But* $\quad \boldsymbol{H} = -\nabla V_m \quad (\boldsymbol{J} = 0)$

$$\frac{I}{2\pi\rho}\boldsymbol{a}_\phi = -\frac{1}{\rho}\frac{\partial V_m}{\partial \phi}\,\mathbf{a}_\varphi \quad \rightarrow \quad V_m = -\frac{I}{2\pi}\phi + C$$

At $(10, 60^o, 7)$, $\phi = 60^o = \dfrac{\pi}{3}$, $V_m = 0 \rightarrow 0 = -\dfrac{I}{2\pi}\cdot\dfrac{\pi}{3} + C$

or $\quad C = \dfrac{I}{6}$

$$V_m = -\frac{I}{2\pi}\phi + \frac{I}{6}$$

At $(4, 30^o, -2)$, $\phi = 30^o = \dfrac{\pi}{6}$,

$$V_m = -\frac{I}{2\pi}\cdot\frac{\pi}{6} + \frac{I}{6} = \frac{I}{12} = \frac{12}{12}$$

$$\underline{\underline{V_m = 1\text{ A}}}$$

**Prob. 7.56**

For an infinite current sheet,

$$\bar{H} = \frac{1}{2}\bar{K} \times \bar{a}_n = \frac{1}{2}50\bar{a}_y \times \bar{a}_n = 25\bar{a}_x$$

But $\bar{H} = -\nabla V_m \quad (\bar{J} = 0)$

$$25\,\bar{a}_x = -\frac{\partial V_m}{\partial x}\,\bar{a}_n \rightarrow V_m = -25x + c$$

At the origin, $x = 0$, $V_m = 0$, $c = 0$, i.e.

$$V_m = -25x$$

(*a*) At $(-2, 0, 5)$, $\underline{\underline{V_m = 50\text{A}}}$.

(*b*) At $(10, 3, 1)$, $\underline{\underline{V_m = -250\text{A}}}$.

**Prob. 7.57**

(a)
$$\nabla \times \nabla V = \nabla \times \left( \frac{\partial V}{\partial \rho}\,\boldsymbol{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\,\boldsymbol{a}_\phi + \frac{\partial V}{\partial z}\boldsymbol{a}_z \right)$$

$$= \left( \frac{1}{\rho}\frac{\partial^2 V}{\partial\phi\partial z} - \frac{1}{\rho}\frac{\partial^2 V}{dz\partial\phi} \right)\boldsymbol{a}_\rho + \left( \frac{\partial^2 V}{\partial z\partial\rho} - \frac{\partial^2 V}{\partial\rho\partial z} \right)\boldsymbol{a}_\phi$$

$$+ \frac{1}{\rho}\left( \frac{\partial^2 V}{\partial\rho\partial\phi} - \frac{\partial^2 V}{\partial\phi d\rho} \right)\boldsymbol{a}_z = 0$$

(b)
$$\nabla\cdot(\nabla\times\boldsymbol{A}) = \nabla\cdot\left[\left( \frac{1}{\rho}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z} \right)\boldsymbol{a}_\rho \right.$$

$$\left. + \left( \frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho} \right)\boldsymbol{a}_\phi + \frac{1}{\rho}\left( \frac{\partial}{\partial\rho}(\rho A_\phi) - \frac{\partial A_\rho}{\partial\phi} \right)\boldsymbol{a}_z \right]$$

$$= \frac{1}{\rho}\frac{\partial^2 A_z}{\partial\rho\partial\phi} - \frac{1}{\rho}\frac{\partial}{\partial\rho}\left( \rho\frac{\partial A_\phi}{\partial z} \right) + \frac{1}{\rho}\frac{\partial^2 A_\rho}{\partial\phi\partial z} - \frac{1}{\rho}\frac{\partial^2 A_z}{\partial\phi\partial\rho}$$

$$+ \frac{\partial}{\partial z}\left( \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\phi) \right) - \frac{\partial}{\partial z}\left( \frac{1}{\rho}\frac{\partial A_\rho}{\partial\phi} \right)$$

$$= -\frac{\partial^2 A_\phi}{\partial\rho\partial z} - \frac{1}{\rho}\frac{\partial A_\phi}{\partial z} + \frac{\partial^2 A_\phi}{\partial z\partial\rho} + \frac{1}{\rho}\frac{\partial A_\phi}{\partial z} = 0$$

**Prob. 7.58**

$$\nabla'\frac{1}{\text{R}} = \left(\frac{\partial}{\partial x'}\,\boldsymbol{a}_x + \frac{\partial}{\partial y'}\,\boldsymbol{a}_y + \frac{\partial}{\partial \text{z}'}\,\boldsymbol{a}_z\right)\left[(\text{x}-\text{x}')^2+(\text{y}-y')^2+(\text{z}-\text{z}')^2\right]^{-1/2}$$

$$= \left(-\frac{1}{2}\right)(-2)\,(\text{x}-\text{x}')\boldsymbol{a}_x\left[(\text{x}-\text{x}')^2+(\text{y}-y')^2+(\text{z}-\text{z}')^2\right]^{-3/2} + \boldsymbol{a}_y \text{ and } \boldsymbol{a}_z \text{ terms}$$

$$= \frac{\boldsymbol{R}}{R^3}$$

$$R = |\boldsymbol{r} - \boldsymbol{r}'| = \left[(\text{x}-\text{x}')^2+(\text{y}-y')^2+(\text{z}-\text{z}')^2\right]^{1/2}$$

$$\nabla\frac{1}{\text{R}} = \left(\frac{\partial}{\partial x}\,\boldsymbol{a}_x + \frac{\partial}{\partial y}\,\boldsymbol{a}_y + \frac{\partial}{\partial \text{z}}\boldsymbol{a}_z\right)\left[(\text{x}-\text{x}')^2+(\text{y}-y')^2+(\text{z}-\text{z}')^2\right]^{-1/2}$$

$$= -\frac{1}{2}\,2\,(\text{x}-\text{x}')\,\boldsymbol{a}_x\left[(\text{x}-\text{x}')^2+(\text{y}-y')^2+(\text{z}-\text{z}')^2\right]^{-3/2} + \boldsymbol{a}_y \text{ and } \boldsymbol{a}_z \text{ terms}$$

$$= -\left[(\text{x}-\text{x}')\boldsymbol{a}_z+(\text{y}-y')\boldsymbol{a}_y+(\text{z}-\text{z}')\boldsymbol{a}_z\right]\Big/\text{R}^3 = -\frac{\overline{\text{R}}}{R^3}$$

# CHAPTER 8

**P.E. 8.1**

(a) $\boldsymbol{F} = m\frac{\partial \boldsymbol{u}}{\partial t} = Q\boldsymbol{E} = \underline{\underline{6\boldsymbol{a}_z N}}$

(b) $\frac{\partial \boldsymbol{u}}{\partial t} = 6\boldsymbol{a}_z = \frac{\partial}{\partial t}(u_x, u_y, u_z) \quad \Rightarrow$

$$\frac{\partial u_x}{\partial t} = 0 \rightarrow u_x = A$$

$$\frac{\partial u_y}{\partial t} = 0 \rightarrow u_y = B$$

$$\frac{\partial u_z}{\partial t} = 6 \rightarrow u_z = 6t + C$$

Since $\boldsymbol{u}(t=0) = 0$, A = B = C = 0

$u_x = 0 = u_y$, $u_z = 6t$

$$u_x = \frac{\partial x}{\partial t} = 0 \rightarrow x = A$$

$$u_y = \frac{\partial y}{\partial t} = 0 \rightarrow y = B$$

$$u_z = \frac{\partial z}{\partial t} = 6t \rightarrow z = 3t^2 + C_1$$

At t = 0, (x,y,z) = (0,0,0) $\rightarrow A_1 = 0 = B_1 = C_1$

Hence , (x,y,z) = $(0,0,3t^2)$,

$\boldsymbol{u} = 6t\boldsymbol{a}_z$ at any time. At P(0,0,12), z = 12 $=3t^2 \rightarrow$ t =2s

$\underline{\underline{t = 2s}}$

(c) $\boldsymbol{u} = 6t\boldsymbol{a}_z = 12\boldsymbol{a}_z$ m/s .

$$\boldsymbol{a} = \frac{\partial \boldsymbol{u}}{\partial t} = \underline{\underline{6\boldsymbol{a}_z \, m/s^2}}$$

(d) $K.E = \frac{1}{2}m|\boldsymbol{u}|^2 = \frac{1}{2}(1)(144) = \underline{\underline{72J}}$

**P.E. 8.2**

(a) $m\boldsymbol{a} = e\boldsymbol{u} \times \boldsymbol{B}$ = $(eB_o u_y, -eB_o u_x, 0)$

$$\frac{d^2x}{dt^2} = \frac{eB_o}{m}\frac{dy}{dt} = \omega\frac{dy}{dt} \qquad (1)$$

$$\frac{d^2y}{dt^2} = -\frac{eBo}{m}\frac{dx}{dt} = -\omega\frac{dx}{dt} \qquad (2)$$

$$\frac{d^2z}{dt^2} = 0; \Rightarrow \frac{dz}{dt} = C_1 \qquad (3)$$

From (1) and (2),

$$\frac{d^3x}{dt^3} = \omega\frac{d^2y}{dt^2} = -\omega^2\frac{dx}{dt}$$

$(D^2 + w^2 D)x = 0 \rightarrow Dx = (0, \pm j\omega)x$

$x = c_2 + c_3\cos\omega t + c_4\sin\omega t$

$$\frac{dy}{dt} = \frac{1}{\omega}\frac{d^2x}{dt^2} = -c_3\omega\cos\omega t - c_4\omega\sin\omega t$$

At t = 0, $\boldsymbol{u} = (\alpha, 0, \beta)$ . Hence,

$$c_1 = \beta, c_3 = 0, c_4 = \frac{\alpha}{\omega}$$

$$\underline{\underline{\frac{dx}{dt} = \alpha\cos\omega t, \frac{dy}{dt} = -\alpha\sin\omega t, \frac{dz}{dt} = \beta}}$$

(b) Solving these yields

$$x = \frac{a}{\omega}\sin\omega t, y = \frac{\alpha}{\omega}\cos\omega t, z = \beta t$$

The starting point of the particle is $(0, \frac{\alpha}{\omega}, 0)$

(c) $x^2 + y^2 = \frac{\alpha^2}{\omega^2}$, $z=\beta t$

showing that the particles move along a helix of radius $\alpha/\omega$ placed along the z-axis.

**P.E. 8.3**

(a) From Example 8.3, QuB = QE regardless of the sign of the charge.

$E = uB = 8 \times 10^6 \times 0.5 \times 10^{-3} = \underline{\underline{4\text{ kV/m}}}$

(b) Yes, since QuB = QE holds for any Q and m.

**P.E. 8.4**

By Newton's 3rd law, $\boldsymbol{F}_{12} = \boldsymbol{F}_{21}$, the force on the infinitely long wire is:

$$\boldsymbol{F}_l = -\boldsymbol{F} = \frac{\mu_o I_1 I_2 b}{2\pi}\left(\frac{1}{\rho_o} - \frac{1}{\rho_o + a}\right)\boldsymbol{a}_\rho$$

$$= \frac{4\pi \times 10^{-7} \times 50 \times 3}{2\pi}\left(\frac{1}{2} - \frac{1}{3}\right)\boldsymbol{a}_\rho = \underline{\underline{5\boldsymbol{a}_\rho \quad \mu N}}$$

**P.E. 8.5**

$$\boldsymbol{m} = IS\boldsymbol{a}_n = 10 \times 10^{-4} \times 50 \frac{(2, 6, -3)}{7}$$

$$= 7.143 \text{ x } 10^{-3} (2, 6, -3)$$

$$= \underline{\underline{(1.429\boldsymbol{a}_x + 4.286\boldsymbol{a}_y - 2.143\boldsymbol{a}_z) \times 10^{-2} \text{ A-m}^2}}$$

**P.E. 8.6**

(a)
$$\boldsymbol{T} = \boldsymbol{m} \times \boldsymbol{B} = \frac{10 \times 10^{-4} \times 50}{7 \times 10}\begin{vmatrix} 2 & 6 & -3 \\ 6 & 4 & 5 \end{vmatrix}$$

$$= \underline{\underline{0.03\boldsymbol{a}_x - 0.02\boldsymbol{a}_y - 0.02\boldsymbol{a}_z \text{ N-m}}}$$

(b) $|\boldsymbol{T}| = ISB\sin\theta \rightarrow \quad |\boldsymbol{T}|_{\max} = ISB$

$$|\boldsymbol{T}|_{max} = \frac{50 \times 10^{-3}}{10} |6\boldsymbol{a}_x + 4\boldsymbol{a}_y + 5\boldsymbol{a}_z| = \underline{\underline{0.04387 \text{ Nm}}}$$

**P.E. 8.7**

(a) $\mu_r = \dfrac{\mu}{\mu_o} = 4.6, \; \chi_m = \mu_r - 1 = \underline{\underline{3.6}}$

(b) $\boldsymbol{H} = \dfrac{\boldsymbol{B}}{\mu} = \dfrac{10 \times 10^{-3} e^{-y}}{4\pi \times 10^{-7} \times 4.6}\boldsymbol{a}_z \; A/m \quad = \underline{\underline{1730 e^{-y}\boldsymbol{a}_z}} \text{ A/m}$

(c) $\boldsymbol{M} = \chi_m \boldsymbol{H} = \underline{\underline{6228 e^{-y}\boldsymbol{a}_z}} \text{ A/m}$

**P.E. 8.8**

$$\boldsymbol{a}_n = \frac{3\boldsymbol{a}_x + 4\boldsymbol{a}_y}{5} = \frac{6\boldsymbol{a}_x + 8\boldsymbol{a}_y}{10}$$

$$\boldsymbol{B}_{1n} = (\boldsymbol{B}_1 \bullet \boldsymbol{a}_n)\boldsymbol{a}_n = \frac{(6 + 32)(6\boldsymbol{a}_x + 8\boldsymbol{a}_y)}{1000}$$

$$= 0.228\boldsymbol{a}_x + 0.304\boldsymbol{a}_y = \boldsymbol{B}_{2n}$$

$$\boldsymbol{B}_{1t} = \boldsymbol{B}_1 - \boldsymbol{B}_{1n} = -0.128\boldsymbol{a}_x + 0.096\boldsymbol{a}_y + 0.2\boldsymbol{a}_z$$

$$\boldsymbol{B}_{2t} = \frac{\mu_2}{\mu_1}\boldsymbol{B}_{1t} = 10\boldsymbol{B}_{1t} = -1.28\boldsymbol{a}_x + 0.96\boldsymbol{a}_y + 2\boldsymbol{a}_z$$

$$\boldsymbol{B}_2 = \boldsymbol{B}_{2n} + \boldsymbol{B}_{2t} = \underline{\underline{-1.052\boldsymbol{a}_x + 1.264\boldsymbol{a}_y + 2\boldsymbol{a}_z}} \text{ Wb/m}^2$$

**P.E. 8.9**

(a) $\boldsymbol{B}_{1n} = \boldsymbol{B}_{2n} \rightarrow \mu_1\boldsymbol{H}_{1n} =_z \mu_2\boldsymbol{H}_{2n}$

or $\mu_1\boldsymbol{H}_1 \bullet \boldsymbol{a}_{n21} = \mu_2\boldsymbol{H}_2 \bullet \boldsymbol{a}_{n21}$

$$\mu_o \frac{(60+2-36)}{7} = 2\mu_o \frac{(6H_{2x} - 10 - 12)}{7}$$

$$35 = 6H_{2x}$$

$$\underline{\underline{H_{2x} = 5.833 \text{ A/m}}}$$

(b) $\boldsymbol{K} = (\boldsymbol{H}_1 - \boldsymbol{H}_2) \times \boldsymbol{a}_{n12} = \boldsymbol{a}_{n21} \times (\boldsymbol{H}_1 - \boldsymbol{H}_2)$

$$= \boldsymbol{a}_{n21} \times \left[(10,1,12) - (35/6, -5, 4)\right]$$

$$= \frac{1}{7}\begin{vmatrix} 6 & 2 & -3 \\ 25/6 & 6 & 8 \end{vmatrix}$$

$$\boldsymbol{K} = \underline{\underline{4.86\boldsymbol{a}_x - 8.64\boldsymbol{a}_y + 3.95\boldsymbol{a}_z}} \text{ A/m}$$

(c) Since $\boldsymbol{B} = \mu\boldsymbol{H}$, $\boldsymbol{B}_1$ and $\boldsymbol{H}_1$ are parallel, i.e. they make the same angle with the normal to the interface.

$$\cos\theta_1 = \frac{\boldsymbol{H}_1 \bullet \boldsymbol{a}_{n21}}{|\boldsymbol{H}_1|} = \frac{26}{7\sqrt{100+1+144}} = 0.2373$$

$$\underline{\underline{\theta_1 = 76.27^o}}$$

$$\cos\theta_2 = \frac{\boldsymbol{H}_2 \bullet \boldsymbol{a}_{n21}}{|\boldsymbol{H}_2|} = \frac{13}{7\sqrt{(5.833)^2 + 25 + 16}} = 0.2144$$

$$\underline{\underline{\theta_2 = 77.62^o}}$$

**P.E. 8.10**

(a) $L' = \mu_o\mu_r n^2 S = 4\pi \times 10^{-7} \times 1000 \times 16 \times 10^6 \times 4 \times 10^{-4}$

= $\underline{\underline{\text{8.042 H/m}}}$

(b) $W_m' = \frac{1}{2} L' I^2 = \frac{1}{2}(8.042)(0.5^2) = \underline{\underline{1.005}}$ J/m

**P.E. 8.11** From Example 8.11**,**

$$L_{in} = \frac{\mu_o l}{8\pi}$$

$$L_{ext} = \frac{2w_m}{I^2} = \frac{1}{I^2}\iiint \frac{\mu I^2}{4\pi^2\rho^2}\rho d\rho d\phi dz$$

$$= \frac{1}{4\pi^2}\int_0^l dz \int_0^{2\pi} d\phi \int_a^b \frac{2\mu_o}{(1+\rho)\rho} d\rho$$

$$= \frac{2\mu_o}{4\pi^2} \bullet 2\pi l \int_a^b \left[\frac{1}{\rho} - \frac{1}{(1+\rho)}\right] d\rho$$

$$= \frac{\mu_o l}{\pi}\left[\ln\frac{b}{a} - \ln\frac{1+b}{1+a}\right]$$

$$L = L_{in} + L_{ext} = \underline{\underline{\frac{\mu_o l}{8\pi} + \frac{\mu_o l}{\pi}\left[\ln\frac{b}{a} - \ln\frac{1+b}{1+a}\right]}}$$

**P.E. 8.12**

(a) $L'_{in} = \frac{\mu_o}{8\pi} = \frac{4\pi\times10^{-7}}{8\pi} = \underline{\underline{0.05\ \mu H/m}}$

$L'_{ext} = L' - L'_{in} = 1.2 - 0.05 = \underline{\underline{1.15\ \mu H/m}}$

(b) $L' = \frac{\mu_o}{2\pi}\left[\frac{1}{4} + \ln\frac{d-a}{a}\right]$

$$\ln\frac{d-a}{a} = \frac{2\pi L'}{\mu_o} - 0.25 = \frac{2\pi\times1.2\times10^{-6}}{4\pi x 10^{-7}} - 0.25$$

$$= 6 - 0.25 = 5.75$$

$$\frac{d-a}{a} = e^{5.75} = 314.19$$

$$d - a = 314.19a = 314.19\times\frac{2.588\times10^{-3}}{2} = 406.6mm$$

$$d = 407.9mm = \underline{\underline{40.79cm}}$$

**P.E. 8.13**
This is similar to Example 8.13. In this case, however, h=0 so that

$$\mathbf{A}_1 = \frac{\mu_o I_1 a^2 b}{4b^3}\mathbf{a}_\phi$$

$$\phi_{12} = \frac{\mu_o I_1 a^2}{4b^2} \bullet 2\pi b = \frac{\mu_o \pi I_1 a^2}{2b}$$

$$m_{12} = \frac{\phi_{12}}{I_1} = \frac{\mu_o \pi a^2}{2b} = \frac{4\pi \times 10^{-7} \times \pi \times 4}{2 \times 3}$$

= <u>2.632 μH</u>

**P.E. 8.14**

$$L_{in} = \frac{\mu_o}{8\pi} l = \frac{\mu_o 2\pi\rho_o}{8\pi} = \frac{4\pi \times 10^{-7} \times 10 \times 10^{-2}}{4}$$

= <u>31.42 nH</u>

**P.E. 8.15**

(a) From Example 7.6,

$$B_{ave} = \frac{\mu_o NI}{l} = \frac{\mu_o NI}{2\pi\rho_o}$$

$$\phi = B_{ave} \bullet S = \frac{\mu_o NI}{2\pi\rho_o} \bullet \pi a^2$$

$$\text{or } I = \frac{2\rho_o \phi}{\mu a^2 N} = \frac{2 \times 10 \times 10^{-2} \times 0.5 \times 10^{-3}}{4\pi \times 10^{-7} \times 10^{-4} \times 10^3}$$

= <u>795.77A</u>

Alternatively, using circuit approach

$$R = \frac{l}{\mu S} = \frac{2\pi\rho_o}{\mu_o S} = \frac{2\pi\rho_o}{\mu_o \pi a^2}$$

$$\Im = NI = \frac{\phi\Re}{N} = \frac{2\rho_o\phi}{\mu a^2 N},$$ as obtained before.

$$\Re = \frac{2\rho_o}{\mu a^2} = \frac{2 \times 10 \times 10^{-2}}{4\pi \times 10^{-7} \times 10^{-4}} = 1.591 \times 10^9$$

$\Im = \phi\Re$ = 0.5x10$^{-3}$x1.591x10$^{9}$=7.9577x10$^{5}$

$$I = \frac{\Im}{N} = 795.77A$$ as obtained before.

(b) If $\mu = 500\mu_o$,

$$I = \frac{795.77}{500} = \underline{\underline{1.592}}\text{ A}$$

**P.E. 8.16**

$$\Im = \frac{B^2{}_a S}{2\mu_o} = \frac{(1.5)^2 \times 10 \times 10^{-4}}{2 \times 4\pi \times 10^{-7}} = \frac{22500}{8\pi} = \underline{\underline{895.25\text{N}}}$$

---

**Prob. 8.1**

At P, x = 2, y = 5, z = -3

$$\boldsymbol{E} = 2(2)(5)(-3)\boldsymbol{a}_x + (2)^2(-3)\boldsymbol{a}_y + (2)^2(5)\boldsymbol{a}_z = -60\boldsymbol{a}_x - 12\boldsymbol{a}_y + 20\boldsymbol{a}_z$$

$$\boldsymbol{B} = (5)^2\boldsymbol{a}_x + (-3)^2\boldsymbol{a}_y + 2^2\boldsymbol{a}_z = 25\boldsymbol{a}_x + 9\boldsymbol{a}_y + 4\boldsymbol{a}_z$$

$$\boldsymbol{F} = Q(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B})$$

$$\boldsymbol{u} \times \boldsymbol{B} = \begin{vmatrix} 1.4 & -3.2 & -1 \\ 25 & 9 & 4 \end{vmatrix} = -3.8\boldsymbol{a}_x - 30.6\boldsymbol{a}_y + \boldsymbol{92.6a}_z$$

$$\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B} = (-60, -12, 20) + (-3.8, -30.6, 92.6) = (-63.8, -42.6, 112.6)$$

$$\boldsymbol{F} = Q(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}) = 4(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B})\text{ mN}$$

$$= \underline{\underline{-255.2\boldsymbol{a}_x - 170.4\boldsymbol{a}_y + 450.4\boldsymbol{a}_z\text{ mN}}}$$

**Prob. 8.2**

$$F = m\omega^2 r = 9.11 \times 10^{-31} \times (2 \times 10^{16})^2 (0.4 \times 10^{-10}) = \underline{\underline{14.576\text{ nN}}}$$

**Prob. 8.3**

(a)

$$\boldsymbol{F} = Q(\boldsymbol{u} \times \boldsymbol{B}) = 10^{-3}\begin{vmatrix} 10 & -2 & 6 \\ 0 & 0 & 25 \end{vmatrix} = 10^{-3}(-50\boldsymbol{a}_x - 250\boldsymbol{a}_y)$$

$$= \underline{\underline{-0.05\boldsymbol{a}_x - 0.25\boldsymbol{a}_y\text{ N}}}$$

(b) Constant velocity implies that acceleration **a** = **0**.

$$\boldsymbol{F} = m\boldsymbol{a} = 0 = Q(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B})$$

$$\boldsymbol{E} = -\boldsymbol{u} \times \boldsymbol{B} = \underline{\underline{50\boldsymbol{a}_x + 250\boldsymbol{a}_y\text{ V/m}}}$$

**Prob. 8.4**

$$\boldsymbol{F}_e = q\boldsymbol{E}, \qquad \boldsymbol{F}_m = q\boldsymbol{u}\times\boldsymbol{B}$$

$$\frac{F_e}{F_m} = \frac{E}{uB} = \frac{20\times10^3}{0.5\times10^8\times5\times10^{-6}} = \underline{\underline{80}}$$

**Prob. 8.5**

$$m\boldsymbol{a} = Q\boldsymbol{u}\times\boldsymbol{B}$$

$$10^{-3}\boldsymbol{a} = -2\times10^{-3}\begin{vmatrix} u_x & u_y & u_z \\ 0 & 6 & 0 \end{vmatrix}$$

$$\frac{d}{dt}(u_x, u_y, u_z) = (12u_z, 0, -12u_x)$$

i.e. $\frac{du_x}{dt} = 12u_z$ (1)

$$\frac{du_y}{dt} = 0 \rightarrow u_y = A_1 \qquad (2)$$

$$\frac{du_z}{dt} = -12u_x \qquad (3)$$

From (1) and (3),

$$\ddot{u}_x = 12\dot{u}_z = -144u_x$$

or

$$\ddot{u}_x + 144u_x = 0 \rightarrow u_x = c_1\cos12t + c_2\sin12t$$

From (1), $u_z = - c_1\sin12t + c_2\cos12t$

At t=0,

$u_x=5,\ u_y=0,\ u_z=0 \rightarrow A_1=0=c_2,\ c_1=5$

Hence,

$$\boldsymbol{u} = (5\cos12t, 0, -5\sin12t)$$

$$\boldsymbol{u}(t=10s) = (5\cos120, 0, -5\sin120) = \underline{\underline{4.071\boldsymbol{a}_x - 2.903\boldsymbol{a}_z}} \text{ m/s}$$

$$u_x = \frac{dx}{dt} = 5\cos12t \rightarrow x = \frac{5}{12}\sin12t + B_1$$

$$u_y = \frac{dy}{dt} = 0 \rightarrow y = B_2$$

$$u_z = \frac{dz}{dt} = -5\sin12t \rightarrow z = \frac{5}{12}\cos12t + B_3$$

At t=0, (x, y, z) = (0, 1, 2) $\rightarrow$ $B_1$=0, $B_2$=1, $B_3$= $\frac{19}{12}$

$$(x,y,z) = \left(\frac{5}{12}\sin 12t, 1, \frac{5}{12}\cos 12t + \frac{19}{12}\right) \qquad (4)$$

At t=10s,

$$(x,y,z) = \left(\frac{5}{12}\sin 120, 1, \frac{5}{12}\cos 120 + \frac{19}{12}\right) = \underline{(0.2419, 1, 1.923)}$$

By eliminating t from (4),

$x^2 + (z - 19/12)^2 = (5/12)^2$, $y = 1$ which is a circle in the y=1 plane with center at (0,1,19/12). The particle gyrates.

**Prob. 8.6**

(a) $m\boldsymbol{a} = -e(\boldsymbol{u} \times \boldsymbol{B})$

$$-\frac{m}{e}\frac{d}{dt}(u_x, u_y, u_z) = \begin{vmatrix} u_x & u_y & u_z \\ 0 & 0 & B_o \end{vmatrix} = u_y B_o \vec{a}_x - B_o u_x \vec{a}_y$$

$$\frac{du_z}{dt} = 0 \rightarrow u_z = c = 0$$

$$\frac{du_x}{dt} = -u_y \frac{B_o e}{m} = -u_y w, \text{ where } w = \frac{B_o e}{m}$$

$$\frac{du_y}{dt} = u_x w$$

Hence,

$$\ddot{u}_x = -w\dot{u}_y = -w^2 u_x$$

or $\ddot{u}_x + w^2 u_x = 0 \rightarrow u_x = A\cos wt + B\sin wt$

$$u_y = -\frac{\dot{u}_x}{w} = A\sin wt - B\cos wt$$

At t=0, $u_x = u_o$, $u_y = 0 \rightarrow A = u_o$, B=0

Hence,

$$u_x = u_o \cos wt = \frac{dx}{dt} \rightarrow x = \frac{u_o}{w}\sin wt + c_1$$

$$u_y = u_o \sin wt = \frac{dy}{dt} \rightarrow y = -\frac{u_o}{w}\cos wt + c_2$$

At t=0, x = 0 = y $\rightarrow$ $c_1$=0, $c_2$=$\frac{u_o}{w}$. Hence,

$$x = \frac{u_o}{w}\sin wt,\; y = \frac{u_o}{w}(1-\cos wt)$$

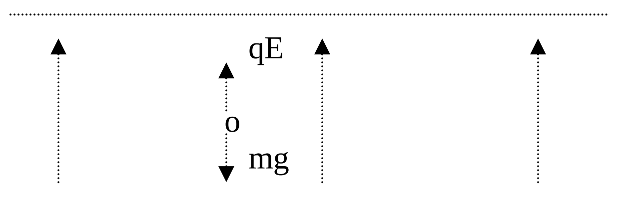
$$\frac{u^2{}_o}{w^2}(\cos^2 wt + \sin^2 wt) = \left(\frac{u_o}{w}\right)^2 = x^2 + (y - \frac{u_o}{w})^2$$

showing that the electron would move in a circle centered at (0, $\frac{u_o}{w}$). But since the field does not exist throughout the circular region, the electron passes through a semi-circle and leaves the field horizontally.

(b) d = twice the radius of the semi-circle

$$= \frac{2u_o}{w} = \frac{2u_o m}{B_o e}$$

**Prob. 8.7**




$$mg = qE \quad \longrightarrow \quad q = \frac{mg}{E} = \frac{0.4\times10^{-3}\times9.81}{1.5\times10^5} = \underline{\underline{26.67\text{ nC}}}$$

**Prob. 8.8**

$$\boldsymbol{F} = \int I d\boldsymbol{l}\times\boldsymbol{B} = I\boldsymbol{L}\times\boldsymbol{B}$$

$$= 4.5(0.2)\boldsymbol{a}_x \times (2.5)(\boldsymbol{a}_y + \boldsymbol{a}_z)10^{-3} = 4.5(0.5)\begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{vmatrix}\text{ mN}$$

$$\boldsymbol{F} = \underline{\underline{2.25(-\boldsymbol{a}_y + \boldsymbol{a}_z)\text{ mN}}}$$

**Prob. 8.9**

$$qE = quB \quad \rightarrow \quad B = \frac{E}{u} = \frac{12\times10^3}{140} = \frac{1200}{14} = \underline{\underline{85.714\text{ Wb/m}^2}}$$

**Prob. 8.10**

$$\mathfrak{F} = I\mathbf{L}\times\mathbf{B} \rightarrow \mathfrak{F} = \frac{\mathbf{F}}{L} = I_1\mathbf{a}_l \times \mathbf{B}_2 = \frac{\mu_o I_1 I_2 \mathbf{a}_l \times \mathbf{a}_\phi}{2\pi\rho}$$

(a) $$\mathbf{F}_{21} = \frac{\mathbf{a}_z \times(-\mathbf{a}_y)4\pi\times10^{-7}(-100)(200)}{2\pi} = \underline{\underline{4\mathbf{a}_x \text{ mN/m}}} \quad \text{(repulsive)}$$

(b) $\mathbf{F}_{12} = -\mathbf{F}_{21} = -4\mathbf{a}_x$ mN/m (repulsive)

(c) $$\mathbf{a}_l \times \mathbf{a}_\phi = \mathbf{a}_z \times(-\frac{4}{5}\mathbf{a}_x + \frac{3}{5}\mathbf{a}_y) = -\frac{3}{5}\mathbf{a}_x - \frac{4}{5}\mathbf{a}_y, \rho = 5$$

$$\mathbf{F}_{31} = \frac{4\pi\times10^{-7}(-3\times10^4)}{2\pi(5)}\left(-\frac{3}{5}\mathbf{a}_x - \frac{4}{5}\mathbf{a}_y\right)$$

$$= \underline{\underline{0.72\mathbf{a}_x + 0.96\mathbf{a}_y}} \text{ mN/m (attractive)}$$

(d) $\mathbf{F}_3 = \mathbf{F}_{31} + \mathbf{F}_{32}$

$$\mathbf{F}_{32} = \frac{4\pi\times10^{-7}\times6\times10^4)}{2\pi(3)}\left(\mathbf{a}_z \times \mathbf{a}_y\right) = -4\mathbf{a}_x \text{ mN/m(attractive)}$$

$$\mathbf{F}_3 = \underline{\underline{-3.28\mathbf{a}_x + 0.96\mathbf{a}_y}} \text{ mN/m}$$

(attractive due to $L_2$ and repulsive due to $L_1$)

**Prob. 8.11**

$$F = \frac{\mu_o I_1 I_2}{2\pi\rho} = \frac{4\pi\times10^{-7}(10)10}{2\pi(20\times10^{-2})} = \underline{\underline{100\ \mu\text{N}}}$$

**Prob. 8.12**

$$W = -\int \mathbf{F}\bullet d\mathbf{l}, \qquad \mathbf{F} = \int Ld\mathbf{l}\times\mathbf{B} = 3(2\mathbf{a}_z)\times\cos\phi/3\ \mathbf{a}_\phi$$

$$F = 6\cos\phi/3\ \vec{a}_\phi \text{ N}$$

$$W = -\int_0^{2\pi} 6\cos\phi/3\ \rho_o d\phi = -6\rho_o\times3\sin\phi/3\Big|_0^{2\pi} \text{ J}$$

$$= -1.8\sin\frac{2\pi}{3} = \underline{\underline{-1.559 \text{ J}}}$$

**Prob. 8.13**

(a) $$\mathbf{F}_1 = \int_{\rho=2}^{6} \frac{\mu_o I_1 I_2}{2\pi\rho} d\rho\mathbf{a}_\rho \times \mathbf{a}_\phi = \frac{4\pi\times10^{-7}}{2\pi}(2)(5)\ln 6/2\ \mathbf{a}_z$$

$$= 2\ln3\mathbf{a}_z\ \mu\text{N} = \underline{\underline{2.197\mathbf{a}_z\ \mu\text{N}}}$$

(b) $$F_2 = \int I_2 d\boldsymbol{l}_2 \times \boldsymbol{B}_1$$
$$= \frac{\mu_o I_1 I_2}{2\pi}\int \frac{1}{\rho}\left[d\rho \boldsymbol{a}_\rho + dz\boldsymbol{a}_z\right]\times \boldsymbol{a}_\phi$$
$$= \frac{\mu_o I_1 I_2}{2\pi}\int \frac{1}{\rho}\left[d\rho \boldsymbol{a}_z - dz\boldsymbol{a}_\rho\right]$$

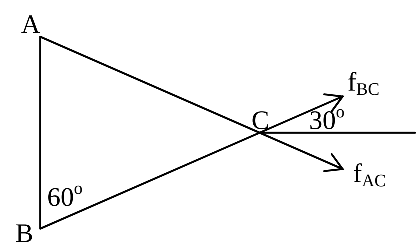
But ρ = z+2, dz=dρ

$$\boldsymbol{F}_2 = \frac{4\pi\times 10^{-7}}{2\pi}(5)(2)\int_{\rho=4}^{2}\frac{1}{\rho}\left[d\rho \boldsymbol{a}_z - dz\boldsymbol{a}_\rho\right]$$
$$2\ln 2/4\,(\boldsymbol{a}_z - \boldsymbol{a}_\rho)\mu N = 1.386\boldsymbol{a}_\rho - 1.386\boldsymbol{a}_z\boldsymbol{a}_z\boldsymbol{a}_z\mu N$$
$$\boldsymbol{F}_3 = \frac{\mu_o I_1 I_2}{2\pi}\int \frac{1}{\rho}\left[d\rho \boldsymbol{a}_z - dz\boldsymbol{a}_\rho\right]$$

But z = -ρ + 6, dz = -dρ
$$\boldsymbol{F}_3 = \frac{4\pi\times 10^{-7}}{2\pi}(5)(2)\int_{\rho=6}^{4}\frac{1}{\rho}\left[d\rho \boldsymbol{a}_z - dz\boldsymbol{a}_\rho\right]$$
$$2\ln 4/6\,(\boldsymbol{a}_z + \boldsymbol{a}_\rho)\mu N = -0.8109\boldsymbol{a}_\rho - 0.8109\boldsymbol{a}_z\mu N$$
$$\boldsymbol{F} = \boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3$$
$$= \boldsymbol{a}_\rho(\ln 4 + \ln 4 - \ln 9) + \boldsymbol{a}_z(\ln 9 - \ln 4 + \ln 4 - \ln 9)$$
$$= \underline{\underline{0.575\boldsymbol{a}_\rho\ \mu\text{N}}}$$

**Prob. 8.14**

From Prob. 8.7,




$$\boldsymbol{f} = \frac{\mu_o I_1 I_2}{2\pi\rho}\boldsymbol{a}_\rho$$
$$\boldsymbol{f} = \boldsymbol{f}_{AC} + \boldsymbol{f}_{BC}$$
$$|\boldsymbol{f}_{AC}| = |\boldsymbol{f}_{BC}| = \frac{4\pi\times 10^{-7}\times 75\times 150}{2\pi\times 2} = 1.125\times 10^{-3}$$
$$\boldsymbol{f} = 2\times 1.125\cos 30^o\,\boldsymbol{a}_x \text{ mN/m}$$

$$= \underline{\underline{1.949\boldsymbol{a}_x}}\text{ mN/m}$$

**Prob. 8.15**

The field due to the current sheet is

$$\boldsymbol{B} = \frac{\mu}{2}\boldsymbol{K}\times\boldsymbol{a}_n = \frac{\mu_o}{2}10\boldsymbol{a}_x\times(-\boldsymbol{a}_z) = 5\mu_o\boldsymbol{a}_y$$

$$\boldsymbol{F} = I_2\int d\boldsymbol{l}_2\times\boldsymbol{B} = 2.5\int_0^L dx\boldsymbol{a}_x\times(5\mu_o\boldsymbol{a}_y) = 2.5L\times5\mu_o(\boldsymbol{a}_z)$$

$$\frac{\boldsymbol{F}}{L} = 12.5\times4\pi\times10^{-7}(\boldsymbol{a}_z) = \underline{\underline{15.71\boldsymbol{a}_z \quad \mu\text{N/m}}}$$

**Prob. 8.16**

$$\boldsymbol{F} = \int Id\boldsymbol{l}\times\boldsymbol{B} = I\boldsymbol{L}\times\boldsymbol{B} = 5(2\boldsymbol{a}_z)\times40\boldsymbol{a}_x10^{-3} = \underline{\underline{0.4\boldsymbol{a}_y\ \text{N}}}$$

**Prob. 8.17**

$$\boldsymbol{m} = IS\boldsymbol{a}_n = 10(2\times6)(\boldsymbol{a}_x) = 120\boldsymbol{a}_x$$

$$\boldsymbol{T} = \boldsymbol{m}\times\boldsymbol{B} = 120\boldsymbol{a}_x\times4.5(\boldsymbol{a}_y-\boldsymbol{a}_z) = 540\begin{vmatrix}1 & 0 & 0\\ 0 & 1 & -1\end{vmatrix} = \underline{\underline{540(\boldsymbol{a}_y+\boldsymbol{a}_z)\ \text{N.m}}}$$

**Prob. 8.18**

$$f(x,y,z) = x+2y-5z-12 = 0 \quad\longrightarrow\quad \nabla f = \boldsymbol{a}_x+2\boldsymbol{a}_y-5\boldsymbol{a}_z$$

$$\boldsymbol{a}_n = \frac{\nabla f}{|\nabla f|} = \frac{\boldsymbol{a}_x+2\boldsymbol{a}_y-5\boldsymbol{a}_z}{\sqrt{30}}$$

$$\boldsymbol{m} = NIS\boldsymbol{a}_n = 2\times60\times8\times10^{-4}\frac{(\boldsymbol{a}_x+2\boldsymbol{a}_y-5\boldsymbol{a}_z)}{\sqrt{30}} = \underline{\underline{17.53\boldsymbol{a}_x+35.05\boldsymbol{a}_y-87.64\boldsymbol{a}_z\ \text{mAm}}}$$

**Prob. 8.19**

$$m = IS \quad\longrightarrow\quad I = \frac{m}{S} = \frac{m}{\pi r^2}$$

$$I = \frac{8\times10^{22}}{\pi(6370\times10^3)^2} = 6.275\times10^8 = \underline{\underline{627.5\ \text{MA}}}$$

**Prob. 8.20**

Let $\boldsymbol{F} = \boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3$

$$\boldsymbol{F}_1 = \int I d\boldsymbol{l} \times \boldsymbol{B} = \int_5^0 2dx\boldsymbol{a}_x \times 30\boldsymbol{a}_z \text{ mN}$$

$$= -60\boldsymbol{a}_y x \Big|_5^0 = 300\boldsymbol{a}_y \text{ mN}$$

$$\boldsymbol{F}_2 = \int_0^5 2dy\boldsymbol{a}_y \times 30\boldsymbol{a}_z \text{ mN}$$

$$= 60\boldsymbol{a}_x y \Big|_0^5 = 300\boldsymbol{a}_x \text{ mN}$$

$$\boldsymbol{F}_3 = \int_0^5 2(dx\boldsymbol{a}_x + dz\boldsymbol{a}_z) \times 30\boldsymbol{a}_z \text{ mN}$$

$$= 60(-\boldsymbol{a}_y)x \Big|_0^5 = -300\boldsymbol{a}_y \text{ mN}$$

$$\boldsymbol{F} = \boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3 = 300\boldsymbol{a}_y + 300\boldsymbol{a}_x - 300\boldsymbol{a}_y \text{ mN} = \underline{\underline{300\boldsymbol{a}_x \text{ mN}}}$$

$$\boldsymbol{T} = \boldsymbol{m} \times \boldsymbol{B} = IS\boldsymbol{a}_n \times \boldsymbol{B} = 2(\frac{1}{2})(5)(5)\boldsymbol{a}_y \times 30\boldsymbol{a}_z 10^{-3} = \underline{\underline{0.75\boldsymbol{a}_x \text{ N.m}}}$$

**Prob. 8.21**

For each turn, $\boldsymbol{T} = \boldsymbol{m} \times \boldsymbol{B}$, $\boldsymbol{m} = \text{IS}\boldsymbol{a}_n$

For N turns,

$$T = NISB = 50 \times 4 \times 12 \times 10^{-4} \times 100 \times 10^{-3} = \underline{\underline{24 \text{ mNm}}}$$

**Prob. 8.22**

$$\boldsymbol{F} = \int I d\boldsymbol{l} \times \boldsymbol{B} \longrightarrow F = IB\ell = 520 \times 0.4 \times 10^{-3} \times 30 \times 10^{-3}$$

$$F = \underline{\underline{6.24 \text{ mN}}}$$

**Prob. 8.23**

$$M = \chi_m H = \chi_m \frac{B}{\mu_o \mu_r} = \frac{\chi_m B}{\mu_o (1 + \chi_m)}$$

**Prob. 8.24**

(a) $$\boldsymbol{M} = \chi_m \boldsymbol{H} = \chi_m \frac{\boldsymbol{B}}{\mu_o \mu}$$

$$M = \frac{4999}{5000} \times \frac{1.5}{4\pi \times 10^{-7}} = \underline{\underline{1.193 \times 10^6 \text{ A/m}}}$$

(b) $$\boldsymbol{M} = \frac{\sum_{k=1}^{N} \boldsymbol{m}_k}{\Delta v}$$

If we assume that all $\boldsymbol{m}_k$ align with the applied **B** field,

$$M = \frac{Nm_k}{\Delta v} \rightarrow m_k = \frac{M}{N/\Delta v} = \frac{1.193 \times 10^6}{8.5 \times 10^{28}}$$

$$m_k = \underline{\underline{1.404 \times 10^{-23} \text{ A} \cdot \text{m}^2}}$$

**Prob. 8.25**

$$\mu_r = \chi_m + 1 = 6.5 + 1 = \underline{\underline{7.5}}$$

$$\boldsymbol{M} = \chi_m \boldsymbol{H} \longrightarrow \boldsymbol{H} = \frac{\boldsymbol{M}}{\chi_m} = \frac{24y^2}{6.5}\boldsymbol{a}_z$$

At y = 2cm,

$$\boldsymbol{H} = \frac{24 \times 4 \times 10^{-4}}{6.5}\boldsymbol{a}_z = \underline{\underline{1.477\boldsymbol{a}_z \text{ mA/m}}}$$

$$\boldsymbol{J} = \nabla \times \boldsymbol{H} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & \frac{24y^2}{6.5} \end{vmatrix} = \frac{48y}{6.5}\mathbf{a}_x$$

At y=2cm,

$$\boldsymbol{J} = \frac{48 \times 2 \times 10^{-2}}{6.5}\mathbf{a}_x = \underline{\underline{0.1477\boldsymbol{a}_x \text{ A/m}^2}}$$

**Prob. 8.26**

(*a*) $\mu = 80\mu_o \rightarrow \mu_r = 80$

(*b*) $\chi_m = \mu_r - 1 = 79$

(*c*) $$\boldsymbol{H} = \frac{\boldsymbol{B}}{\mu} = \frac{20x\boldsymbol{a}_y 10^{-3}}{80(4\pi \times 10^{-7})} = 198.9x\boldsymbol{a}_y \text{ A/m}$$

(d) $\boldsymbol{M} = \chi_m \boldsymbol{H} = 15.713x\boldsymbol{a}_y$ kA/m

(e) $\boldsymbol{J}_b = \nabla \times \boldsymbol{M} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 15.713x & 0 \end{vmatrix} = 15.713\boldsymbol{a}_z$ kA/m

**Prob. 8.27**

When H = 250,

$$B = \frac{2H}{100+H} = \frac{2(250)}{100+250} = 1.4286 \text{ mWb/m}^2$$

But $B = \mu_o \mu_r H$

$$\mu_r = \frac{B}{\mu_o H} = \frac{1.4286 \times 10^{-3}}{4\pi \times 10^{-7} \times 250} = \underline{\underline{4.54}}$$

**Prob. 8.28**

$$\oint \boldsymbol{H} \cdot d\boldsymbol{l} = I_{enc}$$

$$H_\phi \cdot 2\pi\rho = \frac{\pi\rho^2}{\pi a^2} \cdot I \rightarrow H_\phi = \frac{I\rho}{2\pi a^2}$$

$$\boldsymbol{M} = \chi_m \boldsymbol{H} = \underline{\underline{(\mu_r - 1)\frac{I\rho}{2\pi a^2}\boldsymbol{a}_\phi}}$$

$$\boldsymbol{J}_b = \nabla \times \boldsymbol{M} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho M_\phi)\boldsymbol{a}_z = \underline{\underline{(\mu_r - 1)\frac{I}{\pi a^2}\boldsymbol{a}_z}}$$

**Prob. 8.29**

(a) From $H_{1t} - H_{2t} = K$ and $M = \chi_m H$, we obtain:

$$\underline{\underline{\frac{M_{1t}}{\chi_{m1}} - \frac{M_{2t}}{\chi_{m2}} = K}}$$

Also from $B_{1n} - B_{2n} = 0$ and $B = \mu H = (\mu/\chi_m)M$, we get:

$$\underline{\underline{\frac{\mu_1 M_{1n}}{\chi_{m1}} = \frac{\mu_2 M_{2n}}{\chi_{m2}}}}$$

(b) From $B_1\cos\theta_1 = B_{1n} = B_{2n} = B_2\cos\theta_2$ (1)

and $\frac{B_1 \sin\theta_1}{\mu_1} = H_{1t} = K + H_{2t} = K + \frac{B_2 \sin\theta_2}{\mu_2}$ (2)

Dividing (2) by (1) gives

$$\frac{\tan\theta_1}{\mu_1} = \frac{k}{B_2\cos\theta_2} + \frac{\tan\theta_2}{\mu_2} = \frac{\tan\theta_2}{\mu_2}\left(1 + \frac{k\mu_2}{B_2\sin\theta_2}\right)$$

i.e. $$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\mu_1}{\mu_2}\left(1 + \frac{k\mu_2}{B_2\sin\theta_2}\right)$$

**Prob. 8.30**

$$\mathbf{B}_{2n} = \mathbf{B}_{1n} = 1.8\mathbf{a}_z$$

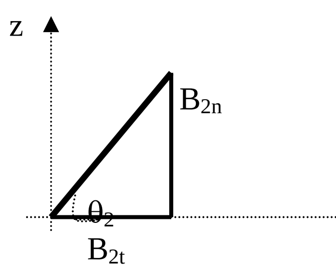
$$\mathbf{H}_{2t} = \mathbf{H}_{1t} \longrightarrow \frac{\mathbf{B}_{2t}}{\mu_2} = \frac{\mathbf{B}_{1t}}{\mu_1}$$

$$\mathbf{B}_{2t} = \frac{\mu_2}{\mu_1}\mathbf{B}_{1t} = \frac{4\mu_o}{2.5\mu_o}(6\mathbf{a}_x - 4.2\mathbf{a}_y) = 9.6\mathbf{a}_x - 6.72\mathbf{a}_y$$

$$\mathbf{B}_2 = \mathbf{B}_{2n} + \mathbf{B}_{2t} = 9.6\mathbf{a}_x - 6.72\mathbf{a}_y + 1.8\mathbf{a}_z \quad \text{mWb/m}^2$$

$$\mathbf{H}_2 = \frac{\mathbf{B}_2}{\mu_2} = \frac{10^{-3}(9.6, -6.72, 1.8)}{4\times 4\pi\times 10^{-7}}$$

$$= 1{,}909.86\,\mathbf{a}_x - 1{,}336.9\,\mathbf{a}_y + 358.1\,\mathbf{a}_z \quad \text{A/m}$$




$$\tan\theta_2 = \frac{B_{2n}}{B_{2t}} = \frac{1.8}{\sqrt{9.6^2 + 6.72^2}} = 0.1536$$

$$\theta_2 = 8.73^o$$

**Prob. 8.31**

$$\boldsymbol{B}_{2n} = \boldsymbol{B}_{1n} = 12\boldsymbol{a}_z$$

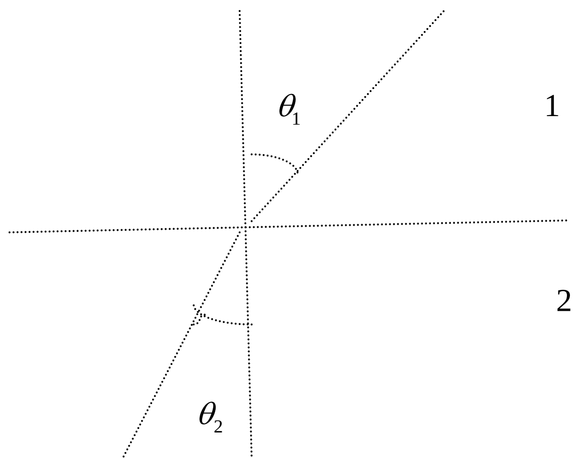
$$\boldsymbol{H}_{2t} = \boldsymbol{H}_{1t} \quad \longrightarrow \quad \frac{\boldsymbol{B}_{2t}}{\mu_2} = \frac{\boldsymbol{B}_{1t}}{\mu_1}$$

$$\boldsymbol{B}_{2t} = \frac{\mu_2}{\mu_1}\boldsymbol{B}_{1t} = \frac{2\mu_o}{5\mu_o}(4\boldsymbol{a}_x - 10\boldsymbol{a}_y) = 1.6\boldsymbol{a}_x - 4\boldsymbol{a}_y$$

$$\boldsymbol{B}_2 = \boldsymbol{B}_{2n} + \boldsymbol{B}_{2t} = \underline{\underline{1.6\boldsymbol{a}_x - 4\boldsymbol{a}_y + 12\boldsymbol{a}_z \text{ mWb/m}^2}}$$

$$w_2 = \frac{1}{2}\mu_2 H_2^3 = \frac{B_2^3}{2\mu_2} = \frac{(1.6^2 + 4^2 + 12^2)\times 10^{-6}}{2(2)(4\pi\times 10^{-7})} = \underline{\underline{32.34 \text{ J/m}^3}}$$

**Prob. 8.32**




$$\tan\theta_1 = \frac{H_{1t}}{H_{1n}}, \qquad \tan\theta_2 = \frac{H_{2t}}{H_{2n}}$$

But $\quad H_{1t} = H_{2t}$

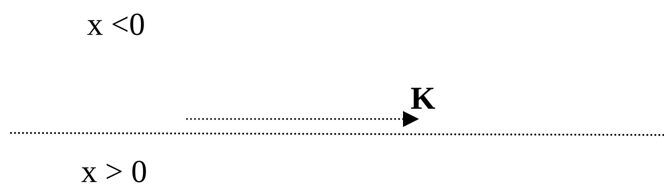
$$B_{1n} = B_{2t} \quad \rightarrow \quad \mu_1 H_{1n} = \mu_2 H_{2n} \quad \rightarrow \quad H_{2n} = \frac{\mu_1}{\mu_2} H_{1n}$$

$$\frac{H_{2t}}{H_{2n}} = \frac{H_{1t}}{\frac{1}{6.5}H_{1n}} = \frac{6.5H_{1t}}{H_{1n}}$$

If $\quad \theta_1 = 42^o, \quad \tan 42^o = \frac{H_{1t}}{H_{1n}} \quad \rightarrow \quad \frac{H_{1t}}{H_{1n}} = 0.9004$

$$\tan\theta_2 = 6.5(0.9004) = 5.832 \quad \rightarrow \quad \underline{\underline{\theta_2 = 80.3^o}}$$

**Prob. 8.33**




$$(\boldsymbol{H}_1 - \boldsymbol{H}_2)\times \boldsymbol{a}_{n12} = \boldsymbol{K} \quad \rightarrow \quad \boldsymbol{H}_1 \times \mathbf{a}_x = \boldsymbol{K} + \boldsymbol{H}_2 \times \boldsymbol{a}_x$$

$$\boldsymbol{H}_1 \times \mathbf{a}_x = (10\boldsymbol{a}_x + 6\boldsymbol{a}_z)\times \boldsymbol{a}_x = 6\boldsymbol{a}_y$$

$$\boldsymbol{H}_2 \times \boldsymbol{a}_x = (H_{2x}, H_{2y}, H_{2z})\times \boldsymbol{a}_x = H_{2z}\boldsymbol{a}_y - H_{2y}\boldsymbol{a}_z$$

$$6\boldsymbol{a}_y = 12\boldsymbol{a}_y + H_{2z}\boldsymbol{a}_y - H_{2y}\boldsymbol{a}_z$$

Equating components,

$$6 = 12 + H_{2z} \quad \rightarrow \quad H_{2z} = -6,$$

$$\mathrm{H}_{2y} = 0$$

Also, $\boldsymbol{B}_{1n} = \boldsymbol{B}_{2n} \quad \rightarrow \quad \mu_1 \boldsymbol{H}_{1n} = \mu_2 \boldsymbol{H}_{2n}$

$$\boldsymbol{H}_{2n} = \frac{\mu_1}{\mu_2}\boldsymbol{H}_{1n} = \frac{2\mu_o}{4\mu_o}(10\boldsymbol{a}_x) = 5\boldsymbol{a}_x$$

$$\underline{\underline{\boldsymbol{H}_2 = 5\boldsymbol{a}_x - 6\boldsymbol{a}_z \text{ A/m}}}$$

**Prob. 8.34**

$$\boldsymbol{H}_{2t} = \boldsymbol{H}_{1t} = \alpha\boldsymbol{a}_x + \delta\boldsymbol{a}_z$$

$$\boldsymbol{B}_{2n} = \boldsymbol{B}_{1n} \longrightarrow \mu_2 \boldsymbol{H}_{2n} = \mu_1 \boldsymbol{H}_{1n}$$

$$\boldsymbol{H}_{2n} = \frac{\mu_1}{\mu_2}\boldsymbol{H}_{1n} = \frac{\mu_{r1}}{\mu_{r2}}\beta\boldsymbol{a}_y$$

$$\boldsymbol{H} = \underline{\underline{\alpha\boldsymbol{a}_x + \frac{\mu_{r1}}{\mu_{r2}}\beta\boldsymbol{a}_y + \delta\boldsymbol{a}_z}}$$

**Prob. 8.35**

(a) $\mathbf{B}_{1n} = \mathbf{B}_{2n} = 15\boldsymbol{a}_\phi$

$$\mathbf{H}_{1t} = \mathbf{H}_{2t} \rightarrow \frac{\mathbf{B}_{1t}}{\mu_1} = \frac{\boldsymbol{B}_{2t}}{\mu_2}$$

$$\boldsymbol{B}_{1t} = \frac{\mu_1}{\mu_2}\boldsymbol{B}_{2t} = \frac{2}{5}\left(10\boldsymbol{a}_\rho - 20\boldsymbol{a}_z\right) = 4\boldsymbol{a}_\rho - 8\boldsymbol{a}_z$$

Hence,

$$\boldsymbol{B}_1 = \underline{\underline{4\boldsymbol{a}_\rho + 15\boldsymbol{a}_\phi - 8\boldsymbol{a}_z \text{ mWb/m}^2}}$$

(b) $$\text{w}_{m1} = \frac{1}{2}\boldsymbol{B}_1 \cdot \boldsymbol{H}_1 = \frac{\text{B}_1^2}{2\mu_1} = \frac{\left(4^2 + 15^2 + 8^2\right)\times 10^{-6}}{2\times 2\times 4\pi \times 10^{-7}}$$

$$\text{w}_{m1} = \underline{\underline{60.68 \text{ J}/m^3}}$$

$$\text{w}_{m2} = \frac{\text{B}_2^2}{2\mu_2} = \frac{\left(10^2 + 15^2 + 20^2\right)\times 10^{-6}}{2\times 5\times 4\pi \times 10^{-7}} = \underline{\underline{57.7 \text{ J}/m^3}}$$

**Prob. 8.36**

$$f(x,y,z) = x - y + 2z$$

$$\nabla f = \boldsymbol{a}_x - \boldsymbol{a}_y + 2\boldsymbol{a}_z$$

$$a_n = \frac{\nabla f}{|\nabla f|} = \frac{1}{\sqrt{6}}(\boldsymbol{a}_x - \boldsymbol{a}_y + 2\boldsymbol{a}_z)$$

(a)

$$\boldsymbol{H}_{1n} = (\boldsymbol{H}_1 \cdot \boldsymbol{a}_n)\boldsymbol{a}_n = (40 - 20 - 60)\frac{(\boldsymbol{a}_x - \boldsymbol{a}_y + 2\boldsymbol{a}_z)}{6}$$

$$= \underline{\underline{-6.667\boldsymbol{a}_x + 6.667\boldsymbol{a}_y - 13.333\boldsymbol{a}_z \text{ A/m}}}$$

(b)

$$\boldsymbol{H}_2 = \boldsymbol{H}_{2n} + \boldsymbol{H}_{2t}$$

But $\boldsymbol{B}_{2n} = \boldsymbol{B}_{1n} \longrightarrow \mu_2\boldsymbol{H}_{2n} = \mu_1\boldsymbol{H}_{1n}$

$$\boldsymbol{B}_2 = \mu_2\boldsymbol{H}_2 = \mu_2\boldsymbol{H}_{2n} + \mu_2\boldsymbol{H}_{2t} = \mu_1\boldsymbol{H}_{1n} + \mu_2\boldsymbol{H}_{2t} = \mu_o(2\boldsymbol{H}_{1n} + 5\boldsymbol{H}_{2t})$$

$$= 4\pi \times 10^{-7}\left[(-13.333, 13.333, -26.667) + (233.333, 66.666, -83.333)\right]$$

$$= 4\pi \times 10^{-7}(220, 80, -110)$$

$$= \underline{\underline{276.5\boldsymbol{a}_x + 100.5\boldsymbol{a}_y - 138.2\boldsymbol{a}_z \ \mu\text{Wb/m}^2}}$$

**Prob. 8.37**

$$\boldsymbol{a}_n = \boldsymbol{a}_\rho$$

$$\boldsymbol{B}_{2n} = \boldsymbol{B}_{1n} = 22\mu_o \boldsymbol{a}_\rho$$

$$\boldsymbol{H}_{2t} = \boldsymbol{H}_{1t} \longrightarrow \frac{\boldsymbol{B}_{2t}}{\mu_2} = \frac{\boldsymbol{B}_{1t}}{\mu_1}$$

$$\boldsymbol{B}_{2t} = \frac{\mu_2}{\mu_1}\boldsymbol{B}_{1t} = \frac{\mu_o}{800\mu_o}(45\mu_o \boldsymbol{a}_\phi) = 0.05625\mu_o \boldsymbol{a}_\phi$$

$$\boldsymbol{B}_2 = \underline{\underline{\mu_o(22\boldsymbol{a}_\rho + 0.05625\boldsymbol{a}_\phi)\ \text{Wb/m}^2}}$$

**Prob. 8.38**

$$\boldsymbol{H}_{1n} = -3\boldsymbol{a}_z, \qquad \boldsymbol{H}_{1t} = 10\boldsymbol{a}_x + 15\boldsymbol{a}_y$$

$$\boldsymbol{H}_{2t} = \boldsymbol{H}_{1t} = 10\boldsymbol{a}_x + 15\boldsymbol{a}_y$$

$$\boldsymbol{H}_{2n} = \frac{\mu_1}{\mu_2}\boldsymbol{H}_{1n} = \frac{1}{200}(-3\boldsymbol{a}_z) = -0.015\boldsymbol{a}_z$$

$$\boldsymbol{H}_2 = 10\boldsymbol{a}_x + 15\boldsymbol{a}_y - 0.015\boldsymbol{a}_z$$

$$\boldsymbol{B}_2 = \mu_2\boldsymbol{H}_2 = 200 \times 4\pi \times 10^{-7}(10, 15, -0.015)$$

$$\boldsymbol{B}_2 = \underline{\underline{2.51\boldsymbol{a}_x + 3.77\boldsymbol{a}_y - 0.0037\boldsymbol{a}_z}}\ \text{mWb/m}^2$$

$$\tan\alpha = \frac{B_{2n}}{B_{2t}}$$

or $$\alpha = \tan^{-1}\frac{0.0037}{\sqrt{2.51^2 + 3.77^2}} = \underline{\underline{0.047^o}}$$

**Prob. 8.39**

(a) $$\boldsymbol{H} = \tfrac{1}{2}\boldsymbol{K} \times \boldsymbol{a}_n = \tfrac{1}{2}(30 - 40)\boldsymbol{a}_x \times (-\boldsymbol{a}_z) = \underline{\underline{-5\boldsymbol{a}_y}}\ \text{A/m}$$

$$\boldsymbol{B} = \mu_o\boldsymbol{H} = 4\pi \times 10^{-7}(-5\boldsymbol{a}_y) = \underline{\underline{-6.28\boldsymbol{a}_y}}\ \mu\text{Wb/m}^2$$

(b) $$\boldsymbol{H} = \tfrac{1}{2}(-30 - 40)\boldsymbol{a}_y = \underline{\underline{-35\boldsymbol{a}_y}}\ \text{A/m}$$

$$\boldsymbol{B} = \mu_o\mu_r\boldsymbol{H} = 4\pi \times 10^{-7}(2.5)(-35\boldsymbol{a}_y) = \underline{\underline{-110\boldsymbol{a}_y\ \ \mu\text{Wb/m}^2}}$$

(c) $\boldsymbol{H} = \frac{1}{2}(-30+40)\boldsymbol{a}_y = \underline{\underline{5\boldsymbol{a}_y}}$

$\boldsymbol{B} = \mu_o \boldsymbol{H} = \underline{\underline{6.283\boldsymbol{a}_y}}\mu$ Wb/m$^2$

**Prob. 8.40**

r = a is the interface between the two media.

$\boldsymbol{B}_{2n} = \boldsymbol{B}_{1n} \longrightarrow B_{o1}(1+1.6)\cos\theta \boldsymbol{a}_r = B_{o2}\cos\theta \boldsymbol{a}_r$

$2.6B_{o1} = B_{o2}$ (1)

$\boldsymbol{H}_{2t} = \boldsymbol{H}_{1t} \longrightarrow \frac{\boldsymbol{B}_{2t}}{\mu_2} = \frac{\boldsymbol{B}_{1t}}{\mu_1}$

$\mu_2 \boldsymbol{B}_{1t} = \mu_1 \boldsymbol{B}_{2t}$

$\mu_2 B_{o1}(-0.2)\sin\theta \boldsymbol{a}_\theta = \mu_o B_{o2}(-\sin\theta)\boldsymbol{a}_\theta$

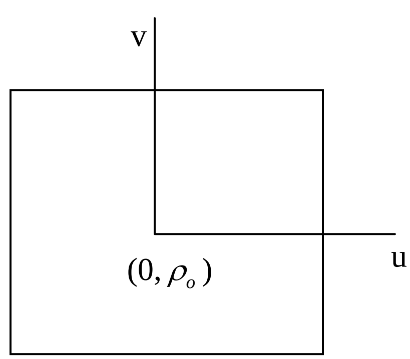
$\mu_2 = \frac{\mu_o B_{o2}}{0.2 B_{o1}}$ (2)

Substituting (1) into (2) gives

$\mu_2 = \frac{\mu_o}{0.2}(2.6) = \underline{\underline{13\mu_o}}$

**Prob. 8.41**

(a) The square cross-section of the toroid is shown below. Let (u,v) be the local coordinates and $\rho_o$ =mean radius. Using Ampere's law around a circle passing through P, we get




$H(2\pi)(\rho_o + v) = NI \longrightarrow H = \frac{NI}{2\pi(\rho_o + v)}$

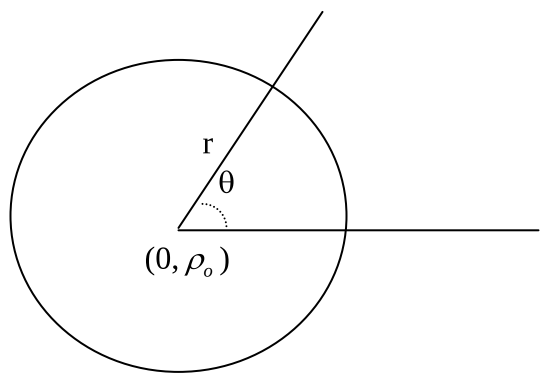
The flux per turn is

$$\Psi = \int_{u=-a/2}^{a/2} \int_{v=-a/2}^{a/2} B du dv = \frac{\mu_o NIa}{2\pi} \ln\left(\frac{\rho_o + a/2}{\rho_o - a/2}\right)$$

$$L = \frac{N\Psi}{I} = \frac{\mu_o N^2 a}{2\pi} \ln\left(\frac{2\rho_o + a}{2\rho_o - a}\right)$$

(b) The circular cross-section of the toroid is shown below. Let (r,θ) be the local coordinates. Consider a point P( $r\cos\theta, \rho_o + r\sin\theta$ ) and apply Ampere's law around a circle that passes through P.

$$H(2\pi)(\rho_o + r\sin\theta) = NI \quad \longrightarrow \quad H = \frac{NI}{2\pi(\rho_o + r\sin\theta)} \approx \frac{NI}{2\pi\rho_o}\left(1 - \frac{r\sin\theta}{\rho_o}\right)$$




Flux per turn $\Psi = \int_{r=0}^{a} \int_{\theta}^{2\pi} \frac{\mu NI}{2\pi\rho_o}\left(1 - \frac{r\sin\theta}{\rho_o}\right) r dr d\theta = \frac{\mu NI}{2\pi\rho_o} \frac{a^2}{2}(2\pi)$

$$L = \frac{N\Psi}{I} = \frac{\mu N^2 a^2}{2\rho_o}$$

Or from Example 8.10,

$$L = L'l = \frac{\mu_o N^2 l S}{l^2} = \frac{\mu_o N^2 \pi a^2}{2\pi\rho_o} = \frac{\mu_o N^2 a^2}{2\rho_o}$$

**Prob. 8.42**

$$\rho_o = \frac{1}{2}(3+5) = 4\text{cm}$$

$$a = 2 \text{ cm}$$

$$L = \frac{\mu_o N^2 a}{2\pi} \ln\left[\frac{2\rho_o + a}{2\rho_o - a}\right]$$

$$N^2 = \frac{2\pi L}{\mu_o a \ln\left[\frac{2\rho_o + a}{2\rho_o - a}\right]} = \frac{2\pi(45\times10^{-6})}{4\pi\times10^{-7}(2\times10^{-2})\ln\left(\frac{8+2}{8-2}\right)} = 22,023.17$$

$$\underline{\underline{N = 148.4 \text{ or } 148}}$$

**Prob. 8.43**

$$L = \frac{\mu_o \ell}{8\pi} = \frac{4\pi\times10^{-7}(40)}{8\pi} = 20\times10^{-7} = \underline{\underline{2\ \mu\text{H}}}$$

**Prob. 8.44**

$$L_{in} = \frac{\mu_o \ell}{8\pi}, \qquad L_{ext} = \frac{\mu_o \ell}{2\pi}\ln(b/a)$$

If $L_{in} = 2L_{ext}$ $\longrightarrow$ $\frac{\mu_o \ell}{8\pi} = \frac{\mu_o \ell}{\pi}\ln(b/a)$

$$\ln(b/a) = \frac{1}{8} \qquad \frac{b}{a} = e^{1/8} = 1.1331$$

$$b = 1.1331a = \underline{\underline{7.365 \text{ mm}}}$$

**Prob. 8.45**

$$L = \frac{\mu_o \ell}{2\pi}\left[\ln\frac{2\ell}{a} - 1\right] = \frac{4\pi\times10^{-7}(10)}{2\pi}\left[\ln\frac{2\times10}{2\times10^{-2}} - 1\right] = 2\times10^{-6}(\ln 1000 - 1)$$

$$= 2(5.908)\ \mu H = \underline{\underline{11.82\ \ \mu\text{H}}}$$

**Prob. 8.46**

$$\psi_{12} = \int \boldsymbol{B}_1 \bullet d\boldsymbol{S} = \int_{\rho=\rho_o}^{\rho_o+a} \int_{z=0}^{b} \frac{\mu_o I}{2\pi\rho} dz d\rho = \frac{\mu_o I b}{2\pi} \ln\frac{a+\rho_o}{\rho_o}$$

$$M_{12} = \frac{N\psi_{12}}{I} = \frac{N\mu_o b}{2\pi}\ln\frac{a+\rho_o}{\rho_o}$$

For N = 1,

$$M_{12} = \frac{\psi_{12}}{I_1} = \frac{\mu_o b}{2\pi}\ln\frac{a+\rho_o}{\rho_o}$$

$$= \frac{4\pi\times10^{-7}}{2\pi}(1)\ln 2 = \underline{\underline{0.1386\mu\text{ H}}}$$

**Prob. 8.47**

We may approximate the longer solenoid as infinite so that $B_1 = \frac{\mu_o N_1 I_1}{l_1}$. The flux linking the second solenoid is:

$$\psi_2 = N_2 B_1 S_1 = \frac{\mu_o N_1 I_1}{l_1} \bullet \pi r_1^2 N_2$$

$$M = \frac{\psi_2}{I_1} = \frac{\mu_o N_1 N_2}{l_1} \bullet \pi r_1^2$$

Here we assume air-core solenoids.

**Prob. 8.48**

For a straight infinitely long conductor,

$$B_\phi = \frac{\mu_o I}{2\pi\rho}$$

$$\Psi = \int_S \boldsymbol{B}\bullet d\boldsymbol{S} = \frac{\mu_o I}{2\pi}\left[\int_{z=0}^{h}\int_{\rho=b}^{b+w}\frac{1}{\rho}dzd\rho - \int_{z=0}^{h}\int_{\rho=a+b}^{a+b+w}\frac{1}{\rho}dzd\rho\right]$$

$$= \frac{\mu_o Ih}{2\pi}\left[\ln\rho\Big|_{b}^{b+w} - \ln\rho\Big|_{a+b}^{a+b+w}\right] = \frac{\mu_o Ih}{2\pi}\left[\ln\frac{b+w}{b} - \ln\frac{a+b+w}{a+b}\right]$$

$$\underline{\underline{\Psi = \frac{\mu_o Ih}{2\pi}\ln\left[\frac{(a+b)(\text{b}+w)}{b(a+b+w)}\right]}}$$

**Prob. 8.49**

$$\boldsymbol{H} = \frac{I}{2\pi\rho}\boldsymbol{a}_\rho$$

$$w_m = \frac{1}{2}\mu|\boldsymbol{H}|^2 = \frac{1}{2}\mu\frac{I^2}{4\pi^2\rho^2}$$

$$W = \int w_m dv = \iiint\frac{1}{2}\mu\frac{I^2}{4\pi^2\rho^2}\rho d\phi d\rho dz = \frac{1}{4\pi}\mu I^2 L\ln(b/a)$$

$$= \frac{1}{4\pi} \times 4 \times 4\pi \times 10^{-7} (625 \times 10^{-6}) 3 \ln(18/12) = \underline{\underline{304.1 \text{ pJ}}}$$

Alternatively,

$$W = \frac{1}{2} L I^2 = \frac{1}{2} \frac{\mu L}{2\pi} \ln \frac{b}{a} \times I^2 = \frac{\mu I^2 L}{4\pi} \ln \frac{b}{a}$$

**Prob. 8.50**

$$\mu_r = \chi_m + 1 = 20$$

$$w_m = \frac{1}{2} \boldsymbol{B}_1 \cdot \boldsymbol{H}_1 = \frac{1}{2} \mu \mathbf{H} \cdot \mathbf{H}$$

$$= \frac{1}{2} \mu \left( 25x^4 y^2 z^2 + 100 x^2 y^4 z^2 + 225 x^2 y^2 z^4 \right)$$

$$W_m = \int w_m dv$$

$$= \frac{1}{2} \mu \left[ 25 \int_0^1 x^4 dx \int_0^2 y^2 dy \int_{-1}^2 z^2 dz + 100 \int_0^1 x^2 dx \int_0^2 y^4 dy \int_{-1}^2 z^2 dz \right.$$
$$\left. + 225 \int_0^1 x^2 dx \int_0^2 y^2 dy \int_{-1}^2 z dz \right]$$

$$= \frac{25\mu}{2} \left[ \left. \frac{x^5}{5} \right|_0^1 \left. \frac{y^3}{3} \right|_0^2 \left. \frac{z^3}{3} \right|_{-1}^2 + 4 \left. \frac{x^3}{3} \right|_0^1 \left. \frac{y^5}{5} \right|_0^2 \left. \frac{z^3}{3} \right|_{-1}^2 \right.$$
$$\left. + 9 \left. \frac{x^3}{3} \right|_0^1 \left. \frac{y^3}{3} \right|_0^2 \left. \frac{z^5}{5} \right|_{-1}^2 \right]$$

$$= \frac{25\mu}{2} \left( \frac{1}{5} \cdot \frac{8}{3} \cdot \frac{9}{3} + \frac{4}{3} \cdot \frac{32}{3} \cdot \frac{9}{3} + \frac{9}{3} \cdot \frac{8}{3} \cdot \frac{33}{5} \right)$$

$$= \frac{25}{2} \times 4\pi \times 10^{-7} \times 20 \times \frac{3600}{45}$$

$$W_m = \underline{\underline{25.13 \text{ mJ}}}$$

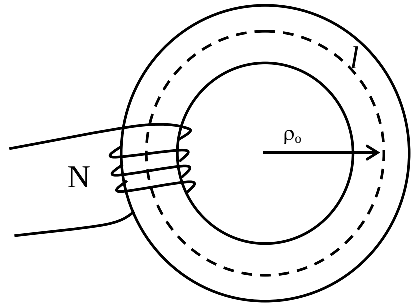
**Prob. 8.51**

$$W = \frac{1}{2} \int_v \boldsymbol{B} \bullet \boldsymbol{H} dv = \int_v \frac{|\boldsymbol{B}|^2}{2\mu} dv = \frac{1}{2(15) 4\pi \times 10^{-7}} \int_{z=0}^{4} \int_{y=0}^{3} \int_{x=0}^{2} (4^2 + 12^2) 10^{-6} dx dy dz$$

$$= \frac{10^{-6}}{(30) 4\pi \times 10^{-7}} (16 + 144)(2)(3)(4) = \frac{320}{\pi} = \underline{\underline{101.86 \text{ J}}}$$

**Prob. 8.52**

$$NI = Hl = \frac{Bl}{\mu}$$




$$N = \frac{Bl}{\mu_o \mu_r I} = \frac{1.5 \times 0.6\pi}{4\pi \times 10^{-7} \times 600 \times 12}$$

$$= \underline{\underline{313 \text{ turns}}}$$

**Prob. 8.53**

F = NI = 400 x 0.5 = 200 A.t

$R_a = \frac{100}{4\pi}$ MAt/Wb, $R_1 = R_2 = \frac{6}{4\pi}$ MAt/Wb, $R_3 = \frac{1.8}{4\pi}$ MAt/Wb

$$F_a = \frac{R_a F}{R_a + R_3 + R_1 // R_2} = \underline{\underline{190.8}} \text{ A.t}$$

$$H_a = \frac{F_a}{l_a} = \frac{190.8}{1 \times 10^{-2}} = \underline{\underline{19080}} \text{ A/m}$$

**Prob. 8.54**

Total F = NI = 2000 x 10 = 20,000 A.t

$$R_c = \frac{l_c}{\mu_o \mu_r S} = \frac{(24 + 20 - 0.6) \times 10^{-2}}{4\pi \times 10^{-7} \times 1500 \times 2 \times 10^{-4}} = \underline{\underline{0.115 \times 10^7 \text{ A.t/m}}}$$

$$R_a = \frac{l_a}{\mu_o \mu_r S} = \frac{0.6 \times 10^{-2}}{4\pi \times 10^{-7} (1) \times 2 \times 10^{-4}} = \underline{\underline{2.387 \times 10^7 \text{ A.t/m}}}$$

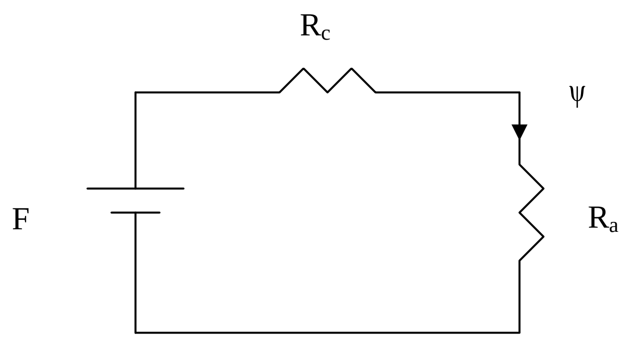
R = $R_a + R_c$ = 2.502 x $10^7$ A.t/m

$$\psi = \frac{\Im}{R} = \psi_a = \psi_c = \frac{20,000}{2.502 \times 10^7} = \underline{\underline{8 \times 10^{-4} \text{ Wb/m}^2}}$$

$$\Im_a = \frac{R_a}{R_a + R_c} \Im = \frac{2.387 \times 20,000}{2.502} = \underline{\underline{19,081}} \text{ A.t}$$

$$\Im_c = \frac{R_c}{R_a + R_c} \Im = \frac{0.115 \times 20,000}{2.502} = \underline{\underline{919}} \text{ A.t}$$

**Prob. 8.55**




F = NI = 500 x 0.2 = 100 A.t

$$R_c = \frac{l_c}{\mu S} = \frac{42\times10^{-2}}{4\pi\times10^{-7}\times10^{3}\times4\times10^{-4}} = \frac{42\times10^{6}}{16\pi}$$

$$R_a = \frac{l_a}{\mu_o S} = \frac{10^{-3}}{4\pi\times10^{-7}\times4\times10^{-4}} = \frac{10^{8}}{16\pi}$$

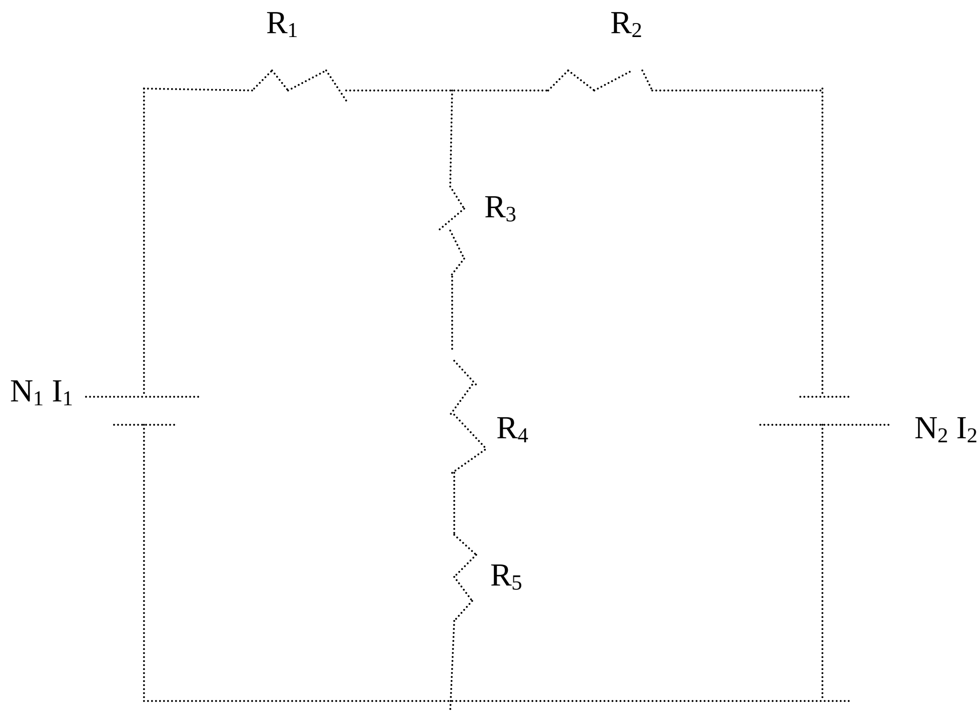
$$R_a + R_c = \frac{1.42\times10^{8}}{16\pi}$$

$$\psi = \frac{F}{R_a + R_c} = \frac{16\pi\times100}{1.42\times10^{8}} = \frac{16\pi}{1.42} \quad \mu\text{Wb}$$

$$B_a = \frac{\psi}{S} = \frac{16\pi\times10^{-6}}{1.42\times4\times10^{-4}} = \underline{\underline{88.5\text{ mWb/m}^2}}$$

**Prob. 8.56**

The equivalent circuit is shown below.




**Prob. 8.57**

$$R = \frac{\ell}{\mu_o S} = \frac{4.4\times10^{-3}}{4\pi\times10^{-7}(4.82\times10^{-2})} = \frac{0.2282\times10^{6}}{\pi} = \underline{\underline{7.2643\times10^{4}\ \text{A.t/Wb}}}$$

**Prob. 8.58**

$$F = \frac{B^2 S}{2\mu_o} = \frac{\psi^2}{2\mu_o S} = \frac{4\times10^{-6}}{2\times4\pi\times10^{-7}\times0.3\times10^{-4}} = \underline{\underline{53.05}}\ \text{kN}$$

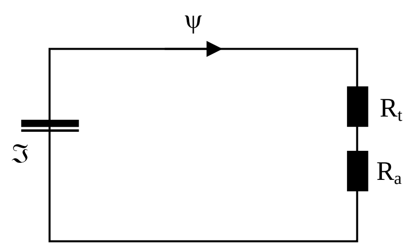
**Prob. 8.59**

(a) F = NI = 200 x $10^{-3}$ x 750 = 150 A.t.

$$R_a = \frac{l_a}{\mu_o S} = \frac{10^{-3}}{25\times10^{-6}\mu_o} = 3.183\times10^{7}$$

$$R_t = \frac{l_t}{\mu_o\mu_r S} = \frac{2\pi\times0.1}{\mu_o\times300\times25\times10^{-6}} = 6.7 \text{ x } 10^{7}$$

$$\psi = \frac{\mathfrak{I}}{R_a + R_t} = \frac{150}{10^{7}(3.183 + 20/3)} = 15.23\times10^{-7}$$



$$F = \frac{B^2 S}{2\mu_o} = \frac{\psi^2}{2\mu_o S} = \frac{2.32\times10^{-12}}{2\times4\pi\times10^{-7}\times25\times10^{-6}}$$

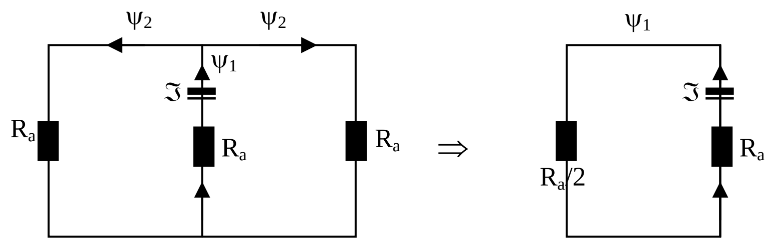
= <u>37 mN</u>

(b)If $\mu_t \to \infty,\ R_t = 0,\ \psi = \frac{\Im}{R_a} = \frac{150}{3.183\times10^7}$

$$F_2 = I_2 dl_2 \bullet B_1 = I_2 dl_2 \frac{\psi_1}{S} = \frac{2\times10^{-3}\times5\times10^{-3}\times150}{3.183\times10^7\times25\times10^{-6}}$$

$F_2$ = <u>1.885 μN</u>

**Prob. 8.60**



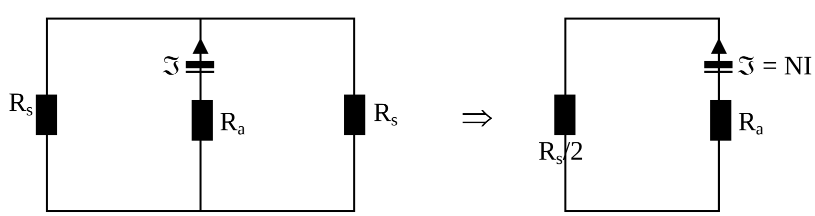

$$\psi_1 = 2\psi_2, \psi_1 = \frac{\Im}{\frac{3}{2}R_a} = \frac{2\Im}{3R_a} \to \psi_2 = \frac{\Im}{3R_a}$$

$$\Im = 2\left(\frac{\psi_2^{\,2}}{2\mu_o S}\right) + \frac{\psi_1}{2\mu_o S} = \frac{3\psi_1^{\,2}}{4\mu_o S} = \frac{\Im^2}{3R_a^{\,2}\mu_o S}$$

$$= \frac{\mu_o S \Im^2}{3 l_a^{\,2}} = \frac{4\pi\times10^{-7}\times200\times10^{-4}\times9\times10^6}{3\times10^{-6}}$$

$$= 24\pi\times10^3 = mg \to m = \frac{24\pi\times10^3}{9.8} = \underline{7694}\ \text{kg}$$

**Prob. 8.61**




Since $\mu \to \infty$ for the core (see Figure), $R_c = 0$.

$$\Im = NI = \psi\left(R_a + \frac{R_s}{2}\right) = \frac{\psi(a/2 + x)}{\mu_o S}$$

$$= \frac{\psi(2x+a)}{2\mu_o S}$$

$$\Im = \frac{B^2 S}{2\mu_o} = \psi^2 \frac{1}{2\mu_o S} = \frac{1}{2\mu_o S} \bullet \frac{N^2 I^2 4\mu_o^2 S^2}{(a+2x)^2}$$

$$= \frac{2N^2 I^2 \mu_o S}{(a+2x)^2}$$

$\boldsymbol{F} = -F\boldsymbol{a}_x$ since the force is attractive, i.e.

$$\boldsymbol{F} = \frac{-2N^2 I^2 \mu_o S \boldsymbol{a}_x}{(a+2x)^2}$$

# CHAPTER 9

**P.E. 9.1**

(a) $V_{emf} = \int (\boldsymbol{u} \times \boldsymbol{B}) \cdot d\boldsymbol{l} = uBl = 8(0.5)(0.1) = \underline{\underline{0.4}}$ V

(b) $I = \dfrac{V_{emf}}{R} = \dfrac{0.4}{20} = \underline{\underline{20}}$ mA

(c) $\boldsymbol{F}_m = I\boldsymbol{l} \times \boldsymbol{B} = 0.02(-0.1\boldsymbol{a}_y \times 0.5\boldsymbol{a}_z) = \underline{\underline{-\boldsymbol{a}_x}}$ mN

(d) $P = FU = I^2R = 8$ mW

or $P = \dfrac{V_{emf}}{R} = \dfrac{(0.4)^2}{20} = \underline{\underline{8}}$ mW

**P.E. 9.2**

(a) $V_{emf} = \int (\boldsymbol{u} \times \boldsymbol{B}) \cdot d\boldsymbol{l}$

where $\boldsymbol{B} = B_o \boldsymbol{a}_y = B_o(\sin\phi \boldsymbol{a}_\rho + \cos\phi \boldsymbol{a}_\phi)$ , $B_o = 0.05\ \text{Wb/m}^2$

$$(\boldsymbol{u} \times \boldsymbol{B}) \cdot d\boldsymbol{l} = -\rho\omega B_o \sin\phi dz = -0.2\pi \sin\left(\omega t + \pi/2\right) dz$$

$$V_{emf} = \int_0^{0.03} (\boldsymbol{u} \times \boldsymbol{B}) \cdot d\boldsymbol{l} = -6\pi \cos(100\pi t) \text{ mV}$$

At t = 1ms,

$V_{emf} = -6\pi \cos 0.1\pi = \underline{\underline{-17.93}}$ mV

$$i = \frac{V_{emf}}{R} = -60\pi \cos(100\pi .t) \text{ mA}$$

At t = 3ms, $i = -60\pi \cos 0.3\pi = \underline{\underline{-110.8 \text{ mA}}}$

(b) Method 1:

$$\Psi = \int \boldsymbol{B} \cdot d\boldsymbol{S} = \int B_o t(\cos\phi \boldsymbol{a}_\rho - \sin\phi \boldsymbol{a}_\phi) \cdot d\rho dz \boldsymbol{a}_\phi = -\int_0^{\rho_o}\int_0^{z_o} B_o t \sin\phi d\rho dz = -B_o \rho_o z_o t \sin\phi$$

where $B_o = 0.02$, $\rho_o = 0.04$, $z_o = 0.03$

$\phi = \omega t + \pi/2$

$\Psi = -B_o \rho_o z_o t \cos\omega t$

$$V_{emf} = -\frac{\partial \Psi}{\partial t} = B_o \rho_o z_o \cos\omega t - B_o \rho_o z_o t\omega \sin\omega t$$

$$=(0.02)(0.04)(0.03)\left[\cos\omega t-\omega t\sin\omega t\right]$$

$$=24\left[\cos\omega t-\omega t\sin\omega t\right]\mu V$$

Method 2:

$$V_{emf}=-\int\frac{\partial \boldsymbol{B}}{\partial t}\bullet d\boldsymbol{S}+\int(\boldsymbol{u}\times\boldsymbol{B}).d\boldsymbol{l}$$

$$\boldsymbol{B}=B_o t\boldsymbol{a}_x=B_o t(\cos\phi\boldsymbol{a}_\rho-\sin\phi\boldsymbol{a}_\phi),\phi=\omega t+\pi/2$$

$$\frac{\partial \boldsymbol{B}}{\partial t}=B_o(\cos\phi\boldsymbol{a}_\rho-\sin\phi\boldsymbol{a}_\phi)$$

Note that only explicit dependence of ***B*** on time is accounted for, i.e. we make ϕ = constant because it is transformer (stationary) emf. Thus,

$$V_{emf}=-B_o\int_0^{\rho_o}\int_0^{z_o}(\cos\phi\boldsymbol{a}_\rho-\sin\phi\boldsymbol{a}_\phi)\cdot d\rho dz\boldsymbol{a}_\phi+\int_{z_o}^{0}-\rho_o\omega B_o t\cos\phi dz$$

$$=B_o\rho_o z_o(\sin\phi-\omega t\cos\phi),\phi=\omega t+\pi/2$$

$$=B_o\rho_o z_o(\cos\omega t-\omega t\sin\omega t)$$ as obtained earlier.

At t = 1ms,

$$V_{emf}=24[\cos 18^o-100\pi\times10^{-3}\sin 18^o]\mu V$$

$$=\underline{\underline{20.5\mu V}}$$

At t = 3ms,

$$i=240[\cos 54^o-.03\pi\sin 54^o]mA$$

$$=\underline{\underline{-41.93\text{ mA}}}$$

**P.E. 9.3**

$$V_1=-N_1\frac{d\psi}{dt},V_2=-N_2\frac{d\psi}{dt}$$

$$\frac{V_2}{V_1}=\frac{N_2}{N_1}\to V_2=\frac{N_2}{N_1}V_1=\frac{300\times120}{500}=\underline{\underline{72V}}$$

**P.E. 9.4**

(a) $$\boldsymbol{J}_d=\frac{\partial\boldsymbol{D}}{\partial t}=\underline{\underline{-20\omega\varepsilon_o\sin(\omega t-50x)\boldsymbol{a}_y A/m^2}}$$

(b) $\nabla \times \boldsymbol{H} = \boldsymbol{J}_d \rightarrow -\frac{\partial H_z}{\partial x}\boldsymbol{a}_y = -20\omega\varepsilon_o \sin(\omega t - 50x)\boldsymbol{a}_y$

or $\boldsymbol{H} = \frac{20\omega\varepsilon_o}{50}\cos(\omega t - 50x)\boldsymbol{a}_z$

$= \underline{\underline{0.4\omega\varepsilon_o \cos(\omega t - 50x)\boldsymbol{a}_z}}$ A/m

(c) $\nabla \times \boldsymbol{E} = -\mu_o \frac{\partial \boldsymbol{H}}{\partial t} \rightarrow \frac{\partial E_y}{\partial x}\boldsymbol{a}_z = 0.4\mu_o\omega^2\varepsilon_o \sin(\omega t - 50x)\boldsymbol{a}_z$

$1000 = 0.4\mu_o\varepsilon_o\omega^2 = 0.4\frac{\omega^2}{c^2}$

or ω = $\underline{\underline{1.5 \text{ x } 10^{10} \text{ rad/s}}}$

**P.E. 9.5**

(a) $j^3\left(\frac{1+j}{2-j}\right)^2 = -j\left[\frac{\sqrt{2}\angle 45^o}{\sqrt{5}\angle -26.56^o}\right]^2 = -j\left(2/5\angle 143.13^o\right)$

= $\underline{\underline{0.24 + j0.32}}$

(b) $6\angle 30^o + j5 - 3 + e^{j45^o} = 5.196 + j3 + j5 - 3 + 0.7071(1+j)$

= $\underline{\underline{2.903 + j8.707}}$

**P.E. 9.6**

$\boldsymbol{P} = 2\sin(10t + x - \pi/4)\boldsymbol{a}_y \quad = 2\cos\left(10t + x - \pi/4 - \pi/2\right)\boldsymbol{a}_y, w = 10$

$= R_e\left(2e^{j(x-3\pi/4)}\boldsymbol{a}_y e^{jwt}\right) = R_e\left(\boldsymbol{P}_s e^{jwt}\right)$

i.e. $\boldsymbol{P}_s = \underline{\underline{2e^{j(x-3\pi/4)}\boldsymbol{a}_y}}$

$\boldsymbol{Q} = R_e\left(\boldsymbol{Q}_s e^{jwt}\right) = R_e\left(e^{j(x+wt)}(\boldsymbol{a}_x - \boldsymbol{a}_z)\right)\sin \pi y$

$= \underline{\underline{\sin \pi y \cos(wt + x)(\boldsymbol{a}_x - \boldsymbol{a}_z)}}$

**P.E. 9.7**

$-\mu\frac{\partial \boldsymbol{H}}{\partial t} = \nabla \times \boldsymbol{E} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(E_\phi \sin\theta)\boldsymbol{a}_r - \frac{1}{r}\frac{\partial}{\partial r}(rE_\phi)\boldsymbol{a}_\theta$

$$= \frac{2\cos\theta}{r^2}\cos(\omega t - \beta r)\boldsymbol{a}_r - \frac{\beta}{r}\sin\theta\sin(\omega t - \beta r)\boldsymbol{a}_\theta$$

$$\boldsymbol{H} = -\frac{2\cos\theta}{\mu\omega r^2}\sin(\omega t - \beta r)\boldsymbol{a}_r - \frac{\beta}{\mu\omega r}\sin\theta\cos(\omega t - \beta r)\boldsymbol{a}_\theta$$

$$\beta = \frac{\omega}{c} = \frac{6\times10^7}{3\times10^8} = \underline{\underline{0.2}} \text{ rad/m}$$

$$\boldsymbol{H} = \underline{\underline{-\frac{1}{12\pi r^2}\cos\theta\sin(6\times10^7 - 0.2r)\boldsymbol{a}_r - \frac{1}{120\pi r}\sin\theta\cos(6\times10^7 - 0.2r)\boldsymbol{a}_\theta}}$$

**P.E. 9.8**

$$\omega = \frac{3}{\sqrt{\mu\varepsilon}} = \frac{3c}{\sqrt{\mu_r\varepsilon_r}} = \frac{9\times10^8}{\sqrt{10}} = \underline{\underline{2.846\times10^8}} \text{ rad/s}$$

$$\boldsymbol{E} = \frac{1}{\varepsilon}\int\nabla\times\boldsymbol{H}dt = -\frac{6}{\omega\varepsilon}\cos(\omega t - 3y)\boldsymbol{a}_x$$

$$= \frac{-6}{\frac{9\times10^8}{\sqrt{10}}\bullet\frac{10^{-9}}{36\pi}(5)}\cos(\omega t - 3y)\boldsymbol{a}_x$$

$$\boldsymbol{E} = \underline{\underline{-476.86\cos(2.846\times10^8 t - 3y)\boldsymbol{a}_x \text{ V/m}}}$$

---

**Prob. 9.1**

$$V = -\frac{\partial\psi}{\partial t} = -\frac{\partial}{\partial t}\int\boldsymbol{B}\bullet d\boldsymbol{S} = -\frac{\partial\boldsymbol{B}}{\partial t}\bullet\boldsymbol{S}$$

= 3770 sin377t x $\pi(0.2)^2$ x $10^{-3}$

= $\underline{\underline{\text{0.4738 sin377t V}}}$

**Prob. 9.2**

$$V_{emf} = -\frac{\partial \psi}{\partial t}, \qquad \psi = \iint \boldsymbol{B} \bullet d\boldsymbol{S} = BS$$

$$V_{emf} = -\frac{\partial B}{\partial t} S$$

$$i(t) = \frac{V_{emf}}{R} = \frac{(4\times 20\sin 20t)(2\times 10^{-4})}{20+30} = \frac{160}{50}\sin 20t(10^{-4})$$

$$\underline{\underline{i(t) \quad = 0.32\sin 20t \text{ mA}}}$$

**Prob.9.3**

$$\psi = \boldsymbol{B} \cdot \boldsymbol{S} = (0.2)^2 \pi \cdot 40\times 10^{-3} \sin 10^4 t$$

$$V = -\frac{\partial \psi}{\partial t} = -16\pi \cos 10^4 t$$

$$i = \frac{V}{R} = \frac{16\pi}{4}\cos 10^4 t$$

$$\underline{\underline{\quad = -12.57\cos 10^4 t \text{ A}}}$$

**Prob.9.4**

Measuring the induced emf in the clockwise direction,

$$V_{\text{emf}} = \oint (\boldsymbol{u}\times\boldsymbol{B}) \cdot d\boldsymbol{l}$$

$$= \int_0^{1.2} (5\boldsymbol{a}_x \times 0.2\boldsymbol{a}_z) \cdot dy\boldsymbol{a}_y + \int_{1.2}^{0} (15\boldsymbol{a}_x \times 0.2\boldsymbol{a}_z) dy\boldsymbol{a}_x$$

$$= -\int_0^{1.2} (1)\, dy - \int_{1.2}^{0} (3) dy$$

$$= -1.2 + 1.2\times 3 = -1.2 + 3.6$$

$$\underline{\underline{\quad = 2.4 \text{ V}}}$$

**Prob. 9.5**

$$V_{emf} = \int (\boldsymbol{u}\times\boldsymbol{B}) \bullet d\boldsymbol{l} = \int_{y=0}^{1.6} (2\boldsymbol{a}_x \times 10\cos\beta y\boldsymbol{a}_z) \bullet dy\boldsymbol{a}_y = -\int_{y=0}^{1.6} 20\cos\beta y dy$$

$$= -\frac{20\sin\beta y}{\beta}\Bigg|_0^{1.6} = \underline{\underline{-\frac{20\sin 1.6\beta}{\beta}}}$$

**Prob. 9.6**

$$\boldsymbol{B} = \frac{\mu_o I}{2\pi y}(-\boldsymbol{a}_x)$$

$$\psi = \int \boldsymbol{B} \bullet d\boldsymbol{S} = \frac{\mu_o I}{2\pi} \int_{z=0}^{a} \int_{y=\rho}^{\rho+a} \frac{dzdy}{y} = \frac{\mu_o Ia}{2\pi} \ln \frac{\rho + a}{\rho}$$

$$V_{emf} = -\frac{\partial \psi}{\partial t} = -\frac{\partial \psi}{\partial \rho} \bullet \frac{\partial \rho}{\partial t} = -\frac{\mu_o Ia}{2\pi} u_o \frac{d}{d\rho}[\ln(\rho + a) - \ln \rho]$$

$$= -\frac{\mu_o Ia}{2\pi} u_o \left[ \frac{1}{\rho + a} - \frac{1}{\rho} \right] = \frac{\mu_o a^2 I u_o}{2\pi\rho(\rho + a)}$$

where $\rho = \rho_o + u_o t$

**Prob. 9.7**

$$V_{emf} = \int_{\rho}^{\rho+a} 3\boldsymbol{a}_z \times \frac{\mu_o I}{2\pi\rho} \boldsymbol{a}_\phi \bullet d\rho \boldsymbol{a}_\rho = -\frac{3\mu_o I}{2\pi} \ln \frac{\rho + a}{\rho}$$

$$= -\frac{4\pi \times 10^{-7}}{2\pi} \times 15 \times 3 \ln \frac{60}{20} = -9.888\mu V$$

Thus the induced emf = 9.888μV, point A at higher potential.

**Prob. 9.8**

$$V = \int (\boldsymbol{u} \times \boldsymbol{B}) \cdot d\boldsymbol{l}$$

$$u = \rho\omega \boldsymbol{a}_\phi, \qquad B = B_o \boldsymbol{a}_z$$

$$V = \int_{\rho=0}^{\ell} \rho\omega B_o d\rho = \frac{1}{2}\omega B_o \rho^2 \Big|_0^{\ell} = \frac{\omega B_o \ell^2}{2}$$

$$= \frac{30}{2} \times 60 \times 10^{-3} (8 \times 10^{-2})^2 = 5.76 \text{ mV}$$

**Prob. 9.9**

$$V_{emf} = -N \frac{\partial \psi}{\partial t} = -N \frac{\partial}{\partial t} \int \boldsymbol{B} \cdot d\boldsymbol{S} = -NB \frac{dS}{dt}$$

$$= -NB\ell \frac{d}{dt}(\rho\phi) = -NB\ell\rho \frac{d\phi}{dt} = -NB\ell\rho\omega$$

$$= -50(0.2)(30 \times 10^{-4})(60) = -1.8\text{V}$$

**Prob. 9.10**

Method 1*:*
We assume that the sliding rode is on $-\ell < y < \ell$

$\ell = x/\sqrt{3} = 5t/\sqrt{3}$

$$V_{emf} = \int (\boldsymbol{u} \times \boldsymbol{B}) \cdot d\boldsymbol{l} = \int 5\boldsymbol{a}_x \times 0.6\boldsymbol{a}_z \bullet dy\boldsymbol{a}_y = -3x\int_{-\ell}^{\ell} dy = -3x(2\ell) = -6\times\left(\frac{25t^2}{\sqrt{3}}\right) = \underline{\underline{-86.6025t^2}}$$

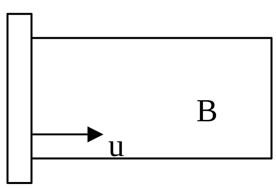
Method 2*:*

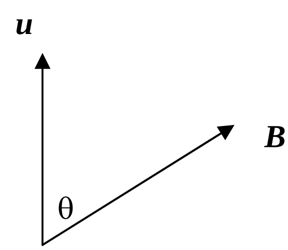
The flux linkage  is given by

$$\psi = \int_{x=o}^{5t} \int_{y=-x/\sqrt{3}}^{x.\sqrt{3}} 0.6xdxdy = 0.6\times\frac{2}{\sqrt{3}}\times 125t^3/3 = 28,8675t^3$$

$$V_{emf} = -\frac{d\psi}{dt} = \underline{\underline{-86.602t^2}}$$

**Prob. 9.11**




$$V_{emf} = \int (\boldsymbol{u} \times \boldsymbol{B}) \cdot d\boldsymbol{l} = uBl\cos\theta$$

$$= \left(\frac{120\times10^3}{3600} m/s\right)\left(4.3\times10^{-5}\right)\left(1.6\right) cos65^o$$

$$= 2.293 cos65^o = \underline{\underline{0.97}} \text{ mV}$$

**Prob. 9.12**

$$V_{emf} = uB\ell = 410\times0.4\times10^{-6}\times36 = \underline{\underline{5.904 \text{ mV}}}$$

**Prob. 9.13**

dψ = 0.64 – 0.45 = 0.19, dt = 0.02

$$V_{emf} = N\frac{d\psi}{dt} = 10\left(\frac{0.19}{0.02}\right) = 95V$$

$$I = \frac{V_{emf}}{R} = \left(\frac{95}{15}\right) = \underline{\underline{6.33\ \text{A}}}$$

Using Lenz's law, the direction of the induced current is counterclockwise.

**Prob. 9.14**

$$V = \int (\boldsymbol{u}\times\boldsymbol{B})\bullet d\boldsymbol{l}\text{, where } \boldsymbol{u} = \rho\omega\boldsymbol{a}_\phi, \quad \boldsymbol{B} = B_o\boldsymbol{a}_z$$

$$V = \int_{\rho_1}^{\rho_2} \rho\omega B_o d\rho = \frac{\omega B_o}{2}(\rho^2{}_2 - \rho^2{}_1)$$

$$V = \frac{60\times 15}{2}\bullet 10^{-3}(100-4)\bullet 10^{-4} = \underline{\underline{4.32\text{ mV}}}$$

**Prob. 9.15**

$$J_{ds} = j\omega D_s \rightarrow |J_{ds}|_{\max} = \omega\varepsilon E_s = \omega\varepsilon\frac{V_s}{d}$$

$$= \frac{10^{-9}}{36\pi}\times\frac{2\pi\times 20\times 10^6\times 50}{0.2\times 10^{-3}}$$

$$= \underline{\underline{277.8}}\ \text{A/m}^2$$

$$I_{ds} = J_{ds}\bullet S = \frac{1000}{3.6}\times 2.8\times 10^{-4} = \underline{\underline{77.78}}\ \text{mA}$$

**Prob. 9.16**

$$J_c = \sigma E, \qquad J_d = \frac{\partial D}{\partial t} = \varepsilon\frac{\partial E}{\partial t}$$

$$|J_c| = \sigma|E|, \qquad |J_d| = \varepsilon\omega|E|$$

$$\text{If} \quad I_c = I_d, \ \text{then} \quad |J_c| = |J_d| \quad \longrightarrow \quad \sigma = \varepsilon\omega$$

$$\omega = 2\pi f = \frac{\sigma}{\varepsilon}$$

$$f = \frac{\sigma}{2\pi\varepsilon} = \frac{4}{2\pi\times 9\times\frac{10^{-9}}{36\pi}} = \underline{\underline{8\text{ GHz}}}$$

**Prob. 9.17** $\frac{J_c}{J_d} = \frac{\sigma E}{\omega\varepsilon E} = \frac{\sigma}{\omega\varepsilon}$

(a) $$\frac{\sigma}{\omega\varepsilon} = \frac{2\times10^{-3}}{2\pi\times10^9\times81\times\frac{10^{-9}}{36\pi}} = \underline{\underline{0.444\times10^{-3}}}$$

(b) $$\frac{\sigma}{\omega\varepsilon} = \frac{25}{2\pi\times10^9\times81\times\frac{10^{-9}}{36\pi}} = \underline{\underline{5.555}}$$

(c) $$\frac{\sigma}{\omega\varepsilon} = \frac{2\times10^{-4}}{2\pi\times10^9\times5\times\frac{10^{-9}}{36\pi}} = \underline{\underline{7.2\times10^{-4}}}$$

**Prob. 9.18**

$$J_c = \sigma E \quad \rightarrow \quad J_{cs} = \sigma E_s$$

$$J_d = \varepsilon\frac{\partial E}{\partial t} \quad \rightarrow \quad J_{ds} = j\omega\varepsilon E_s$$

$$\frac{|J_{cS}|}{|J_{ds}|} = \frac{\sigma}{\omega\varepsilon} = \frac{4}{2\pi\times10^7\times81\times\frac{10^{-9}}{36\pi}} = \frac{400(18)}{81} = \underline{\underline{88.89}}$$

**Prob. 9.19**

$$\frac{J_d}{J} = \frac{\omega\varepsilon E}{\sigma E} = \frac{\omega\varepsilon}{\sigma} = 1 \quad \longrightarrow \quad \omega = \frac{\sigma}{\varepsilon} = \frac{10^{-4}}{3\times\frac{10^{-9}}{36\pi}} = 12\pi\times10^5$$

$$2\pi f = 12\pi\times10^5 \quad \longrightarrow \quad f = \underline{\underline{600 \text{ kHz}}}$$

**Prob. 9.20**

$$J_c = \sigma E = 0.4\cos(2\pi\times10^8 t)$$

$$E = \frac{0.4}{\sigma}\cos(2\pi\times10^8 t)$$

$$J_d = \varepsilon\frac{\partial E}{\partial t} = -\frac{0.4\varepsilon}{\sigma}(2\pi\times10^8)\sin(2\pi\times10^8 t)$$

$$= -\frac{0.4\times4.5\times\frac{10^{-9}}{36\pi}}{10^{-4}}(2\pi\times10^8)\sin(2\pi\times10^8 t)$$

$$\underline{\underline{= -100\sin(2\pi\times10^8 t)\ \text{A/m}^2}}$$

**Prob. 9.21**

$$\nabla\times\boldsymbol{H} = \boldsymbol{J} + \frac{\partial\boldsymbol{D}}{\partial t} = \boldsymbol{J} + \boldsymbol{J}_d$$

Since the regin is source-free, **J** = **0**.

$$\boldsymbol{J}_d = \nabla\times\boldsymbol{H} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x(z) & 0 & 0 \end{vmatrix} = \frac{\partial H_x}{\partial z}\boldsymbol{a}_y = -H_o(-\beta)\sin(\omega t - \beta z)\boldsymbol{a}_y$$

$$= \underline{\underline{\beta H_o\sin(\omega t - \beta z)\boldsymbol{a}_y}}$$

**Prob. 9.22**

$$J_d = \frac{\partial D}{\partial t} = \varepsilon\frac{\partial E}{\partial t}$$

$$I = \int_S \boldsymbol{J}_d\bullet d\boldsymbol{S} = J_d S = \varepsilon\frac{\partial E}{\partial t}S = \frac{10^{-9}}{36\pi}25\times10^3\cos10^3 t(2\times5) = 2.2105\times10^{-8}\cos10^3 t$$

$$\underline{\underline{I\ = 22.1\cos10^3 t\ \text{nA}}}$$

**Prob. 9.23**

(a) $\nabla\bullet\boldsymbol{E}_s = \rho_s/\varepsilon, \nabla\bullet\boldsymbol{H}_s = 0$

$$\underline{\underline{\nabla\times\boldsymbol{E}_s = j\omega\mu\boldsymbol{H}_s, \quad \nabla\times\boldsymbol{H}_s = (\sigma - j\omega\varepsilon)\boldsymbol{E}_s}}$$

(b) $$\nabla \bullet \boldsymbol{D} = \rho_v \rightarrow \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \rho_v \quad (1)$$

$$\nabla \bullet \boldsymbol{B} = 0 \rightarrow \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0 \quad (2)$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \rightarrow \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{\partial B_x}{\partial t} \quad (3)$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{\partial B_y}{\partial t} \quad (4)$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\frac{\partial B_z}{\partial t} \quad (5)$$

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \rightarrow \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = J_x + \frac{\partial D_x}{\partial t} \quad (6)$$

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = J_y + \frac{\partial D_y}{\partial t} \quad (7)$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = J_z + \frac{\partial D_z}{\partial t} \quad (8)$$

**Prob. 9.24**

If $\boldsymbol{J} = 0 = \rho_v$, then

$$\nabla \bullet \boldsymbol{B} = 0 \quad (1)$$

$$\nabla \bullet \boldsymbol{D} = \rho_v \quad (2)$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \quad (3)$$

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \quad (4)$$

Since $\nabla \bullet \nabla \times \boldsymbol{A} = 0$ for any vector field $\boldsymbol{A}$,

$$\nabla \bullet \nabla \times \boldsymbol{E} = -\frac{\partial}{\partial t} \nabla \bullet \boldsymbol{B} = 0$$

$$\nabla \bullet \nabla \times \boldsymbol{H} = -\frac{\partial}{\partial t} \nabla \bullet \boldsymbol{D} = 0$$

showing that (1) and (2) are incorporated in (3) and (4). Thus Maxwell's equations can be reduced to (3) and (4), i.e.

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \quad \nabla \times \boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t}$$

**Prob. 9.25**

$$\nabla \cdot \boldsymbol{E} = 0 \quad \longrightarrow \quad (1)$$

$$\nabla \cdot \boldsymbol{H} = 0 \quad \longrightarrow \quad (2)$$

$$\nabla \times \boldsymbol{E} = -\mu \frac{\partial \boldsymbol{H}}{\partial t} \quad \longrightarrow \quad (3)$$

$$\nabla \times \boldsymbol{E} = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & E_y(x,t) & 0 \end{bmatrix}$$

$$= \frac{\partial E_y}{\partial x} \boldsymbol{a}_z = -E_o \sin x \cos t \boldsymbol{a}_z$$

$$H = -\frac{1}{\mu} \int \nabla \times \boldsymbol{E} dt = \frac{E_o}{\mu_o} \sin \times \sin t \boldsymbol{a}_z$$

$$\nabla \times \boldsymbol{H} = \varepsilon \frac{\partial \boldsymbol{E}}{\partial t} \quad \longrightarrow \quad (4)$$

$$\nabla \times \boldsymbol{H} = \begin{bmatrix} \frac{\partial}{\partial \boldsymbol{x}} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & H_z(x,t) \end{bmatrix}$$

$$= -\frac{\partial H_z}{\partial \boldsymbol{x}} \boldsymbol{a}_y = -\frac{E_o}{\mu_o} \cos x \sin t \boldsymbol{a}_y$$

$$\boldsymbol{E} = \frac{\boldsymbol{1}}{\varepsilon} \int \nabla \times \boldsymbol{H} dt = \frac{E_o}{\mu_o \varepsilon} \cos x \cos t \boldsymbol{a}_y$$

which is off the given $\boldsymbol{E}$ by a factor. Thus, Maxwell's equations (1) to (3) are satisfied, but (4) is not. The only way (4) is satisfied is for $\mu_o \varepsilon = 1$ which is not true.

**Prob. 9.26**

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\nabla \times \nabla \times \boldsymbol{E} = -\frac{\partial}{\partial t} \nabla \times \boldsymbol{B} = -\mu \frac{\partial}{\partial t} \nabla \times \boldsymbol{H} = -\mu \frac{\partial \boldsymbol{J}}{\partial t} - \mu\varepsilon \frac{\partial^2 \boldsymbol{E}}{\partial t^2}$$

But

$$\nabla \times \nabla \times \boldsymbol{E} = \nabla(\nabla \bullet \boldsymbol{E}) - \nabla^2 \boldsymbol{E}$$

$$\nabla(\nabla \bullet \boldsymbol{E}) - \nabla^2 \boldsymbol{E} = -\mu \frac{\partial \boldsymbol{J}}{\partial t} - \mu\varepsilon \frac{\partial^2 \boldsymbol{E}}{\partial t^2}, \qquad \boldsymbol{J} = \sigma \boldsymbol{E}$$

In a source-free region, $\nabla \bullet \boldsymbol{E} = \rho_v / \varepsilon = 0$. Thus,

$$\underline{\underline{\nabla^2 E = \mu\sigma \frac{\partial E}{\partial t} + \mu\varepsilon \frac{\partial^2 E}{\partial t^2}}}$$

**Prob. 9.27**

$$\nabla \bullet J = (0 + 0 + 3z^2)\sin 10^4 t = -\frac{\partial \rho_v}{\partial t}$$

$$\rho_v = -\int \nabla \bullet J dt = -\int 3z^2 \sin 10^4 t dt = \frac{3z^2}{10^4}\cos 10^4 t + C_o$$

If $\rho_v |_{z=0} = 0$, then $C_o = 0$ and

$$\underline{\underline{\rho_v = 0.3z^2 \cos 10^4 t \text{ mC/m}^3}}$$

**Prob. 9.28**

$$\nabla \bullet \boldsymbol{D} = \varepsilon \nabla \bullet \boldsymbol{E} = \rho_v$$

$$\rho_v = \varepsilon \nabla \bullet \boldsymbol{E} = \varepsilon_o \frac{\partial E_z}{\partial z} = \underline{\underline{-\varepsilon_o E_o \sin z \cos t}}$$

**Prob. 9.29**

Let $\omega = 10^8$

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} = \boldsymbol{0} + \varepsilon \frac{\partial \boldsymbol{E}}{\partial t} \quad \rightarrow \quad \boldsymbol{E} = \frac{1}{\varepsilon}\int \nabla \times \boldsymbol{H} dt$$

$$\nabla \times \boldsymbol{H} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 40\sin(\omega t + \beta x) & 0 \end{vmatrix} = 40\beta \cos(\omega t + \beta x)\boldsymbol{a}_z$$

$$\boldsymbol{E} = \frac{1}{\varepsilon}\int \nabla \times \boldsymbol{H} dt = \underline{\underline{\frac{40\beta}{\omega \varepsilon_o}\sin(\omega t + \beta x)\boldsymbol{a}_z}}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} = -\mu \frac{\partial \boldsymbol{H}}{\partial t}$$

$$\nabla \times \boldsymbol{E} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & \frac{40\beta}{\omega\varepsilon_o}\sin(\omega t + \beta x) \end{vmatrix} = -\frac{40\beta^2}{\omega\varepsilon_o}\cos(\omega t + \beta x)\boldsymbol{a}_y \qquad (1)$$

$$-\mu_o \frac{\partial \boldsymbol{H}}{\partial t} = -40\mu_o\omega\cos(\omega t + \beta x)\boldsymbol{a}_y \qquad (2)$$

Equating (1) and (2) gives

$$40\mu_o\omega = \frac{40\beta^2}{\omega\varepsilon_o} \quad \rightarrow \quad \beta^2 = \mu_o\varepsilon_o\omega^2$$

$$\beta = \omega\sqrt{\varepsilon_o\mu_o} = 10^8\sqrt{4\pi\times10^{-7}\times\frac{10^{-9}}{36\pi}} = \frac{1}{3} = \underline{\underline{0.333\text{ rad/m}}}$$

**Prob. 9.30**

$$\boldsymbol{J}_d = \frac{\partial \boldsymbol{D}}{\partial t} = \varepsilon_o\frac{\partial \boldsymbol{E}}{\partial t} = \frac{50\varepsilon_o}{\rho}(-10^8)\sin(10^8 t - kz)\boldsymbol{a}_\rho = \underline{\underline{-\frac{4.421\times10^{-2}}{\rho}\sin(10^8 t - kz)\boldsymbol{a}_\rho \text{ A/m}^2}}$$

$$\nabla \times \boldsymbol{E} = -\mu_o\frac{\partial \boldsymbol{H}}{\partial t}$$

$$\nabla \times \boldsymbol{E} = \frac{\partial E_\rho}{\partial z}\boldsymbol{a}_\phi = \frac{50k}{\rho}\sin(10^8 t - kz)\boldsymbol{a}_\phi$$

$$\boldsymbol{H} = -\frac{1}{\mu_o}\int \nabla\times\boldsymbol{E}\,dt = \frac{1}{4\pi\times10^{-7}}\frac{50k}{10^8\rho}\cos(10^8 t - kz)\boldsymbol{a}_\phi$$

$$\boldsymbol{H} = \underline{\underline{\frac{2.5k}{2\pi\rho}\cos(10^8 t - kz)\boldsymbol{a}_\phi}} \text{ A/m}$$

$$\nabla \times \boldsymbol{H} = -\frac{\partial H_\phi}{\partial z}\boldsymbol{a}_\rho = -\frac{2.5k^2}{2\pi\rho}\sin(10^8 t - kz)\boldsymbol{a}_\rho$$

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}_d \quad \longrightarrow \quad -\frac{4.421x10^{-2}}{\rho}\sin(10^8 t - kz)\boldsymbol{a}_\rho = \frac{-2.5k^2}{2\pi\rho}\sin(10^8 t - kz)\boldsymbol{a}_\rho$$

$$k^2 = \frac{2\pi}{2.5}\times4.421\times10^{-2} \quad \longrightarrow \quad \underline{\underline{k = 0.333}}$$

**Prob. 9.31**

$$\nabla \times \boldsymbol{E} = -\mu_o \frac{\partial \boldsymbol{H}}{\partial t} \quad \longrightarrow \quad \boldsymbol{H} = -\frac{1}{\mu_o}\int \nabla \times \boldsymbol{E} dt$$

$$\nabla \times \boldsymbol{E} = \frac{1}{r}\frac{\partial}{\partial r}(rE_\theta)\boldsymbol{a}_\phi = \frac{1}{r}\frac{\partial}{\partial r}\left[10\sin\theta\cos(\omega t - \beta r)\right]\boldsymbol{a}_\phi$$

$$= \frac{10\beta}{r}\sin\theta\cos(\omega t - \beta r)\boldsymbol{a}_\phi$$

$$\boldsymbol{H} = -\frac{10\beta}{\mu r}\sin\theta\int \sin(\omega t - \beta r)dt\boldsymbol{a}_\phi$$

$$= \frac{10\beta}{\omega\mu_o r}\sin\theta\cos(\omega t - \beta r)\boldsymbol{a}_\phi$$

**Prob. 9.32**

(a) $\boldsymbol{J}_d = \frac{\partial \boldsymbol{D}}{dt} \quad \longrightarrow \quad \boldsymbol{D} = \int \boldsymbol{J}_d \, dt$

$$\boldsymbol{D} = \frac{-60\times 10^{-3}}{109}\cos(10^9 t - \beta z)\boldsymbol{a}_x = -60\times 10^{-12}\cos(10^9 t - \beta z)\boldsymbol{a}_x \text{ C/m}^2$$

$$\nabla \times \boldsymbol{E} = \mu\frac{\partial \boldsymbol{H}}{\partial t} \quad \longrightarrow \quad \nabla \times \frac{\boldsymbol{D}}{\varepsilon} = -\mu\frac{\partial \boldsymbol{H}}{\partial t}$$

$$\nabla \times \frac{\boldsymbol{D}}{\varepsilon} = \frac{1}{\varepsilon}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ & & \\ D_x & 0 & 0 \end{vmatrix} = \frac{1}{\varepsilon}(-60)(-1)\times 10^{-12}\sin(10^9 t - \beta z)\boldsymbol{a}_x$$

$$= \frac{60\beta}{\varepsilon}\times 10^{-12}\sin(10^9 t - \beta z)\boldsymbol{a}_y$$

$$\boldsymbol{H} = -\frac{1}{\mu}\int \nabla \times \frac{\boldsymbol{D}}{\varepsilon}dt = -\frac{1}{\mu}(-1)\frac{60\beta}{\varepsilon}\times\frac{10^{-12}}{10^9}\cos(10^9 t - \beta z)\boldsymbol{a}_y$$

$$= \frac{60\beta}{\mu\varepsilon}\times 10^{-21}\cos(10^9 t - \beta z)\boldsymbol{a}_y \text{ A/m}$$

(b) $\nabla \times \boldsymbol{H} = \boldsymbol{J} + \boldsymbol{J}_\phi = 0 + \boldsymbol{J_d}$

$$\boldsymbol{J}_d = \nabla \times \boldsymbol{H} = \begin{vmatrix} \frac{\partial}{\partial_x} & \frac{\partial}{\partial_y} & \frac{\partial}{\partial_z} \\ \\ 0 & H_y & 0 \end{vmatrix} = \frac{(-\beta)(-1)60\beta}{\mu\varepsilon} \times (10^{-21}) \sin(10^9 t - \beta z)\boldsymbol{a}_x$$

Equating this with the given $\boldsymbol{J_d}$

$$60 \times 10^{-3} = \frac{60\beta^2}{\mu\varepsilon} \times 10^{-21}$$

$$\beta^2 = \mu\varepsilon \cdot 10^{18} = 2 \times 4\pi \times 10^{-7} \times 10 \times \frac{10^{-9}}{36\pi} = \frac{2000}{9}$$

$\beta = \underline{\underline{14.907 \text{ rad/m}}}$

**Prob. 9.33**

$$\nabla \bullet \boldsymbol{D} = \rho_v, \quad \nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}$$

$$\nabla \bullet \nabla \times \boldsymbol{H} = \nabla \bullet \boldsymbol{J} + \frac{\partial \nabla \bullet \boldsymbol{D}}{\partial t} = 0 \quad \rightarrow \quad \nabla \bullet \boldsymbol{J} + \frac{\partial \rho_v}{\partial t} = 0$$

Or $\underline{\underline{\nabla \bullet \boldsymbol{J} = -\frac{\partial \rho_v}{\partial t}}}$

**Prob. 9.34**

From Maxwell's equations,

$$\nabla \bullet \boldsymbol{D} = \rho_v = 0, \quad \nabla \bullet \boldsymbol{B} = 0, \quad \nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \quad \nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}$$

$$\nabla \times \nabla \times \boldsymbol{E} = -\mu \frac{\partial}{\partial t} \nabla \times \boldsymbol{H} \quad \rightarrow \quad \nabla(\nabla \bullet \boldsymbol{E}) - \nabla^2 \boldsymbol{E} = -\mu \frac{\partial}{\partial t}\left(\boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}\right)$$

But $\nabla \bullet \boldsymbol{E} = 0, \quad \boldsymbol{J} = \sigma \boldsymbol{E}, \quad \boldsymbol{D} = \varepsilon \boldsymbol{E}$

$$-\nabla^2 \boldsymbol{E} = -\mu \frac{\partial}{\partial t}\left(\sigma \boldsymbol{E} + \frac{\partial \varepsilon \boldsymbol{E}}{\partial t}\right) = -\mu\sigma \frac{\partial \boldsymbol{E}}{\partial t} - \mu\varepsilon \frac{\partial^2 \boldsymbol{E}}{\partial t^2}$$

$$\underline{\underline{\nabla^2 \boldsymbol{E} - \mu\sigma \frac{\partial \boldsymbol{E}}{\partial t} - \mu\varepsilon \frac{\partial^2 \boldsymbol{E}}{\partial t^2} = 0}}$$

**Prob. 9.35**

(a) $\nabla \bullet \boldsymbol{A} = 0$

$$\nabla \times \boldsymbol{A} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & E_z(x,t) \end{vmatrix} = -\frac{\partial E_z(x,t)}{\partial x}\boldsymbol{a}_y \neq 0$$

Yes, $A$ is a possible EM field.

(b) $\nabla \bullet B = 0$

$$\nabla \times \boldsymbol{B} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left[10\cos(\omega t - 2\rho)\right]\boldsymbol{a}_z \neq 0$$

Yes, $B$ is a possible EM field.

(c) $$\nabla \bullet \boldsymbol{C} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(3\rho^3 \cot\phi \sin\omega t\right) - \frac{\sin\phi \sin\omega t}{\rho^2} \neq 0$$

$$\nabla \times \boldsymbol{C} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\cos\phi \sin\omega t\right)\boldsymbol{a}_z - 3\rho^2 \frac{\partial}{\partial \phi}(\cot\phi \sin\omega t)\boldsymbol{a}_z \neq 0$$

No, $C$ cannot be an EM field.

(d) $$\nabla \bullet \boldsymbol{D} = \frac{1}{r^2 \sin\theta}\sin(\omega t - 5r)\frac{\partial}{\partial \theta}(\sin^2\theta) \neq 0$$

$$\nabla \times \boldsymbol{D} = -\frac{\partial D_\theta}{\partial \phi}\boldsymbol{a}_r + \frac{1}{r}\frac{\partial}{\partial r}(rD_\theta)\boldsymbol{a}_\phi = \frac{1}{r}\sin\theta(-5)\sin(\omega t - 5r)\boldsymbol{a}_\phi \neq 0$$

No, $\boldsymbol{D}$ cannot be an EM field.

**Prob. 9.36**

From Maxwell's equations,

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \qquad (1)$$

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \qquad (2)$$

Dotting both sides of (2) with $\vec{E}$ gives:

$$\boldsymbol{E} \bullet (\nabla \times \boldsymbol{H}) = \boldsymbol{E} \bullet \boldsymbol{J} + \boldsymbol{E} \bullet \frac{\partial \boldsymbol{D}}{\partial t} \qquad (3)$$

But for any arbitrary vectors $\vec{A}$ and $\vec{B}$,

$$\nabla \bullet (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{B} \bullet (\nabla \times \boldsymbol{A}) - \boldsymbol{A} \bullet (\nabla \times \boldsymbol{B})$$

Applying this on the left-hand side of (3) by letting $\boldsymbol{A} \equiv \boldsymbol{H}$ and $\boldsymbol{B} \equiv \boldsymbol{E}$, we get

$$\boldsymbol{H} \bullet (\nabla \times \boldsymbol{E}) + \nabla \bullet (\boldsymbol{H} \times \boldsymbol{E}) = \boldsymbol{E} \bullet \boldsymbol{J} + \tfrac{1}{2} \frac{\partial}{\partial t} (\boldsymbol{D} \bullet \boldsymbol{E}) \qquad (4)$$

From (1),

$$\boldsymbol{H} \bullet (\nabla \times \boldsymbol{E}) = \boldsymbol{H} \bullet \left( -\frac{\partial \boldsymbol{B}}{\partial t} \right) = \tfrac{1}{2} \frac{\partial}{\partial t} (\boldsymbol{B} \bullet \boldsymbol{H})$$

Substituting this in (4) gives:

$$-\tfrac{1}{2} \frac{\partial}{\partial t} (\boldsymbol{B} \bullet \boldsymbol{H}) - \nabla \bullet (\boldsymbol{E} \times \boldsymbol{H}) = \boldsymbol{J} \bullet \boldsymbol{E} + \tfrac{1}{2} \frac{\partial}{\partial t} (\boldsymbol{D} \bullet \boldsymbol{E})$$

Rearranging terms and then taking the volume integral of both sides:

$$\int_v \nabla \bullet (\boldsymbol{E} \times \boldsymbol{H}) dv = -\frac{\partial}{\partial t} \tfrac{1}{2} \int_v (\boldsymbol{E} \bullet \boldsymbol{D} + \boldsymbol{H} \bullet \boldsymbol{B}) dv - \int_v \boldsymbol{J} \bullet \boldsymbol{E} dv$$

Using the divergence theorem, we get

$$\oint_s (\boldsymbol{E} \times \boldsymbol{H}) \bullet d\boldsymbol{S} = -\frac{\partial W}{\partial t} - \int_v \boldsymbol{J} \bullet \boldsymbol{E} dv$$

or $\dfrac{\partial W}{\partial t} = -\oint_s (\boldsymbol{E} \times \boldsymbol{H}) \bullet d\boldsymbol{S} - \int_v \boldsymbol{E} \bullet \boldsymbol{J} dv$ as required.

**Prob. 9.37**

$$\nabla \times \boldsymbol{E} = -\mu_o \frac{\partial \boldsymbol{H}}{\partial t} \quad \rightarrow \quad \boldsymbol{H} = -\frac{1}{\mu_o}\int \nabla \times \boldsymbol{E} dt$$

$$\nabla \times \boldsymbol{E} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & E_z(x,y) \end{vmatrix} = \frac{\partial E_z}{\partial y}\boldsymbol{a}_x - \frac{\partial E_z}{\partial x}\boldsymbol{a}_y$$

$$= -\alpha \cos(12\pi x)\cos(10^{11}t - \alpha y)\boldsymbol{a}_x + 12\pi \sin(12\pi x)\sin(10^{11}t - \alpha y)\boldsymbol{a}_y$$

$$\boldsymbol{H} = -\frac{1}{\mu_o}\int \nabla \times \boldsymbol{E} dt$$

$$= -\frac{1}{\mu_o}\left[-\alpha \cos(12\pi x)\int \cos(10^{11}t - \alpha y)dt\boldsymbol{a}_x + 12\pi \sin(12\pi x)\int \sin(10^{11}t - \alpha y)dt\boldsymbol{a}_y\right]$$

$$= \frac{\alpha}{\mu_o 10^{11}}\cos(12\pi x)\sin(10^{11}t - \alpha y)\boldsymbol{a}_x + \frac{12\pi}{\mu_o 10^{10}}\sin(12\pi x)\cos(10^{11}t - \alpha y)\boldsymbol{a}_y$$

But $\nabla \times \boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t} = \varepsilon_o \frac{\partial \boldsymbol{E}}{\partial t}$

$$\varepsilon_o \frac{\partial \boldsymbol{E}}{\partial t} = \varepsilon_o (10^{11})\cos(12\pi x)\cos(10^{11}t - \alpha t)\boldsymbol{a}_z \qquad (1)$$

$$\nabla \times \boldsymbol{H} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x(x,y) & H_y(x,y) & 0 \end{vmatrix} = \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\boldsymbol{a}_z$$

$$= \left[\frac{(12\pi)^2}{\mu_o 10^{11}}\cos(12\pi x)\cos(10^{11}t - \alpha y) + \frac{\alpha^2}{\mu_o 10^{11}}\cos(12\pi x)\cos(10^{11}t - \alpha y)\right]\boldsymbol{a}_z \qquad (2)$$

Equating (1) and (2),

$$\varepsilon_o (10^{11}) = \frac{(12\pi)^2}{\mu_o 10^{11}} + \frac{\alpha^2}{\mu_o 10^{11}} \quad \rightarrow \quad \mu_o \varepsilon_o (10^{22}) = (12\pi)^2 + \alpha^2$$

$$4\pi \times 10^{-7} \times \frac{10^{-9}}{36\pi}(10^{22}) = \frac{10^6}{9} = 144\pi^2 + \alpha^2 \quad \rightarrow \quad \underline{\underline{\alpha = 331.2}}$$

$$\frac{\alpha}{\mu_o 10^{11}} = \frac{331.2 \times 10^{-11}}{4\pi \times 10^{-7}} = 2.636 \times 10^{-3}, \qquad \frac{12\pi}{\mu_o 10^{11}} = \frac{12\pi \times 10^{-11}}{4\pi \times 10^{-7}} = 3 \times 10^{-3}$$

Thus,

$$\underline{\underline{\boldsymbol{H} = 2.636\cos(12\pi x)\sin(10^{11}t - \alpha y)\boldsymbol{a}_x + 3\sin(12\pi x)\cos(10^{11}t - \alpha y)\boldsymbol{a}_y \text{ mA/m}}}$$

**Prob. 9.38**

$$\nabla \times \boldsymbol{E} = -\mu_o \frac{\partial \boldsymbol{H}}{\partial t} \quad \rightarrow \quad \boldsymbol{H} = -\frac{1}{\mu_o}\int \nabla \times \boldsymbol{E} dt$$

$$\nabla \times \boldsymbol{E} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x(z) & 0 & 0 \end{vmatrix} = \frac{\partial E_x}{\partial z}\boldsymbol{a}_y = -\beta E_o \cos(1200\pi t - \beta z)\boldsymbol{a}_y \qquad (1)$$

$$-\mu_o \frac{\partial \boldsymbol{H}}{\partial t} = -\frac{\mu_o E_o}{\eta}(1200\pi)\cos(1200\pi t - \beta z)\boldsymbol{a}_y \qquad (2)$$

Setting (1) and (2) equal,

$$\beta E_o = \frac{\mu_o E_o}{\eta}(1200\pi) \quad \rightarrow \quad \beta = \frac{1200\pi\mu_o}{\eta} \qquad (3)$$

But $\nabla \times \boldsymbol{H} = \varepsilon_o \frac{\partial \boldsymbol{E}}{\partial t}$

$$\nabla \times \boldsymbol{H} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & H_y(z) & 0 \end{vmatrix} = -\frac{\partial H_y}{\partial z}\boldsymbol{a}_x = \frac{\beta E_o}{\eta}\cos(1200\pi t - \beta z)\boldsymbol{a}_x \qquad (4)$$

$$\varepsilon_o \frac{\partial \boldsymbol{E}}{\partial t} = \varepsilon_o (1200\pi) E_o \cos(1200\pi t - \beta z)\boldsymbol{a}_x \qquad (5)$$

Setting (4) and (5) equal,

$$\beta = \varepsilon_o (1200\pi)\eta \qquad (6)$$

From (3) and (6),

$$\frac{1200\pi\mu_o}{\eta} = \varepsilon_o(1200\pi)\eta \quad \rightarrow \quad \eta^2 = \frac{\mu_o}{\varepsilon_o}$$

$$\eta = \sqrt{\frac{\mu_o}{\varepsilon_o}} = \sqrt{\frac{4\pi \times 10^{-7}}{\frac{10^{-9}}{36\pi}}} = 120\pi = \underline{\underline{377}}$$

From (3),

$$\beta = \frac{1200\pi\mu_o}{\eta} = \frac{1200\pi \times 4\pi \times 10^{-7}}{120\pi} = 40\pi \times 10^{-7} = \underline{\underline{1.257 \times 10^{-5} \text{ rad/m}}}$$

**Prob. 9.39** Using Maxwell's equations,

$$\nabla \times \boldsymbol{H} = \sigma \boldsymbol{E} + \varepsilon \frac{\partial \boldsymbol{E}}{\partial t} \qquad (\sigma = 0) \qquad \longrightarrow \qquad \boldsymbol{E} = \frac{1}{\varepsilon}\int \nabla \times \boldsymbol{H} dt$$

But

$$\nabla \times \boldsymbol{H} = -\frac{1}{r\sin\theta}\frac{\partial H_\theta}{\partial \phi}\boldsymbol{a}_r + \frac{1}{r}\frac{\partial}{\partial r}(rH_\theta)\boldsymbol{a}_\phi = \frac{12\sin\theta}{r}\beta\sin(2\pi\times10^8 t - \beta r)\boldsymbol{a}_\phi$$

$$\boldsymbol{E} = \frac{12\sin\theta}{\varepsilon_o}\beta\int \sin(2\pi\times10^8 t - \beta r)dt\boldsymbol{a}_\phi$$

$$= -\frac{12\sin\theta}{\omega\varepsilon_o r}\beta\cos(\omega t - \beta r)\boldsymbol{a}_\phi, \qquad \omega = 2\pi\times10^8$$

**Prob. 9.40**

With the given **A**, we need to prove that

$$\nabla^2\boldsymbol{A} = \mu\varepsilon\frac{\partial^2\boldsymbol{A}}{\partial t^2}$$

$$\nabla^2\boldsymbol{A} = \mu\varepsilon(j\omega)(j\omega)\boldsymbol{A} = -\omega^2\mu\varepsilon\boldsymbol{A}$$

Let $\beta^2 = \omega^2\mu\varepsilon$, then $\nabla^2\boldsymbol{A} = -\beta^2\boldsymbol{A}$ is to be proved. We recognize that

$$\boldsymbol{A} = \frac{\mu_o}{4\pi r}e^{j\omega t}e^{-j\beta r}\boldsymbol{a}_z$$

Assume $\varphi = \dfrac{e^{-j\beta r}}{r}$, $\quad \boldsymbol{A} = \dfrac{\mu_o}{4\pi}e^{j\omega t}\varphi\boldsymbol{a}_z$

$$\nabla^2\varphi = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2\sin\theta\frac{\partial\varphi}{dr})\right] = \frac{1}{r^2}\left[\frac{\partial}{\partial r}(r^2)\left(\frac{-j\beta}{r} - \frac{1}{r^2}\right)e^{-j\beta r}\right]$$

$$= \frac{1}{r^2}\left(-\beta^2 r + j\beta - j\beta\right)e^{-j\beta r} = -\beta^2\frac{e^{-j\beta r}}{r} = -\beta^2\varphi$$

Therefore, $\quad \nabla^2\boldsymbol{A} = -\beta^2\boldsymbol{A}$

We can find V using Lorentz gauge.

$$V = \frac{-1}{\mu_o\varepsilon_o}\int\nabla\bullet\boldsymbol{A}dt = \frac{-1}{j\omega\mu_o\varepsilon_o}\nabla\bullet\boldsymbol{A}$$

$$= \frac{-1}{j\omega\mu_o\varepsilon_o}\frac{\partial}{\partial r}\left(\frac{\mu_o}{4\pi r}e^{-j\beta r}e^{j\omega t}\right) = \frac{-1}{j\omega\varepsilon_o(4\pi)}\left(\frac{-j\beta}{r} - \frac{1}{r^2}\right)e^{-j\beta r}e^{j\omega t}\cos\theta$$

$$V = \frac{\cos\theta}{j4\pi\omega\varepsilon_o r}\left(j\beta + \frac{1}{r}\right)e^{j(\omega t - \beta r)}$$

**Prob. 9.41**

Take the curl of both sides of the equation.

$$\nabla \times \boldsymbol{E} = -\nabla \times \nabla V - \frac{\partial}{\partial t}\nabla \times \boldsymbol{A}$$

But $\nabla \times \nabla V = \mathbf{0}$ and $\boldsymbol{B} = \nabla \times \boldsymbol{A}.$ Hence,

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

which is Faraday's law.

**Prob. 9.42**

(a)
$$\nabla \cdot \boldsymbol{A} = \frac{\partial A_z}{\partial z} = \frac{x}{c}, \qquad \frac{\partial V}{\partial t} = -xc, \qquad -\mu_o \varepsilon_o \frac{\partial V}{\partial t} = \frac{x}{c^2} c = \frac{x}{c}$$

Hence, $\nabla \cdot \boldsymbol{A} = -\mu_o \varepsilon_o \dfrac{\partial V}{\partial t}$

(b)
$$\boldsymbol{E} = -\nabla V - \frac{\partial A}{\partial t} = -\left(\frac{\partial V}{\partial x}\boldsymbol{a}_x + \frac{\partial V}{\partial z}\boldsymbol{a}_z\right) + x\boldsymbol{a}_z = -(z\boldsymbol{a}_x + x\boldsymbol{a}_z) + x\boldsymbol{a}_z$$

$$\underline{\underline{\boldsymbol{E} = -z\boldsymbol{a}_x}}$$

**Prob. 9.43**

$$\nabla \cdot \boldsymbol{A} = 0 = -\mu\varepsilon \frac{\partial V}{\partial t} \longrightarrow \underline{\underline{V = \text{constant}}}$$

(a) $\boldsymbol{E} = -\nabla V - \dfrac{\partial \boldsymbol{A}}{\partial t} = \mathbf{0} - A_o \omega \cos(\omega t - \beta z)\boldsymbol{a}_x$

$$= \underline{\underline{-A_o \omega \cos(\omega t - \beta z)\boldsymbol{a}_x}}$$

(b) Using Maxwell’s equations, we can show that

$$\underline{\underline{\beta = \omega\sqrt{\mu_o \varepsilon_o}}}$$

**Prob. 9.44**

(a)

$$z = 4\angle 30^o - 10\angle 50^o = 3.464 + 2j - 6.427 - j7.66 = -2.963 - j5.66$$

$$= 6.389\angle -117.64^o$$

$$z^{1/2} = \underline{\underline{2.5277\angle -58.82^o}}$$

(b)

$$\frac{1 + j2}{6 - j8 - 7\angle 15^o} = \frac{2.236\angle 63.43^o}{6 - j8 - 7.761 - j1.812} = \frac{2.236\angle 63.43^o}{9.841\angle 265.57^o}$$

$$= \underline{\underline{0.2272\angle -202.1^o}}$$

(c) $z = \dfrac{(5\angle 53.13^o)^2}{12 - j7 - 6 - j10} = \dfrac{25\angle 106.26^o}{18.028\angle -70.56^o}$

$= \underline{\underline{1.387\angle 176.8^o}}$

(d)

$$\frac{1.897\angle -100^o}{(5.76\angle 90^o)(9.434\angle -122^o)} = \underline{\underline{0.0349\angle -68^o}}$$

**Prob. 9.45**

$(a)\quad \boldsymbol{H} = \text{Re}\left[\boldsymbol{H}_s e^{j\omega t}\right], \qquad \omega = 10^6$

$\boldsymbol{H} = \text{Re}\left[-10e^{j(\omega t+\pi/3)}\boldsymbol{a}_x\right] \quad \rightarrow \quad \underline{\underline{\boldsymbol{H}_s = -10e^{j\pi/3}\boldsymbol{a}_x}}$

$(b)\underline{\underline{\boldsymbol{E}_s = 4\cos(4y)e^{-j2x}\boldsymbol{a}_z}}$

$(c)\,\sin\text{A} = \cos(\text{A}-90^o)$

$\boldsymbol{D} = 5\cos(\omega t - \pi/2 + \pi/3)\boldsymbol{a}_x - 8\cos(\omega t - \pi/4)\boldsymbol{a}_y$

$\underline{\underline{\boldsymbol{D}_s = 5e^{-j\pi/6}\boldsymbol{a}_x - 8e^{-j\pi/4}\boldsymbol{a}_y}}$

**Prob. 9.46**

$(a)\qquad \boldsymbol{A}_s = 10e^{j\pi/2}\boldsymbol{a}_x + 20e^{-j\pi/2}\boldsymbol{a}_y$

$\boldsymbol{A} = \text{Re}[\boldsymbol{A}_s e^{j\omega t}] = \text{Re}\left[10e^{j(\omega t+\pi/2)}\boldsymbol{a}_x + 20e^{j(\omega t-\pi/2)}\boldsymbol{a}_y\right]$

$\quad = 10\cos(\omega t + \pi/2)\boldsymbol{a}_x + 20\cos(\omega t - \pi/2)\boldsymbol{a}_y$

$\underline{\underline{\boldsymbol{A} = -10\sin\omega t\boldsymbol{a}_x + 20\sin\omega t\boldsymbol{a}_y}}$

(b)

$\boldsymbol{B} = \text{Re}[\boldsymbol{B}_s e^{j\omega t}] = \text{Re}\left[4e^{j(\omega t-2x+\pi/2)}\boldsymbol{a}_x + 6e^{j(\omega t+2x)}\boldsymbol{a}_y\right] = 4\cos(\omega t - 2x + \pi/2)\boldsymbol{a}_x + 6\cos(\omega t + 2x)\boldsymbol{a}_y$

$\underline{\underline{= -4\sin(\omega t - 2x)\boldsymbol{a}_x + 6\sin(\omega t + 2x)\boldsymbol{a}_y}}$

$(c)\mathbf{C}_s = 2e^{j\pi/2}e^{-10z}e^{-j\pi/4}\boldsymbol{a}_z$

$\boldsymbol{C} = \text{Re}[\boldsymbol{C}_s e^{j\omega t}] = 2\,\text{Re}[\text{e}^{j(\omega t+\pi/2-\pi/4)}\,e^{-20z}\boldsymbol{a}_z] = 2\,\text{e}^{-20z}\cos(\omega t - \pi/4 + \pi/2)\boldsymbol{a}_z$

$= \underline{\underline{-2\,\text{e}^{-20z}\sin(\omega t - \pi/4)\boldsymbol{a}_z}}$

**Prob. 9.47**

(a) $$\underline{\underline{\boldsymbol{H}_s = \frac{1}{\rho} e^{-j3z} \boldsymbol{a}_\phi}}$$

(b) $$\nabla \times \boldsymbol{H} = \varepsilon_o \frac{\partial \boldsymbol{E}}{\partial t} \quad \rightarrow \quad \boldsymbol{E} = \frac{1}{\varepsilon_o} \int \nabla \times \boldsymbol{H} dt$$

But $$\nabla \times \boldsymbol{H} = -\frac{\partial H_\phi}{\partial z} \boldsymbol{a}_\rho + \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho H_\phi) \boldsymbol{a}_z = -\frac{3}{\rho} \sin(\omega t - 3z) \boldsymbol{a}_\rho + \mathbf{0}$$

$$\boldsymbol{E} = -\frac{1}{\varepsilon_o} \int \frac{3}{\rho} \sin(\omega t - 3z) \boldsymbol{a}_\rho dt = \underline{\underline{\frac{3}{\rho \varepsilon_o \omega} \cos(\omega t - 3z) \boldsymbol{a}_\rho}} \text{ V/m}$$

(c) $$\nabla \times \boldsymbol{E} = \frac{\partial E_\rho}{\partial z} \boldsymbol{a}_\phi = \frac{9}{\rho \varepsilon_o \omega} \sin(\omega t - 3z) \boldsymbol{a}_\phi$$

$$-\frac{\partial \boldsymbol{B}}{\partial t} = -\mu_o \frac{\partial \boldsymbol{H}}{\partial t} = \frac{\mu_o \omega}{\rho} \sin(\omega t - 3z) \boldsymbol{a}_\phi$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \quad \rightarrow \quad \frac{9}{\rho \varepsilon_o \omega} = \frac{\mu_o \omega}{\rho}$$

$$\omega^2 = \frac{9}{\mu_o \varepsilon_o} \quad \rightarrow \quad \omega = \frac{3}{\sqrt{\mu_o \varepsilon_o}} = 3 \times 3 \times 10^8 = \underline{\underline{9 \times 10^8 \text{ rad/s}}}$$

**Prob. 9.48**

We can use Maxwell's equations or borrow ideas from chapter 10.

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} = \eta_o \sqrt{\frac{1}{\varepsilon_r}} = \frac{120\pi}{9}$$

$$H_o = \frac{E_o}{\eta} = \frac{10 \times 9}{120\pi} = \underline{\underline{0.2387}}$$

$$\beta = \omega \sqrt{\mu \varepsilon} = \frac{\omega}{c} \sqrt{\varepsilon_r} = \frac{2\pi \times 10^9}{3 \times 10^8} \sqrt{81} = 60\pi = \underline{\underline{188.5 \text{ rad/m}}}$$

**Prob. 9.49**

$$\nabla \times \boldsymbol{H}_s = j\omega\varepsilon_o \boldsymbol{E}_s$$

$$\nabla \times \boldsymbol{H}_s = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & 12e^{j\alpha x} \end{vmatrix} = -j12\alpha e^{j\alpha x}\boldsymbol{a}_y$$

$$\boldsymbol{E}_s = \frac{\nabla \times \boldsymbol{H}_s}{j\omega\varepsilon_o} = -\frac{12\alpha}{\omega\varepsilon_o} e^{j\alpha x}\boldsymbol{a}_y$$

But $\quad \nabla \times \boldsymbol{E}_s = -j\omega\mu_o \boldsymbol{H}_s$

$$\nabla \times \boldsymbol{E}_s = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & E_{sy}(x) & 0 \end{vmatrix} = \frac{\partial E_{sy}}{\partial x}\boldsymbol{a}_z = -j\frac{12\alpha^2}{\omega\varepsilon_o} e^{j\alpha x}\boldsymbol{a}_z$$

$$\boldsymbol{H}_s = -\frac{\nabla \times \boldsymbol{E}_s}{j\omega\mu_o} = \frac{12\alpha^2}{\omega^2\mu_o\varepsilon_o} e^{j\alpha x}\boldsymbol{a}_z$$

Equating this with the given **H**,

$$12 = \frac{12\alpha^2}{\omega^2\mu_o\varepsilon_o} \quad \rightarrow \quad \alpha^2 = \omega^2\mu_o\varepsilon_o \quad \rightarrow \quad \alpha^2 = (10^9)^2 \times 4\pi \times 10^{-7} \times \frac{10^{-9}}{36\pi} = (10^9)^2 \times \frac{10^{-16}}{9}$$

$$\alpha = \frac{10}{3} = \underline{\underline{3.333}}$$

$$\boldsymbol{E}_s = -\frac{12\alpha}{\omega\varepsilon_o} e^{j\alpha x}\boldsymbol{a}_y = -\frac{12(10/3)}{10^9 \times \frac{10^{-9}}{36\pi}} e^{j10x/3}\boldsymbol{a}_y = -40(36\pi)e^{j10x/3}\boldsymbol{a}_y = \underline{\underline{-4.533e^{j3.33x}\boldsymbol{a}_y \text{ kV/m}}}$$

**Prob. 9.50**

$$(j\omega)^2 Y + 4j\omega Y + Y = 2\angle 0^o, \qquad \omega = 3$$

$$Y(-\omega^2 + 4j\omega + 1) = 2$$

$$Y = \frac{2}{-\omega^2 + 4j\omega + 1} = \frac{2}{-9 + j12 + 1} = \frac{2}{-8 + j12} = -0.0769 - j0.1154$$

$$= 0.1387\angle -123.7^o$$

$$y(t) = \text{Re}(Ye^{j\omega t}) = \underline{\underline{0.1387\cos(3t - 123.7^o)}}$$

**Prob. 9.51** We begin with Maxwell's equations:

$$\nabla \bullet \boldsymbol{D} = \rho_v / \varepsilon = 0, \qquad \nabla \bullet \boldsymbol{B} = 0$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \qquad \nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}$$

We write these in phasor form and in terms of $E_s$ and $H_s$ only.

$$\nabla \bullet \boldsymbol{E}_s = 0 \qquad (1)$$

$$\nabla \bullet \boldsymbol{H}_s = 0 \qquad (2)$$

$$\nabla \times \boldsymbol{E}_s = -j\omega\mu \boldsymbol{H}_s \qquad (3)$$

$$\nabla \times \boldsymbol{H}_s = (\sigma + j\omega\varepsilon)\boldsymbol{E}_s \qquad (4)$$

Taking the curl of (3),

$$\nabla \times \nabla \times \boldsymbol{E}_s = -j\omega\mu \nabla \times \boldsymbol{H}_s$$

$$\nabla(\nabla \bullet \boldsymbol{E}_s) - \nabla^2 \boldsymbol{E}_s = -j\omega\mu(\sigma + j\omega\varepsilon)\boldsymbol{E}_s$$

$$\nabla^2 \boldsymbol{E}_s + (\omega^2 \mu\varepsilon - j\omega\mu\sigma)\boldsymbol{E}_s = 0 \quad \longrightarrow \quad \underline{\underline{\nabla^2 \boldsymbol{E}_s + \gamma^2 \boldsymbol{E}_s = 0}}$$

Similarly, by taking the curl of (4),

$$\nabla \times \nabla \times \boldsymbol{H}_s = (\sigma + j\omega\varepsilon)\nabla \times \boldsymbol{E}_s$$

$$\nabla(\nabla \bullet \boldsymbol{H}_s) - \nabla^2 \boldsymbol{H}_s = -j\omega\mu(\sigma + j\omega\varepsilon)\boldsymbol{H}_s$$

$$\nabla^2 \boldsymbol{H}_s + (\omega^2 \mu\varepsilon - j\omega\mu\sigma)\boldsymbol{H}_s = 0 \quad \longrightarrow \quad \underline{\underline{\nabla^2 \boldsymbol{H}_s + \gamma^2 \boldsymbol{H}_s = 0}}$$

## CHAPTER 10

**P. E. 10.1** (a)

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{2\times10^8} = \underline{\underline{31.42\ \text{ns}}},$$

$$\lambda = uT = 3\times10^8 \times 31.42\times10^{-9} = \underline{\underline{9.425\ \text{m}}}$$

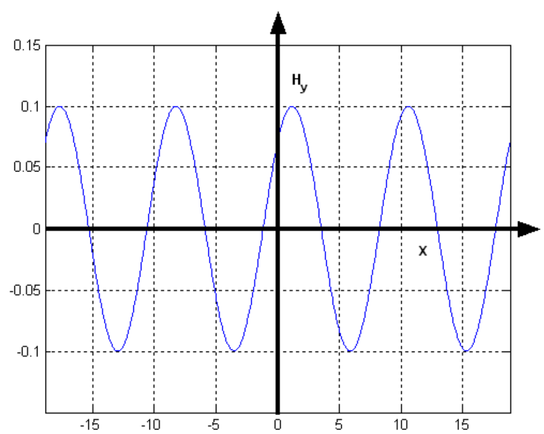
$$k = \beta = 2\pi/\lambda = \underline{\underline{0.6667\ \text{rad/m}}}$$

(b) $t_1 = T/8 = \underline{\underline{3.927\ \text{ns}}}$

(c )

$$\boldsymbol{H}(t=t_1) = 0.1\cos(2\times10^8 \frac{\pi}{8\times10^8} - 2x/3)\boldsymbol{a}_y = 0.1\cos(2x/3 - \pi/4)\boldsymbol{a}_y$$

as sketched below.




**P. E. 10.2** Let $x_o = \sqrt{1 + (\sigma/\omega\varepsilon)^2}$ , then

$$\alpha = \omega\sqrt{\frac{\mu_o\varepsilon_o}{2}\mu_r\varepsilon_r(x_o - 1)} = \frac{\omega}{c}\sqrt{\frac{16}{2}}\sqrt{x_o - 1}$$

or $\sqrt{x_o - 1} = \dfrac{\alpha c}{\omega\sqrt{8}} = \dfrac{1/3\times3\times10^8}{10^8\sqrt{8}} = \dfrac{1}{\sqrt{8}}$ $\longrightarrow$ $x_o = 9/8$

$x_o^{\,2} = \dfrac{81}{64} = 1 + (\sigma/\omega\varepsilon)^2$ $\longrightarrow$ $\dfrac{\sigma}{\omega\varepsilon} = 0.5154$

$\tan 2\theta_\eta = 0.5154 \longrightarrow \theta_\eta = 13.63^o$

$$\frac{\beta}{\alpha} = \sqrt{\frac{x_o + 1}{x_o - 1}} = \sqrt{17}$$

(a) $\beta = \alpha\sqrt{17} = \dfrac{\sqrt{17}}{3} = \underline{\underline{1.374 \text{ rad/m}}}$

(b) $\dfrac{\sigma}{\omega\varepsilon} = \underline{\underline{0.5154}}$

(c ) $|\eta| = \dfrac{\sqrt{\mu/\varepsilon}}{\sqrt{x_o}} = \dfrac{120\pi\sqrt{2/8}}{\sqrt{9/8}} = 177.72$

$\eta = \underline{\underline{177.72\angle 13.63^o\,\Omega}}$

(d) $u = \dfrac{\omega}{\beta} = \dfrac{10^8}{1.374} = \underline{\underline{7.278 \times 10^7}}$ m/s

(e) $\mathbf{a}_H = \mathbf{a}_k \times \mathbf{a}_E \longrightarrow \mathbf{a}_z \times \mathbf{a}_x = \mathbf{a}_H \longrightarrow \mathbf{a}_H = \mathbf{a}_y$

$$\mathbf{H} = \frac{0.5}{177.5} e^{-z/3} \sin(10^8 t - \beta z - 13.63^o)\mathbf{a}_y = \underline{\underline{2.817 e^{-z/3} \sin(10^8 t - \beta z - 13.63^o)\mathbf{a}_y \text{ mA/m}}}$$

**P. E. 10.3** (a) Along $\underline{\underline{\text{-z direction}}}$

(b) $\lambda = \dfrac{2\pi}{\beta} = 2\pi/2 = \underline{\underline{3.142 \text{ m}}}$

$f = \dfrac{\omega}{2\pi} = \dfrac{10^8}{2\pi} = \underline{\underline{15.92 \text{ MHz}}}$

$$\beta = \omega\sqrt{\mu\varepsilon} = \omega\sqrt{\mu_o\varepsilon_o}\sqrt{\mu_r\varepsilon_r} = \frac{\omega}{c}\sqrt{(1)\varepsilon_r}$$

or $\sqrt{\varepsilon_r} = \beta c/\omega = \dfrac{3\times 10^8 \times 2}{10^8} = 6 \longrightarrow \underline{\underline{\varepsilon_r = 36}}$

(c ) $\theta_\eta = 0, |\eta| = \sqrt{\mu/\varepsilon} = \sqrt{\mu_o/\varepsilon_o}\sqrt{1/\varepsilon_r} = \dfrac{120\pi}{6} = 20\pi$

$\mathbf{a}_k = \mathbf{a}_E \times \mathbf{a}_H \longrightarrow -\mathbf{a}_z = \mathbf{a}_y \times \mathbf{a}_H \longrightarrow \mathbf{a}_H = \mathbf{a}_x$

$$\boldsymbol{H} = \frac{50}{20\pi}\sin(\omega t + \beta z)\boldsymbol{a}_x = \underline{\underline{795.8\sin(10^8 t + 2z)\boldsymbol{a}_x}} \quad \text{mA/m}$$

**P. E. 10.4** (a)

$$\frac{\sigma}{\omega\varepsilon} = \frac{10^{-2}}{10^9\pi \times 4 \times \frac{10^{-9}}{36\pi}} = 0.09$$

$$\alpha \cong \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1\right]} = \frac{\omega}{2c}\sqrt{\mu_r\varepsilon_r}\,\frac{\sigma}{\omega\varepsilon} = \frac{10^9\pi}{2\times3\times10^8}(2)(0.09) = 0.9425 \text{ Np/m}$$

$$\beta \cong \omega\sqrt{\frac{\mu\varepsilon}{2}\left[1+\frac{1}{2}\left(\frac{\sigma}{\omega\varepsilon}\right)^2+1\right]} = \frac{10^9\pi}{3\times10^8}\sqrt{2[2+0.5(0.09)^2]} = 20.965 \text{ rad/m}$$

$$\boldsymbol{E} = 30e^{-0.9425y}\cos(10^9\pi t - 20.96y + \pi/4)\boldsymbol{a}_z$$

At t = 2ns, y = 1m,

$$\boldsymbol{E} = 30e^{-0.9425}\cos(2\pi - 20.96 + \pi/4)\boldsymbol{a}_z = \underline{\underline{2.844\boldsymbol{a}_z}} \quad \text{V/m}$$

(b) $\beta y = 10^o = \frac{10\pi}{180}$ rad

or

$$y = \frac{\pi}{18}\frac{1}{\beta} = \frac{\pi}{18\times20.965} = \underline{\underline{8.325 \text{ mm}}}$$

(c ) $30(0.6) = 30\ e^{-\alpha y}$

$$y = \frac{1}{\alpha}\ln(1/0.6) = \frac{1}{0.9425}\ln\frac{1}{0.6} = \underline{\underline{542 \text{ mm}}}$$

(d)

$$|\eta| \cong \frac{\sqrt{\mu/\varepsilon}}{[1+\frac{1}{4}(0.09)^2]} = \frac{60\pi}{1.002} = 188.11\ \Omega$$

$2\theta_\eta = \tan^{-1} 0.09 \longrightarrow \theta_\eta = 2.571^o$

$\boldsymbol{a}_H = \boldsymbol{a}_k \times \boldsymbol{a}_E = \boldsymbol{a}_y \times \boldsymbol{a}_z = \boldsymbol{a}_x$

$$\boldsymbol{H} = \frac{30}{188.11} e^{-0.9425y} \cos(10^9 \pi t - 20.96y + \pi/4 - 2.571^o)\boldsymbol{a}_x$$

At y = 2m, t = 2ns,

$$\boldsymbol{H} = (0.1595)(0.1518)\ \cos(-34.8963rad)\boldsymbol{a}_x = \underline{\underline{-22.83\boldsymbol{a}_x}}\ \text{mA/m}$$

**P. E. 10.5**

$$I_s = \int_0^w \int_0^\infty J_{xs} dydz = J_{xs}(0)\int_0^w dy \int_0^\infty e^{-z(1+j)/\delta} dz = \frac{J_{xs}(0)w\delta}{1+j}$$

$$\underline{\underline{|I_s| = \frac{J_{xs}(0)w\delta}{\sqrt{2}}}}$$

**P. E. 10.6** (a)

$$\frac{R_{ac}}{R_{dc}} = \frac{a}{2\delta} = \frac{a}{2}\sqrt{\pi f \mu\sigma} = \frac{1.3\times10^{-3}}{2}\sqrt{\pi\times10^7\times4\pi\times10^{-7}\times3.5\times10^7} = \underline{\underline{24.16}}$$

(b)

$$\frac{R_{ac}}{R_{dc}} = \frac{1.3\times10^{-3}}{2}\sqrt{\pi\times2\times10^9\times4\pi\times10^{-7}\times3.5\times10^7} = \underline{\underline{341.7}}$$

**P. E. 10.7**

$$E = \text{Re}[\boldsymbol{E}_s e^{j\omega t}] = \text{Re}\left[E_o e^{j\omega t} e^{-j\beta z}\boldsymbol{a}_x + E_o e^{-j\pi/2} e^{j\omega t} e^{-j\beta z}\boldsymbol{a}_y\right]$$
$$= E_o \cos(\omega t - \beta z)\boldsymbol{a}_x + E_o \cos(\omega t - \beta z - \pi/2)\boldsymbol{a}_y$$
$$= E_o \cos(\omega t - \beta z)\boldsymbol{a}_x + E_o \sin(\omega t - \beta z)\boldsymbol{a}_y$$

At z = 0, $E_x = E_o \cos\omega t, E_y = E_o \sin\omega t$

$$\cos^2 \omega t + \sin^2 \omega t = 1 \longrightarrow \left(\frac{E_x}{E_o}\right)^2 + \left(\frac{E_y}{E_o}\right)^2 = 1$$

which describes a circle. Hence the polarization is circular.

**P. E. 10.8**

$$\mathcal{P}_{ave} = \frac{1}{2}\eta H_o^{\,2}\boldsymbol{a}_x$$

(a) Let $f(x,z) = x + y - 1 = 0$

$$\boldsymbol{a}_n = \frac{\nabla f}{|\nabla f|} = \frac{\boldsymbol{a}_x + \boldsymbol{a}_y}{\sqrt{2}}, \qquad d\boldsymbol{S} = dS\boldsymbol{a}_n$$

$$P_t = \int \mathbf{P}.d\mathbf{S} = \mathbf{P}.S\mathbf{a_n} = \frac{1}{2}\eta H_o^{\,2}\boldsymbol{a}_x.\frac{\boldsymbol{a}_x + \boldsymbol{a}_y}{\sqrt{2}}$$

$$= \frac{1}{2\sqrt{2}}(120\pi)(0.2)^2(0.1)^2 = \underline{\underline{53.31 \text{ mW}}}$$

(b) $d\boldsymbol{S} = dydz\boldsymbol{a}_x$, $P_t = \int \mathcal{P}.d\mathbf{S} = \frac{1}{2}\eta H_o^{\,2}S$

$$P_t = \frac{1}{2}(120\pi)(0.2)^2\pi(0.05)^2 = \underline{\underline{59.22 \text{ mW}}}$$

**P. E. 10.9** $\eta_1 = \eta_o = 120\pi, \eta_2 = \sqrt{\frac{\mu}{\varepsilon}} = \frac{\eta_o}{2}$

$$\tau = \frac{2\eta_2}{\eta_2 + \eta_1} = 2/3, \Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = -1/3$$

$$E_{ro} = \Gamma E_{io} = -\frac{10}{3}$$

$$\underline{\underline{\boldsymbol{E}_{rs} = -\frac{10}{3}e^{j\beta_1 z}\boldsymbol{a}_x \text{ V/m}}}$$

where $\beta_1 = \omega / c = 100\pi / 3$.

$$E_{to} = \tau E_{io} = \frac{20}{3}$$

$$\underline{\underline{\boldsymbol{E}_{ts} = \frac{20}{3}e^{-j\beta_2 z}\boldsymbol{a}_x \text{ V/m}}}$$

where $\beta_2 = \omega\sqrt{\varepsilon_r} / c = 2\beta_1 = 200\pi / 3$.

**P. E. 10.10**

$$\alpha_1 = 0, \quad \beta_1 = \frac{\omega}{c}\sqrt{\mu_r \varepsilon_r} = \frac{2\omega}{c} = 5 \longrightarrow \omega = 5c/2 = 7.5\times10^8$$

$$\frac{\sigma_2}{\omega\varepsilon_2} = \frac{0.1}{7.5\times10^8 \times 4\times\frac{10^{-9}}{36\pi}} = 1.2\pi$$

$$\alpha_2 = \frac{\omega}{c}\sqrt{\frac{4}{2}\left[\sqrt{1+1.44\pi^2} - 1\right]} = 6.021$$

$$\beta_2 = \frac{\omega}{c}\sqrt{\frac{4}{2}\left[\sqrt{1+1.44\pi^2} + 1\right]} = 7.826$$

$$|\eta_2| = \frac{60\pi}{\sqrt[4]{1+1.44\pi^2}} = 95.445, \eta_1 = 120\pi\sqrt{\varepsilon_{r1}} = 754$$

$$\tan 2\theta_{\eta_2} = 1.2\pi \longrightarrow \theta_{\eta_2} = 37.57^o$$

$$\eta_2 = 95.445\angle 37.57^o$$

(a)

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{95.445\angle 37.57^o - 754}{95.445\angle 37.57^o + 754} = \underline{\underline{0.8186\angle 171.08^o}}$$

$$\tau = 1 + \Gamma = \underline{\underline{0.2295\angle 33.56^o}}$$

$$s = \frac{1+|\Gamma|}{1-|\Gamma|} = \frac{1+0.8186}{1-0.8186} = \underline{\underline{10.025}}$$

(b) $\boldsymbol{E}_i = 50\sin(\omega t - 5x)\boldsymbol{a}_y = \text{Im}(\boldsymbol{E}_{is}e^{j\omega t})$, where $E_{is} = 50e^{-j5x}a_y$ .

$$E_{ro} = \Gamma E_{io} = 0.8186e^{j171.08^o}(50) = 40.93e^{j171.08^o}$$

$$\boldsymbol{E}_{rs} = 40.93e^{j5x + j171.08^o}\boldsymbol{a}_y$$

$$\boldsymbol{E}_r = \text{Im}(\boldsymbol{E}_{rs}e^{j\omega t}) = \underline{\underline{40.93\sin(\omega t + 5x + 171.1^o)\boldsymbol{a}_y}} \text{ V/m}$$

$$\boldsymbol{a}_H = \boldsymbol{a}_k \times \boldsymbol{a}_E = -\boldsymbol{a}_x \times \boldsymbol{a}_y = -\boldsymbol{a}_z$$

$$\boldsymbol{H}_r = -\frac{40.93}{754}\sin(\omega t + 5x + 171.1^o)\boldsymbol{a}_z = \underline{\underline{-0.0543\sin(\omega t + 5x + 171.1^o)\boldsymbol{a}_z}} \quad \text{A/m}$$

(c )

$$E_{to} = \tau E_{io} = 0.229 e^{j33.56^o}(50) = 11.475 e^{j33.56^o}$$

$$\boldsymbol{E}_{ts} = 11.475 e^{-j\beta_2 x + j33.56^o} e^{-\alpha_2 x}\boldsymbol{a}_y$$

$$\boldsymbol{E}_t = \text{Im}(\boldsymbol{E}_{ts}e^{j\omega t}) = \underline{\underline{11.475e^{-6.021x}\sin(\omega t - 7.826x + 33.56^o)\boldsymbol{a}_y}} \quad \text{V/m}$$

$$\boldsymbol{a}_H = \boldsymbol{a}_k \times \boldsymbol{a}_E = \boldsymbol{a}_x \times \boldsymbol{a}_y = \boldsymbol{a}_z$$

$$\boldsymbol{H}_t = \frac{11.495}{95.445}e^{-6.021x}\sin(\omega t - 7.826x + 33.56^o - 37.57^o)\boldsymbol{a}_z$$

$$= \underline{\underline{0.1202e^{-6.021x}\sin(\omega t - 7.826x - 4.01^o)\boldsymbol{a}_z}} \quad \text{A/m}$$

(d)

$$\mathscr{P}_{1\text{ave}} = \frac{E_{io}^{\ 2}}{2\eta_1}\boldsymbol{a}_x + \frac{E_{ro}^{\ 2}}{2\eta_1}(-\boldsymbol{a}_x) = \frac{1}{2(240\pi)}[50^2\boldsymbol{a}_x - 40.93^2\boldsymbol{a}_x] = \underline{\underline{0.5469\boldsymbol{a}_x}} \quad \text{W/m}^2$$

$$\boldsymbol{P}_{2\text{ave}} = \frac{E_{to}^{\ 2}}{2|\eta_2|}e^{-2\alpha_2 x}\cos\theta_{\eta_2}\boldsymbol{a}_x = \frac{(11.475)^2}{2(95.445)}\cos 37.57^o e^{-2(6.021)x}\boldsymbol{a}_x = \underline{\underline{0.5469e^{-12.04x}\boldsymbol{a}_x}} \quad \text{W/m}^2$$

**P. E. 10.11** (a)

$$\boldsymbol{k} = -2\boldsymbol{a}_y + 4\boldsymbol{a}_z \longrightarrow k = \sqrt{2^2 + 4^2} = \sqrt{20}$$

$$\omega = kc = 3\times 10^8\sqrt{20} = \underline{\underline{1.342\times 10^9 \text{ rad/s}}},$$

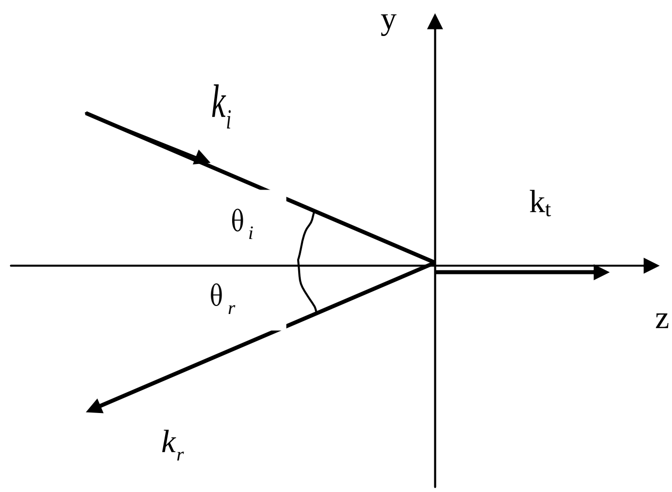
$$\lambda = 2\pi / k = \underline{\underline{1.405\text{m}}}$$

(b) $$\boldsymbol{H} = \frac{\boldsymbol{a}_k \times \boldsymbol{E}}{\eta_o} = \frac{(-2\boldsymbol{a}_y + 4\boldsymbol{a}_z)}{\sqrt{20}(120\pi)} \times (10\boldsymbol{a}_y + 5\boldsymbol{a}_z)\cos(\omega t - \boldsymbol{k}.\boldsymbol{r})$$

$$= \underline{\underline{-29.66\cos(1.342x10^9 t + 2y - 4z)\boldsymbol{a}_x}} \quad \text{mA/m}$$

(c ) $\mathscr{P}_{ave} = \frac{|E_o|^2}{2\eta_o}\boldsymbol{a}_k = \frac{125}{2(120\pi)}\frac{(-2\boldsymbol{a}_y + 4\boldsymbol{a}_z)}{\sqrt{20}} = \underline{\underline{-74.15\boldsymbol{a}_y + 148.9\boldsymbol{a}_z}}$ mW/m$^2$

**P. E. 10.12** (a)




$$\tan\theta_i = \frac{k_{iy}}{k_{iz}} = \frac{2}{4} \longrightarrow \underline{\underline{\theta_i = 26.56 = \theta_r}}$$

$$\sin\theta_t = \sqrt{\frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}}\sin\theta_i = \frac{1}{2}\sin 26.56^o \longrightarrow \underline{\underline{\theta_t = 12.92^o}}$$

(b) $\eta_1 = \eta_o, \eta_2 = \eta_o/2$ **E** is parallel to the plane of incidence. Since $\mu_1 = \mu_2 = \mu_o$,

we may use the result of Prob. 10.42, i.e.

$$\Gamma_{\backslash\backslash} = \frac{\tan(\theta_t - \theta_i)}{\tan(\theta_t + \theta_i)} = \frac{\tan(-13.64^o)}{\tan(39.48^o)} = \underline{\underline{-0.2946}}$$

$$\tau_{\backslash\backslash} = \frac{2\cos 26.56^o \sin 12.92^o}{\sin 39.48^o \cos(-13.64^o)} = \underline{\underline{0.6474}}$$

(c) $\boldsymbol{k}_r = -\beta_1 \sin\theta_r \boldsymbol{a}_y - \beta_1 \cos\theta_r \boldsymbol{a}_z$ . Once $k_r$ is known, $E_r$ is chosen such that

$\boldsymbol{k}_r.\boldsymbol{E}_r = 0$ or $\nabla.\boldsymbol{E}_r = 0$. Let

$$\boldsymbol{E}_r = \pm E_{or}(-\cos\theta_r \boldsymbol{a}_y + \sin\theta_r \boldsymbol{a}_z)\cos(\omega t + \beta_1 \sin\theta_r y + \beta_1 \cos\theta_r z)$$

Only the positive sign will satisfy the boundary conditions. It is evident that

$$\boldsymbol{E}_i = E_{oi}(\cos\theta_i \boldsymbol{a}_y + \sin\theta_i \boldsymbol{a}_z)\cos(\omega t + 2y - 4z)$$

Since $\theta_r = \theta_i$,

$$E_{or}\cos\theta_r = \Gamma_{//}E_{oi}\cos\theta_i = 10\Gamma_{//} = -2.946$$

$$E_{or}\sin\theta_r = \Gamma_{//}E_{oi}\sin\theta_i = 5\Gamma_{//} = -1.473$$

$$\beta_1\sin\theta_r = 2, \beta_1\cos\theta_r = 4$$

i.e.

$$E_r = -(2.946a_y - 1.473a_z)\cos(\omega t + 2y + 4z)$$

$$\boldsymbol{E}_1 = \boldsymbol{E}_i + \boldsymbol{E}_r = \underline{\underline{(10\boldsymbol{a}_y + 5\boldsymbol{a}_z)\cos(\omega t + 2y - 4z) + (-2.946\boldsymbol{a}_y + 1.473\boldsymbol{a}_z)\cos(\omega t + 2y + 4z)}}$$

V/m

(d) $\boldsymbol{k}_t = -\beta_2\sin\theta_t \boldsymbol{a}_y + \beta_2\cos\theta_t \boldsymbol{a}_z$.   Since $k_r \bullet E_r = 0$, let

$$\boldsymbol{E}_t = E_{ot}(\cos\theta_t \boldsymbol{a}_y + \sin\theta_t \boldsymbol{a}_z)\cos(\omega t + \beta_2 y\sin\theta_t - \beta_2 z\cos\theta_t)$$

$$\beta_2 = \omega\sqrt{\mu_2\varepsilon_2} = \beta_1\sqrt{\varepsilon_{r2}} = 2\sqrt{20}$$

$$\sin\theta_t = \frac{1}{2}\sin\theta_i = \frac{1}{2\sqrt{5}}, \qquad \cos\theta_t = \frac{\sqrt{9}}{\sqrt{20}}$$

$$\beta_2\cos\theta_t = 2\sqrt{20}\sqrt{\frac{19}{20}} = 8.718$$

$$E_{ot}\cos\theta_t = \tau_{//}E_{oi}\cos\theta_t = 0.6474\sqrt{125}\sqrt{\frac{19}{20}} = 7.055$$

$$E_{ot}\sin\theta_t = \tau_{//}E_{oi}\sin\theta_t = 0.6474\sqrt{125}\sqrt{\frac{1}{20}} = 1.6185$$

Hence

$$\boldsymbol{E}_2 = \boldsymbol{E}_t = \underline{\underline{(7.055\boldsymbol{a}_y + 1.6185\boldsymbol{a}_z)\cos(\omega t + 2y - 8.718z)}} \quad \text{V/m}$$

(d) $\tan\theta_{B//} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} = 2 \longrightarrow \underline{\underline{\theta_{B//} = 63.43^o}}$

**P.E. 10.13**

$S_i = \frac{1+0.4}{1-0.4} = \frac{1.4}{0.6} = \underline{\underline{2.333}}$

$S_o = \frac{1+0.2}{1-0.2} = \frac{1.2}{0.8} = \underline{\underline{1.5}}$

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**Prob. 10.1** (a) Wave propagates $\underline{\underline{\text{along } +\mathbf{a}_x}}$.

(b)

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{2\pi \times 10^6} = \underline{\underline{1\mu s}}$$

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{6} = \underline{\underline{1.047\text{m}}}$$

$$u = \frac{\omega}{\beta} = \frac{2\pi \times 10^6}{6} = \underline{\underline{1.047 \times 10^6 \text{ m/}s}}$$

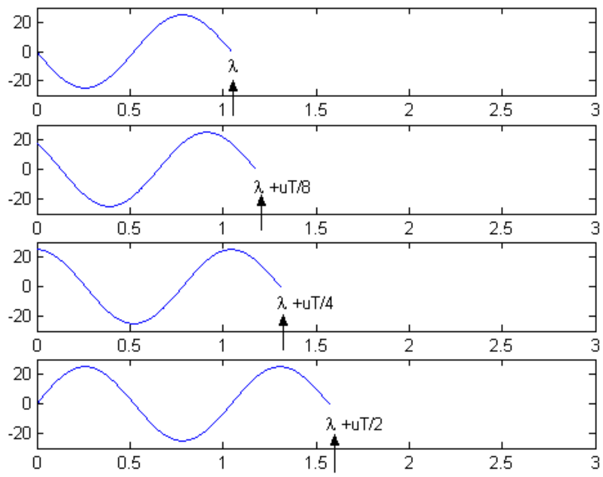
(c ) At $t$=0, $E_z = 25\sin(-6x) = -25\sin 6x$

At $t$=$T$/8, $E_z = 25\sin(\frac{2\pi}{T}\frac{T}{8} - 6x) = 25\sin(\frac{\pi}{4} - 6x)$

At $t$=$T$/4, $E_z = 25\sin(\frac{2\pi}{T}\frac{T}{4} - 6x) = 25\sin(-6x + 90^o) = 25\cos 6x$

At $t$=$T$/2, $E_z = 25\sin(\frac{2\pi}{T}\frac{T}{2} - 6x) = 25\sin(-6x + \pi) = 25\sin 6x$

These are sketched below.




**Prob. 10.2**

(a) $\lambda = \frac{c}{f} = \frac{3\times10^8}{60} = \underline{\underline{5\times10^6 \text{ m}}}$

(b) $\lambda = \frac{3\times10^8}{2\times10^6} = \underline{\underline{150 \text{ m}}}$

(c ) $\lambda = \frac{3\times10^8}{120\times10^6} = \underline{\underline{2.5 \text{ m}}}$

(d) $\lambda = \frac{3\times10^8}{2.4\times10^9} = \underline{\underline{0.125 \text{ m}}}$

**Prob. 10.3**

(a) $\omega = \underline{\underline{10^8 \text{ rad/s}}}$

(b) $\beta = \frac{\omega}{c} = \frac{10^8}{3\times10^8} = \underline{\underline{0.333 \text{ rad/m}}}$

(c ) $\lambda = \frac{2\pi}{\beta} = 6\pi = \underline{\underline{18.85 \text{ m}}}$

(d) Along **-a**$_y$

At y=1, t=10ms,

(e) $H = 0.5\cos(10^8 t\times10\times10^{-9} + \frac{1}{3}\times3) = 0.5\cos(1+1)$

$= \underline{\underline{-0.1665 \text{ A/m}}}$

**Prob. 10.4**

(a)

$$\frac{\partial E}{\partial x} = -\sin(x+\omega t) - \sin(x-\omega t)$$

$$\frac{\partial^2 E}{\partial x^2} = -\cos(x+\omega t) - \cos(x-\omega t) = -E$$

$$\frac{\partial E}{\partial t} = -\omega \sin(x+\omega t) - \omega \sin(x-\omega t)$$

$$\frac{\partial^2 E}{\partial t^2} = -\omega^2 \cos(x+\omega t) - \omega^2 \cos(x-\omega t) = -\omega^2 E$$

$$\frac{\partial^2 E}{\partial t^2} - u^2 \frac{\partial^2 E}{\partial x^2} = -\omega^2 E + u^2 E = 0$$

if $\omega^2 = u^2$ and hence, eq. (10.1) is satisfied.

(b) $\underline{\underline{u = \omega}}$

**Prob. 10.5** If

$$\gamma^2 = j\omega\mu(\sigma + j\omega\varepsilon) = -\omega^2\mu\varepsilon + j\omega\mu\sigma \quad \text{and}\ \gamma = \alpha + j\beta,\ \text{then}$$

$$|\gamma^2| = \sqrt{(\alpha^2 - \beta^2) + 4\alpha^2\beta^2} = \sqrt{(\alpha^2 + \beta^2)^2} = \alpha^2 + \beta^2$$

i.e.

$$\alpha^2 + \beta^2 = \omega\mu\sqrt{(\sigma^2 + \omega^2\varepsilon^2)} \qquad (1)$$

$$\text{Re}(\gamma^2) = \alpha^2 - \beta^2 = -\omega^2\mu\varepsilon$$

$$\beta^2 - \alpha^2 = \omega^2\mu\varepsilon \qquad (2)$$

Subtracting and adding (1) and (2) lead respectively to

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right]}$$

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1\right]}$$

(b) From eq. (10.25), $\boldsymbol{E}_s(z) = E_o e^{-\gamma z}\boldsymbol{a}_x$.

$$\nabla \times \boldsymbol{E} = -j\omega\mu \boldsymbol{H}_s \longrightarrow \boldsymbol{H}_s = \frac{j}{\omega\mu}\nabla \times \boldsymbol{E}_s = \frac{j}{\omega\mu}(-\gamma E_o e^{-\gamma z}\boldsymbol{a}_y)$$

But $\boldsymbol{H}_s(z) = H_o e^{-\gamma z}\boldsymbol{a}_y$, hence $H_o = \dfrac{E_o}{\eta} = -\dfrac{j\gamma}{\omega\mu}E_o$

$$\eta = \frac{j\omega\mu}{\gamma}$$

(c) From (b),

$$\eta = \frac{j\omega\mu}{\sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = \frac{\sqrt{\mu/\varepsilon}}{\sqrt{1 - j\dfrac{\sigma}{\omega\varepsilon}}}$$

$$|\eta| = \frac{\sqrt{\mu/\varepsilon}}{\sqrt[4]{1+\left(\dfrac{\sigma}{\omega\varepsilon}\right)^2}}, \tan 2\theta_\eta = \left(\frac{\omega\varepsilon}{\sigma}\right)^{-1} = \frac{\sigma}{\omega\varepsilon}$$

**Prob. 10.6** (a)

From eq. (10.18),

$$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)} = \sqrt{j\omega\mu\, j\omega\varepsilon(1 - \frac{j\sigma}{\omega\varepsilon})} = j\omega\sqrt{\mu\varepsilon}\left(1 - \frac{j\sigma}{\omega\varepsilon}\right)^{1/2}$$

Assuming that $\dfrac{\sigma}{\omega\varepsilon} \ll 1$, we include up to the second power in $\dfrac{\sigma}{\omega\varepsilon}$ and neglect higher-order terms.

$$\gamma = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}\left(1 - \frac{\sigma^2}{\omega^2\varepsilon}\right) + j\omega\sqrt{\mu\varepsilon}\left(1 + \frac{\sigma^2}{8\omega^2\varepsilon^2}\right) = \alpha + j\beta$$

Thus,

$$\beta = \omega\sqrt{\mu\varepsilon}\left(1 + \frac{\sigma^2}{8\omega^2\varepsilon^2}\right)$$

**Prob. 10.7**

(a)

$$\frac{\sigma}{\omega\varepsilon} = \frac{8\times10^{-2}}{2\pi\times50\times10^6\times3.6\times\dfrac{10^{-9}}{36\pi}} = 8$$

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1\right]} = \frac{2\pi\times 50\times 10^6}{3\times 10^8}\sqrt{\frac{2.1\times 3.6}{2}[\sqrt{65}-1]} = 5.41$$

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1\right]} = 6.129$$

$\gamma = \alpha + j\beta = \underline{\underline{5.41 + j6.129}}$ /m

(b) $\lambda = \dfrac{2\pi}{\beta} = \dfrac{2\pi}{6.129} = \underline{\underline{1.025}}$ m

(c) $u = \dfrac{\omega}{\beta} = \dfrac{2\pi\times 50\times 10^6}{6.129} = \underline{\underline{5.125\times 10^7}}$ m/s

(d) $|\eta| = \dfrac{\sqrt{\dfrac{\mu}{\varepsilon}}}{\sqrt[4]{1+\left(\dfrac{\sigma}{\omega\varepsilon}\right)^2}} = \dfrac{120\pi\sqrt{\dfrac{2.1}{3.6}}}{\sqrt[4]{65}} = 101.4$

$\tan 2\theta_\eta = \dfrac{\sigma}{\omega\varepsilon} = 8 \longrightarrow \theta_\eta = 41.44^o$

$\eta = \underline{\underline{101.41\angle 41.44^o\ \Omega}}$

(e) $\boldsymbol{H}_s = \boldsymbol{a}_k \times \dfrac{\boldsymbol{E}_s}{\eta} = \boldsymbol{a}_x \times \dfrac{6}{\eta}e^{-\gamma z}\boldsymbol{a}_z = -\dfrac{6}{\eta}e^{-\gamma z}\boldsymbol{a}_y = \underline{\underline{-59.16e^{-j41.44^o}e^{-\gamma z}\boldsymbol{a}_y}}$ mA/m

**Prob. 10.8**

(a) $\tan\theta = \dfrac{\sigma}{\omega\varepsilon} = \dfrac{10^{-2}}{2\pi\times 12\times 10^6\times 10\times\dfrac{10^{-9}}{36\pi}} = \underline{\underline{1.5}}$

(b) $\tan\theta = \dfrac{10^{-4}}{2\pi\times 12\times 10^6\times 4\times\dfrac{10^{-9}}{36\pi}} = \underline{\underline{3.75\times 10^{-2}}}$

(c ) $\tan\theta = \dfrac{4}{2\pi\times 12\times 10^{6}\times 81\times \dfrac{10^{-9}}{36\pi}} = \underline{\underline{74.07}}$

**Prob. 10.9**

(a)

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1\right]} = \frac{2\pi\times 15\times 10^{9}}{3\times 10^{8}}\sqrt{\frac{1\times 9.6}{2}\left[\sqrt{1+9\times 10^{-8}}-1\right]}$$

$$= 100\pi\sqrt{4.8\left(\frac{1}{2}\times 9\times 10^{-8}\right)} = 0.146$$

$$\delta = \frac{1}{\alpha} = \underline{\underline{6.85\ \text{m}}}$$

(b) $A = \alpha\ell = 0.146\times 5\times 10^{-3} = \underline{\underline{0.73\times 10^{-3}\ \text{Np}}}$

**Prob. 10.10**

If $\sigma = \varepsilon\omega$, the loss tangent is $\dfrac{\sigma}{\varepsilon\omega} = 1$

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1\right]} = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+1^2}-1\right]} = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{2}-1\right]}$$

But $\lambda_o = \dfrac{2\pi}{\beta} = \dfrac{2\pi c}{\omega} \quad\rightarrow\quad \omega = \dfrac{2\pi c}{\lambda_o}$

$$\alpha = \frac{2\pi c}{\lambda_o}\sqrt{\frac{\mu\varepsilon}{2}}\sqrt{\sqrt{2}-1} = \frac{2\pi\times 3\times 10^{8}}{12\times 10^{-2}}\sqrt{\frac{1}{2}\times 4\pi\times 10^{-7}\times 4\times \frac{10^{-9}}{36\pi}(0.6436)} = \underline{\underline{47.66\ \ \text{Np/m}}}$$

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1\right]} = \frac{2\pi c}{\lambda_o}\sqrt{\frac{\mu\varepsilon}{2}}\sqrt{\sqrt{2}+1}$$

$$= \frac{2\pi\times 3\times 10^{8}}{12\times 10^{-2}}\sqrt{\frac{1}{2}\times 4\pi\times 10^{-7}\times 4\times \frac{10^{-9}}{36\pi}(1.5538)}$$

$$= \underline{\underline{115.06\ \text{rad/m}}}$$

$$u = \frac{\omega}{\beta} = \frac{2\pi c}{\lambda_o\beta} = \frac{2\pi\times 3\times 10^{8}}{12\times 10^{-2}\times 115.06} = \underline{\underline{1.3652\times 10^{8}\ \text{m/s}}}$$

**Prob. 10.11**

For silver, the loss tangent is

$$\frac{\sigma}{\omega\varepsilon} = \frac{6.1\times10^{7}}{2\pi\times10^{8}\times\frac{10^{-9}}{36\pi}} = 6.1\times18\times10^{8} \gg 1$$

Hence, silver is a good conductor
For rubber,

$$\frac{\sigma}{\omega\varepsilon} = \frac{10^{-15}}{2\pi\times10^{8}\times3.1\times\frac{10^{-9}}{36\pi}} = \frac{18}{3.1}\times10^{-14} \ll 1$$

Hence, rubber is a poor conductor or a good insulator.

**Prob. 10.12**

$$\frac{\sigma}{\omega\varepsilon} = \frac{4}{2\pi\times10^{5}\times80\times10^{-9}/36\pi} = 9,000 >> 1$$

$$\alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\frac{2\pi\times10^{5}}{2}\times4\pi\times10^{-7}\times4} = 0.4\pi$$

(a) $u = \omega/\beta = \frac{2\pi\times10^{5}}{0.4\pi} = \underline{\underline{5\times10^{5}}}$ m/s

(b) $\lambda = 2\pi/\beta = \frac{2\pi}{0.4\pi} = \underline{\underline{5}}$ m

(c) $\delta = 1/\alpha = \frac{1}{0.4\pi} = \underline{\underline{0.796}}$ m

(d) $\eta = |\eta|\angle\theta_{\eta}, \theta_{\eta} = 45^{o}$

$$|\eta| = \frac{\sqrt{\frac{\mu}{\varepsilon}}}{\sqrt[4]{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^{2}}} \cong \sqrt{\frac{\mu}{\varepsilon}\frac{\omega\varepsilon}{\sigma}} = \sqrt{\frac{4\pi\times10^{-7}\times2\pi\times10^{5}}{4}} = 0.4443$$

$$\eta = \underline{\underline{0.4443\angle45^{o} \quad \Omega}}$$

**Prob. 10.13** (a)

$$T = 1/f = 2\pi/\omega = \frac{2\pi}{\pi x 10^8} = \underline{\underline{20}}\ \text{ns}$$

(b) Let $x = \sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2}$

$$\frac{\alpha}{\beta} = \left(\frac{x-1}{x+1}\right)^{1/2}$$

But $\alpha = \frac{\omega}{c}\sqrt{\frac{\mu_r \varepsilon_r}{2}}\sqrt{x-1}$

$$\sqrt{x-1} = \frac{\alpha c}{\omega\sqrt{\frac{\mu_r\varepsilon_r}{2}}} = \frac{0.1\times3\times10^8}{\pi\times10^8\sqrt{2}} = 0.06752 \longrightarrow x = 1.0046$$

$$\beta = \left(\frac{x+1}{x-1}\right)^{1/2}\alpha = \left(\frac{2.0046}{0.0046}\right)^{1/2} 0.1 = 2.088$$

$$\lambda = 2\pi/\beta = \frac{2\pi}{2.088} = \underline{\underline{3}}\ \text{m}$$

(c ) $|\eta| = \frac{\sqrt{\mu/\varepsilon}}{\sqrt{x}} = \frac{377}{2\sqrt{1.0046}} = 188.1$

$$x = \sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} = 1.0046$$

$$\frac{\sigma}{\omega\varepsilon} = 0.096 = \tan 2\theta_\eta \longrightarrow \theta_\eta = 2.74^o$$

$$\eta = 188.1\angle 2.74^o \quad \Omega$$

$$E_o = \eta H_o = 12\times188.1 = 2257.2$$

$$\boldsymbol{a}_E \times \boldsymbol{a}_H = \boldsymbol{a}_k \longrightarrow \boldsymbol{a}_E \times \boldsymbol{a}_x = \boldsymbol{a}_y \longrightarrow \boldsymbol{a}_E = \boldsymbol{a}_z$$

$$\boldsymbol{E} = \underline{\underline{2.257e^{-0.1y}\sin(\pi\times10^8 t - 2.088y + 2.74^o)\boldsymbol{a}_z}}\ \text{kV/m}$$

(d) The phase difference is $\underline{\underline{2.74^o}}$.

**Prob. 10.14**

This is a lossy medium in which $\mu=\mu_o$.

Let $x = \left(\dfrac{\sigma}{\omega\varepsilon}\right)^2$

$$\eta = \frac{j\omega\mu}{\gamma} = \frac{j2\pi\times10^9\times4\pi}{100+j200} = 35.31\angle 26.57^o$$

$$E_o = 0.05\times35.31 = 1.765$$

$$\mathbf{a}_E = \mathbf{a}_H\times\mathbf{a}_k = -\mathbf{a}_z$$

Thus, we obtain

$$\underline{\underline{\mathbf{E} = -1.765\cos(2\pi\times10^9 t - 200x + 26.57^o)\mathbf{a}_z \text{ V/m}}}$$

$$\sqrt{\varepsilon_r(1/3)} = \frac{\alpha c}{\omega} = \frac{100\times3\times10^8}{2\pi\times10^9} = \frac{15}{\pi}$$

$$\varepsilon_r\frac{1}{3} = 4.776 \longrightarrow \varepsilon_r = 14.32$$

$$\tan 2\theta_\eta = \frac{\sigma}{\omega\varepsilon} = \frac{4}{3} \longrightarrow \theta_\eta = 26.57^o$$

$$|\eta| = \frac{\sqrt{\mu/\varepsilon}}{\sqrt[4]{1+x}} = \frac{\dfrac{377}{\sqrt{14.32}}}{\sqrt{5/3}} = 77.175$$

$$E_o = |\eta|H_o = 77.175\times50\times10^{-3} = 3.858$$

$$\mathbf{a}_E = -(\mathbf{a}_k\times\mathbf{a}_H) = -(\mathbf{a}_x\times\mathbf{a}_y) = -\mathbf{a}_z$$

$$\mathbf{E} = \underline{\underline{-3.858e^{-100x}\cos(2\pi\times10^9 t - 200x + 26.57^o)\mathbf{a}_z \text{ V/m}}}$$

**Prob. 10.15**

$$\frac{\sigma}{\omega\varepsilon} = \frac{1}{2\pi\times 10^9 \times 4 \times \frac{10^{-9}}{36\pi}} = 4.5$$

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right]}$$

$$= 2\pi\times 10^9\sqrt{\frac{4\pi}{2}\times 10^{-7}\times 4\times 9\times\frac{10^{-9}}{36\pi}\left[\sqrt{1+4.5^2} - 1\right]}$$

$$= 20\pi\sqrt{2[\sqrt{21.25} - 1]} = \underline{\underline{168.8 \text{ Np/m}}}$$

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1\right]} = 20\pi\sqrt{2[\sqrt{21.25} + 1]} = \underline{\underline{210.5 \text{ rad/m}}}$$

$$\tan 2\theta_\eta = \frac{\sigma}{\omega\varepsilon} = 4.5 \quad \longrightarrow \quad \theta_\eta = 38.73^o$$

$$|\eta| = \frac{\sqrt{\mu/\varepsilon}}{\sqrt[4]{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}} = \frac{120\pi\sqrt{9/4}}{\sqrt[4]{1+4.5^2}} = 263.38$$

$$\eta = \underline{\underline{263.38\angle 38.73^o \ \ \Omega}}$$

$$u = \frac{\omega}{\beta} = \frac{2\pi\times 10^9}{210.5} = \underline{\underline{2.985\times 10^7 \text{ m/s}}}$$

**Prob. 10.16**

(a) $\beta = 6.5 = \omega\sqrt{\mu_o\varepsilon_o} = \frac{\omega}{c}$

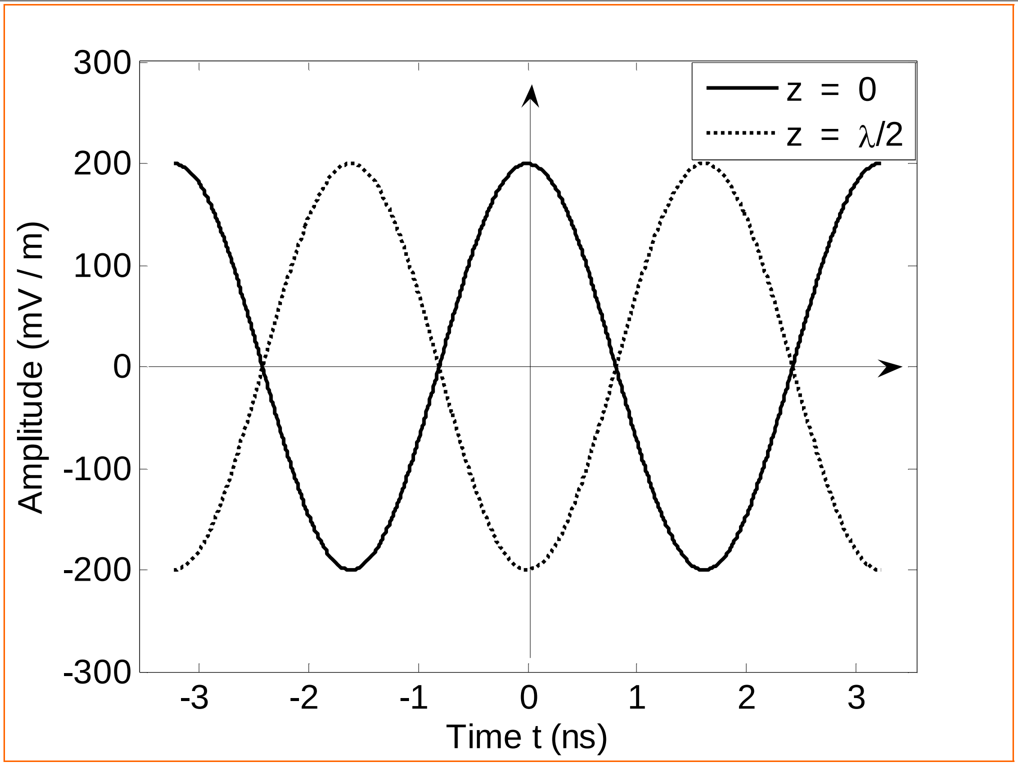
$$\omega = \beta c = 6.5\times 3\times 10^8 = \underline{\underline{1.95\times 10^9 \text{ rad/s}}}$$

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{6.5} = \underline{\underline{0.9666 \text{ m}}}$$

(b) For z=0, $E_z = 0.2\cos\omega t$

For z=$\lambda$/2, $E_z = 0.2\cos(\omega t - \frac{2\pi}{\lambda}\frac{\lambda}{2}) = -0.2\cos\omega t$

The two waves are sketched below.




(c) $\boldsymbol{H} = H_o \cos(\omega t - 6.5z)\boldsymbol{a}_H$

$$H_o = \frac{E_o}{\eta_o} = \frac{0.2}{377} = 5.305\times10^{-4}$$

$$\boldsymbol{a}_E \times \boldsymbol{a}_H = \boldsymbol{a}_k \longrightarrow \boldsymbol{a}_x \times \boldsymbol{a}_H = \boldsymbol{a}_z \longrightarrow \boldsymbol{a}_H = \boldsymbol{a}_y$$

$$\boldsymbol{H} = \underline{\underline{0.5305\cos(\omega t - 6.5z)\boldsymbol{a}_y \text{ mA/m}}}$$

**Prob. 10.17**

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega\sqrt{\mu\varepsilon}}$$

$$\frac{\lambda_o}{\lambda} = \frac{\sqrt{\mu_o\varepsilon_o\varepsilon_r}}{\sqrt{\mu_o\varepsilon_o}} = \sqrt{\varepsilon_r}$$

$$\sqrt{\varepsilon_r} = \frac{\lambda_o}{\lambda} = \frac{6.4}{2.8} = 2.286 \quad \rightarrow \quad \underline{\underline{\varepsilon_r = 5.224}}$$

**Prob. 10.18** (a) Along <u>-x direction.</u>

(b) $\beta = 6, \quad \omega = 2\times10^8$ ,

$$\beta = \omega\sqrt{\mu\varepsilon} = \frac{\omega}{c}\sqrt{\mu_r\varepsilon_r}$$

$$\sqrt{\varepsilon_r} = \beta c/\omega = \frac{6\times3\times10^8}{2\times10^8} = 9 \quad \longrightarrow \quad \varepsilon_r = 81$$

$$\varepsilon = \varepsilon_o\varepsilon_r = \frac{10^{-9}}{36\pi}\times81 = \underline{\underline{7.162\times10^{-10}}} \text{ F/m}$$

(c ) $\eta = \sqrt{\mu/\varepsilon} = \sqrt{\mu_o/\varepsilon_o}\sqrt{\mu_r/\varepsilon_r} = \frac{120\pi}{9} = 41.89\ \Omega$

$$E_o = H_o\eta = 25\times10^{-3}\times41.88 = 1.047$$

$$\boldsymbol{a}_E\times\boldsymbol{a}_H = \boldsymbol{a}_k \longrightarrow \boldsymbol{a}_E\times\boldsymbol{a}_y = -\boldsymbol{a}_x \longrightarrow \boldsymbol{a}_E = \boldsymbol{a}_z$$

$$\boldsymbol{E} = \underline{\underline{1.047\sin(2\times10^8 t + 6x)\boldsymbol{a}_z}} \text{ V/m}$$

**Prob. 10.19** (a) $\frac{\sigma}{\omega\varepsilon} = \frac{10^{-6}}{2\pi\times10^7\times5\times\frac{10^{-9}}{36\pi}} = 3.6\times10^{-4} << 1$

Thus, the material is <u>lossless</u> at this frequency.

(b) $\beta = \omega\sqrt{\mu\varepsilon} = \frac{2\pi\times10^7}{3\times10^8}\sqrt{5\times750} = \underline{\underline{12.83}}$ rad/m

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{12.83} = \underline{\underline{0.49}} \text{ m}$$

(c ) Phase difference = $\beta l$ = <u>25.66 rad</u>

(d) $\eta = \sqrt{\mu/\varepsilon} = 120\pi\sqrt{\frac{\mu_r}{\varepsilon_r}} = 120\pi\sqrt{\frac{750}{5}} = \underline{\underline{4.62\text{ k}\Omega}}$

**Prob. 10.20**

(a)

$$\beta = \omega\sqrt{\mu\varepsilon} = \frac{\omega}{c} = \frac{10^8\pi}{3\times10^8} = \frac{\pi}{3} = \underline{\underline{1.0472 \text{ rad/m}}}$$

(b)

$$E = 0 \quad \longrightarrow \quad \sin(10^8\pi t_o - \beta x_o) = 0 = \sin(n\pi), n = 1, 2, 3, \ldots$$

$$10^8\pi t_o - \beta x_o = \pi$$

$$10^8\pi\times5\times10^{-3} - \frac{\pi}{3}x_o = \pi \quad \longrightarrow \quad x_o \simeq \underline{\underline{5\times10^5 \text{ m}}}$$

(c )

$$\boldsymbol{H} = H_o\sin(10^8\pi t - \beta x)\boldsymbol{a}_H$$

$$H_o = \frac{E_o}{\eta} = \frac{50\times10^{-3}}{120\pi} = 132.63\ \mu\text{A/m}$$

$$\boldsymbol{a}_H = \boldsymbol{a}_k\times\boldsymbol{a}_E = \boldsymbol{a}_x\times\boldsymbol{a}_z = -\boldsymbol{a}_y$$

$$\underline{\underline{\boldsymbol{H} = -132.63\sin(10^8\pi t - 1.0472x)\boldsymbol{a}_y\ \mu\text{A/m}}}$$

**Prob. 10.21**

This is a lossless material.

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} = 377\sqrt{\frac{\mu_r}{\varepsilon_r}} = 105 \qquad (1)$$

$$u = \frac{\omega}{\beta} = \frac{\omega}{\omega\sqrt{\mu\varepsilon}} = \frac{c}{\sqrt{\mu_r\varepsilon_r}} = 7.6\times10^7 \qquad (2)$$

From (1),

$$\sqrt{\frac{\mu_r}{\varepsilon_r}} = \frac{105}{377} = 0.2785 \qquad (1)a$$

From (2),

$$\frac{1}{\sqrt{\mu_r\varepsilon_r}} = \frac{7.6\times10^7}{3\times10^8} = 0.2533 \qquad (2)a$$

Multiplying (1)a by (2)a,

$$\frac{1}{\varepsilon_r} = 0.2785\times0.2533 = 0.07054 \quad \longrightarrow \quad \varepsilon_r = \underline{\underline{14.175}}$$

Dividing (1)a by (2)a,

$$\mu_r = \frac{0.2785}{0.2533} = \underline{\underline{1.0995}}$$

**Prob. 10.22**

(a) $$\nabla \times \boldsymbol{E} = \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x(z,t) & E_y(z,t) & 0 \end{vmatrix} = -\frac{\partial E_y}{\partial z}\boldsymbol{a}_x + \frac{\partial E_x}{\partial z}\boldsymbol{a}_y$$

$$= -6\beta\cos(\omega t - \beta z)\boldsymbol{a}_x + 8\beta\sin(\omega t - \beta z)\boldsymbol{a}_y$$

But $\nabla \times \boldsymbol{E} = -\mu\frac{\partial \boldsymbol{H}}{\partial t} \longrightarrow \boldsymbol{H} = -\frac{1}{\mu}\int \nabla \times \boldsymbol{E}dt$

$$\underline{\underline{\boldsymbol{H} = \frac{6\beta}{\mu\omega}\sin(\omega t - \beta z)\boldsymbol{a}_x + \frac{8\beta}{\mu\omega}\cos(\omega t - \beta z)\boldsymbol{a}_y}}$$

(b) $\beta = \omega\sqrt{\mu\varepsilon} = \frac{2\pi f}{c}\sqrt{4.5} = \frac{2\pi \times 40 \times 10^6}{3 \times 10^8}\sqrt{4.5} = \underline{\underline{1.777 \text{ rad/m}}}$

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{1.777} = \underline{\underline{3.536 \text{ m}}}$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} = \frac{120\pi}{\sqrt{4.5}} = \underline{\underline{177.72\ \Omega}}$$

$$u = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{\sqrt{4.5}} = \frac{3 \times 10^8}{\sqrt{4.5}} = \underline{\underline{1.4142 \times 10^8 \text{ m/s}}}$$

**Prob. 10.23**

(a) $\boldsymbol{E} = \text{Re}[\boldsymbol{E}_s e^{j\omega t}] = (5\boldsymbol{a}_x + 12\boldsymbol{a}_y)e^{-0.2z}\cos(\omega t - 3.4z)$

At $z$ = 4m, $t = T/8$, $\omega t = \frac{2\pi}{T}\frac{T}{8} = \frac{\pi}{4}$

$$\boldsymbol{E} = (5\boldsymbol{a}_x + 12\boldsymbol{a}_y)e^{-0.8}\cos(\pi/4 - 13.6)$$

$$|E| = 13e^{-0.8}|\cos(\pi/4 - 13.6)| = \underline{\underline{5.662}} \text{ V/m}$$

(b) loss $= \alpha \Delta z = 0.2(3) = 0.6$ Np. Since 1 Np = 8.686 dB,

loss = 0.6 x 8.686 = $\underline{\underline{5.212 \text{ dB}}}$

(c ) Let $x = \sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}$

$$\frac{\alpha}{\beta} = \left(\frac{x-1}{x+1}\right)^{1/2} = 0.2/3.4 = \frac{1}{17}$$

$$\frac{x-1}{x+1} = 1/289 \quad \longrightarrow \quad x = 1.00694$$

$$\alpha = \omega\sqrt{\mu\varepsilon/2}\sqrt{x-1} = \frac{\omega}{c}\sqrt{\varepsilon_r/2}\sqrt{x-1}$$

$$\sqrt{\frac{\varepsilon_r}{2}} = \frac{\alpha c}{\omega\sqrt{x-1}} = \frac{0.2\times3\times10^8}{10^8\sqrt{0.00694}} = 7.2 \quad \longrightarrow \quad \varepsilon_r = 103.68$$

$$|\eta| = \frac{\sqrt{\frac{\mu_o}{\varepsilon_o}}\cdot\frac{1}{\sqrt{\varepsilon_r}}}{\sqrt{x}} = \frac{120\pi}{\sqrt{103.68\times1.00694}} = 36.896$$

$$\tan 2\theta_\eta = \frac{\sigma}{\omega\varepsilon} = \sqrt{x^2-1} = 0.118 \quad \longrightarrow \quad \theta_\eta = 3.365^o$$

$$\eta = \underline{\underline{36.896\angle 3.365^o\ \Omega}}$$

**Prob. 10.24**

$$\alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}} \quad \longrightarrow \quad \sigma = \frac{2\alpha^2}{\omega\mu} = \frac{2\times12^2}{2\pi\times10^6\times4\pi\times10^{-7}} = \underline{\underline{36.48}}$$

$$\eta = |\eta|\angle\theta_\eta = \sqrt{\frac{\omega\mu}{\sigma}}\angle 45^o$$

$$|\eta| = \sqrt{\frac{\omega\mu}{\sigma}} = \sqrt{\frac{2\pi\times10^6\times4\pi\times10^{-7}}{36.48}} = 0.4652$$

$$E_o = |\eta| H_o = 0.4652\times20\times10^{-3} = 9.305\times10^{-3}$$

$$\mathbf{a}_E = \mathbf{a}_H\times\mathbf{a}_k = \mathbf{a}_y\times(-\mathbf{a}_z) = -\mathbf{a}_x$$

$$\mathbf{E} = E_o e^{-\alpha z}\sin(\omega t+\beta z)\mathbf{a}_E$$

$$\underline{\underline{= -9.305 e^{-12z}\sin(2\pi\times10^6 t + 12z + 45^o)\mathbf{a}_x \text{ mV/m}}}$$

**Prob. 10.25** For a good conductor, $\frac{\sigma}{\omega\varepsilon} >> 1$, say $\frac{\sigma}{\omega\varepsilon} > 100$

(a) $$\frac{\sigma}{\omega\varepsilon} = \frac{10^{-2}}{2\pi \times 8\times 10^6 \times 15\times \frac{10^{-9}}{36\pi}} = 1.5 \longrightarrow \text{lossy}$$

No, not conducting.

(b) $$\frac{\sigma}{\omega\varepsilon} = \frac{0.025}{2\pi \times 8\times 10^6 \times 16\times \frac{10^{-9}}{36\pi}} = 3.515 \longrightarrow \text{lossy}$$

No, not conducting.

(c ) $$\frac{\sigma}{\omega\varepsilon} = \frac{25}{2\pi \times 8\times 10^6 \times 81\times \frac{10^{-9}}{36\pi}} = 694.4 \longrightarrow \text{conducting}$$

Yes, conducting.

**Prob. 10.26**

$$\alpha = \beta = \frac{1}{\delta}$$

But $u = \frac{\omega}{\beta} = \delta\omega = 0.02\times 10^{-3} \times 2\pi \times 100\times 10^6 = 4\pi \times 10^3 = \underline{\underline{1.256\times 10^4 \text{ m/s}}}$

**Prob. 10.27 (a)**

$$R_{dc} = \frac{l}{\sigma S} = \frac{l}{\sigma\pi a^2} = \frac{600}{5.8\times 10^7 \times \pi \times (1.2)^2 \times 10^{-6}} = \underline{\underline{2.287\ \Omega}}$$

(b) $R_{ac} = \frac{l}{\sigma 2\pi a\delta}$. At 100 MHz, $\delta = 6.6\times 10^{-3}$ mm $=6.6\times 10^{-6}$ m mm for copper (see Table 10.2).

$$R_{ac} = \frac{600}{5.8\times 10^7 \times 2\pi \times (1.2\times 10^{-3}) \times 6.6\times 10^{-6}} = \underline{\underline{207.61\ \Omega}}$$

(c ) $$\frac{R_{ac}}{R_{dc}} = \frac{a}{2\delta} = 1 \longrightarrow \delta = a/2 = \frac{66.1\times 10^{-3}}{\sqrt{f}}$$

$$\sqrt{f} = \frac{66.1 \times 2 \times 10^{-3}}{a} = \frac{66.1 \times 2}{1.2} \longrightarrow f = \underline{\underline{12.137}} \text{ kHz}$$

**Prob. 10.28**

(a) Copper is a good conductor.

$$\alpha = \sqrt{\pi f \mu \sigma} = \sqrt{\pi \times 10^{10} \times 4\pi \times 10^{-7} \times 5.8 \times 10^{7}} = 2\pi \times 10^{5}\sqrt{5.8}$$

$$\underline{\underline{\alpha = 1.513 \times 10^{6} \text{ Np/m}}}$$

(b) $\delta = \frac{1}{\alpha} = \frac{1}{1.513 \times 10^{6}} = \underline{\underline{6.609 \times 10^{-7} \text{ m}}}$

(c) $\eta = \frac{1+j}{\sigma\delta} = \frac{1+j}{5.8 \times 10^{7} \times 6.609 \times 10^{-7}} = \underline{\underline{26.09(1+j) \times 10^{-3} \ \Omega}}$

**Prob. 10.29**

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}} \longrightarrow f = \frac{1}{\delta^{2} \pi \mu \sigma}$$

$$f = \frac{1}{4 \times 10^{-6} \times \pi \times 4\pi \times 10^{-7} \times 6.1 \times 10^{7}} = \underline{\underline{1.038 \text{ kHz}}}$$

**Prob. 10.30**

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}} = \frac{1}{\sqrt{\pi \times 5.8 \times 10^{7} (60) 4\pi \times 10^{-7}}} = \frac{1}{2\pi\sqrt{5.8(60)}} = \underline{\underline{8.531 \text{ mm}}}$$

**Prob. 10.31**

This is a good conductor.

$$\eta = \sqrt{\frac{\omega\mu}{\sigma}}\angle 45^o$$

$$24.6 = \sqrt{\frac{2\pi \times 12 \times 10^6 \times 4\pi \times 10^{-7}}{\sigma}} = \sqrt{\frac{16\pi^2(0.6)}{\sigma}}$$

$$\sigma = \frac{16\pi^2(0.6)}{24.6^2} = \underline{\underline{0.1566 \text{ S/m}}}$$

$$\beta = \alpha = \frac{1}{\delta} = \frac{1}{0.12} = \underline{\underline{8.333 \text{ rad/m}}}$$

$$\lambda = \frac{2\pi}{\beta} = 2\pi(0.12) = \underline{\underline{0.754 \text{ m}}}$$

$$u = \frac{\omega}{\beta} = \omega\delta = 2\pi \times 12 \times 10^6 \times 0.12 = \underline{\underline{9.05 \times 10^6 \text{ m/s}}}$$

**Prob. 10.32**

$$\frac{\sigma}{\omega\varepsilon} = \frac{0.12}{2\pi \times 2.42 \times 10^9 \times 5.5 \times \frac{10^{-9}}{36\pi}} = 0.1623$$

This is a lossy material.

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right]}$$

$$= 2\pi \times 2.42 \times 10^9 \sqrt{\frac{4\pi}{2} \times 10^{-7} \times 5.5 \times \frac{10^{-9}}{36\pi}\left[\sqrt{1+0.1623^2} - 1\right]}$$

$$= 15.21(10^9)(10^{-8})\sqrt{0.01308}$$

$$= 17.39$$

$$\delta = \frac{1}{\alpha} = \underline{\underline{57.5 \text{ mm}}}$$

**Prob. 10.33**

$$t = 5\delta = \frac{5}{\sqrt{\pi f \mu\sigma}} = \frac{5}{\sqrt{\pi \times 12 \times 10^9 \times 4\pi \times 10^{-7} \times 6.1 \times 10^7}} = \underline{\underline{2.94 \times 10^{-6} \text{ m}}}$$

**Prob. 10.34**

$$\frac{\sigma}{\omega\varepsilon} = \frac{4}{2\pi\times 2\times 10^9 \times 24\times \frac{10^{-9}}{36\pi}} = 1.5$$

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right]} = 2\pi\times 2\times 10^9 \sqrt{\frac{4\pi\times 10^{-7}\times 24\times \frac{10^{-9}}{36\pi}}{2}\left[\sqrt{1+(1.5)^2} - 1\right]}$$

$$= 130.01 \text{ Np/m}$$

$$10^{-5} E_o = E_o e^{-\alpha d}$$

Taking the log of both sides gives

$$\text{-5}\ln 10 = -\alpha d \qquad \longrightarrow \qquad d = \frac{5\ln 10}{\alpha} = \frac{5\ln 10}{130.01} = \underline{\underline{0.0886 \text{ m}}}$$

**Prob. 10.35**

(a) Linearly polarized along $\mathbf{a}_z$

(b) $\omega = 2\pi f = 2\pi \times 10^7 \qquad \longrightarrow \qquad f = 10^7 = \underline{\underline{10 \text{ MHz}}}$

(c )

$$\beta = \omega\sqrt{\mu\varepsilon} = \omega\sqrt{\mu_o\varepsilon_o}\sqrt{\varepsilon_r} = \frac{\omega}{c}\sqrt{\varepsilon_r}$$

$$\sqrt{\varepsilon_r} = \frac{\beta c}{\omega} = \frac{3\times 3\times 10^8}{2\pi\times 10^7} = 14.32 \qquad \longrightarrow \qquad \varepsilon_r = \underline{\underline{205.18}}$$

Let $\boldsymbol{H} = H_o \sin(\omega t - 3y)\boldsymbol{a}_H$

$$H_o = \frac{E_o}{\eta}, \qquad \eta = \sqrt{\frac{\mu}{\varepsilon}} = \frac{\eta_o}{\sqrt{\varepsilon_r}} = \frac{120\pi}{14.32} = 26.33$$

(d) $H_o = \frac{12}{26.33} = 0.456$

$$\boldsymbol{a}_H = \boldsymbol{a}_k \times \boldsymbol{a}_E = \boldsymbol{a}_y \times \boldsymbol{a}_z = \boldsymbol{a}_x$$

$$\boldsymbol{H} = \underline{\underline{0.456\sin(2\pi\times 10^7 t - 3y)\boldsymbol{a}_x \text{ A/m}}}$$

**Prob. 10.36**

$$\boldsymbol{E} = (2\boldsymbol{a}_y - 5\boldsymbol{a}_z)\sin(\omega t - \beta x)$$

The ratio $E_y / E_z$ remains the same as t changes. Hence the wave is <u>linearly</u> polarized

**Prob. 10.37**

(a)

$E_x = E_o\cos(\omega t + \beta y), \qquad E_y = E_o\sin(\omega t + \beta y)$

$$E_x(0,t) = E_o\cos\omega t \quad \longrightarrow \quad \cos\omega t = \frac{E_x(0,t)}{E_o}$$

$$E_y(0,t) = E_o\sin\omega t \quad \longrightarrow \quad \sin\omega t = \frac{E_y(0,t)}{E_o}$$

$$\cos^2\omega t + \sin^2\omega t = 1 \quad \longrightarrow \quad \left(\frac{E_x}{E_o}\right)^2 + \left(\frac{E_y}{E_o}\right)^2 = 1$$

Hence, we have circular polarization.

(b)

$E_x = E_o\cos(\omega t - \beta y), \qquad E_y = -3E_o\sin(\omega t - \beta y)$

In the y=0 plane,

$$E_x(0,t) = E_o\cos\omega t \quad \longrightarrow \quad \cos\omega t = \frac{E_x(0,t)}{E_o}$$

$$E_y(0,t) = E_o\sin\omega t \quad \longrightarrow \quad \sin\omega t = \frac{-E_y(0,t)}{3E_o}$$

$$\cos^2\omega t + \sin^2\omega t = 1 \quad \longrightarrow \quad \left(\frac{E_x}{E_o}\right)^2 + \frac{1}{9}\left(\frac{E_y}{E_o}\right)^2 = 1$$

Hence, we have elliptical polarization.

**Prob. 10.38**

(a) We can write

$\boldsymbol{E} = \mathrm{Re}(E_s e^{j\omega t}) = (40\boldsymbol{a}_x + 60\boldsymbol{a}_y)\cos(\omega t - 10z)$

Since $E_x / E_y$ does not change with time, the wave is linearly polarized.

(b) This is elliptically polarized.

**Prob. 10.39**

(a) The wave is elliptically polarized.

(b)

Let $\boldsymbol{E} = \boldsymbol{E}_1 + \boldsymbol{E}_2$,

where $\boldsymbol{E}_1 = 40\cos(\omega t - \beta z)\boldsymbol{a}_x$, $\quad \boldsymbol{E}_2 = 60\sin(\omega t - \beta z)\boldsymbol{a}_y$

$$\boldsymbol{H}_1 = H_{o1}\cos(\omega t - \beta z)\boldsymbol{a}_{H1}$$

$$H_{o1} = \frac{40}{\eta_o} = \frac{40}{120\pi} = 0.106$$

$$\boldsymbol{a}_{H1} = \boldsymbol{a}_k \times \boldsymbol{a}_E = \boldsymbol{a}_z \times \boldsymbol{a}_x = \boldsymbol{a}_y$$

$$\boldsymbol{H}_1 = 0.106\cos(\omega t - \beta z)\boldsymbol{a}_y$$

$$\boldsymbol{H}_2 = H_{o2}\sin(\omega t - \beta z)\boldsymbol{a}_{H2}$$

$$H_{o1} = \frac{60}{\eta_o} = \frac{60}{120\pi} = 0.1592$$

$$\boldsymbol{a}_{H2} = \boldsymbol{a}_k \times \boldsymbol{a}_E = \boldsymbol{a}_z \times \boldsymbol{a}_y = -\boldsymbol{a}_x$$

$$\boldsymbol{H}_2 = -0.1592\sin(\omega t - \beta z)\boldsymbol{a}_x$$

$$\boldsymbol{H} = \boldsymbol{H}_1 + \boldsymbol{H}_2 = \underline{\underline{-159.2\sin(\omega t - \beta z)\boldsymbol{a}_x + 106\cos(\omega t - \beta z)\boldsymbol{a}_y \text{ mA/m}}}$$

**Prob. 10.40**

We can write $\mathbf{E}_s$ as

$$\boldsymbol{E}_s = \boldsymbol{E}_1(z) + \boldsymbol{E}_2(z)$$

where

$$\boldsymbol{E}_1(z) = \frac{1}{2}E_o(\boldsymbol{a}_x - j\boldsymbol{a}_y)e^{-j\beta z}$$

$$\boldsymbol{E}_2(z) = \frac{1}{2}E_o(\boldsymbol{a}_x + j\boldsymbol{a}_y)e^{-j\beta z}$$

We recognize that $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$ are circularly polarized waves. The problem is therefore proved.

**Prob. 10.41**

(a)

When $\phi = 0$,

$$\boldsymbol{E}(y,t) = (E_{o1}\boldsymbol{a}_x + E_{o2}\boldsymbol{a}_z)\cos(\omega t - \beta y)$$

The two components are in phase and the wave is linearly polarized.

(b)

When $\phi = \pi/2$,

$$E_z = E_{o2}\cos(\omega t - \beta y + \pi/2) = -E_{o2}\sin(\omega t - \beta y)$$

We can combine $E_x$ and $E_z$ to show that the wave is elliptically polarized.

(c )

When $\phi = \pi$,

$$\boldsymbol{E}(y,t) = E_{o1}\cos(\omega t - \beta y)\boldsymbol{a}_x + E_{o2}\cos(\omega t - \beta y + \pi)\boldsymbol{a}_z$$

$$= (E_{o1}\,\boldsymbol{a}_x - E_{o2}\boldsymbol{a}_y)\cos(\omega t - \beta y)$$

Thus, the wave is linearly polarized.

**Prob. 10.42**

Let $\boldsymbol{E}_s = \boldsymbol{E}_r + j\boldsymbol{E}_i$ and $\boldsymbol{H}_s = \boldsymbol{H}_r + j\boldsymbol{H}_i$

$$\boldsymbol{E} = \text{Re}(\boldsymbol{E}_s e^{j\omega t}) = \boldsymbol{E}_r\cos\omega t - \boldsymbol{E}_i\sin\omega t$$

Similarly,

$$\boldsymbol{H} = \boldsymbol{H}_r\cos\omega t - \boldsymbol{H}_i\sin\omega t$$

$$\mathscr{P} = \boldsymbol{E}\times\boldsymbol{H} = \boldsymbol{E}_r\times\boldsymbol{H}_r\cos^2\omega t + \boldsymbol{E}_i\times\boldsymbol{H}_i\sin^2\omega t - \frac{1}{2}(\boldsymbol{E}_r\times\boldsymbol{H}_i + \boldsymbol{E}_i\times\boldsymbol{H}_r)\sin\ 2\omega t$$

$$\mathscr{P}_{\text{ave}} = \frac{1}{T}\int_0^T \mathscr{P}dt = \frac{1}{T}\int_0^T \cos^2\omega dt(\boldsymbol{E}_r\times\boldsymbol{H}_r) + \frac{1}{T}\int_0^T \sin^2\omega dt(\boldsymbol{E}_i\times\boldsymbol{H}_i) - \frac{1}{2T}\int_0^T \sin 2\omega dt(\boldsymbol{E}_i\times\boldsymbol{H}_i + \boldsymbol{E}_i\times\boldsymbol{H}_r)$$

$$= \frac{1}{2}(\boldsymbol{E}_r\times\boldsymbol{H}_r + \boldsymbol{E}_i\times\boldsymbol{H}_i) = \frac{1}{2}\text{Re}[(\boldsymbol{E}_r + j\boldsymbol{E}_i)\times(\boldsymbol{H}_r - j\boldsymbol{H}_i)]$$

$$\mathscr{P}_{\text{ave}} = \frac{1}{2}\ \text{Re}(\boldsymbol{E}_s\times\boldsymbol{H}_s^*)$$

as required.

**Prob. 10.43**

(a)

$$\beta = \omega\sqrt{\mu\varepsilon} = \omega\sqrt{\mu_o\varepsilon_o}\sqrt{\varepsilon_r} = \frac{\omega}{c}\sqrt{\varepsilon_r}$$

$$\sqrt{\varepsilon_r} = \frac{\beta c}{\omega} = \frac{8\times 3\times 10^8}{10^9} = 2.4$$

$$\varepsilon_r = \underline{\underline{5.76}}$$

(b) $\eta = \sqrt{\dfrac{\mu}{\varepsilon}} = \sqrt{\dfrac{\mu_o}{\varepsilon_o}}\dfrac{1}{\sqrt{\varepsilon_r}} = \dfrac{377}{2.4} = \underline{\underline{157.1\ \Omega}}$

(c ) $u = \dfrac{\omega}{\beta} = \dfrac{10^9}{8} = \underline{\underline{1.25\times 10^8\ \text{m/s}}}$

(d)

Let $\boldsymbol{H} = H_o \cos(10^9 t + 8x)\boldsymbol{a}_H$

$$H_o = \frac{E_o}{\eta} = \frac{150}{157.1} = 0.955$$

$$\boldsymbol{a}_H = \boldsymbol{a}_k \times \boldsymbol{a}_E = -\boldsymbol{a}_x \times \boldsymbol{a}_z = \boldsymbol{a}_y$$

$$\boldsymbol{H} = \underline{\underline{0.955\cos(10^9 t + 8x)\boldsymbol{a}_y \text{ A/m}}}$$

(e)

$$\mathscr{P} = \boldsymbol{E} \times \boldsymbol{H} = -150(0.955)\cos^2(10^9 t + 8x)\boldsymbol{a}_x$$

$$= \underline{\underline{-143.25\cos^2(10^9 t + 8x)\boldsymbol{a}_x \text{ W/m}^2}}$$

**Prob. 10.44**

$$\mathscr{P} = \boldsymbol{E} \times \boldsymbol{H} = \frac{E_o^2}{\eta}\cos^2(\omega t - 10z)\boldsymbol{a}_z$$

$$\mathscr{P}_{ave} = \frac{1}{T}\int_0^T \mathscr{P} dt = \frac{E_o^2}{2\eta}\boldsymbol{a}_z$$

$$P = \int_S \mathscr{P}_{ave} \bullet d\boldsymbol{S} = \frac{E_o^2 S}{2\eta} = \frac{(40)^2 \times \pi(1.5)^2}{2 \times 120\pi} = \frac{(60)^2}{240} = \underline{\underline{15 \text{ W}}}$$

**Prob. 10.45**

(a)

Let $\boldsymbol{H}_s = \dfrac{H_o}{r}\sin\theta e^{-j3r}\boldsymbol{a}_H$

$$H_o = \frac{E_o}{\eta_o} = \frac{10}{120\pi} = \frac{1}{12\pi}$$

$$\boldsymbol{a}_H = \boldsymbol{a}_k \times \boldsymbol{a}_E = \boldsymbol{a}_r \times \boldsymbol{a}_\theta = \boldsymbol{a}_\phi$$

$$\boldsymbol{H}_s = \underline{\underline{\frac{1}{12\pi r}\sin\theta e^{-j3r}\boldsymbol{a}_\phi \text{ A/m}}}$$

(b)

$$\mathscr{P}_{ave} = \frac{1}{2}\text{Re}(\boldsymbol{E}_s \times \boldsymbol{H}_s) = \frac{10}{2 \times 12\pi r^2}\sin^2\theta\boldsymbol{a}_r$$

$$P_{ave} = \int_S \mathscr{P}_{ave} \cdot d\boldsymbol{S}, \qquad d\boldsymbol{S} = r^2 \sin\theta d\theta d\phi \boldsymbol{a}_r$$

$$P_{ave} = \frac{10}{24\pi}\int_{\phi=0}^{\pi}\int_{\theta=0}^{\pi/6} r^2 \sin^3\theta d\theta \Big|_{r=2} = \frac{5}{8} - \frac{5\sqrt{3}}{32} = 0.007145$$

$$= \underline{\underline{7.145 \text{ mW}}}$$

**Prob. 10.46**

(a) $P_{ave} = \frac{1}{2}\text{Re}(E_s H_s^*) = \frac{1}{2}\text{Re}(\frac{|E_s|}{|\eta|}) = \frac{8^2}{2|\eta|}e^{-0.2z}$

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1\right]}$$
$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1\right]}$$

Let $x = \sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}$

$$\frac{\alpha}{\beta} = \frac{\sqrt{x-1}}{\sqrt{x+1}} = 0.1/0.3 = 1/3$$
$$\frac{x-1}{x+1} = \frac{1}{9} \longrightarrow x = 5/4$$

$$\frac{5}{4} = \sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} \longrightarrow \frac{\sigma}{\omega\varepsilon} = 3/4$$

$$|\eta| = \frac{\sqrt{\frac{\mu}{\varepsilon}}}{\sqrt[4]{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}} = \frac{120\pi/\sqrt{81}}{\sqrt{\frac{5}{4}}} = 37.4657$$
$$P_{ave} = \frac{64}{2(37.4657)}e^{-0.2z} = \underline{\underline{0.8541e^{-0.2z}\ \text{W/m}^2}}$$

(b) $20dB = 10\log\frac{P_1}{P_2} \longrightarrow \frac{P_1}{P_2} = 100$

$$\frac{P_2}{P_1} = e^{-0.2z} = \frac{1}{100} \longrightarrow e^{0.2z} = 100$$
$$z = 5\log 100 = \underline{\underline{23\text{ m}}}$$

**Prob. 10.47**

(a) $u = \omega / \beta \quad \longrightarrow \quad \omega = u\beta = \frac{\beta c}{\sqrt{4.5}} = \frac{2 \times 3 \times 10^8}{\sqrt{4.5}} = \underline{\underline{2.828 \times 10^8}}$ rad/s

$$\eta = \frac{120\pi}{\sqrt{4.5}} = 177.7\,\Omega$$

$$\boldsymbol{H} = \boldsymbol{a}_k \times \frac{\boldsymbol{E}}{\eta} = \frac{\boldsymbol{a}_z}{\eta} \times \frac{40}{\rho}\sin(\omega t - 2z)\boldsymbol{a}_\rho = \underline{\underline{\frac{0.225}{\rho}\sin(\omega t - 2z)\boldsymbol{a}_\phi}} \text{ A/m}$$

(b) $\boldsymbol{P} = \boldsymbol{E} \times \boldsymbol{H} = \underline{\underline{\frac{9}{\rho^2}\sin^2(\omega t - 2z)\boldsymbol{a}_z}}$ W/m$^2$

(c) $\mathcal{P}_{\text{ave}} = \frac{4.5}{\rho^2}\boldsymbol{a}_z$, $d\mathbf{S} = \rho d\phi d\rho \boldsymbol{a}_z$

$$P_{\text{ave}} = \int \mathbf{P}_{\text{ave}} \bullet dS = 4.5 \int_{2\text{mm}}^{3\text{mm}} \frac{d\rho}{\rho} \int_0^{2\pi} d\phi = 4.5\ln(3/2)(2\pi) = \underline{\underline{11.46 \text{ W}}}$$

**Prob. 10.48**

$$\mathcal{P} = \boldsymbol{E} \times \boldsymbol{H} = \frac{E_o^2}{\eta r^2}\sin^2\theta \sin^2 \omega(t - r/c)\boldsymbol{a}_r$$

$$\boldsymbol{P}_{ave} = \frac{1}{T}\int_0^T \mathcal{P}\, dt = \underline{\underline{\frac{E_o^2}{2\eta r^2}\sin^2\theta \boldsymbol{a}_r}}$$

**Prob. 10.49**

$\beta = \frac{\omega}{c} \quad \longrightarrow \quad \omega = \beta c = 40(3 \times 10^8) = \underline{\underline{12 \times 10^9 \text{ rad/s}}}$

$$\boldsymbol{E} = \frac{1}{\varepsilon}\int \nabla \times \boldsymbol{H} dt$$

$$\nabla \times \boldsymbol{H} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 10\sin(\omega t - 40x) & -20\sin(\omega t - 40x) \end{vmatrix}$$

$$= -800\cos(\omega t - 40x)\boldsymbol{a}_y - 400\cos(\omega t - 40x)\boldsymbol{a}_z$$

$$\boldsymbol{E} = \frac{1}{\varepsilon}\int \nabla \times \boldsymbol{H} dt = -\frac{800}{\omega\varepsilon}\sin(\omega t - 40x)\boldsymbol{a}_y - \frac{400}{\omega\varepsilon}\sin(\omega t - 40x)\boldsymbol{a}_z$$

$$= -\frac{800}{12\times 10^9 \times \frac{10^{-9}}{36\pi}}\sin(\omega t - 40x)\boldsymbol{a}_y - \frac{400}{12\times 10^9 \times \frac{10^{-9}}{36\pi}}\sin(\omega t - 40x)\boldsymbol{a}_z$$

$$= -7.539\sin(\omega t - 40x)\boldsymbol{a}_y - 3.77\sin(\omega t - 40x)\boldsymbol{a}_z \text{ kV/m}$$

$$\boldsymbol{P} = \boldsymbol{E}\times\boldsymbol{H} = \begin{vmatrix} 0 & E_y & E_z \\ 0 & H_y & H_z \end{vmatrix} = (E_y H_z - E_z H_y)\boldsymbol{a}_x$$

$$= \left[20(7.537)\sin^2(\omega t - 40x) + 37.7\sin^2(\omega t - 40x)\right]\boldsymbol{a}_x 10^3$$

$$\boldsymbol{P}_{\text{ave}} = \frac{1}{2}\left[20(7.537) + 37.7\right]\boldsymbol{a}_x 10^3 = 94.23\,\boldsymbol{a}_x \text{ kW/m}^2$$

**Prob. 10.50**

$$P = \frac{E_o^2}{2\eta_o} \longrightarrow E_o^2 = 2\eta_o P = 2(120\pi)10\times 10^{-3} = 7.539$$

$$E_o = 2.746 \text{ V/m}$$

**Prob. 10.51**

Let $T = \omega t - \beta z.$

$$-\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times \boldsymbol{E} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos T & \sin T & 0 \end{vmatrix}$$

$$-\mu\frac{\partial \boldsymbol{H}}{\partial t} = \beta\cos T\boldsymbol{a}_x + \beta\sin T\boldsymbol{a}_y$$

$$\boldsymbol{H} = -\frac{\beta}{\mu}\int\left[\cos T\boldsymbol{a}_x + \sin T\boldsymbol{a}_y\right]dt = -\frac{\beta}{\mu\omega}\sin T\boldsymbol{a}_x + \frac{\beta}{\mu\omega}\cos T\boldsymbol{a}_y$$

$$\mathscr{P} = \boldsymbol{E}\times\boldsymbol{H} = \begin{vmatrix} \cos T & \sin T & 0 \\ -\frac{\beta}{\mu\omega}\sin T & \frac{\beta}{\mu\omega}\cos T & 0 \end{vmatrix} = \frac{\beta}{\mu\omega}(\cos^2 T + \sin^2 T)\boldsymbol{a}_z$$

$$= \frac{\beta}{\mu\omega}\boldsymbol{a}_z = \sqrt{\frac{\varepsilon}{\mu}}\boldsymbol{a}_z$$

which is constant everywhere.

**Prob. 10.52**

$$\mathcal{P} = \frac{E_o^2}{2\eta_o}$$

$$P = \mathcal{P}S = \frac{E_o^2 S}{2\eta_o} = \frac{(2.4\times10^3)^2 \times 450\times10^{-4}}{2\times377} = \underline{\underline{343.8 \text{ W}}}$$

**Prob. 10.53**

$$\mathscr{P} = \boldsymbol{E}\times\boldsymbol{H} = \frac{V_o I_o}{2\pi\rho^2 \ln(b/a)}\sin^2(\omega t - \beta z)\boldsymbol{a}_z$$

(a) $$\mathscr{P}_{ave} = \frac{1}{T}\int_0^T \mathscr{P}dt = \frac{V_o I_o}{2\pi\rho^2 \ln(b/a)}\frac{1}{T}\int_0^T \sin^2(\omega t - \beta z)dt\boldsymbol{a}_z = \frac{V_o I_o}{2\pi\rho^2 \ln(b/a)}\frac{1}{2}\boldsymbol{a}_z$$

$$= \underline{\underline{\frac{V_o I_o}{4\pi\rho^2 \ln(b/a)}\boldsymbol{a}_z}}$$

(b)

$$P_{ave} = \int_S \mathscr{P}_{ave} \cdot d\boldsymbol{S}, \qquad d\boldsymbol{S} = \rho\, d\rho d\phi \boldsymbol{a}_z$$

$$= \frac{V_o I_o}{4\pi \ln(b/a)}\int_{\phi=0}^{2\pi}\int_{\rho=a}^{b}\frac{1}{\rho^2}\rho d\rho d\phi = \frac{V_o I_o}{4\pi \ln(b/a)}(2\pi)\ln(b/a)$$

$$= \underline{\underline{\frac{1}{2}V_o I_o}}$$

**Prob. 10.54**

(a) $$P_{i,ave} = \frac{E_{io}^{\ 2}}{2\eta_1}, \qquad P_{r,ave} = \frac{E_{ro}^{\ 2}}{2\eta_1}, \qquad P_{t,ave} = \frac{E_{to}^{\ 2}}{2\eta_2}$$

$$R = \frac{P_{r,ave}}{P_{i,ave}} = \frac{E_{ro}^{\ 2}}{E_{io}^{\ 2}} = \Gamma^2 = \underline{\underline{\left(\frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}\right)^2}}$$

$$R = \left(\frac{\sqrt{\frac{\mu_o}{\varepsilon_2}} - \sqrt{\frac{\mu_o}{\varepsilon_1}}}{\sqrt{\frac{\mu_o}{\varepsilon_2}} + \sqrt{\frac{\mu_o}{\varepsilon_1}}}\right)^2 = \left(\frac{\sqrt{\mu_o\varepsilon_1} - \sqrt{\mu_o\varepsilon_2}}{\sqrt{\mu_o\varepsilon_1} + \sqrt{\mu_o\varepsilon_2}}\right)^2$$

Since $n_1 = c\sqrt{\mu_1\varepsilon_1} = c\sqrt{\mu_o\varepsilon_1}, \qquad n_2 = c\sqrt{\mu_o\varepsilon_2},$

$$R = \left( \frac{n_1 - n_2}{n_1 + n_2} \right)^2$$

$$T = \frac{P_{t,ave}}{P_{i,ave}} = \frac{\eta_1}{\eta_2} \frac{E_{to}^2}{E_{io}^2} = \frac{\eta_1}{\eta_2} \tau^2 = \frac{\eta_1}{\eta_2} (1+\Gamma)^2 = \frac{4n_1 n_2}{(n_1 + n_2)^2}$$

(b) If $P_{r,ave} = P_{t,ave} \longrightarrow RP_{i,ave} = TP_{i.ave} \longrightarrow R = T$

i.e. $(n_1 - n_2)^2 = 4n_1 n_2 \longrightarrow n_1^2 - 6n_1 n_2 + n_2^2 = 0$

or $\left( \frac{n_1}{n_2} \right)^2 - 6\left( \frac{n_1}{n_2} \right) + 1 = 0,$ so

$$\frac{n_1}{n_2} = 3 \pm \sqrt{8} = \underline{\underline{5.828 \quad \text{or} \quad 0.1716}}$$

(Note that these values are mutual reciprocals, reflecting the inherent symmetry of the problem.)

**Prob. 10.55**

$$\eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}} = \sqrt{\frac{2\mu_o}{8\varepsilon_o}} = \frac{\eta_o}{2}, \qquad \eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}} = \sqrt{\frac{\mu_o}{16\varepsilon_o}} = \frac{\eta_o}{4}$$

$$\Gamma = \frac{\eta_1 - \eta_2}{\eta_1 + \eta_2} = \frac{\eta_o / 4 - \eta_o / 2}{\eta_o / 4 + \eta_o / 2} = -\frac{1}{3}, \qquad \tau = \Gamma + 1 = \frac{2}{3}$$

$$\boldsymbol{E}_r = -\frac{1}{3}(60)\sin(\omega t + 10z)\boldsymbol{a}_x - \frac{1}{3}(30)\sin(\omega t + 10z + \pi / 6)\boldsymbol{a}_y$$

$$\underline{\underline{= -20\sin(\omega t + 10z)\boldsymbol{a}_x - 10\sin(\omega t + 10z + \pi / 6)\boldsymbol{a}_y \text{ V/m}}}$$

$$\boldsymbol{E}_t = \frac{2}{3}(60)\sin(\omega t - 10z)\boldsymbol{a}_x + \frac{2}{3}(30)\sin(\omega t - 10z + \pi / 6)\boldsymbol{a}_y$$

$$\underline{\underline{= 40\sin(\omega t - 10z)\boldsymbol{a}_x + 20\sin(\omega t - 10z + \pi / 6)\boldsymbol{a}_y \text{ V/m}}}$$

**Prob. 10.56**

$$\eta_1 = \eta_o = 120\pi, \qquad \eta_2 = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_o}{9\varepsilon_o}} = \frac{\eta_o}{3} = 40\pi$$

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{\eta_o/3 - \eta_o}{\eta_o/3 + \eta_o} = -\frac{1}{2}, \qquad \tau = 1 + \Gamma = \frac{1}{2}$$

$$E_r = \Gamma E_o = -E_o/2, \qquad E_t = \tau E_o = E_o/2$$

$$P_{iave} = \frac{|E_o|^2}{2\eta_1} = \frac{E_o^2}{2\eta_o}$$

$$P_{tave} = \frac{|E_t|^2}{2\eta_2} = \frac{\frac{1}{4}E_o^2}{2(\eta_o/3)} = \frac{E_o^2}{2\eta_o}\frac{3}{4}$$

$$\frac{P_{tave}}{P_{iave}} = \frac{3}{4} = \underline{\underline{0.75}}$$

**Prob. 10.57**

(a) $\eta_1 = \eta_o$

$$\boldsymbol{E}_i = E_{io}\sin(\omega t - 5x)\boldsymbol{a}_E$$

$$E_{io} = H_{io}\eta_o = 120\pi \times 4 = 480\pi$$

$$\boldsymbol{a}_E \times \boldsymbol{a}_H = \boldsymbol{a}_k \longrightarrow \boldsymbol{a}_E \times \boldsymbol{a}_y = \boldsymbol{a}_x \longrightarrow \boldsymbol{a}_E = -\boldsymbol{a}_z$$

$$\boldsymbol{E}_i = -480\pi\sin(\omega t - 5x)\boldsymbol{a}_z$$

$$\eta_2 = \sqrt{\frac{\mu_o}{4\varepsilon_o}} = \frac{120\pi}{\sqrt{4}} = 60\pi$$

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{60\pi - 120\pi}{60\pi + 120\pi} = -1/3, \qquad \tau = 1 + \Gamma = 2/3$$

$$E_{ro} = \Gamma E_{io} = (-1/3)(-480\pi) = 160\pi$$

$$\boldsymbol{E}_r = 160\pi\sin(\omega t + 5x)\boldsymbol{a}_z$$

$$\boldsymbol{E}_1 = \boldsymbol{E}_i + \boldsymbol{E}_r = \underline{\underline{-1.508\sin(\omega t - 5x)\boldsymbol{a}_z + 0.503\sin(\omega t + 5x)\boldsymbol{a}_z}} \text{ kV/m}$$

(b) $E_{to} = \tau E_{io} = (2/3)(480\pi) = 320\pi$

$$\mathcal{P} = \frac{E_{to}^{\ 2}}{2\eta_2}\boldsymbol{a}_x = \frac{(320\pi)^2}{2(60\pi)}\boldsymbol{a}_x = \underline{\underline{2.68\boldsymbol{a}_x \text{ kW/m}^2}}$$

(c) $s = \dfrac{1+|\Gamma|}{1-|\Gamma|} = \dfrac{1+1/3}{1-1/3} = \underline{\underline{2}}$

**Prob. 10.58** $\eta_1 = \sqrt{\dfrac{\mu_1}{\varepsilon_1}} = \eta_o/2, \qquad \eta_2 = \eta_o$

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = 1/3, \qquad \tau = 1 + \Gamma = 4/3$$

$$E_{or} = \Gamma E_{io} = (1/3)(5) = 5/3, \qquad E_{ot} = \tau E_{io} = 20/3$$

$$\beta = \frac{\omega}{c}\sqrt{\mu_r \varepsilon_r} = \frac{10^8}{3\times 10^8}\sqrt{4} = 2/3$$

(a) $\boldsymbol{E}_r = \dfrac{5}{3}\cos(10^8 t - 2y/3)\boldsymbol{a}_z$

$$\boldsymbol{E}_1 = \boldsymbol{E}_i + \boldsymbol{E}_r = \underline{\underline{5\cos(10^8 t + \frac{2}{3}y)\boldsymbol{a}_z + \frac{5}{3}\cos(10^8 t - \frac{2}{3}y)\boldsymbol{a}_z}} \text{ V/m}$$

(b) $\mathcal{P}_{\text{ave1}} = \dfrac{E_{io}^{\ 2}}{2\eta_1}(-\boldsymbol{a}_y) + \dfrac{E_{ro}^{\ 2}}{2\eta_1}(+\boldsymbol{a}_y) = \dfrac{25}{2(60\pi)}(1-\dfrac{1}{9})(-\boldsymbol{a}_y) = \underline{\underline{-0.0589\boldsymbol{a}_y \text{ W/m}^2}}$

(c ) $\mathcal{P}_{\text{ave2}} = \dfrac{E_{to}^{\ 2}}{2\eta_2}(-\boldsymbol{a}_y) = \dfrac{400}{9(2)(120\pi)}(-\boldsymbol{a}_y) = \underline{\underline{-0.0589\boldsymbol{a}_y \text{ W/m}^2}}$

**Prob. 10.59**

$$\beta_1 = \omega\sqrt{\mu_1\varepsilon_1} = \omega\sqrt{\mu_o\varepsilon_o}\sqrt{\mu_{r1}\varepsilon_{r1}} = \frac{\omega}{c}\sqrt{16} = \frac{90\times10^9(4)}{3\times10^8} = 300(4) = 1200$$

$$\beta_2 = \omega\sqrt{\mu_o\varepsilon_o} = \frac{90\times10^9}{3\times10^8} = 300$$

$$\eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}} = \sqrt{\frac{\mu_o}{\varepsilon_o}}\sqrt{\frac{\mu_{r1}}{\varepsilon_{r1}}} = \frac{\eta_o}{4} = 30\pi, \qquad \eta_2 = \eta_o = 120\pi$$

$$\Gamma = \frac{\eta_2-\eta_1}{\eta_2+\eta_1} = \frac{\eta_o-\eta_o/4}{\eta_o+\eta_o/4} = \frac{3}{5}, \qquad \tau = 1+\Gamma = \frac{8}{5}$$

$$\mathscr{P} = \boldsymbol{E}\times\boldsymbol{H}$$

$$\mathscr{P}_1 = \mathscr{P}_i + \mathscr{P}_r = \frac{60^2}{\eta_1}\cos^2(\omega t-\beta_1 z)\boldsymbol{a}_x + \frac{(60\times\frac{3}{5})^2}{\eta_1}\cos^2(\omega t+\beta_1 z)\boldsymbol{a}_x$$

$$\frac{60^2}{\eta_1} = \frac{60^2}{30\pi} = 38.197, \qquad \frac{(60\times\frac{3}{5})^2}{\eta_1} = 13.75$$

$$\underline{\underline{\mathscr{P}_1 = 38.197\cos^2(\omega t-\beta_1 z)\boldsymbol{a}_x + 13.75\cos^2(\omega t+\beta_1 z)\boldsymbol{a}_x \text{ W/m}^2,}} \quad \text{where } \beta_1 = 1200$$

$$\mathscr{P}_2 = \mathscr{P}_t = \frac{(60\times\frac{8}{5})^2}{\eta_o}\cos^2(\omega t-\beta_1 z)\boldsymbol{a}_x$$

$$\frac{(60\times\frac{8}{5})^2}{\eta_o} = \frac{96^2}{120\pi} = 24.46$$

$$\underline{\underline{\mathscr{P}_2 = 24.46\cos^2(\omega t-\beta_2 z)\boldsymbol{a}_x \text{ W/m}^2,}} \quad \text{where } \beta_2 = 300$$

**Prob. 10.60**

(a) In air, $\beta_1 = 1, \lambda_1 = 2\pi/\beta_1 = 2\pi = \underline{\underline{6.283 \text{ m}}}$

$$\omega = \beta_1 c = \underline{\underline{3\times10^8 \text{ rad/s}}}$$

In the dielectric medium, $\omega$ is the same .

$$\omega = \underline{\underline{3\times10^8 \text{ rad/s}}}$$

$$\beta_2 = \frac{\omega}{c}\sqrt{\varepsilon_{r2}} = \beta_1\sqrt{\varepsilon_{r2}} = \sqrt{3}$$

$$\lambda_2 = \frac{2\pi}{\beta_2} = \frac{2\pi}{\sqrt{3}} = \underline{\underline{3.6276 \text{ m}}}$$

(b) $H_o = \dfrac{E_o}{\eta_o} = \dfrac{10}{120\pi} = 0.0265$

$$\mathbf{a}_H = \mathbf{a}_k \times \mathbf{a}_E = \mathbf{a}_z \times \mathbf{a}_y = -\mathbf{a}_x$$

$$\mathbf{H}_i = \underline{\underline{-26.5\cos(\omega t - z)\mathbf{a}_x \text{ mA/m}}}$$

(c ) $\eta_1 = \eta_o, \quad \eta_2 = \eta_o / \sqrt{3}$

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{(1/\sqrt{3}) - 1}{(1/\sqrt{3}) + 1} = \underline{\underline{-0.268}}, \qquad \tau = 1 + \Gamma = \underline{\underline{0.732}}$$

(d) $E_{to} = \tau E_{io} = 7.32, \quad E_{ro} = \Gamma E_{io} = -2.68$

$$\mathbf{E}_1 = \mathbf{E}_i + \mathbf{E}_r = \underline{\underline{10\cos(\omega t - z)\mathbf{a}_y - 2.68\cos(\omega t + z)\mathbf{a}_y \text{ V/m}}}$$

$$\mathbf{E}_2 = \mathbf{E}_t = \underline{\underline{7.32\cos(\omega t - z)\mathbf{a}_y \text{ V/m}}}$$

$$\mathcal{P}_{\text{ave1}} = \frac{1}{2\eta_1}(\mathbf{a}_z)[E_{io}^{\ 2} - E_{ro}^{\ 2}] = \frac{1}{2(120\pi)}(\mathbf{a}_z)(10^2 - 2.68^2) = \underline{\underline{0.1231\mathbf{a}_z \text{ W/m}^2}}$$

$$\mathcal{P}_{\text{ave2}} = \frac{E_{to}^{\ 2}}{2\eta_2}(\mathbf{a}_z) = \frac{\sqrt{3}}{2\times 120\pi}(7.32)^2(\mathbf{a}_z) = \underline{\underline{0.1231\mathbf{a}_z \text{ W/m}^2}}$$

**Prob. 10.61**

$\eta_1 = \eta_o = 120\pi$

For seawater (lossy medium),

$$\eta_2 = \sqrt{\frac{j\omega\mu_o}{\sigma + j\omega\varepsilon}} = \sqrt{\frac{j2\pi\times 10^8 \times 4}{4 + j2\pi\times 10^8 \times 81 \times \dfrac{10^{-9}}{36\pi}}} = 10.44 + j9.333$$

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = 0.9461\angle 177.16$$

$|\Gamma|^2 = 0.8952, \quad 1 - |\Gamma| = 0.1084$

$$\frac{P_r}{P_i} = \underline{\underline{89.51\%}}, \qquad \frac{P_t}{P_i} = \underline{\underline{10.84\%}},$$

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{7.924\angle 43.975 - 377}{7.924\angle 43.975 + 377} = 0.9702\angle 178.2^o$$

The fraction of the incident power reflected is

$$\frac{P_r}{P_i} = |\Gamma|^2 = 0.9702^2 = \underline{\underline{0.9413}}$$

The transmitted fraction is

$$\frac{P_t}{P_i} = 1 - |\Gamma|^2 = 1 - 0.9702^2 = \underline{\underline{0.0587}}$$

**Prob. 10.62**

(a)

$$\eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}} = \frac{120\pi}{\sqrt{4}} = 188.5, \qquad \eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}} = \frac{120\pi}{\sqrt{3.2}} = 210.75$$

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{210.75 - 188.5}{210.75 + 188.5} = 0.0557, \qquad \tau = \frac{2\eta_2}{\eta_2 + \eta_1} = \frac{2 \times 210.75}{210.75 + 188.5} = 1.0557$$

$$E_{ro} = \Gamma E_{io} = (0.0557)(12) = 0.6684 \qquad E_{to} = \tau E_{io} = 1.0557(12) = 12.668$$

$$\beta_1 = \omega\sqrt{\mu_1\varepsilon_1} = \frac{\omega}{c}\sqrt{4} \longrightarrow \omega = \frac{\beta_1 c}{2} = \frac{40\pi(3\times 10^8)}{2} = \underline{\underline{6\pi \times 10^9 \text{ rad/s}}}$$

(b)

$$\beta_2 = \omega\sqrt{\mu_2\varepsilon_2} = \frac{\omega}{c}\sqrt{3.2} = \frac{6\pi \times 10^9\sqrt{3.2}}{3\times 10^8} = 112.4$$

$$\mathbf{E}_r = E_{ro}\cos(\omega t + 40\pi x)\mathbf{a}_z = \underline{\underline{0.6684\cos(6\pi \times 10^9 t + 40\pi x)\mathbf{a}_z \text{ V/m}}}$$

$$\mathbf{E}_t = E_{to}\cos(\omega t - \beta_2 x)\mathbf{a}_z = \underline{\underline{12.668\cos(6\pi \times 10^9 t - 112.4x)\mathbf{a}_z \text{ V/m}}}$$

**Prob. 10.63** (a) $\omega = \beta c = 3 \times 3 \times 10^8 = \underline{\underline{9 \times 10^8 \text{ rad/s}}}$

(b) $\lambda = 2\pi / \beta = 2\pi / 3 = \underline{\underline{2.094 \text{ m}}}$

(c) $\dfrac{\sigma}{\omega\varepsilon} = \dfrac{4}{9\times 10^8 \times 80 \times 10^{-9} / 36\pi} = 2\pi = \underline{\underline{6.288}}$

$$\tan 2\theta_\eta = \frac{\sigma}{\omega\varepsilon} = 6.288 \longrightarrow \theta_\eta = 40.47^o$$

$$|\eta_2| = \frac{\sqrt{\mu_2/\varepsilon_2}}{\sqrt[4]{1 + \left(\frac{\sigma_2}{\omega\varepsilon_2}\right)^2}} = \frac{377/\sqrt{80}}{\sqrt[4]{1 + 4\pi^2}} = 16.71$$

$\eta_2 = \underline{\underline{16.71\angle 40.47^o \ \Omega}}$

(d) $\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{16.71\angle 40.47^o - 377}{16.71\angle 40.47^o + 377} = 0.935\angle 179.7^o$

$E_{or} = \Gamma E_{oi} = 9.35\angle 179.7^o$

$\underline{\underline{\boldsymbol{E}_r = 9.35\sin(\omega t - 3z + 179.7)\boldsymbol{a}_x \ \text{V/m}}}$

$$\alpha_2 = \frac{\omega}{c}\sqrt{\frac{\mu_{r2}\varepsilon_{r2}}{2}\left[\sqrt{1+\left(\frac{\sigma_2}{\omega\varepsilon_2}\right)^2} - 1\right]} = \frac{9\times10^8}{3\times10^8}\sqrt{\frac{80}{2}\left[\sqrt{1+4\pi^2} - 1\right]} = 43.94 \ \text{Np/m}$$

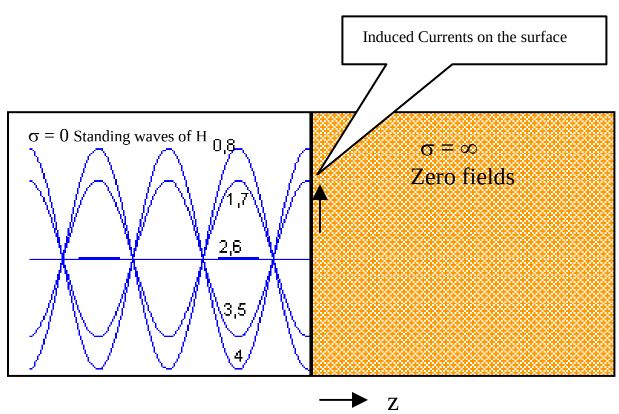
$$\beta_2 = \frac{9\times10^8}{3\times10^8}\sqrt{\frac{80}{2}\left[\sqrt{1+4\pi^2} + 1\right]} = 51.48 \ \text{rad/m}$$

$$\tau = \frac{2\eta_2}{\eta_2 + \eta_1} = \frac{2\times 16.71\angle 40.47^o}{16.71\angle 40.47^o + 377} = 0.0857\angle 38.89^o$$

$E_{ot} = \tau E_o = 0.857\angle 38.89^o$

$\underline{\underline{\boldsymbol{E}_t = 0.857 e^{43.94z}\sin(9\times10^8 t + 51.48z + 38.89^o)\boldsymbol{a}_x \ \text{V/m}}}$

**Prob. 10.64**




Curve 0 is at t = 0; curve 1 is at t = T/8; curve 2 is at t = T/4; curve 3 is at t = 3T/8, etc.

**Prob. 10.65** Since $\mu_o = \mu_1 = \mu_2$,

$$\sin\theta_{t1} = \sin\theta_i\sqrt{\frac{\varepsilon_o}{\varepsilon_1}} = \frac{\sin 45^o}{\sqrt{4.5}} = 0.3333 \longrightarrow \underline{\underline{\theta_{t1} = 19.47^o}}$$

$$\sin\theta_{t2} = \sin\theta_{t1}\sqrt{\frac{\varepsilon_1}{\varepsilon_2}} = \frac{1}{3}\sqrt{\frac{4.5}{2.25}} = 0.4714 \longrightarrow \underline{\underline{\theta_{t2} = 28.13^o}}$$

**Prob. 10.66**

$$\boldsymbol{E}_s = \frac{20(e^{jk_x x} - e^{-jk_x x})}{j2}\frac{(e^{jk_y y} + e^{-jk_y y})}{2}\boldsymbol{a}_z$$

$$= -j5\left[e^{j(k_x x + k_y y)} + e^{j(k_x x - k_y y)} - e^{-j(k_x x - k_y y)} - e^{-j(k_x x + k_y y)}\right]\boldsymbol{a}_z$$

which consists of four plane waves.

$$\nabla\times\boldsymbol{E}_s = -j\omega\mu_o\boldsymbol{H}_s \longrightarrow \boldsymbol{H}_s = \frac{j}{\omega\mu_o}\nabla\times\boldsymbol{E}_s = \frac{j}{\omega\mu_o}\left(\frac{\partial E_z}{\partial y}\boldsymbol{a}_x - \frac{\partial E_z}{\partial x}\boldsymbol{a}_y\right)$$

$$\underline{\underline{\boldsymbol{H}_s = -\frac{j20}{\omega\mu_o}\left[k_y\sin(k_x x)\sin(k_y y)\boldsymbol{a}_x + k_x\cos(k_x x)\cos(k_y y)\boldsymbol{a}_y\right]}}$$

**Prob. 10.67**

$\eta_1 = \eta_o = 377\Omega$

For $\eta_2$,

$$\frac{\sigma_2}{\omega\varepsilon_2} = \frac{4}{2\pi\times1.2\times10^9\times50\times\frac{10^{-9}}{36\pi}} = 1.2$$

$$\tan 2\theta_{\eta_2} = \frac{\sigma_2}{\omega\varepsilon_2} = 1.2 \longrightarrow \theta_{\eta_2} = 25.1^o$$

$$|\eta_2| = \frac{\sqrt{\mu/\varepsilon}}{\sqrt[4]{1+\left(\frac{\sigma_2}{\omega\varepsilon_2}\right)^2}} = \frac{120\pi\sqrt{1/50}}{\sqrt[4]{1+1.2^2}} = 42.658$$

$$\eta_2 = 42.658\angle 25.1^o$$

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{42.658\angle 25.1^o - 377}{42.658\angle 25.1^o + 377} = \underline{\underline{0.8146\angle 174.4^o}}$$

**Prob. 10.68**

(a)

$$P_t = (1-|\Gamma|^2)P_i$$

$$s = \frac{1+|\Gamma|}{1-|\Gamma|} \longrightarrow |\Gamma| = \frac{s-1}{s+1}$$

$$\frac{P_t}{P_i} = 1 - \left(\frac{s-1}{s+1}\right)^2 = \underline{\underline{\frac{4s}{(s+1)^2}}}$$

(b) $P_i = P_r + P_t \longrightarrow \frac{P_r}{P_i} = 1 - \frac{P_t}{P_i} = \underline{\underline{\left(\frac{s-1}{s+1}\right)^2}}$

**Prob. 10.69**

If **A** is a uniform vector and $\Phi(r)$ is a scalar,

$$\nabla \times (\Phi \boldsymbol{A}) = \nabla\Phi \times \boldsymbol{A} + \Phi(\nabla \times \boldsymbol{A}) = \nabla\Phi \times \boldsymbol{A}$$

since $\nabla \times \boldsymbol{A} = \boldsymbol{0}$.

$$\nabla \times \boldsymbol{E} = (\frac{\partial}{\partial x}\boldsymbol{a}_x + \frac{\partial}{\partial y}\boldsymbol{a}_y + \frac{\partial}{\partial z}\mathrm{a}_z) \times \boldsymbol{E}_o e^{j(k_x x + k_y y + k_z z - \omega t)} = j(k_x \boldsymbol{a}_x + k_y \boldsymbol{a}_y + k_z \boldsymbol{a}_z)e^{j(\boldsymbol{k}\bullet\mathbf{r} - \omega t)} \times \boldsymbol{E}_o$$

$$= j\boldsymbol{k} \times \boldsymbol{E}_o e^{j(\boldsymbol{k}\bullet\mathbf{r} - \omega t)} = j\boldsymbol{k} \times \boldsymbol{E}$$

Also, $-\frac{\partial \boldsymbol{B}}{\partial t} = j\omega\mu\boldsymbol{H}$ . Hence $\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$ becomes $\boldsymbol{k} \times \boldsymbol{E} = \omega\mu\boldsymbol{H}$

From this, $\underline{\underline{\boldsymbol{a}_k \times \boldsymbol{a}_E = \boldsymbol{a}_H}}$

**Prob. 10.70**

$$k = |\mathrm{k}| = \sqrt{124^2 + 124^2 + 263^2} = 316.1$$

$$\lambda = \frac{2\pi}{k} = \underline{\underline{19.88 \text{ mm}}}$$

$$k = \omega\sqrt{\mu\varepsilon} = \frac{2\pi f}{c} \longrightarrow f = \frac{kc}{2\pi} = \frac{316.1 \times 3 \times 10^8}{2\pi} = \underline{\underline{15.093 \text{ GHz}}}$$

$$\boldsymbol{k} \bullet \boldsymbol{a}_x = k\cos\theta_x \longrightarrow \cos\theta_x = \frac{124}{316.1} \longrightarrow \theta_x = 66.9^o = \theta_y$$

$$\theta_z = \cos^{-1}\frac{263}{361.1} = 33.69^o$$

Thus,

$$\underline{\underline{\theta_x = \theta_y = 66.9^o, \quad \theta_z = 33.69^o}}$$

**Prob. 10.71**

$\boldsymbol{k} = -3.4\boldsymbol{a}_x + 4.2\boldsymbol{a}_y$

$\boldsymbol{k}\cdot\boldsymbol{E} = 0 \longrightarrow 0 = -3.4E_o + 4.2$

$E_o = \dfrac{4.2}{3.4} = \underline{\underline{1.235}}$

$k = |\boldsymbol{k}| = \beta = \sqrt{(-3.4)^2 + (4.2)^2} = 5.403$

$\lambda = \dfrac{2\pi}{\beta} = \dfrac{2\pi}{5.403} = 1.162$

$f = \dfrac{c}{\lambda} = \dfrac{3\times10^8}{1.162} = \underline{\underline{258\text{ MHz}}}$

$$\boldsymbol{H}_s = \frac{1}{\mu\omega}\boldsymbol{k}\times\boldsymbol{E}_s = \frac{1}{\mu kc}\boldsymbol{k}\times\boldsymbol{E}_s$$

$$= \frac{1}{4\pi\times10^{-7}\times5.403\times3\times10^8}\begin{vmatrix} -3.4 & 4.2 & 0 \\ E_o & 1 & 3+j4 \end{vmatrix} A_o$$

where $A_o = e^{-j3.4x+4.2y}$

$$\boldsymbol{H}_s = 4.91A_o\times10^{-4}\left[4.2(3+j4)\boldsymbol{a}_x + 3.4(3+j4)\boldsymbol{a}_y + (-3.4-4.2E_o)\boldsymbol{a}_z\right]$$

$$= 0.491\left[(12.6+j16.8)\boldsymbol{a}_x + (10.2+j13.6)\boldsymbol{a}_y - 8.59\boldsymbol{a}_z\right]e^{-j3.4x+4.2y}\ \text{mA/m}$$

**Prob.10.72**

$$\nabla\bullet\boldsymbol{E} = \left(\frac{\partial}{\partial x}\boldsymbol{a}_x + \frac{\partial}{\partial y}\boldsymbol{a}_y + \frac{\partial}{\partial z}\boldsymbol{a}_z\right)\bullet\boldsymbol{E}_o e^{j(k_x x+k_y y+k_z z-\omega t)} = j(k_x\boldsymbol{a}_x + k_y\boldsymbol{a}_y + k_z\boldsymbol{a}_z)e^{j(\mathbf{k}\bullet\mathbf{r}-\omega t)}\bullet\boldsymbol{E}_o$$

$$= j\boldsymbol{k}\bullet\boldsymbol{E}_o e^{j(\mathbf{k}\bullet\mathbf{r}-\omega t)} = j\boldsymbol{k}\bullet\boldsymbol{E} = 0 \longrightarrow \boldsymbol{k}\bullet\boldsymbol{E} = 0$$

Similarly,

$$\nabla\bullet\boldsymbol{H} = j\boldsymbol{k}\bullet\boldsymbol{H} = 0 \longrightarrow \boldsymbol{k}\bullet\boldsymbol{H} = 0$$

It has been shown in the previous problem that

$$\nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t} \longrightarrow \boldsymbol{k}\times\boldsymbol{E} = \omega\mu\boldsymbol{H}$$

Similarly,

$$\nabla\times\boldsymbol{H} = \frac{\partial\boldsymbol{D}}{\partial t} \longrightarrow kxH = -\varepsilon\omega E$$

From $\boldsymbol{k} \times \boldsymbol{E} = \omega\mu\boldsymbol{H}$ , $\boldsymbol{a}_k \times \boldsymbol{a}_E = \boldsymbol{a}_H$ and

From $\boldsymbol{k} \times \boldsymbol{H} = -\varepsilon\omega\boldsymbol{E}$ , $\boldsymbol{a}_k \times \boldsymbol{a}_H = -\boldsymbol{a}_E$

**Prob. 10.73**

If $\mu_o = \mu_1 = \mu_2, \quad \eta_1 = \dfrac{\eta_o}{\sqrt{\varepsilon_{r1}}}, \eta_2 = \dfrac{\eta_o}{\sqrt{\varepsilon_{r2}}}$

$$\Gamma_{\backslash\backslash} = \frac{\dfrac{1}{\sqrt{\varepsilon_{r2}}}\cos\theta_t - \dfrac{1}{\sqrt{\varepsilon_{r1}}}\cos\theta_i}{\dfrac{1}{\sqrt{\varepsilon_{r2}}}\cos\theta_t + \dfrac{1}{\sqrt{\varepsilon_{r1}}}\cos\theta_i}$$

$$\sqrt{\varepsilon_{r1}}\sin\theta_i = \sqrt{\varepsilon_{r2}}\sin\theta_t \quad \longrightarrow \quad \frac{\sqrt{\varepsilon_{r2}}}{\sqrt{\varepsilon_{r1}}} = \frac{\sin\theta_i}{\sin\theta_t}$$

$$\Gamma_{\backslash\backslash} = \frac{\cos\theta_t - \dfrac{\sin\theta_i}{\sin\theta_t}\cos\theta_i}{\cos\theta_t + \dfrac{\sin\theta_i}{\sin\theta_t}\cos\theta_i} = \frac{\sin\theta_t\cos\theta_t - \sin\theta_i\cos\theta_i}{\sin\theta_t\cos\theta_t + \sin\theta_i\cos\theta_i}$$

$$= \frac{\sin 2\theta_t - \sin 2\theta_i}{\sin 2\theta_t + \sin 2\theta_i} = \frac{\sin(\theta_t - \theta_i)\cos(\theta_t + \theta_i)}{\cos(\theta_t - \theta_i)\sin(\theta_t + \theta_i)} = \frac{\tan(\theta_t - \theta_i)}{\tan(\theta_t + \theta_i)}$$

Similarly,

$$\tau_{\backslash\backslash} = \frac{\dfrac{2}{\sqrt{\varepsilon_{r2}}}\cos\theta_i}{\dfrac{1}{\sqrt{\varepsilon_{r2}}}\cos\theta_t + \dfrac{1}{\sqrt{\varepsilon_{r1}}}\cos\theta_i} = \frac{2\cos\theta_i}{\cos\theta_t + \dfrac{\sin\theta_i}{\sin\theta_t}\cos\theta_i}$$

$$= \frac{2\cos\theta_i\sin\theta_t}{\sin\theta_t\cos\theta_t(\sin^2\theta_i + \cos^2\theta_i) + \sin\theta_i\cos\theta_i(\sin^2\theta_t + \cos^2\theta_t)}$$

$$= \frac{2\cos\theta_i\sin\theta_t}{(\sin\theta_i\cos\theta_t + \sin\ \theta_t\cos\ \theta_i)(\cos\theta_i\cos\theta_t + \sin\ \theta_i\sin\ \theta_t)}$$

$$= \frac{2\cos\theta_i\sin\theta_t}{\sin(\theta_i + \theta_t)\cos(\theta_i - \theta_t)}$$

$$\Gamma_\perp = \frac{\frac{1}{\sqrt{\varepsilon_{r2}}}\cos\theta_i - \frac{1}{\sqrt{\varepsilon_{r1}}}\cos\theta_t}{\frac{1}{\sqrt{\varepsilon_{r2}}}\cos\theta_i + \frac{1}{\sqrt{\varepsilon_{r1}}}\cos\theta_t} = \frac{\cos\theta_i - \frac{\sin\theta_i}{\sin\theta_t}\cos\theta_t}{\cos\theta_i + \frac{\sin\theta_i}{\sin\theta_t}\cos\theta_t} = \frac{\sin(\theta_t - \theta_i)}{\sin(\theta_t + \theta_i)}$$

$$\tau_\perp = \frac{\frac{2}{\sqrt{\varepsilon_{r2}}}\cos\theta_i}{\frac{1}{\sqrt{\varepsilon_{r2}}}\cos\theta_i + \frac{1}{\sqrt{\varepsilon_{r1}}}\cos\theta_t} = \frac{2\cos\theta_i}{\cos\theta_i + \frac{\sin\theta_i}{\sin\theta_t}\cos\theta_t} = \frac{2\cos\theta_i \sin\theta_i}{\sin(\theta_t + \theta_i)}$$

**Prob. 10.74**

(a) $n_1 = 1, \quad n_2 = c\sqrt{\mu_2\varepsilon_2} = c\sqrt{6.4\varepsilon_o \times \mu_o} = \sqrt{6.4} = 2.5298$

$$\sin\theta_t = \frac{n_1}{n_2}\sin\theta_i = \frac{1}{2.5298}\sin 12^o = 0.082185 \longrightarrow \theta_t = 4.714^o$$

$$\eta_1 = 120\pi, \qquad \eta_2 = 120\pi\sqrt{\frac{1}{6.4}} = 47.43\pi$$

$$\frac{E_{ro}}{E_{io}} = \Gamma = \frac{\eta_2\cos\theta_t - \eta_1\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i} = \frac{47.43\pi\cos 4.714^o - 120\pi\cos 12^o}{47.43\pi\cos 4.714^o + 120\pi\cos 12^o}$$

$$= \frac{47.27 - 117.38}{47.27 + 117.38} = \underline{\underline{-0.4258}}$$

$$\frac{E_{to}}{E_{io}} = \tau = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i} = \frac{2x47.43\cos 12^o}{47.27 + 117.33} = \frac{92.787}{164.65} = \underline{\underline{0.5635}}$$

**Prob. 10.75**

(a) $\boldsymbol{k}_i = 4\boldsymbol{a}_y + \mathbf{3}a_z$

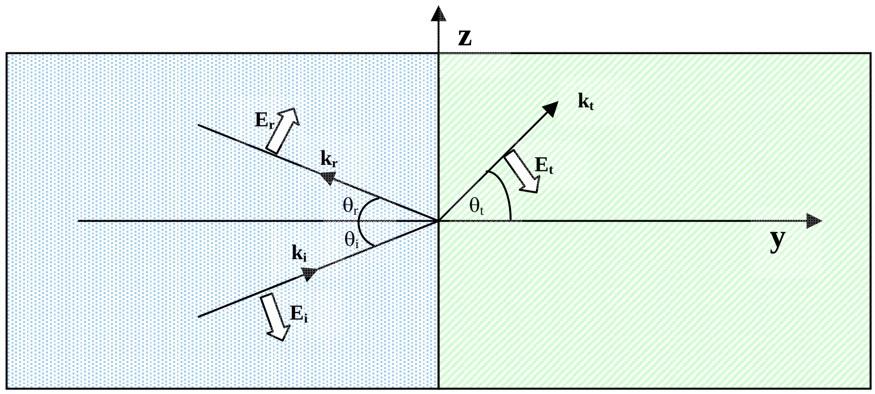
$\boldsymbol{k}_i \bullet \boldsymbol{a}_n = k_i\cos\theta_i \longrightarrow \cos\theta_i = 4/5 \longrightarrow \underline{\underline{\theta_i = 36.87^o}}$

(b)

$$\boldsymbol{P}_{\text{ave}} = \frac{1}{2}\,\text{Re}(\boldsymbol{E}_s \times \boldsymbol{H}_s^*) = \frac{E_o^2}{2\eta}\boldsymbol{a}_k = \frac{(\sqrt{8^2+6^2})^2}{2\times 120\pi}\frac{(4\boldsymbol{a}_y + 3\boldsymbol{a}_z)}{5} = \underline{\underline{106.1\boldsymbol{a}_y + 79.58\boldsymbol{a}_z \text{ mW/m}^2}}$$

(c ) $\theta_r = \theta_i = 36.87^o$. Let

$$\boldsymbol{E}_r = (E_{ry}\boldsymbol{a}_x + E_{rz}\boldsymbol{a}_z)\sin(\omega t - \boldsymbol{k}_r \bullet \boldsymbol{r})$$




From the figure, $\boldsymbol{k}_r = k_{rz}\boldsymbol{a}_z - k_{ry}\boldsymbol{a}_y$. But $k_r = k_i = 5$

$$k_{rz} = k_r \sin\theta_r = 5(3/5) = 3, \quad k_{ry} = k_r \cos\theta_r = 5(4/5) = 4,$$

Hence, $\boldsymbol{k}_r = -4\boldsymbol{a}_y + 3\boldsymbol{a}_z$

$$\sin\theta_t = \frac{n_1}{n_2}\sin\theta_i = \frac{c\sqrt{\mu_1\varepsilon_1}}{c\sqrt{\mu_2\varepsilon_2}}\sin\theta_i = \frac{3/5}{\sqrt{4}} = 0.3$$

$$\theta_t = 17.46, \quad \cos\theta_t = 0.9539, \quad \eta_1 = \eta_o = 120\pi, \eta_2 = \eta_o/2 = 60\pi$$

$$\Gamma_{//} = \frac{E_{ro}}{E_{io}} = \frac{\eta_2\cos\theta_t - \eta_1\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i} = \frac{\frac{\eta_o}{2}(0.9539) - \eta_o(0.8)}{\frac{\eta_o}{2}(0.9539) + \eta_o(0.8)} = -0.253$$

$$E_{ro} = \Gamma_{//}E_{io} = -0.253(10) = -2.53$$

But $(E_{ry}\boldsymbol{a}_y + E_{rz}\boldsymbol{a}_z) = E_{ro}(\sin\theta_r\boldsymbol{a}_y + \cos\theta_r\boldsymbol{a}_z) = -2.53(\frac{3}{5}\boldsymbol{a}_y + \frac{4}{5}\boldsymbol{a}_z)$

$$\boldsymbol{E}_r = -(1.518\boldsymbol{a}_y + 2.024\boldsymbol{a}_z)\sin(\omega t + 4y - 3z) \quad \text{V/m}$$

Similarly, let

$$\boldsymbol{E}_t = (E_{ty}\boldsymbol{a}_y + E_{tz}\boldsymbol{a}_z)\sin(\omega t - \boldsymbol{k}_t \bullet \boldsymbol{r})$$

$$k_t = \beta_2 = \omega\sqrt{\mu_2\varepsilon_2} = \omega\sqrt{4\mu_o\varepsilon_o}$$

But $k_i = \beta_1 = \omega\sqrt{\mu_o\varepsilon_o}$

$$\frac{k_t}{k_i} = 2 \longrightarrow k_t = 2k_i = 10$$

$$k_{ty} = k_t\cos\theta_t = 9.539, \quad k_{tz} = k_t\sin\theta_t = 3,$$

$$k_t = 9.539a_y + 3a_z$$

Note that $k_{iz} = k_{rz} = k_{tz} = 3$

$$\tau_{\backslash\backslash} = \frac{E_{to}}{E_{io}} = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i} = \frac{\eta_o(0.8)}{\frac{\eta_o}{2}(0.9539) + \eta_o(0.8)} = 0.6265$$

$$E_{to} = \tau_{\backslash\backslash}E_{io} = 6.265$$

But

$$(E_{ty}\boldsymbol{a}_y + E_{tz}\boldsymbol{a}_z) = E_{to}(\sin\theta_t\boldsymbol{a}_y - \cos\theta_t\boldsymbol{a}_z) = 6.256(0.3\boldsymbol{a}_y - 0.9539\boldsymbol{a}_z)$$

Hence,

$$\underline{\underline{\boldsymbol{E}_t = (1.879\boldsymbol{a}_y - 5.968\boldsymbol{a}_z)\sin(\omega t - 9.539y - 3z)}} \text{ V/m}$$

**Prob. 10.76**

(a)

$$\tan\theta_i = \frac{k_{ix}}{k_{iz}} = \frac{1}{\sqrt{8}} \longrightarrow \underline{\underline{\theta_i = \theta_r = 19.47^o}}$$

$$\sin\theta_t = \sin\theta_i\sqrt{\frac{\varepsilon_{r1}}{\varepsilon_{r2}}} = \frac{1}{3}(3) = 1 \longrightarrow \underline{\underline{\theta_t = 90^o}}$$

(b) $\beta_1 = \frac{\omega}{c}\sqrt{\varepsilon_{r1}} = \frac{10^9}{3\times10^8}\times 3 = 10 = k\sqrt{1+8} = 3k \longrightarrow \underline{\underline{k = 3.333}}$

(c ) $\lambda = 2\pi/\beta, \quad \lambda_1 = 2\pi/\beta_1 = 2\pi/10 = \underline{\underline{0.6283}}$ m

$$\beta_2 = \omega/c = 10/3, \quad \lambda_2 = 2\pi/\beta_2 = 2\pi\times 3/10 = \underline{\underline{1.885}} \text{ m}$$

(d) $$\boldsymbol{E}_i = \eta_1 \boldsymbol{H}_x \times \boldsymbol{a}_k = 40\pi(0.2)\cos(\omega t - \boldsymbol{k}\bullet\boldsymbol{r})\boldsymbol{a}_y \times \frac{(\boldsymbol{a}_x + \sqrt{8}\boldsymbol{a}_z)}{3}$$

$$= (23.6954\boldsymbol{a}_x - 8.3776\boldsymbol{a}_z)\cos(10^9 t - kx - k\sqrt{8}z) \text{ V/m}$$

(e) $$\tau_{//} = \frac{2\cos\theta_i \sin\theta_t}{\sin(\theta_i + \theta_t)\cos(\theta_t - \theta_i)} = \frac{2\cos 19.47^o \sin 90^o}{\sin 19.47^o \cos 19.47^o} = 6$$

$$\Gamma_{//} = -\frac{\cot 19.47^o}{\cot 19.47^o} = -1$$

Let $\boldsymbol{E}_t = -E_{io}(\cos\theta_t \boldsymbol{a}_x - \sin\theta_t \boldsymbol{a}_z)\cos(10^9 t - \beta_2 x \sin\theta_t - \beta_2 z\cos\theta_t)$

where

$$\boldsymbol{E}_t = -E_{io}(\cos\theta_i \boldsymbol{a}_x - \sin\theta_i \boldsymbol{a}_z)\cos(10^9 t - \beta_1 x \sin\theta_i - \beta_1 z\cos\theta_i)$$

$$\sin\theta_t = 1, \quad \cos\theta_t = 0, \quad \beta_2 \sin\theta_t = 10/3$$

$$E_{to}\sin\theta_t = \tau_{\backslash\backslash} E_{io} = 6(24\pi)(3)(1) = 1357.2$$

Hence,

$$\boldsymbol{E}_t = 1357\cos(10^9 t - 3.333x)\boldsymbol{a}_z \text{ V/m}$$

Since $\Gamma = -1$, $\theta_r = \theta_i$

$$\boldsymbol{E}_r = (213.3\boldsymbol{a}_x + 75.4\boldsymbol{a}_z)\cos(10^9 t - kx + k\sqrt{8}z) \text{ V/m}$$

(f) $$\tan\theta_{B//} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} = \sqrt{\frac{\varepsilon_o}{9\varepsilon_o}} = 1/3 \longrightarrow \theta_{B//} = 18.43^o$$

**Prob. 10.77**

$(a)$ $$\boldsymbol{E}_{i\perp} = 5\cos(\omega t - 0.5\pi x - 0.866\pi z)\boldsymbol{a}_y$$

$$\boldsymbol{E}_{i\parallel} = (4\boldsymbol{a}_x - 3\boldsymbol{a}_z)\cos(\omega t - 0.5\pi x - 0.866\pi z)$$

$(b)$ Comparing $\boldsymbol{E}_{i\parallel}$ with eq. (10.115a),

$$4\boldsymbol{a}_x - 3\boldsymbol{a}_z = (\cos\theta_i \boldsymbol{a}_x - \sin\theta_i \boldsymbol{a}_z)E_{io}$$

$$\tan\theta_i = \frac{\sin\theta_i}{\cos\theta_i} = \frac{3}{4} \quad \rightarrow \quad \theta_i = 36.87^o$$

**Prob. 10.78**

$$\tan\theta_B = \sqrt{\frac{\varepsilon_1}{\varepsilon_2}} = \sqrt{\frac{\varepsilon_o\varepsilon_r}{\varepsilon_o}} = \sqrt{\varepsilon_r}$$

$$\varepsilon_r = \tan^2\theta_B = \tan^2 68 = \underline{\underline{6.126}}$$

**Prob. 10.79**

(a) $n = \frac{c}{u} = \sqrt{\mu_r\varepsilon_r} = \sqrt{2.1\times1} = \underline{\underline{1.45}}$

(b) $n = \sqrt{\mu_r\varepsilon_r} = \sqrt{1\times81} = \underline{\underline{9}}$

(c ) $n = \sqrt{\varepsilon_r} = \sqrt{2.7} = \underline{\underline{1.643}}$

**Prob.10.80**

Microwave is used:

(1) For surveying land with a piece of equipment called the *tellurometer*. This radar system can precisely measure the distance between two points.

(2) For guidance. The guidance of missiles, the launching and homing guidance of space vehicles, and the control of ships are performed with the aid of microwaves.

(3) In semiconductor devices. A large number of new microwave semiconductor devices have been developed for the purpose of microwave oscillator, amplification, mixing/detection, frequency multiplication, and switching. Without such achievement, the majority of today's microwave systems could not exist.

**Prob.10.81**

(a) In terms of the S-parameters, the T-parameters are given by

$T_{11} = 1/S_{21}$, $T_{12} = -S_{22}/S_{21}$, $T_{21} = S_{11}/S_{21}$, $T_{22} = S_{12} - S_{11} S_{22}/S_{21}$

(b) $T_{11} = 1/0.4 = 2.5$, $T_{12} = -0.2/0.4$,

$T_{21} = 0.2/0.4$, $T_{22} = 0.4 - 0.2 \times 0.2/0.4 = 0.3$

Hence,

$$T = \underline{\underline{\begin{bmatrix} 2.5 & -0.5 \\ 0.5 & 0.3 \end{bmatrix}}}$$

**Prob. 10.82**

Since $Z_L = Z_o$, $\Gamma_L = 0$.

$\Gamma_i = S_{11} = \underline{0.33 - j0.15}$

$\Gamma_g = (Z_g - Z_o)/(Z_g + Z_o) = (2 - 1)/(2 + 1) = 1/3$

$\Gamma_o = S_{22} + S_{12}S_{21}\Gamma_g /(1 - S_{11}\Gamma_g)$

$= 0.44 - j0.62 + 0.56 x 0.56 x(1/3)/[1 - (0.11 - j0.05)]$

$= \underline{\underline{0.5571 - j0.6266}}$

**Prob. 10.83** The microwave wavelengths are of the same magnitude as the circuit components. The wavelength in air at a microwave frequency of 300 GHz, for example, is 1 mm. The physical dimension of the lumped element must be in this range to avoid interference. Also, the leads connecting the lumped element probably have much more inductance and capacitance than is needed.

**Prob. 10.84**

$$\lambda = c/f = \frac{3\times10^8}{8.4\times10^9} = \underline{\underline{35.71\text{ mm}}}$$

## CHAPTER 11

**P.E. 11.1** Since $Z_o$ is real and $\alpha \neq 0$, this is a distortionless line.

$$Z_o = \sqrt{\frac{R}{G}} \qquad (1)$$

or $$\frac{L}{R} = \frac{C}{G} \qquad (2)$$

$$\alpha = \sqrt{RG} \qquad (3)$$

$$\beta = \omega L\sqrt{\frac{G}{R}} = \frac{\omega L}{Z_o} \qquad (4)$$

$(1) \times (3) \rightarrow R = \alpha Z_o = 0.04 \times 80 = \underline{\underline{3.2\Omega / \text{m}}}$,

$(3) \div (1) \rightarrow G = \frac{\alpha}{Z_o} = \frac{0.04}{80} = \underline{\underline{5 \times 10^{-4} S / \text{m}}}$

$$L = \frac{\beta Z_o}{\omega} = \frac{1.5 \times 80}{2\pi \times 5 \times 10^8} = \underline{\underline{38.2 \text{ nH / m}}}$$

$$C = \frac{LG}{R} = \frac{12}{\pi} \times 10^{-8} \times \frac{0.04}{80} \times \frac{1}{0.04 \times 80} = \underline{\underline{5.97 \text{ pF/m}}}$$

**P.E. 11.2**

(a) $Z_o = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{0.03 + j2\pi \times 0.1 \times 10^{-3}}{0 + j2\pi \times 0.02 \times 10^{-6}}}$

$= 70.73 - j1.688 = \underline{\underline{70.75\angle -1.367^o \Omega}}$

(b) $\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = \sqrt{(0.03 + j0.2\pi)(j0.4 \times 10^{-4}\pi)}$

$\underline{\underline{= 2.121 \times 10^{-4} + j8.888 \times 10^{-3} / \text{m}}}$

(c) $u = \frac{\omega}{\beta} = \frac{2\pi \times 10^3}{8.888 \times 10^{-3}} = \underline{\underline{7.069 \times 10^5 \text{ m/s}}}$

**P.E. 11.3**

(a) $Z_o = Z_l \rightarrow Z_{in} = Z_o = \underline{\underline{30 + j60\Omega}}$

(b) $V_{in} = V_o = \dfrac{Z_{in}}{Z_{in} + Z_o} V_g = \dfrac{V_g}{2} = \underline{\underline{7.5\angle 0^o \text{ V}_{rms}}}$

$$I_{in} = I_o = \frac{V_g}{Z_g + Z_{in}} = \frac{V_g}{2Z_o} = \frac{15\angle 0^o}{2(30 + j60)}$$

$$\underline{\underline{= 0.2236\angle -63.43^o\, A}}$$

assuming that $Z_g = 0$.

(c ) Since $Z_o = Z_r$, $\Gamma = 0 \to V_o^- = 0, V_o^+ = V_o$

The load voltage is $V_L = V_s(z = l) = V_o^+ e^{-\gamma l}$

$$e^{-\gamma l} = \frac{V_o^+}{V_L} = \frac{7.5\angle 0^o}{5\angle -48^o} 1.5\angle 48^o$$

$$e^{\alpha l} e^{j\beta l} = 1.5\angle 48^o$$

$$e^{\alpha l} = 1.5 \to \alpha = \frac{1}{l}\ln(1.5) = \frac{1}{40}\ln(1.5) = 0.0101$$

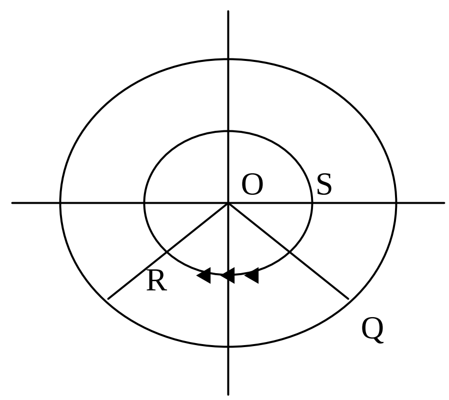
$$e^{j\beta l} = e^{j48^o} \to \beta = \frac{1}{l}\frac{48^o}{180^o}\pi rad = 0.02094$$

$$\underline{\underline{\gamma = 0.0101 + j0.02094 \text{ /m}}}$$

assuming that $Z_g = 0$.

**P.E. 11.4**

(a) Using the Smith chart, locate S at s = 1.6. Draw a circle of radius OS. Locate P where $\theta_\Gamma = 300^o$. At P,




$$|\Gamma| = \frac{OP}{OQ} = \frac{2.1cm}{9.2cm} = 0.228$$

$$\underline{\underline{\Gamma = 0.228\angle 300^o}}$$

Also at P, $\underline{\underline{z_L = 1.15 - j0.48}}$,

$Z_L = Z_o z_L = 70(1.15 - j0.48) = \underline{\underline{80.5 - j33.6\Omega}}$

$\ell = 0.6\lambda \to 0.6 \times 720^o = 432^o = \underline{\underline{360^o + 73^o}}$

From P, move $432^o$ to R. At R, $z_{in} = 0.68 - j025$

$$Z_{in} = Z_o Z_{in} = 70(0.68 - j0.25) = \underline{\underline{47.6 - j17.5\Omega}}$$

(b) The minimum voltage (the only one) occurs at $\theta_\Gamma = 180^o$; its distance from the load is $\frac{180-60}{720}\lambda = \frac{\lambda}{6} = \underline{\underline{0.1667\lambda}}$

Values obtained using formulas are as follows:

$$\Gamma = \frac{s-1}{s+1}\angle 300^o = 0.2308\angle 300^o$$

$Z_L = 80.5755 - j34.018\ \Omega$

$Z_{in} = 48.655 - j17.63\ \Omega$

These are pretty close.

**P.E. 11.5**

(a) $\Gamma = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{60 + j60 - 60}{60 + j60 + 60} = \frac{j}{2 + j} = \underline{\underline{0.4472\angle 63.43^o}}$

$$s = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + 0.4472}{1 - 0.4472} = \underline{\underline{2.618}}$$

Let $x = \tan(\beta l) = \tan\frac{2\pi l}{\lambda}$

$$Z_{in} = Z_o\left[\frac{Z_L + jZ_o\tan(\beta l)}{Z_o + jZ_L\tan(\beta l)}\right]$$

$$120 - j60 = 60\left[\frac{60 + j60 + j60x}{60 + j(60 + j60)x}\right]$$

Or $2 - j = \frac{1 + j(1 + x)}{1 - x + jx} \rightarrow 1 - x + j(2x - 2) = 0$

Or $x = 1 = \tan(\beta l)$

$$\frac{\pi}{4} + n\pi = \frac{2\pi l}{\lambda}$$

i.e $\underline{\underline{l = \frac{\lambda}{8}(1 + 4n), n = 0,1,2,3\ldots}}$

(b) $z_L = \frac{Z_L}{Z_o} \frac{60 + j60}{60} = 1 + j$

Locate the load point P on the Smith chart.

$$|\Gamma| = \frac{OP}{OQ} = \frac{4.1cm}{9.2cm} = 0.4457, \theta_\Gamma = 62^o$$

$$\underline{\underline{\Gamma = 0.4457\angle 62^o}}$$

Locate the point S on the Smith chart. At S, r = $\underline{\underline{s = 2.6}}$

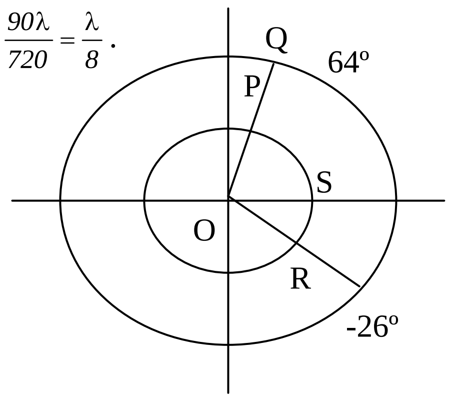
$Z_{in} = \frac{Z_{in}}{Z_o} = \frac{120 + j60}{60} = 2 - j$, which is located at R on the chart. The angle between OP and OR is 64°-(-25°) = 90° which is equivalent to $\frac{90\lambda}{720} = \frac{\lambda}{8}$.

Hence $l = \frac{\lambda}{8} + n\frac{\lambda}{2} = \frac{\lambda}{8}(1+4n), n = 0,1,2........$

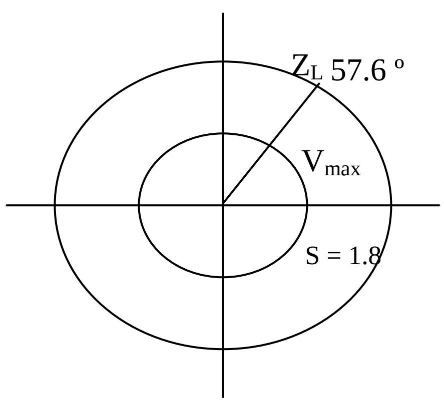
$$(Z_{in})_{max} = sZ_o = 2.618(60) = 157.08\Omega$$

$$(Z_{in})_{min} = Z_o / s = 60 / 2.618 = \underline{\underline{22.92\ \Omega}}$$

$$l = \frac{62^o}{720^o}\lambda = 0.0851\lambda$$




**P.E. 11.6**




$$\frac{\lambda}{2} = 37.5 - 25 = 12.5cm \text{ or } \lambda = 25cm$$

$$l = 37.5 - 35.5 = 2cm = \frac{2\lambda}{25}$$

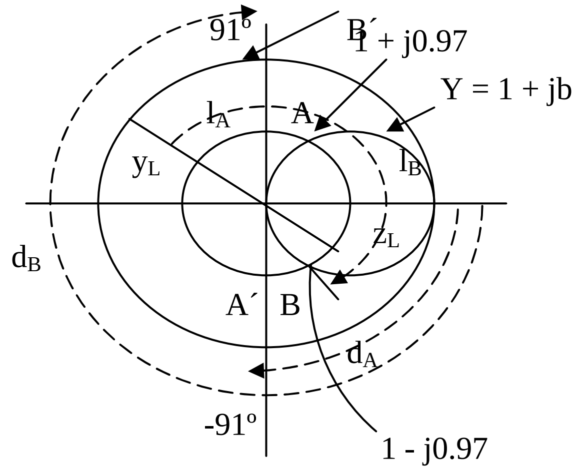
$$l = 0.08\lambda \rightarrow 57.6^o$$

$$z_L = 1.184 + j0.622$$

$$Z_L = Z_o z_L = 50(1.184 - j0.622)$$

$$\underline{\underline{= 59.22 + j31.11\Omega}}$$

**P.E. 11.7** See the Smith chart




$$z_L = \frac{100 - j80}{75} = 1.33 - j1.067$$

$$l_A = \frac{132^o - 65}{72}\lambda = \underline{\underline{0.093\lambda}}$$

$$l_B = \frac{132^o + 64^o}{720^o} = \underline{\underline{0.272\lambda}}$$

$$d_A = \frac{91}{720}\lambda = 0.126\lambda$$

$$d_B = 0.5\lambda - d_A \quad \underline{\underline{= 0.374\lambda}}$$

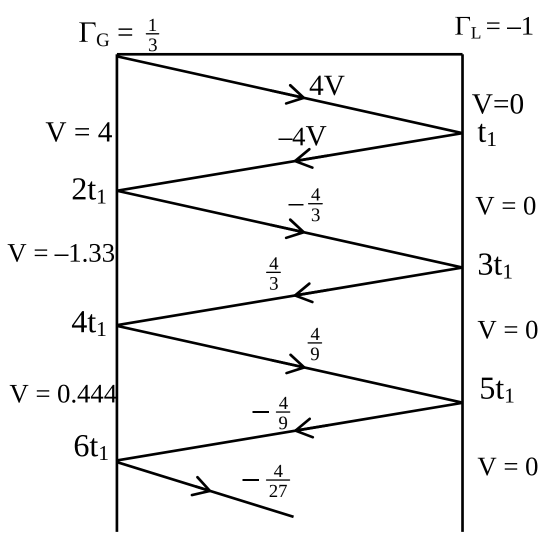
$$Y_s = \pm\frac{j0.95}{75} = \underline{\underline{\pm j12.67 \text{ mS}}}$$

**P.E. 11.8**

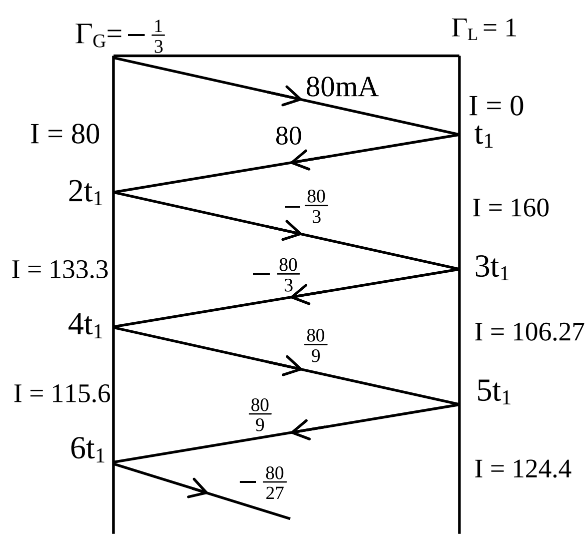
(a) $\Gamma_G = \frac{1}{3}, \Gamma_L = \lim_{Z_L \to 0} \frac{Z_L - Z_o}{Z_L + Z_o} = -1$

$$V_\infty = \lim_{z_L \to 0} \frac{Z_L}{Z_L + Z_g} V_g = 0, \qquad I_\infty = \lim_{z_L \to 0} \frac{V_g}{Z_g + Z_L} = \frac{V_g}{Z_g} = \frac{12}{100} = 120mA$$

Thus the bounce diagrams for current and waves are as shown below.

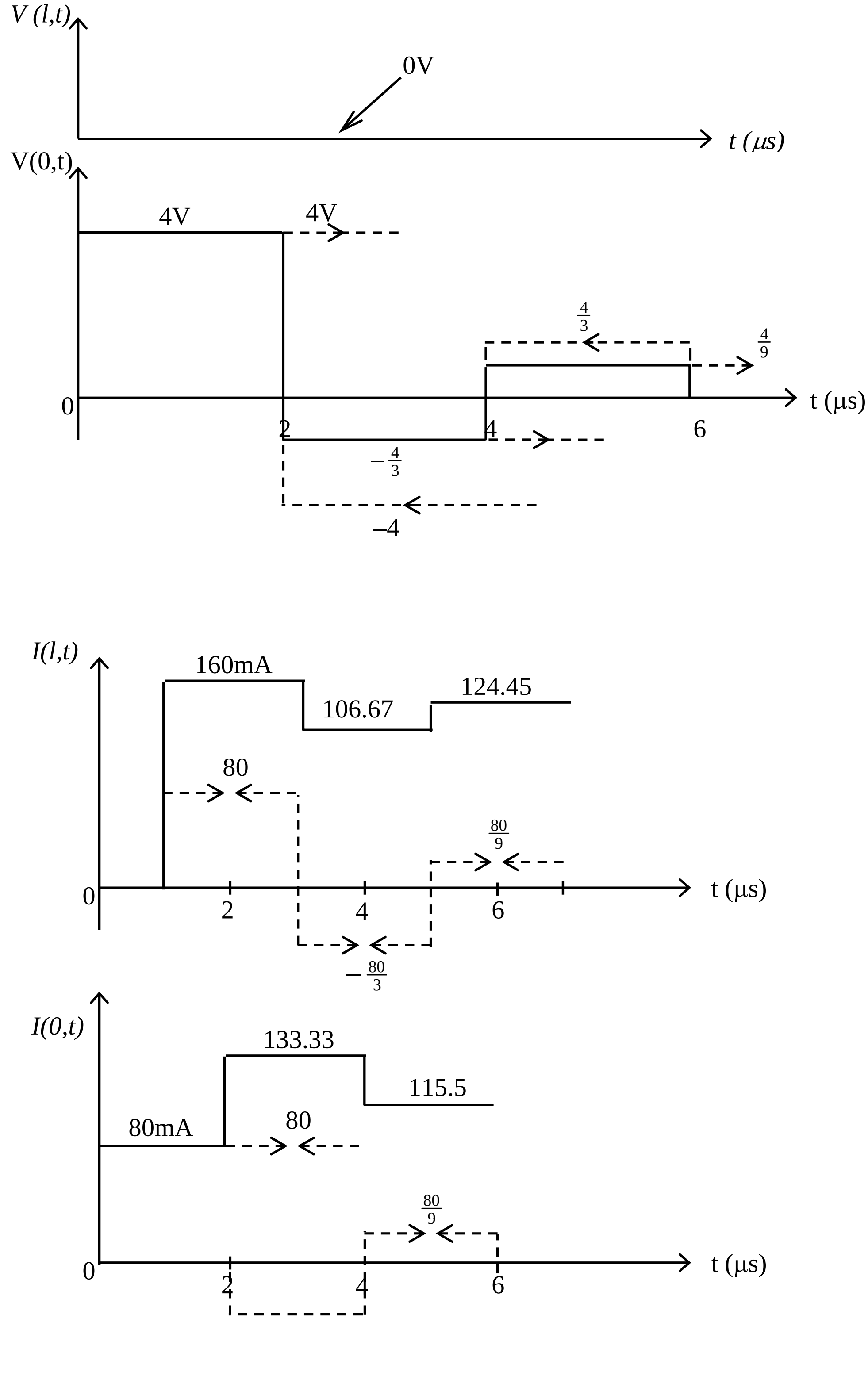



(Voltage)




(Current)

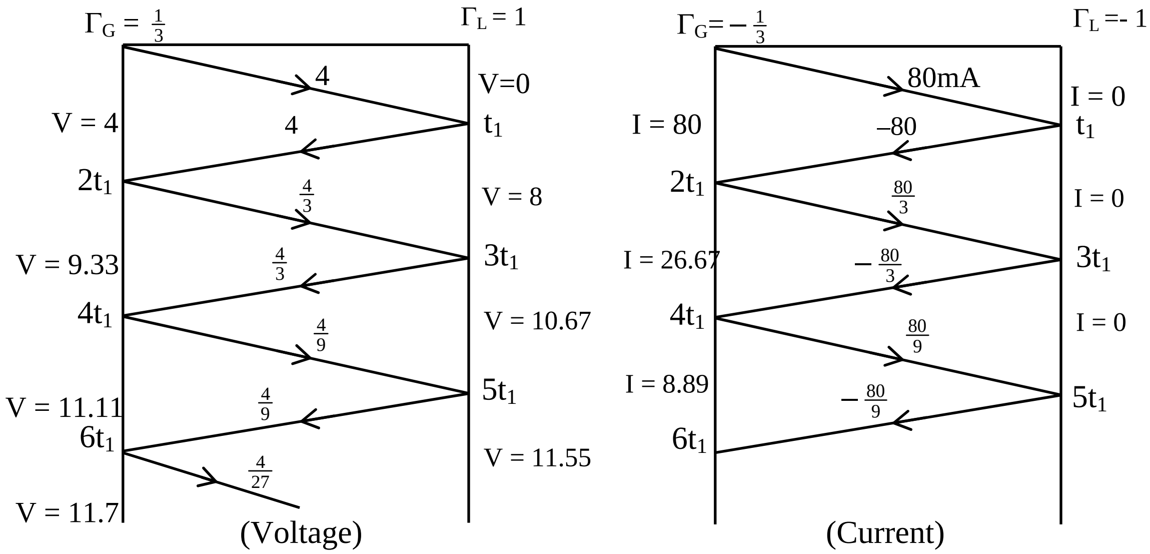

V (l,t)
0V
t (μs)
V(0,t)
4V
4V
4/3
4/9
0
2
4
6
t (μs)
−4/3
−4
I(l,t)
160mA
106.67
124.45
80
80/9
0
2
4
6
t (μs)
−80/3
I(0,t)
133.33
115.5
80mA
80
80/9
0
2
4
6
t (μs)

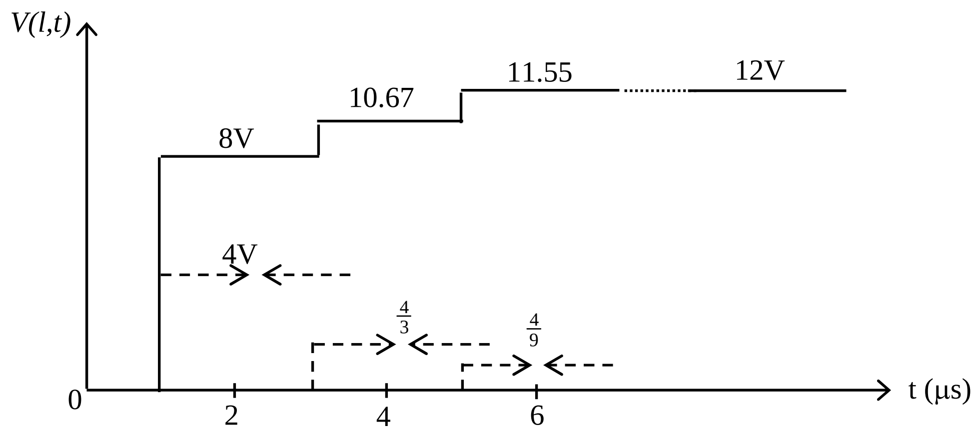
 

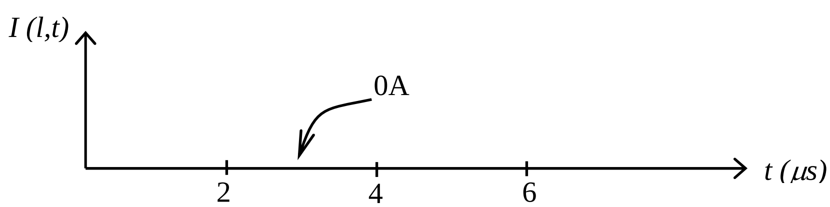
(b) $\Gamma_G = \frac{1}{3}, \Gamma_L = \lim_{Z_L \to \infty} \frac{Z_L - Z_o}{Z_L + Z_o} = 1$

$$V_\infty = \lim_{Z_L \to \infty} \frac{Z_L}{Z_L + Z_g} V_g = V_g = 12V, \qquad I_\infty = \lim_{Z_L \to \infty} \frac{V_g}{Z_L + Z_g} = 0$$

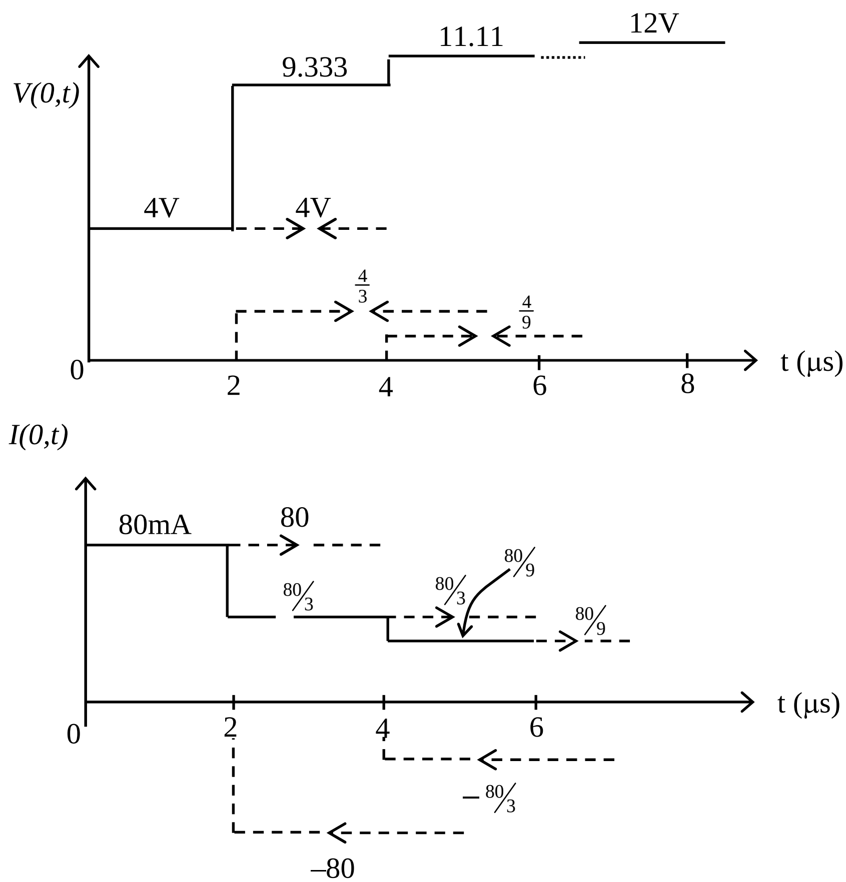
The bounce diagrams for current and voltage waves are as shown below.



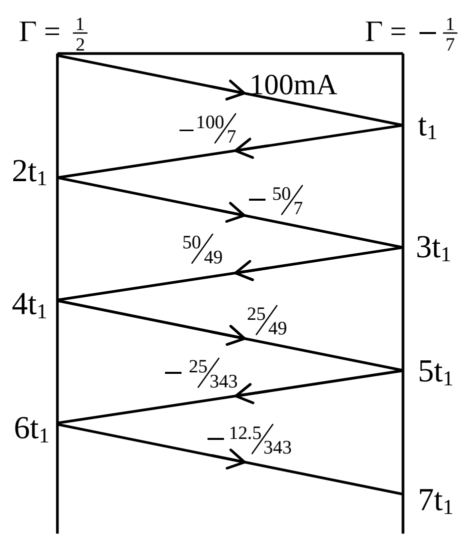


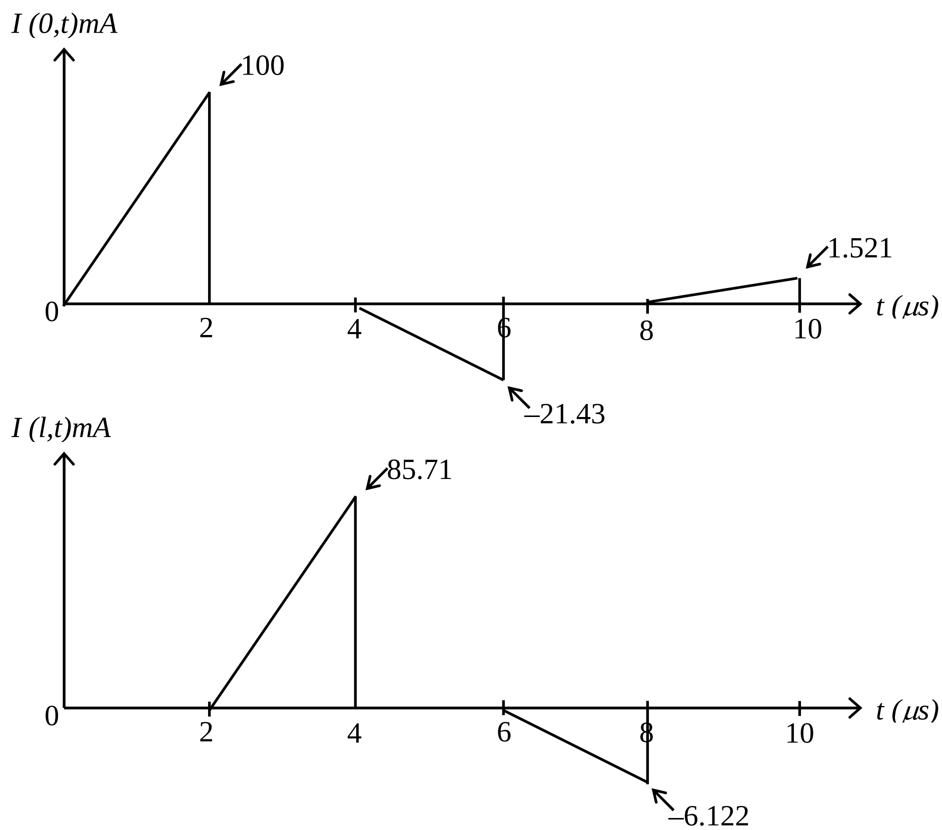

**P.E. 11.9**

$$\Gamma_G = -\frac{1}{2}, \Gamma_L = \frac{1}{7}, t_1 = 2\mu s$$

$$\left(I_o\right)_{\max} = \frac{\left(V_g\right)_{\max}}{Z_g + Z_o} = \frac{10}{100} = 100mA$$

The bounce diagrams for maximum current are as shown below.






**P.E. 11.10**

(a) For $w/h = 0.8$, $\varepsilon_{eff} = \frac{4.8}{2} + \frac{2.8}{2}\left[1 + \frac{12}{0.8}\right]^{-1/2} = \underline{\underline{2.75}}$

(b) $Z_o = \frac{60}{\sqrt{2.75}} \ln\left(\frac{8}{0.8} + \frac{0.8}{4}\right) = 36.18 \ln 10.2 = \underline{\underline{84.03\Omega}}$

(c) $\lambda = \frac{3 \times 10^8}{10^{10}\sqrt{2.75}} = \underline{\underline{18.09 \text{ mm}}}$

**P.E. 11.11**

$$R_s = \sqrt{\frac{\pi f \mu_o}{\sigma_c}} = \sqrt{\frac{\pi \times 20 \times 10^9 \times 4\pi \times 10^{-7}}{5.8 \times 10^7}}$$

$$= 3.69 \times 10^{-2}$$

$$\alpha_c = 8.685 \frac{R_s}{wZ_o} = \frac{8.686 \times 3.69 \times 10^{-2}}{2.5 \times 10^{-3} \times 50}$$

$$= \underline{\underline{2.564 \text{ dB/m}}}$$

**Prob. 11.1**

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma_c}} = \frac{1}{\sqrt{\pi \times 500 \times 10^6 \times 4\pi \times 10^{-7} \times 7 \times 10^7}}$$

$$\delta = 2.6902 \times 10^{-6}$$

$$R = \frac{2}{w\delta\sigma_c} = \frac{2}{0.3 \times 2.6902 \times 10^{-6} \times 7 \times 10^7} = \underline{\underline{0.0354 \Omega / \text{m}}}$$

$$L = \frac{\mu_o d}{w} = \frac{4\pi \times 10^{-7} \times 1.2 \times 10^{-2}}{0.3} = \underline{\underline{50.26 \text{ nH / m}}}$$

$$C = \frac{\varepsilon_o w}{d} = \frac{10^{-9}}{36\pi} \times \frac{0.3}{1.2 \times 10^{-2}} = \underline{\underline{221 \text{ pF / m}}}$$

Since $\sigma = 0$ for air,

$$G = \frac{\sigma w}{d} = \underline{\underline{0}}$$

**Prob. 11.2**

$$\delta = \frac{1}{\sqrt{\pi f \mu_c \sigma_c}} = \frac{1}{\sqrt{\pi \times 80 \times 10^6 \times 4\pi \times 10^{-7} \times 5.28 \times 10^7}} = 7.744 \times 10^{-6}$$

$$R = \frac{1}{2\pi\delta\sigma_c}\left[\frac{1}{a} + \frac{1}{b}\right] = \frac{\left[\frac{1}{0.8 \times 10^{-3}} + \frac{1}{2.6 \times 10^{-3}}\right]}{2\pi \times 7.744 \times 10^{-6} \times 5.28 \times 10^7} = \frac{10^3(1.25 + 0.3836)}{2569.09} = \underline{\underline{0.6359 \ \Omega/\text{m}}}$$

$$L = \frac{\mu}{2\pi} \ln\frac{b}{a} = \frac{4\pi \times 10^{-7}}{2\pi} \ln\frac{2.6}{0.8} = \underline{\underline{2.357 \times 10^{-7} \text{ H/m}}}$$

$$G = \frac{2\pi\sigma}{\ln\frac{b}{a}} = \frac{2\pi \times 10^{-5}}{\ln\frac{2.6}{0.8}} = \underline{\underline{5.33 \times 10^{-5} \text{ S/m}}}$$

$$C = \frac{2\pi\varepsilon}{\ln\frac{b}{a}} = \frac{2\pi \times 3.5 * \frac{10^{-9}}{36\pi}}{\ln\frac{2.6}{0.8}} = \underline{\underline{1.65 \times 10^{-10} \text{ F/m}}}$$

**Prob. 11.3**

Method 1:
Assume a charge per unit length Q on the surface of the inner conductor and –Q on the surface of the outer conductor. Using Gauss's law,

$$E_\rho = \frac{Q}{2\pi\varepsilon\rho}, \qquad a < \rho < b$$

$$V = -\int_a^b \boldsymbol{E} \cdot d\boldsymbol{l} = \frac{Q}{2\pi\varepsilon}\ln(b/a)$$

$$J = \sigma E = \frac{\sigma Q}{2\pi\varepsilon\rho}$$

$$I = \int_S \boldsymbol{J} \cdot d\boldsymbol{S} = \int_{\phi=0}^{2\pi} \frac{\sigma Q}{2\pi\varepsilon\rho}(1)\rho d\phi = \frac{\sigma Q}{\varepsilon}$$

$$G = \frac{I}{V} = \frac{\frac{\sigma Q}{\varepsilon}}{\frac{Q}{2\pi\varepsilon}\ln(b/a)} = \frac{2\pi\sigma}{\ln(b/a)}$$

Method 2:

Consider a section of unit length, Assume that a total current of I flows from inner conductor to outer conductor. At any radius ρ between a and b,

$$\boldsymbol{J} = \frac{I}{2\pi\rho}\boldsymbol{a}_\rho, \qquad a < \rho < b$$

$$\boldsymbol{E} = \frac{\boldsymbol{J}}{\sigma} = \frac{I}{2\pi\sigma\rho}\boldsymbol{a}_\rho$$

$$V = -\int_a^b \boldsymbol{E} \cdot d\boldsymbol{l} = \frac{I}{2\pi\sigma}\ln(b/a)$$

$$G = \frac{I}{V} = \frac{2\pi\sigma}{\ln(b/a)}$$

**Prob. 11.4**

(*a*) $R = \frac{2}{w\delta\sigma_c}$

$$\delta = \frac{1}{\sqrt{\pi f \sigma_c \mu_c}} = \frac{1}{\sqrt{\pi \times 200\times10^6 \times 5.8\times10^7 \times 4\pi\times10^{-7}}} = \frac{10^{-3}}{2\pi\sqrt{200(5.8)}} = 4.67\times10^{-6}$$

$$R = \frac{2}{30\times10^{-3}\times4.67\times10^{-6}\times5.8\times10^7} = \frac{20}{3(4.67)5.8} = \underline{\underline{0.2461\ \Omega/\text{m}}}$$

$$L = \frac{\mu d}{w} = \frac{4\pi\times10^{-7}\times2\times10^{-3}}{30\times10^{-3}} = \frac{4\pi\times10^{-7}}{15} = \underline{\underline{83.77\ \text{nH/m}}}$$

$$G = \frac{\sigma w}{d} = \frac{10^{-3}\times30\times10^{-3}}{2\times10^{-3}} = \underline{\underline{15\ \text{mS/m}}}$$

$$C = \frac{\varepsilon w}{d} = \frac{4\times\frac{10^{-9}}{36\pi}\times30\times10^{-3}}{2\times10^{-3}} = \frac{60\times10^{-9}}{36\pi} = \underline{\underline{0.5305\ \text{nF/m}}}$$

(*b*) $\gamma = \sqrt{(R + j\omega L)(G + j\omega C)}$

$$\gamma^2 = (0.2461 + j2\pi\times200\times10^6\times83.77\times10^{-9})(15\times10^{-3} + j2\pi\times200\times10^6\times0.5305\times10^{-9})$$

$$= (0.2461 + j105.3)(15\times10^{-3} + j0.6667)$$

$$\underline{\underline{\gamma = 0.104 + j8.379\ /\text{m}}}$$

$$Z = \sqrt{\frac{(R + j\omega L)}{(G + j\omega C)}} = \sqrt{\frac{(0.2461 + j105.3)}{(15\times10^{-3} + j0.6667)}} = \underline{\underline{12.565 + j0.1266\ \Omega}}$$

**Prob. 11.5**

$$C = \frac{\pi\varepsilon l}{\cosh^{-1}(d/2a)} \cong \frac{\pi\varepsilon l}{\ln(d/a)}$$

since $(d/2a)^2 = 11.11 >> 1$.

$$C = \frac{\pi\times\frac{10^{-9}}{36\pi}\times16\times10^{-3}}{\ln(2/0.3)} = \underline{\underline{0.2342\ \text{pF}}}$$

$$\delta = \frac{1}{\sqrt{\pi f \mu\sigma_c}} = \frac{1}{\sqrt{\pi\times10^7\times4\pi\times10^{-7}\times5.8\times10^7}} = 2.09\times10^{-5}\ \text{m} << a$$

$$R_{ac} = \frac{l}{\pi a \delta \sigma_c} = \frac{16\times10^{-3}}{\pi\times0.3\times10^{-3}\times2.09\times10^{-5}\times5.8\times10^{7}} = \underline{\underline{1.4\times10^{-2}\ \Omega}}$$

**Prob. 11.6**

We use the two-wire line formulas in Table 11.1

$$\frac{d}{a} = \frac{15mm}{1.2} = 12.5, \qquad \left(\frac{d}{2a}\right)^2 = 39.1 \gg 1$$

Hence, $\cosh^{-1}\frac{d}{2a} \simeq \ln\frac{d}{a} = \ln 12.5 = 2.526$

$$L = \frac{\mu}{\pi}\cosh^{-1}\frac{d}{2a} = \frac{\mu}{\pi}\ln\frac{d}{a} = \frac{4\pi\times10^{-7}\times2.526}{\pi} = 10.103\times10^{-7} = \underline{\underline{1.01\ \mu\text{H/m}}}$$

$$C = \frac{\pi\varepsilon}{\cosh^{-1}\frac{d}{2a}} = \frac{\pi\varepsilon}{\ln\frac{d}{a}} = \frac{\pi}{2.526}4\times\frac{10^{-9}}{36\pi} = \underline{\underline{44.105\ \text{pF/m}}}$$

$$Z_o = \sqrt{\frac{L}{C}} = \sqrt{\frac{1.01\times10^{-6}}{44.105\times10^{-12}}} = \underline{\underline{151.16\ \Omega}}$$

**Prob. 11.7**

$$V(z+\Delta z,t) = V(z,t) - L\Delta z\frac{\partial I(z,t)}{\partial t}$$

or

$$\frac{V(z+\Delta z,t)-V(z,t)}{\Delta z} = -L\frac{\partial I(z,t)}{\partial t} \qquad (1)$$

$$\text{I}(z+\Delta z,t) = I(z,t) - C\Delta z\frac{\partial V(z+\Delta z,t)}{\partial t}$$

or

$$\frac{\text{I}(z+\Delta z,t)-I(z,t)}{\Delta z} = -C\frac{\partial V(z+\Delta z,t)}{\partial t} \qquad (2)$$

As $\Delta z \to 0$, we obtain

$$\frac{\partial V}{\partial z} = -L\frac{\partial I}{\partial t}$$

$$\frac{\partial I}{\partial z} = -C\frac{\partial V}{\partial t}$$

**Prob.11.8**

(a)

$$\gamma = \sqrt{(R+j\omega L)(G+j\omega C)} = j\omega\sqrt{LC}\sqrt{(1+\frac{R}{j\omega L})(1+\frac{G}{j\omega C})}$$

$$= j\omega\sqrt{LC}\sqrt{1-\frac{RG}{\omega^2 LC}+\frac{R}{j\omega L}+\frac{G}{j\omega C}}$$

As R<< ωL and G<<ωC, dropping the $\omega^2$ term gives

$$\gamma \cong j\omega\sqrt{LC}\sqrt{1+\frac{R}{j\omega L}+\frac{G}{j\omega C}} \cong j\omega\sqrt{LC}\left[1+\frac{R}{2j\omega L}+\frac{G}{2j\omega C}\right]$$

$$= \underline{\underline{\left(\frac{R}{2}\sqrt{\frac{C}{L}}+\frac{G}{2}\sqrt{\frac{L}{C}}\right)+j\omega\sqrt{LC}}}$$

(b)

$$Z_o = \sqrt{\frac{R+j\omega L}{G+j\omega C}} = \sqrt{\frac{L}{C}}\sqrt{\frac{1+\frac{R}{j\omega L}}{1+\frac{G}{j\omega C}}} = \sqrt{\frac{L}{C}}\left(1+\frac{R}{j\omega L}\right)^{1/2}\left(1+\frac{G}{j\omega C}\right)^{-1/2}$$

$$\cong \sqrt{\frac{L}{C}}\left(1+\frac{R}{2j\omega L}+...\right)\left(1-\frac{G}{j2\omega C}+...\right) = \sqrt{\frac{L}{C}}\left(1-j\frac{R}{2\omega L}+j\frac{G}{2\omega C}+...\right)$$

$$\cong \underline{\underline{\sqrt{\frac{L}{C}}\left[1+j\left(\frac{G}{2\omega C}-\frac{R}{2\omega L}\right)\right]}}$$

**Prob. 11.9**

(a) $R+j\omega L = 0.2 + j2\pi\times 12\times 10^6\times 40\times 10^{-6} = 0.2 + j24\pi(40) = 0.2 + j3015.93$

$$\mathrm{G}+j\omega C = 4\times 10^{-3} + j2\pi\times 12\times 10^6\times 25\times 10^{-6} = 4\times 10^{-3} + j50\pi(12)$$

$$= 4\times 10^{-3} + j1884.96$$

$$\gamma = \sqrt{(R+j\omega L)(\mathrm{G}+j\omega C)} = \sqrt{(0.2+j3015.93)(4\times 10^{-3}+j1884.96)}$$

$$\underline{\underline{= 8.159\times 10^{-2} + j2.384\times 10^3 \text{ /m}}}$$

($b$) Since R<< $\omega$L and G<<$\omega L$, we may use the result in Problem 11.8(a).

$$\frac{L}{C} = \frac{40\times 10^{-6}}{25\times 10^{-6}} = 1.6, \qquad LC = 40(25)\times 10^{-12} = 10^{-9}$$

$$\gamma = \left(\frac{R}{2}\sqrt{\frac{C}{L}}+\frac{G}{2}\sqrt{\frac{L}{C}}\right)+j\omega\sqrt{LC} = \left(\frac{0.1}{\sqrt{1.6}}+\frac{4\times 10^{-3}}{2}\sqrt{1.6}\right)+j2\pi\times 12\times 10^6\times\sqrt{40\times 10^{-6}\times 25\times 10^{-6}}$$

$$= \underline{\underline{8.159\times 10^{-2} + j2.384\times 10^3 \text{ /m}}}$$

**Prob. 11.10**

(*a*) $I_1(\omega t - \beta z)$ travels along +z direction.

(b) From eq. (11.18),

$$Z_o = \frac{V_o^+}{I_o^+} = -\frac{V_o^-}{I_o^-}$$

Hence,

$$\underline{\underline{V(z,t) = Z_o I_1(\omega t - \beta z) - Z_o I_2(\omega t + \beta z)}}$$

**Prob. 11.11**

$$Z_o = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \qquad (1)$$

$$\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)} \qquad (2)$$

$$\alpha = 0.04 \text{ dB/m} = \frac{0.04}{8.686} \text{ Np/m} = 0.00461 \text{ Np/m}$$

Multiplying (1) and (2),

$$Z_o(\alpha + j\beta) = R + j\omega L \longrightarrow 50(0.00461 + j2.5) = R + j\omega L$$

$$R = 50 \times 0.00461 = \underline{\underline{0.2305\ \Omega\text{/m}}}$$

$$L = \frac{50 \times 2.5}{2\pi \times 60 \times 10^6} = \underline{\underline{0.3316\ \mu\text{H/m}}}$$

Dividing (2) by (1),

$$\frac{\alpha + j\beta}{Z_o} = G + j\omega C$$

$$G = \frac{\alpha}{Z_o} = \frac{0.00461}{50} = \underline{\underline{92.2\ \mu\text{S/m}}}$$

$$C = \frac{\beta}{\omega Z_o} = \frac{2.5}{2\pi \times 60 \times 10^6 \times 50} = \underline{\underline{0.1326\ \text{nF/m}}}$$

**Prob. 11.12**

$$Z_o = \sqrt{\frac{L}{c}} = \sqrt{\frac{\mu d}{w} \cdot \frac{d}{\varepsilon w}} = \frac{d}{w}\sqrt{\frac{\mu}{\varepsilon}}$$

$$Z_o = \eta_o \frac{d}{w} = 78$$

$$Z_o{}' = \eta_o \frac{d}{w'} = 75$$

$$\frac{78}{75} = \frac{w'}{w} \rightarrow w' = 1.04w$$

i.e. the width must be increased by 4%.

**Prob. 11.13**

(a) $$Z_o = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{6.8 + j2\pi \times 10^3 \times 3.4 \times 10^{-3}}{0.42 \times 10^{-6} + j2\pi \times 10^3 \times 8.4 \times 10^{-9}}}$$

$$= 10^3 \sqrt{\frac{6.8 + j21.36}{0.42 + j52.78}} = 644.3 - j97\ \Omega$$

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = 10^{-3}\sqrt{(6.8 + j21.36)(0.42 - j52.78)}$$

$$= (5.415 + j33.96) \times 10^{-3}\ \ /\text{mi}$$

(b) $$u = \frac{\omega}{\beta} = \frac{2\pi \times 10^3}{33.96 \times 10^{-3}} = 1.85 \times 10^5\ \ \text{mi/s}$$

(c) $$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{33.96 \times 10^{-3}} = 185.02\ \ \text{mi}$$

**Prob. 11.14**

Using eq. (11.42a),

$$Z_{in} = -jZ_o \cot \beta\ell$$

$$Z_o = 250, \qquad \lambda = \frac{c}{f} = \frac{3 \times 10^8}{400 \times 10^6} = 0.75\ \text{m}$$

$$\beta\ell = \frac{2\pi}{\lambda}\ell = \frac{2\pi \times 0.1}{0.75} = 48^o$$

$$Z_{in} = -j(250)\cot 48^o = -j225.1\ \Omega$$

**Prob. 11.15**

Assume that the line is lossless.

$$Z_o = \sqrt{\frac{L}{C}}$$

From Table 11.1, $L = \frac{\mu}{2\pi} \ln \frac{b}{a}$, $C = \frac{2\pi\varepsilon}{\ln \frac{b}{a}}$

$$\frac{L}{C} = \frac{\mu}{\varepsilon}\left(\frac{1}{2\pi} \ln \frac{b}{a}\right)^2$$

$$Z_o = \sqrt{\frac{L}{C}} = \frac{1}{2\pi} \ln \frac{b}{a} \times \sqrt{\frac{\mu}{\varepsilon}} = \frac{\eta_o}{2\pi\sqrt{\varepsilon_r}} \ln \frac{b}{a}$$

$$\ln \frac{b}{a} = 2\pi\sqrt{\varepsilon_r} \frac{Z_o}{\eta_o} = 2\pi\sqrt{2.25} \frac{75}{120\pi} = 1.875$$

$$\frac{b}{a} = e^{1.875} \longrightarrow a = be^{-1.875} = 3e^{-1.875} \text{ mm} = \underline{\underline{0.46 \text{ mm}}}$$

**Prob. 11.16**

(a) For a lossless line, R = 0 = G.

$$\gamma = j\omega\sqrt{LC} \longrightarrow \beta = \omega\sqrt{LC} = \omega\sqrt{\mu_o c_o} = \frac{\omega}{c}$$

$$u = \frac{\omega}{\beta} = c = \frac{1}{\sqrt{LC}}$$

(b) For lossless line, R = 0 =G

$$L = \frac{\mu}{\pi} \cosh^{-1} \frac{d}{2a}, C = \frac{\pi\varepsilon}{\cosh^{-1} \frac{d}{2a}}$$

$$Z_o = \sqrt{\frac{L}{C}} = \sqrt{\frac{\mu}{\pi} . \frac{1}{\pi\varepsilon}} \cosh^{-1} \frac{d}{2a} = \frac{120\pi}{\pi\sqrt{\varepsilon_r}} \cosh^{-1} \frac{d}{2a}$$

$$= \underline{\underline{\frac{120}{\sqrt{\varepsilon_r}} \cosh^{-1} \frac{d}{2a}}}$$

Yes, true for other lossless lines.

**Prob. 11.17**

$$L = \frac{\mu}{\pi}\cosh^{-1}\frac{d}{2a} = 4\times 10^{-7}\cosh^{-1}\frac{0.32}{0.12}$$

$$\underline{\underline{L = 0.655\ \mu\text{H / m}}}$$

$$C = \frac{\pi\varepsilon}{Cosh^{-1}\frac{d}{2a}} = \frac{\pi\times\frac{10^{-9}}{36\pi}\times 3.5}{Cosh^{-1}2.667}$$

$$\underline{\underline{C = 59.4\ \text{pF / m}}}$$

$$Z_o = \sqrt{\frac{L}{C}} = \sqrt{\frac{0.655\times10^{-6}}{59.4\times10^{-12}}} = \underline{\underline{105\Omega}}$$

or

$$Z_o = \frac{120}{\sqrt{3.5}}\cosh^{-1} 2.667 = \underline{\underline{105\Omega}}$$

**Prob. 11.18**

For a distortionless cable,

$$\frac{R}{L} = \frac{G}{C} \longrightarrow RC = LG \qquad (1)$$

$$Z_o = \sqrt{\frac{L}{C}} = 60 \qquad (2)$$

$$u = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}} = \frac{\ell}{t_o} = \frac{4}{80\times10^{-6}} \qquad (3)$$

$$\alpha\ell = 0.24dB = \frac{0.24}{8.686}Np = 0.0276$$

$$\alpha = \sqrt{RG} = 0.00069 \qquad (4)$$

From (2) and (3),

$$\frac{1}{\text{C}} = \frac{60\times4}{80\times10^{-6}} \longrightarrow C = \frac{8\times10^{-5}}{240} = \underline{\underline{333.3\text{ nF/m}}}$$

From (2),

$$L = (60)^2 C = 3600\times333.3\times10^{-9} = \underline{\underline{1.20\text{ mH/m}}}$$

From (1) and (4),

$$\frac{C}{G} = \frac{LG}{0.00069^2} \longrightarrow G^2 = \frac{0.00069^2}{60^2}$$

$$G = \frac{0.00069}{60} = \underline{\underline{11.51\ \mu\text{S/m}}}$$

From (4),

$$R = \frac{0.00069^2}{G} = 0.00069 \times 60 = \underline{\underline{0.0414\ \Omega/\text{m}}}$$

**Prob. 11.19**

(a) $\frac{R}{L} = \frac{G}{C} \rightarrow G = \frac{R}{L}C = \frac{20 \times 63 \times 10^{-12}}{0.3 \times 10^{-6}}$

$G = 4.2 \times 10^{-3}$ S / m

$\alpha = \sqrt{RG} = \sqrt{20 \times 4.2 \times 10^{-3}} = 0.2898$

$\beta = \omega \sqrt{LC} = 2\pi \times 120 \times 10^6 \sqrt{0.3 \times 10^{-6} \times 63 \times 10^{-12}} = 3.278$

$\underline{\underline{\gamma = 0.2898 + j3.278 \ /\text{m}}}$

$$u = \frac{\omega}{\beta} = \frac{2\pi \times 120 \times 10^6}{3.278} = \underline{\underline{2.3 \times 10^8\ \text{m/s}}}$$

$$Z_o = \sqrt{\frac{L}{C}} = \sqrt{\frac{0.3 \times 10^{-6}}{63 \times 10^{-12}}} = \underline{\underline{69\ \Omega}}$$

(b) Let $V_o$ be its original magnitude

$V_o e^{-\alpha z} = 0.2V_o \rightarrow e^{\alpha z} = 5$

$z = \frac{1}{\alpha} \ln 5 = \underline{\underline{5.554\ \text{m}}}$

(c) $\beta l = 45^o = \pi/4 \rightarrow l = \frac{\pi}{4\beta} = \frac{4}{4 \times 3.278} = \underline{\underline{0.3051\ \text{m}}}$

**Prob. 11.20**

(a) $\alpha = \underline{\underline{0.0025\ \text{Np/m}}}$, $\beta = \underline{\underline{2\ \text{rad/m}}}$,

$$u = \frac{\omega}{\beta} = \frac{10^8}{2} = \underline{\underline{5 \times 10^7 \text{ m/s}}}$$

(b) $\Gamma = \frac{V_o^-}{V_o^+} = \frac{60}{120} = \frac{1}{2}$

But $\Gamma = \frac{Z_L - Z_o}{Z_L + Z_o} \rightarrow \frac{1}{2} = \frac{300 - Z_o}{300 + Z_o} \rightarrow \underline{\underline{Z_o = 100\Omega}}$

$$I(l') = \frac{120}{Z_o} e^{0.0025l'} \cos(10^8 + 2l') - \frac{60}{Z_o} e^{-0.0025l'} \cos(10^8 t - 2l')$$

$$\underline{\underline{= 1.2e^{0.0025l'} \cos(10^8 + 2l') - 0.6e^{-0.0025l'} \cos(10^8 t - 2l') A}}$$

**Prob. 11.21**

$\alpha = 10^{-3}, \quad \beta = 0.01$

$$\gamma = \alpha + j\beta = 0.001 + j0.01 = \underline{\underline{(1 + j10) \times 10^{-3} \text{ /m}}}$$

$$u = \frac{\omega}{\beta} = \frac{2\pi \times 10^4}{0.01} = \underline{\underline{6.283 \times 10^6 \text{ m/s}}}$$

**Prob. 11.22**

$$Z_o = \sqrt{\frac{L}{C}} = \sqrt{\frac{0.6 \times 10^{-6}}{82 \times 10^{-12}}} = \underline{\underline{85.54\ \Omega}}$$

$$RC = LG \quad \longrightarrow \quad G = \frac{RC}{L}$$

$$\alpha = \sqrt{RG} = \sqrt{R\frac{RC}{L}} = \frac{R}{Z_o} = \frac{10 \times 10^{-3}}{85.54} = 1.169 \times 10^{-4} \text{ Np/m}$$

$$\beta = \omega\sqrt{LC} = 2\pi \times 80 \times 10^6 \sqrt{0.6 \times 10^{-6} \times 82 \times 10^{-12}}$$

$$= 3.5258 \text{ rad/m}$$

$$\gamma = \underline{\underline{1.169 \times 10^{-4} + j3.5258 \text{ /m}}}$$

**Prob. 11.23**

$$R + j\omega L = 6.5 + j2\pi \times 2 \times 10^6 \times 3.4 \times 10^{-6} = 6.5 + j42.73$$

$$G + j\omega C = 8.4 \times 10^{-3} + j2\pi \times 2 \times 10^6 \times 21.5 \times 10^{-12} = (8.4 + j0.27) \times 10^{-3}$$

$$Z_o = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{6.5 + j42.73}{(8.4 + j0.27) \times 10^{-3}}}$$

$Z_o = 71.71\angle 39.75^o = \underline{\underline{55.12 + j45.85\Omega}}$

$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)}$ $= \sqrt{(43.19\angle 81.34^o)(8.4\times 10^{-3}\angle 1.84^o)}$

$= 0.45 + j0.4$ /m

↑ α ↑ β

$t = \frac{l}{u}$, but $u = \frac{\omega}{\beta}$,

$$t = \frac{\beta l}{\omega} = \frac{0.39\times 5.6}{2\pi\times 2\times 10^6} = \underline{\underline{0.1783\mu s}}$$

**Prob. 11.24**

(a) For a lossy line,

$$Z_{in} = Z_o\left[\frac{Z_L + Z_o \tanh\gamma\ell}{Z_0 + Z_L \tanh\gamma\ell}\right]$$

For a short-circuit, $Z_L = 0$.

$$Z_{sc} = Z_{in}\Big|_{Z_L = 0} = Z_o \tanh\gamma\ell$$

$$\tanh\gamma\ell = \frac{Z_{sc}}{Z_o} = \frac{30 - j12}{80 + j60} = 0.168 - j0.276$$

$$\gamma\ell = \alpha\ell + j\beta\ell = \tanh^{-1}(0.168 - j0.276) = 0.1571 - j0.2762$$

$$\alpha = \frac{0.1571}{2.1} = \underline{\underline{0.0748 \text{ Np/m}}}$$

$$\beta = \frac{0.2762}{2.1} = \underline{\underline{0.1316 \text{ rad/m}}}$$

(b)

$$Z_{in} = (80 + j60)\left[\frac{(40 + j30) + (80 + j60)(0.168 - j0.276)}{(80 + j60) + (40 + j30)(0.168 - j0.276)}\right]$$

$$= \underline{\underline{61.46 + j24.43\ \Omega}}$$

**Prob. 11.25**

$$Z_{in} = Z_o\left[\frac{Z_L + Z_o \tanh\gamma\ell}{Z_0 + Z_L \tanh\gamma\ell}\right]$$

$$\gamma\ell = \alpha\ell + j\beta\ell = 1.4\times0.5 + j2.6\times0.5 = 0.7 + j1.3$$

$$\tanh\gamma\ell = 1.4716 + j0.3984$$

$$Z_{in} = (75 + j60)\left[\frac{200 + (75 + j60)(1.4716 + j0.3984)}{(75 + j60) + 200(1.4716 + j0.3984)}\right]$$

$$= \underline{\underline{57.44 + j48.82\ \Omega}}$$

**Prob. 11.26**

(a) $$T_L = \frac{V_L}{V_o^+} = \frac{Z_L I_L}{\frac{1}{2}\left(V_L + Z_o I_L\right)} = \frac{2Z_L I_L}{Z_L I_L + Z_o I_L}$$

$$= \underline{\underline{\frac{2Z_L}{Z_L + Z_o}}}$$

$$1 + \Gamma_L = 1 + \frac{Z_L - Z_o}{Z_L + Z_o} = \underline{\underline{\frac{2Z_L}{Z_L + Z_o}}}$$

(b) (i) $$\tau_L = \frac{2nZ_o}{nZ_o + Z_o} = \frac{2n}{n+1}$$

(ii) $$\tau_L = \lim_{Y_L \to 0} = \frac{2}{1 + Z_o/Z_L} = 2$$

(iii) $$\tau_L = \lim_{Z_L \to 0} = \frac{2Z_L}{Z_L + Z_o} = 0$$

(iv) $$\tau_L = \frac{2Z_o}{2Z_o} = 1$$

**Prob. 11.27**

$(a)$ $$\Gamma = \frac{Z_L - Z_o}{Z_L + Z_o} \quad \rightarrow \quad \Gamma Z_L + \Gamma Z_o = Z_L - Z_o \quad \rightarrow \quad Z_L(\Gamma - 1) = -(\Gamma + 1)Z_o$$

Hence, $$Z_L = \frac{1+\Gamma}{1-\Gamma}Z_o$$

$$(b) \qquad Z_L = \frac{1+0.6\angle 45^o}{1-0.6\angle 45^o}(50) = \frac{1.424 + j0.424}{0.5751 - j0.424}(50) = (1.252 + j1.6586)(50)$$
$$= \underline{\underline{62.6 + j83.03\ \Omega}}$$

**Prob. 11.28**

$$\Gamma_L = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{200 - j240 - 120}{200 - j240 + 120} = \frac{80 - j240}{320 - j240} = 0.52 - j0.36 = \underline{\underline{0.6325\angle -34.7^o}}$$
$$s = \frac{1+|\Gamma_L|}{1-|\Gamma_L|} = \frac{1+0.6325}{1-0.6325} = \underline{\underline{4.4415}}$$

**Prob. 11.29**

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = \sqrt{(3.5 + j2\pi \times 400 \times 10^6 \times 2 \times 10^{-6})(0 + j2\pi \times 400 \times 10^6 \times 120 \times 10^{-12})}$$
$$= \sqrt{(3.5 + j5026.55)(j0.3016)} = 0.0136 + j38.94$$

$$\alpha = \underline{\underline{0.0136\text{ Np/m}}}, \qquad \beta = \underline{\underline{38.94\text{ rad/m}}}$$
$$u = \frac{\omega}{\beta} = \frac{2\pi \times 400 \times 10^6}{38.94} = \underline{\underline{6.452 \times 10^7\ \text{m/s}}}$$
$$Z_o = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{3.5 + j5026.55}{j0.3016}} = \underline{\underline{129.1 - j0.045\ \Omega}}$$

**Prob. 11.30**

From eq. (11.33)
$$Z_{sc} = Z_{in}\Big|_{Z_L=0} = Z_o \tanh \gamma l$$
$$Z_{oc} = Z_{in}\Big|_{Z_L=\infty} = \frac{Z_o}{\tanh \gamma l} = Z_o \coth(\gamma l)$$
For lossless line, $\gamma = j\beta, \tan(\gamma l) = \tanh(j\beta l) = j\tan(\beta l)$

$$Z_{sc} = jZ_o \tan(\beta l), Z_{oc} = -jZ_o \cot(\beta l)$$
$$Z_{sc} = jZ_o \tan(\beta l) = (65 + j38)\tanh[(0.7 + j2.5)0.8] = (65 + j38)\tanh(0.56 + j2) = 75.25\angle 30.3^o \frac{\exp(0.56 -}{\exp(0.56 -}$$
$$= 7.86\angle 60.3^o = 3.89 + j6.83$$

**Prob. 11.31**

$$\Gamma = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{(1+j2)-1}{(1+j2)+1} = \frac{j}{1+j} = \frac{1\angle 90^\circ}{\sqrt{2}\angle 45^\circ} = \underline{\underline{0.7071\angle -45^o}}$$

**Prob. 11.32**

(a) $\Gamma = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{120-50}{170} = \underline{\underline{0.4118}}$

For resistive load, $s = \frac{Z_L}{Z_o} = \underline{\underline{2.4}}$

(b) $Z_{in} = Z_o \frac{Z_L + jZ_o \tan(\beta l)}{Z_o + jZ_L \tan(\beta l)}$

$$\beta l = \frac{2\pi}{\lambda} \cdot \frac{\lambda}{6} = 60^o$$

$$Z_{in} = 50\left[\frac{120 + j50\tan(60^o)}{50 + j120\tan(60^o)}\right] = \underline{\underline{34.63\angle -40.65^o\,\Omega}}$$

**Prob. 11.33**

$(a)$ $\beta l = \frac{1}{4} \times 100 = 25 \text{ rad} = 1432.4^o = 352.4^o$

$$Z_{in} = 60\left[\frac{j40 + j60\tan 352.4^o}{60 - 40\tan 352.4^o}\right] = \underline{\underline{j29.375\Omega}}$$

$$V(z=0) = V_o = \frac{Z_{in}}{Z_{in} + Z_g} V_g = \frac{j29.375(10\angle 0^o}{j29.375 + 50 - j40}$$

$$= \frac{293.75\angle 90^o}{51.116\angle -12^o} = \underline{\underline{5.75\angle 102^o}}$$

$(b)$ $Z_{in} = Z_L = \underline{\underline{j40\Omega.}}$

$V_o^+ = \frac{V_g}{(e^{j\beta l} + \Gamma e^{-j\beta l})}$ (l is from the load)

$$V_L = \frac{V_g(1+\Gamma)}{(e^{j\beta l} + \Gamma e^{-j\beta l})} = \underline{\underline{12.62\angle 0^o \text{ V}}}$$

$(c)\quad \beta l' = \frac{1}{4}\times 4 = 1rad = 57.3^o$

$$Z_{in} = 60\left[\frac{j40 + j60\tan 57.3^o}{60 - 40\tan 57.3^o}\right] = \underline{\underline{-j3471.88\Omega.}}$$

$$V = \frac{V_g(e^{j} + \Gamma e^{-j})}{(e^{j25} + \Gamma e^{-j25})} = \underline{\underline{22.74\angle 0^o \text{ V}}}$$

(d) 3m from the source is the same as 97m from the load., i.e.

$$l' = 100 - 3 = 97m, \quad \beta l' = \frac{1}{4}\times 97 = 24.25rad = 309.42^o$$

$$Z_{in} = 60\left[\frac{j40 + j60\tan 309.42^o}{60 - 40\tan 309.42^o}\right] = \underline{\underline{-j18.2\Omega}}$$

$$V = \frac{V_g(e^{j97/4} + \Gamma e^{-j97/4})}{(e^{j25} + \Gamma e^{-j25})} = \underline{\underline{6.607\angle 180^o \text{ V}}}$$

**Prob. 11.34**

$$V_1 = V_s(z=0) = V_o^+ + V_o^- \qquad (1)$$

$$V_2 = V_s(z=l) = V_o^+ e^{-\gamma l} + V_o^- e^{\gamma l} \qquad (2)$$

$$I_1 = I_s(z=0) = \frac{V_o^+}{Z_o} - \frac{V_o^-}{Z_o} \qquad (3)$$

$$I_2 = -I_s(z=l) = -\frac{V_o^+}{Z_o}e^{-\gamma l} + \frac{V_o^-}{Z_o}e^{\gamma l} \qquad (4)$$

$$(1) + (3) \rightarrow V_o^+ = \frac{1}{2}(V_1 + Z_o I_1)$$

$$(1) - (3) \rightarrow V_o^- = \frac{1}{2}(V_1 - Z_o I_1)$$

Substituting $V_o^+$ and $V_o^-$ in (2) gives

$$V_2 = \frac{1}{2}(V_1 + Z_o I_1)e^{-\gamma l} + \frac{1}{2}(V_1 - Z_o I_1)e^{\gamma l}$$

$$= \frac{1}{2}(e^{\gamma l} + e^{-\gamma l})V_1 + \frac{1}{2}Z_o(e^{-\gamma l} - e^{\gamma l})I_1$$

$$V_2 = \cosh\gamma l V_1 - Z_o \sinh\gamma l I_1 \qquad (5)$$

Substituting $V_o^+$ and $V_o^-$ in (4),

$$I_2 = -\frac{1}{2Z_o}(V_1 + Z_o I_1)e^{-\gamma l} + \frac{1}{2Z_o}(V_1 - Z_o I_1)e^{\gamma l}$$

$$= \frac{1}{2Z_o}(e^{\gamma l} - e^{-\gamma l})V_1 + \frac{1}{2}(e^{\gamma l} + e^{-\gamma l})I_1$$

$$I_2 = \frac{1}{Z_o}\sinh\gamma l V_1 - \cosh\gamma l I_1 \qquad (6)$$

From (5) and (6)

$$\begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} = \begin{bmatrix} \cosh\gamma l & -Z_o\sinh\gamma l \\ -\frac{1}{Z_o}\sinh\gamma l & \cosh\gamma l \end{bmatrix}\begin{bmatrix} V_1 \\ I_1 \end{bmatrix}$$

But

$$\begin{bmatrix} \cosh\gamma l & -Z_o\sinh\gamma l \\ -\frac{1}{Z_o}\sinh\gamma l & \cosh\gamma l \end{bmatrix}^{-1} = \begin{bmatrix} \cosh\gamma l & Z_o\sinh\gamma l \\ \frac{1}{Z_o}\sinh\gamma l & \cosh\gamma l \end{bmatrix}$$

Thus

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} \cosh\gamma l & Z_o\sinh\gamma l \\ \frac{1}{Z_o}\sinh\gamma l & \cosh\gamma l \end{bmatrix}\begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

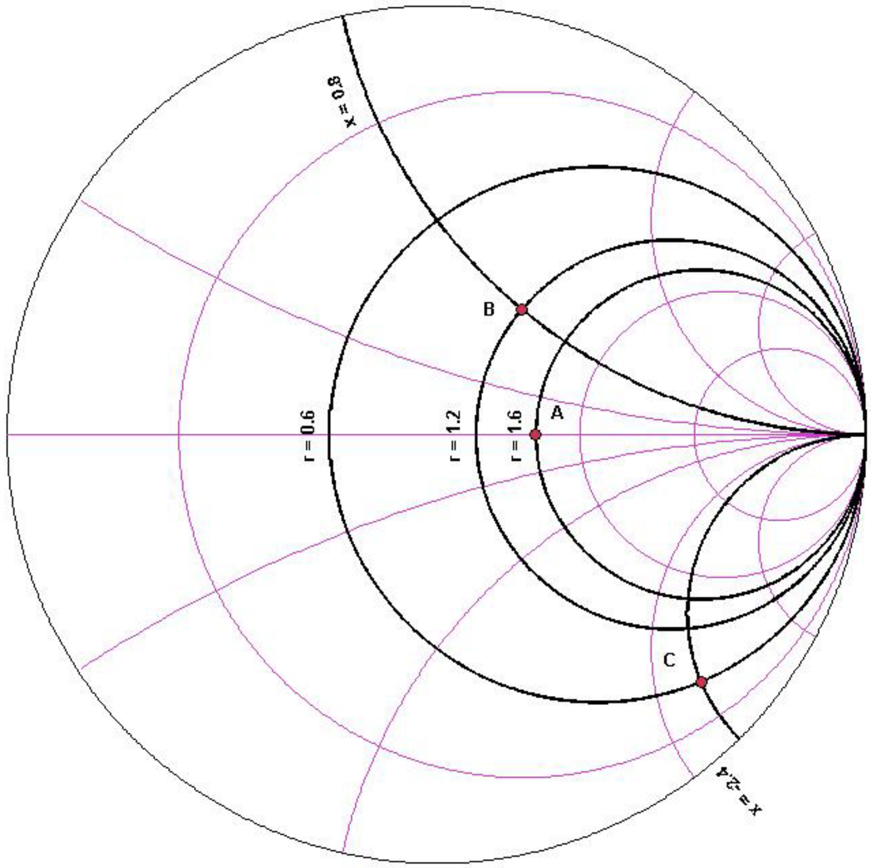
**Prob. 11.35**

(a) $z_a = \frac{Z_a}{Z_o} = \frac{80}{50} = 1.6$

(b) $z_b = \frac{Z_b}{Z_o} = \frac{60 + j40}{50} = 1.2 + j0.8$

(c) $z_c = \frac{Z_c}{Z_o} = \frac{30 - j120}{50} = 0.6 - j2.4$

The three loads are located on the Smith chart, as A, B, and C as shown next.




**Prob. 11.36**

$$z_L = \frac{Z_L}{Z_O} = \frac{210}{100} = 2.1 = s$$

Or $\Gamma = \frac{Z_L - Z_O}{Z_L + Z_O} = \frac{110}{310},$

$$s = \frac{1+|\Gamma|}{1-|\Gamma|} = 2.1$$

But $s = \frac{V_{\max}}{V_{\min}} \to V_{\max} = sV_{\min}$

Since the line is $\frac{\lambda}{4}$ long, $\frac{\lambda}{4} \to \frac{720^o}{4} = 120^o$

Hence the sending end will be $V_{\min}$ , while the receiving end at $V_{\max}$

$V_{\min} = V_{\max} / s = 80 / 2.1 = 38.09$

$V_{sending} = \underline{\underline{38.09\angle 90^o}}$

**Prob. 11.37**

$Z_L = (1+j2)Z_o \quad \longrightarrow \quad z_L = \dfrac{Z_L}{Z_o} = 1+j2$

We locate $z_L$ on the Smith chart.

$\dfrac{\lambda}{4} \quad \longrightarrow \quad \dfrac{720^o}{4} = 180^o$

We move $180^o$ toward the generator and locate point Q at which

$z = 0.2 - j0.4$

$Z = zZ_o = \underline{\underline{(0.2 - j0.4)Z_o}}$

**Prob. 11.38**

(a) Method 1: At Y,

$$Z_{in} = Z_o\left[\frac{Z_L + jZ_o \tan\beta\ell}{Z_o + jZ_L \tan\beta\ell}\right]$$

$$\beta\ell = \frac{2\pi}{\lambda}\frac{\lambda}{2} = \pi, \qquad \tan\pi = 0$$

$$Z_{in} = Z_o\frac{Z_L}{Z_o} = Z_L = 150\ \Omega$$

At X, $\quad Z_L' = 150\Omega$

$$\beta\ell = \frac{2\pi}{\lambda}\frac{\lambda}{4} = \pi/2, \qquad \tan\pi/2 = \infty$$

$$Z_{in} = \lim_{\tan\beta\ell\to\infty} Z_o\left[\frac{jZ_o + \dfrac{Z_L'}{\tan\beta\ell}}{jZ_L' + \dfrac{Z_o}{\tan\beta\ell}}\right] = \frac{Z_o^2}{Z_L'} = \frac{(75)^2}{150} = 37.5\Omega$$

Method 2: Using the Smith chart,

$$z_L = \frac{Z_L'}{Z_o} = \frac{150}{50} = 3$$

Since $\ell = \lambda/2$, we must move $360^o$ toward the generator. We arrive at the same point. Hence,

$$Z_{in} = Z_L = 150$$

$$z_L' = \frac{150}{75} = 2$$

$$\ell = \frac{\lambda}{4} \to 180^o$$

We move $180^o$ toward the generator. $z_{in} = 0.5$

$$Z_{in} = 75(0.6) = \underline{\underline{37.5\Omega}}$$

(b) From the Smith chart,

s = 3 for section XY

s = 2 for section YZ

(c) $\Gamma = \frac{Z_L - Z_o}{Z_L - Z_o} = \frac{150-50}{150+50} = \underline{\underline{0.5}}$

**Prob. 11.39**

$$z_{in} = \frac{Z_{in}}{Z_o} = \frac{100 - j120}{50} = 2 - j2.4$$

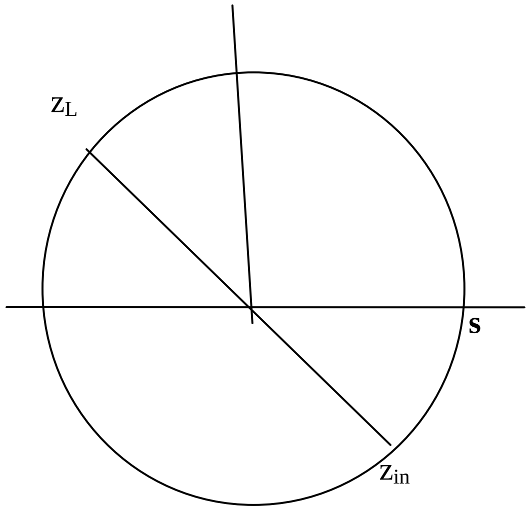
$$\lambda = \frac{u}{f} = \frac{0.8 \times 3 \times 10^8}{6 \times 10^8} = 0.4 \text{ m}, \qquad \ell = 0.1 \text{ m} = \frac{\lambda}{4}$$

If $\lambda \to 720^o$, then $\lambda / 4 \to 180^o$

We move $180^o$ counterclockwise from $z_{in}$ to get $z_L$ on the Smith chart as shown below.

$z_L = 0.2 + j0.25 \quad \to \quad Z_L = Z_o z_L = 50(0.2 + j0.25) = \underline{\underline{10 + j12.5\ \Omega}}$

From the Smith chart, $\underline{\underline{s = 5.19}}$




**Prob. 11.40**

$$z_L = \frac{Z_L}{Z_o} = \frac{40 - j25}{50} = 0.8 - j0.5$$

We locate this at point P on the Smith chart shown below

$$|\Gamma_L| = \frac{OP}{OQ} = \frac{2.4 \text{ cm}}{8 \text{ cm}} = 0.3, \qquad \theta_\Gamma = -96^o$$

$\Gamma_L = 0.3\angle -96^o$

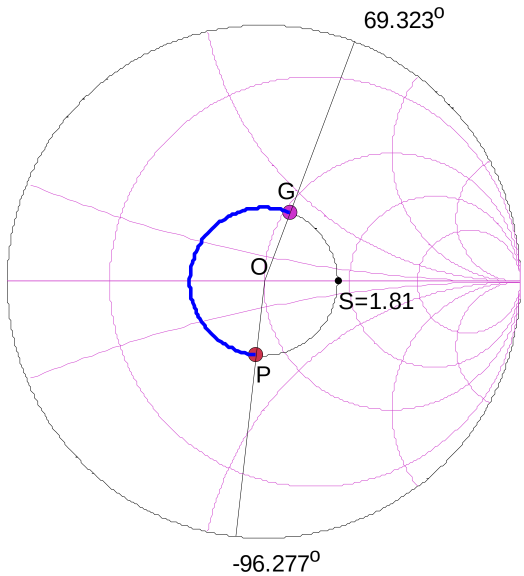
At S, s = r = 1.81

$\ell = 0.27\lambda \longrightarrow 0.27 \times 720^o = 194.4^o$

From P, we move $194.4^o$ toward the generator to G. At G,

$z_{in} = 1.0425 + j0.6133$

$Z_{in} = Z_o z_{in} = 50(1.0425 + j0.6133) = 52.13 - j30.66\ \Omega$

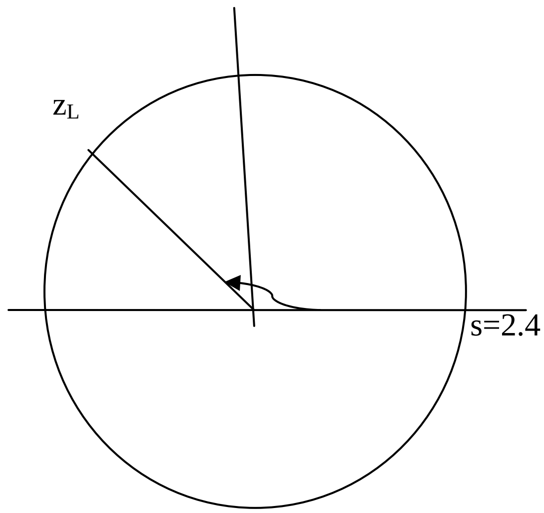



**Prob. 11.41**

$\lambda \rightarrow 720^o$ implies that $0.2\lambda \rightarrow 144^o$. The minimum voltage is located at s on the Smith chart. We move $144^o$ from there to locate the load $z_L$ as shown on the Smith chart below. At $z_L$, we obtain

$$z_L = 0.46 + j0.26, \quad \rightarrow \quad Z_L = 100(0.46 + j0.26) = \underline{\underline{46 + j26\ \Omega}}$$

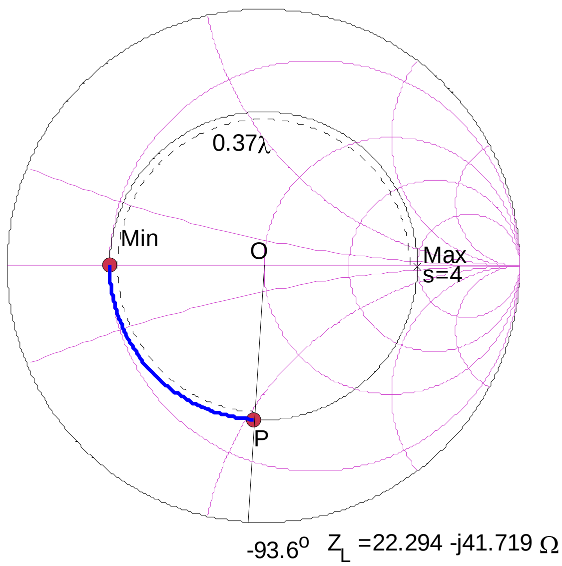
$$\Gamma = |\Gamma| \angle\theta_\Gamma = \frac{OP}{OQ}\angle 144^o = \frac{3.5cm}{8.4cm}\angle 144^o = \underline{\underline{0.4167\angle 144^o}}$$




**Prob. 11.42**

(a) $0.12\lambda \longrightarrow 0.12 \times 720^o = 86.4^o$

We draw the s=4 circle and locate $V_{min}$. We move from that location $86.4^o$ toward the load.




At P, $z_L = 0.45 - j0.83$

$Z = 50(0.45 - j0.83) = \underline{\underline{22.3 - j41.72\ \Omega}}$

(b) the load is $\underline{\underline{\text{capacitive.}}}$

(c) $V_{min}$ and $V_{max}$ are $\lambda/4$ apart. Hence the first maximum occurs at

$0.12\lambda + 0.25\lambda = \underline{\underline{0.37\lambda}}$

**Prob. 11.43**

(a) $z_L = \frac{Z_L}{Z_o} = \frac{75 + j60}{50} = 1.5 + j1.2$

$|\Gamma| = \frac{OP}{OQ} = \frac{3.8\text{cm}}{8\text{cm}} = 0.475, \qquad \theta_\Gamma = 42^o$

$\Gamma = \underline{\underline{0.475\angle 42^0}}$

(Exact value = 0.4688 $\angle 41.76^o$ )

(b) $\underline{s=2.8}$

(Exact value = 2.765)

(c ) $0.2\lambda \rightarrow 0.2x720^o = 144^o$

$z_{in} = 0.55 - j0.65$

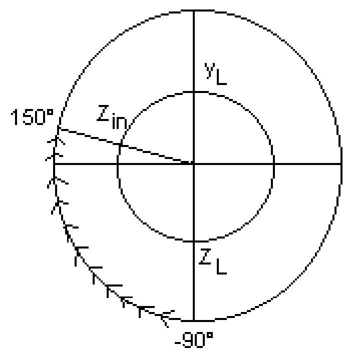
$Z_{in} = Z_o z_{in} = 50(0.55 + j0.65) = \underline{\underline{27.5 + j32.5\ \Omega}}$

(d) Since $\theta_\Gamma = 42^o$, $V_{min}$ occurs at

$$\frac{42}{720}\lambda = \underline{\underline{0.05833\lambda}}$$

(e) same as in (d), i.e.. $\underline{\underline{0.05833\lambda}}$

**Prob. 11.44**




If $\lambda \rightarrow 720^o$, then $\frac{\lambda}{6} \rightarrow 120^o$

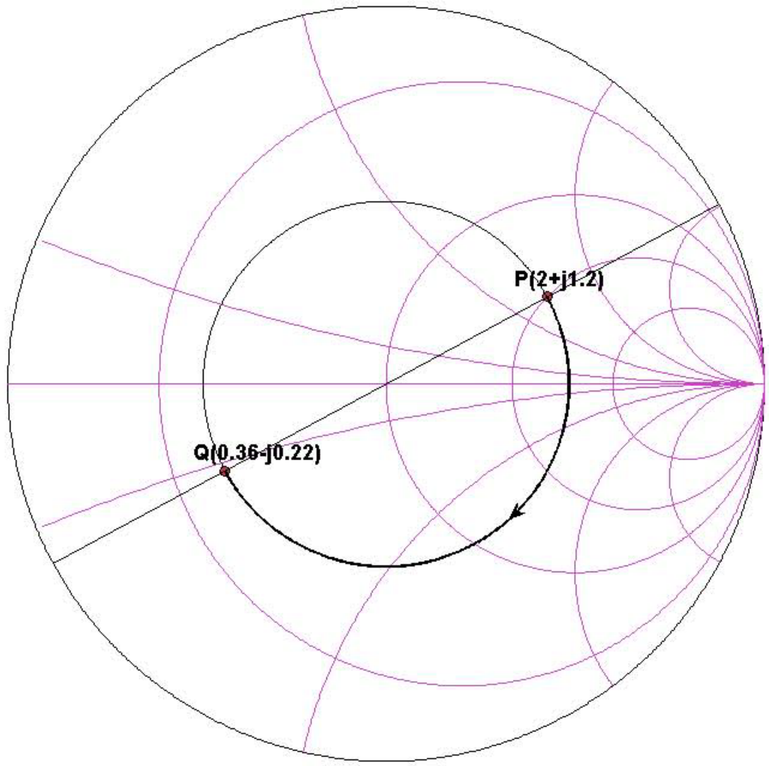
$z_{in} = \underline{\underline{0.35 + j0.24}}$

**Prob. 11.45**

$$z = \frac{Z}{Z_o} = \frac{100 + j60}{50} = 2 + j1.2$$

We locate z on the Smith chart. We move $180^o$ toward the generator to reach point Q. At Q, y = 0.36-j0.22

$$Y = yY_o = \frac{1}{50}(0.36 - j0.22) = \underline{\underline{7.4 - j4.4\text{ mS}}}$$




**Prob. 11.46**

$$\lambda = \frac{u}{f} = \frac{0.5\times 3\times 10^8}{160\times 10^6} = 0.9375 \text{ m}$$

$$z_L = \frac{Z_L}{Z_o} = \frac{50 + j30}{50} = 1 + j0.6$$

We locate z at P on the Smith chart. We draw a circle that passes through P.

We locate point Q as the point where the circle crosses the $\Gamma_r$ – axis. At Q,

$z_{in} = 1.8$

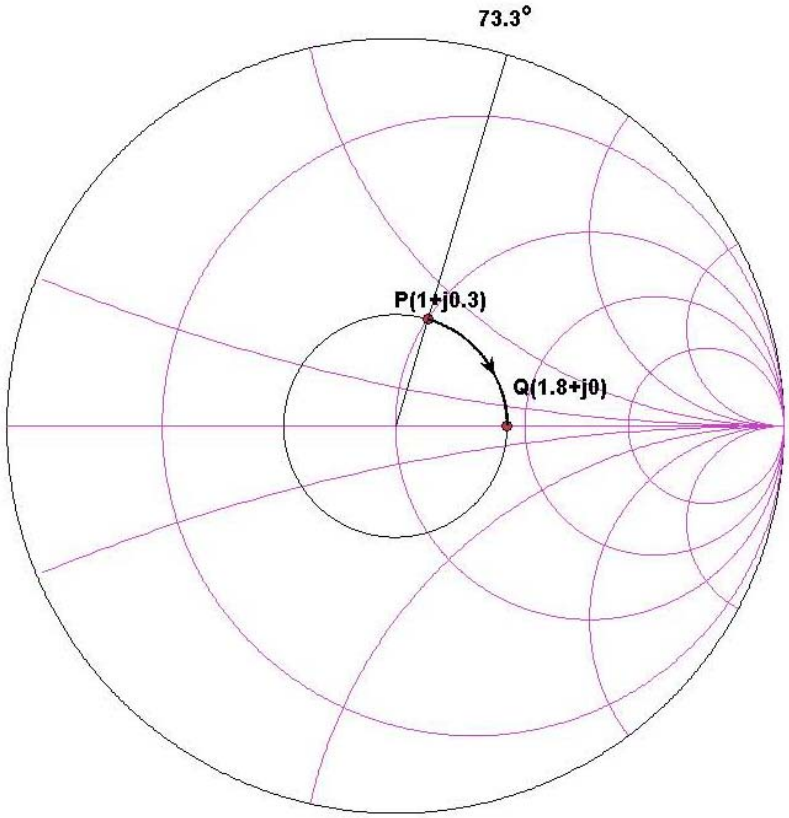
$Z_{in} = Z_o z_{in} = 50(1.8) = \underline{\underline{90\ \Omega}}$

The angular distance between P and Q is $\theta_\Gamma = 73.3^o$.

If $\lambda \to 720^o$, $73.3^o \to \dfrac{\lambda}{720^o}73.3^o$

$$\ell = \frac{73.3^o}{720^o}\times 0.9375 = \underline{\underline{0.0954 \text{ m}}}$$




**Prob. 11.47**

$$z_L = \frac{60}{100} = 0.6$$

If $\lambda \rightarrow 720^o$, then $\lambda / 2 \rightarrow 360^o$. This means that the input impedance is located on the same spot as $z_L$ on the Smith chart.

$Z_{in}^{'} = Z_o z_L = 60 \quad \rightarrow \quad z_{in}^{'} = 60/50 = 1.2$

$\lambda / 4 \rightarrow 180^o, \qquad z_{in} = 0.84$

$Z_{in} = 50(0.84) = \underline{\underline{42\ \Omega}}$

**Prob. 11.48**

(a) $\Gamma = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{0.5 - j - 1}{0.5 - j + 1} = 0.0769 - j0.6154 = \underline{\underline{0.6202\angle -82.87^o}}$

(b)

$$\Gamma = \frac{z_L - 1}{z_L + 1} \longrightarrow z_L = \frac{1+\Gamma}{1-\Gamma} = \frac{1+0.4\angle 25^o}{1-0.4\angle 25^o}$$

$$z_L = 1.931 + j0.7771$$

$$Z_L = \underline{\underline{(1.931 + j0.7771)Z_o}}$$

**Prob. 11.49**

(a) $Z_{in} = \frac{Z_{in}}{Z_o} = \frac{100 - j120}{80} = 1.25 - j1.5$

$\lambda = \frac{u}{f} = \frac{0.8 \times 3 \times 10^8}{12 \times 10^6} = 20\text{m}$

$l_1 = 22\text{m} = \frac{22\lambda}{20} = 1.1\lambda \rightarrow 720° + 72°$

$l_2 = 28\text{m} = \frac{28\lambda}{20} = 1.4\lambda \rightarrow 720° + 72° + 216°$

To locate P(the load), we move 2 revolutions plus 72° toward the load. At P,

$|\Gamma_L| = \frac{OP}{OQ} = \frac{5.1\text{cm}}{9.2\text{cm}} = 0.5543$

$\theta_\Gamma = 72° - 47° = 25°$

$\Gamma_L = \underline{\underline{0.5543\angle 25°}}$

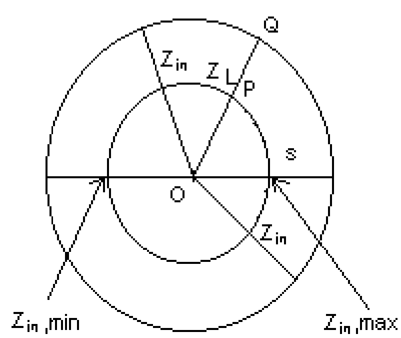
(Exact value = $0.5624\angle 25.15^o$)

$Z_{in}, \max = sZ_o = 3.7(80) = \underline{\underline{296\Omega}}$

(Exact value = 285.59 Ω)

$Z_{in}, \min = \frac{Z_o}{s} = \frac{80}{3.7} = \underline{\underline{21.622\Omega}}$

(Exact value =22.41 Ω)



(b) Also, at P, $Z_L = 2.3 + j1.55$

$$Z_L = 80(2.3 + j1.55) = \underline{\underline{184 + j124\Omega}}$$

(Exact value = 183.45+j128.25 Ω)

At S, $s = \underline{\underline{3.7}}$

To Locate $Z_{in}^{'}$, we move 216° from $Z_{in}$ toward the geneator.

At $Z_{in}^{'}$,

$z_{in}^{'} = 0.48 + j0.76$

$Z_{in}^{'} = 80(0.48 + j0.76) = \underline{\underline{38.4 + j60.8\Omega}}$

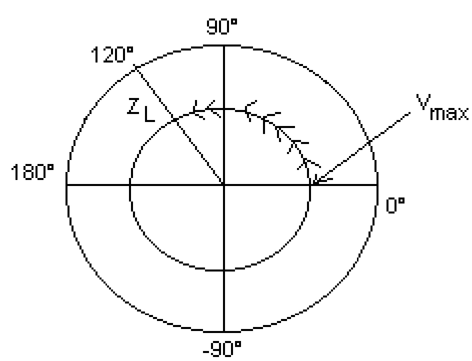
(Exact=37.56+j61.304 Ω)

(c) Between $Z_L$ and $Z_{in}$, we move 2 revolutions and 72°. During the movement, we pass through $Z_{in,\,max}$ 3 times and $Z_{in,min}$ twice.

Thus there are:

$$\underline{\underline{3\ Z_{in,\max} \text{ and } 2\ Z_{in,\min}}}$$

**Prob. 11.50**



(a) $\frac{\lambda}{2} = 120\text{cm} \rightarrow \lambda = 2.4\text{m}$

$u = f\lambda \rightarrow f = \frac{u}{\lambda} = \frac{3\times 10^8}{2.4} = \underline{\underline{125\text{MHz}}}$

(b) 40cm $= \frac{40\lambda}{240} = \frac{\lambda}{6} \rightarrow \frac{720°}{6} = 120°$

$Z_L = Z_o z_L = 150(0.48 + j0.48)$

$= \underline{\underline{72 + j72}}\ \Omega$

(Exact value = 73.308+j70.324 Ω)

(c) $|\Gamma| = \frac{s-1}{s+1} = \frac{1.6}{3.9} = 0.444,$

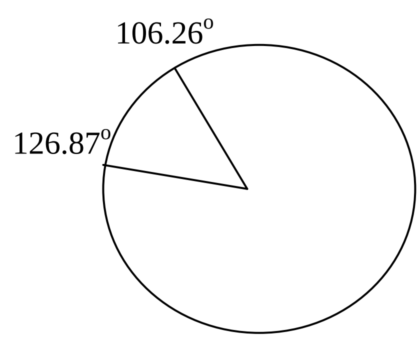
$\Gamma = \underline{\underline{0.444\angle 120°}}$

**Prob. 11.51**

(a)

$z_L = \frac{Z_L}{Z_o} = \frac{j60}{80} = j0.75, \qquad z_{in} = \frac{Z_{in}}{Z_o} = \frac{j40}{80} = j0.5$

The two loads fall on the r=0 circle, the outermost resistance circle. The shortest distance between them is




$$\frac{360^o - (126.87^o - 106.26^o)\lambda}{720^o} = \underline{\underline{0.4714\lambda}}$$

(b) $\underline{\underline{s = \infty}}, \qquad \Gamma_L = \underline{\underline{1\angle 106.26^o}}$

**Prob. 11.52**

$$z_L = \frac{Z_L}{Z_o} = \frac{40 + j25}{50} = 0.8 + j0.5$$

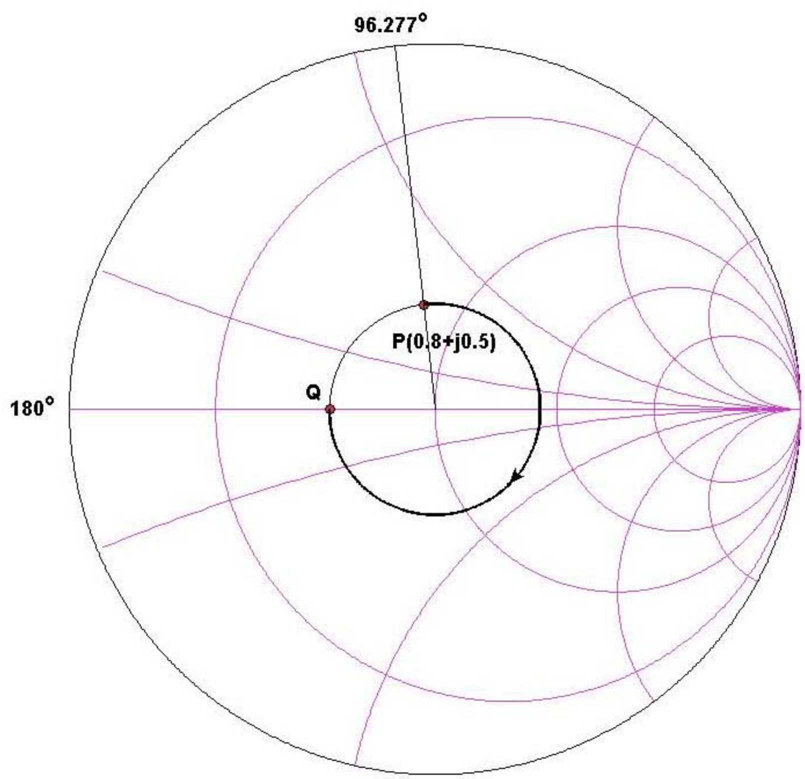
Locate this load at point P on the Smith chart. Draw a circle that passes through P. Locate point Q where the negative $\Gamma_r$ – axis crosses the circle.

$\theta_P = 97^o$. The angular distance between P and Q is

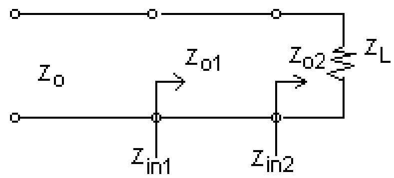
$\theta = 180 + \theta_P = 277^o$.

$720^o \rightarrow \lambda$

$$277^o \rightarrow \ell = \frac{\lambda}{720^o} 277^o = \underline{\underline{0.3847\lambda}}$$



**Prob. 11.53**




(a) From Eq. (11.43), $Z_{in2} = \frac{Z_{o2}^2}{Z_L}$

$Z_{in1} = \frac{Z_{o1}^2}{Z_{in2}} = Z_o$, i.e. $Z_{in2} = \frac{Z_{o1}^2}{Z_o} = \frac{Z_{o2}^2}{Z_L}$

$Z_{o1} = Z_{o2}\sqrt{\frac{Z_o}{Z_L}} = 30\sqrt{\frac{50}{75}} = \underline{\underline{24.5\Omega.}}$

(b) Also, $\frac{Z_o}{Z_{o1}} = \left(\frac{Z_{o2}}{Z_L}\right) \rightarrow Z_{o2} = \frac{Z_o Z_L}{Z_{o1}}$ (1)

Also, $\frac{Z_{o1}}{Z_{o2}} = \left(\frac{Z_{o2}}{Z_L}\right)^2 \rightarrow (Z_{o2})^3 = Z_{o1}Z_L^2$ (2)

From (1) and (2), $(Z_{o2})^3 = Z_{o1}Z_L^2 = \frac{Z_o^3 Z_L^3}{Z_{o1}^3}$ (3)

or $Z_{o1} = \sqrt[4]{Z_o^3 Z_L} = \sqrt[4]{(50)^3(75)} = \underline{\underline{55.33\Omega}}$

From (3), $Z_{o2} = \sqrt[3]{Z_{o1}Z_L^2} = \sqrt[3]{(55.33)(75)^2} = \underline{\underline{67.74\Omega.}}$

**Prob. 11.54**

$$\frac{\lambda}{4} \rightarrow 180°, \quad z_L = \frac{74}{50} = 1.48, \quad \frac{1}{z_L} = 0.6756$$

This acts as the load to the left line. But there are two such loads in parallel due to the two lines on the right. Thus

$$Z_L^{'} = 50\frac{\left(\frac{1}{Z_L}\right)}{2} = 25(0.6756) = 16.892$$

$$z_L^{'} = \frac{16.892}{50} = 0.3378, \quad z_{in} = \frac{1}{z_L^{'}} = 2.96$$

$Z_{in} = 50(2.96) = \underline{\underline{148\Omega.}}$

**Prob. 11.55**

From the previous problem, $Z_{in} = 148\Omega$

$$I_{in} = \frac{V_g}{Z_g + Z_{in}} = \frac{120}{80+148} = 0.5263A$$

$$P_{ave} = \frac{1}{2}|I_{in}|^2 R_{in} = \frac{1}{2}(0.5263)^2(148) = 20.5W$$

Since the lines are lossless, the average power delivered to either antenna is $\underline{\underline{10.25W}}$

**Prob. 11.56**

(a) $\beta l = \frac{2\pi}{4}.\frac{\lambda}{4} = \frac{\pi}{2}, \qquad \tan\beta l = \infty$

$$Z_{in} = Z_o\left(\frac{Z_L + jZ_o\tan\beta l}{Z_o + jZ_L\tan\beta l}\right) = Z_o\frac{\left(\frac{Z_L}{\tan\beta l} + jZ_o\right)}{\left(\frac{Z_o}{\tan\beta l} + jZ_L\right)}$$

As $\tan\beta l \to \infty$,

$$Z_{in} = \frac{Z_o^2}{Z_L} = \frac{(50)^2}{200} = \underline{\underline{12.5\Omega}}$$

(b) If $Z_L = 0$,

$$Z_{in} = \frac{Z_o^2}{0} = \underline{\underline{\infty}} \qquad \text{(open)}$$

(c) $Z_L = 25 // \infty = \frac{25\times\infty}{25+\infty} = \frac{25}{1+\frac{25}{\infty}} = 25\Omega$

$$Z_{in} = \frac{(50)^2}{12.5} = \underline{\underline{200\Omega}}$$

**Prob. 11.57**

$$l_1 = \frac{\lambda}{4} \rightarrow Z_{in1} = \frac{Z_o^2}{Z_L} \quad or \quad y_{in1} = \frac{Z_L}{Z_o^2}$$

$$y_{in1} = \frac{200 + j150}{(100)^2} = \underline{\underline{20 + j15 \text{ mS}}}$$

$$l_2 = \frac{\lambda}{8} \rightarrow Z_{in2} =_{Z_L} \underset{}{\lim}_0 Z_o \left( \frac{Z_L + jZ_o \tan\frac{\pi}{4}}{Z_o + jZ_L \tan\frac{\pi}{4}} \right) = jZ_o$$

$$y_{in2} = \frac{1}{jZ_o} = \frac{1}{j100} = \underline{\underline{-j10 \text{ mS}}}$$

$$l_3 = \frac{7\lambda}{8} \rightarrow Z_{in3} = Z_o \frac{\left( Z_i + jZ_o \tan\frac{7\pi}{4} \right)}{\left( Z_o + jZ_i \tan\frac{7\pi}{4} \right)} = \frac{Z_o\left(Z_i - jZ_o\right)}{\left(Z_o - jZ_i\right)}$$

But

$$y_i = y_{in1} + y_{in2} = 20 + j5 \text{ mS}$$

$$z_i = \frac{1}{y_i} = \frac{1000}{20 + j5} = 47.06 - j11.76$$

$$y_{in3} = \frac{Z_o - jZ_{in}}{Z_o\left(Z_{in} - jZ_o\right)} = \frac{100\text{-j}47.06\text{-}11.76}{100\left(47.06\text{-j}11.76\text{-j}100\right)}$$

$$= \underline{\underline{6.408 + j5.189 \text{ mS}}}$$

If the shorted section were open,

$$y_{in1} = \underline{\underline{20 + j15 \text{ mS}}}$$

$$y_{in2} = \frac{1}{Z_{in2}} = \frac{j\tan{\pi/4}}{Z_o} = \frac{j}{100} = \underline{\underline{j10 \text{ mS}}}$$

$$l_3 = \frac{7\lambda}{8} \rightarrow Z_{in3} = Z_o \frac{\left(Z_i + jZ_o \tan\frac{7\pi}{4}\right)}{\left(Z_o + jZ_i \tan\frac{7\pi}{4}\right)} = \frac{Z_o(Z_i - jZ_o)}{(Z_o - jZ_i)}$$

$$y_i = y_{in1} + y_{in2} = 20 + j15 + j10 = 20 + j25 \text{ mS}$$

$$Z_i = \frac{1}{y_i} = \frac{1000}{20 + j25} = 19.51 - j24.39\Omega$$

$$y_{in3} = \frac{Z_o - jZ_i}{Z_o(Z_i - jZ_o)} = \frac{75.61\text{- j}19.51}{100(19.51 - j124.39)}$$

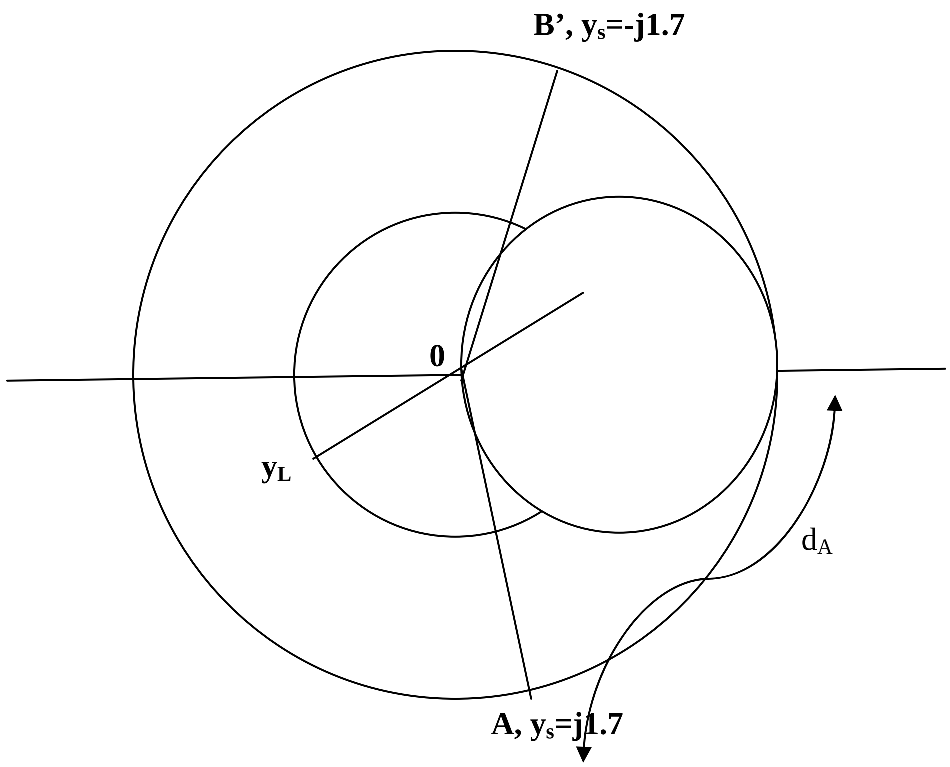
$$= \underline{\underline{2.461 + j5.691 \text{ mS}}}$$

**Prob. 11.58**

$$z_L = \frac{Z_L}{Z_o} = \frac{75 + j100}{50} \quad = 1.5 + j2$$

Locate $z_L$ on the Smith chart. Draw the s-circle passing through $z_L$. Extend the diameter through O to $y_L$. Locate points A and B where the s-circle intersects the g=1 circle. At A, $y_s = -j1.7$ and at B, $y_s = j1.7$. Locate points A' and B' where the stubs admittance is j1.7 and -j1.7 respectively. Calculate

$$d_A = \frac{26.5}{720}\lambda = \underline{\underline{0.0368\lambda}}$$




**Prob. 11.59**

$d_A = 0.12\lambda \quad \rightarrow \quad 0.12 \times 720^o = 86.4^o$

$l_A = 0.3\lambda \quad \rightarrow \quad 0.3 \times 720^o = 216^o$

(a) From the Smith Chart below,

$z_L = 0.57 + j0.69$

$Z_L = 60(0.57 + j0.69)$

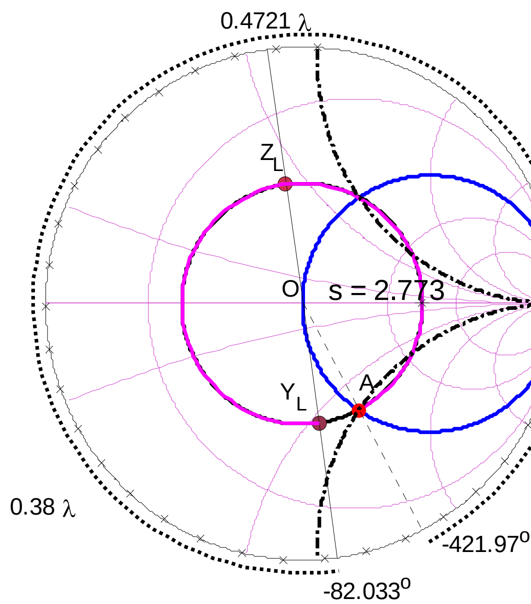
$= \underline{\underline{34.2 + j41.4\Omega}}$

(b) $d_B = \dfrac{360^o - 86.4^o}{720^o}\lambda \quad = \quad \underline{\underline{0.38\lambda}}$

$l_B = \dfrac{\lambda}{2} - \dfrac{(-62.4^o - -82^o)}{720^o}\lambda \quad = \quad \underline{\underline{0.473\lambda}}$

(c) $\underline{\underline{s = 2.65}}$

(Exact value = 2.7734 )




**Prob.11.60**

$$z_L = \frac{Z_L}{Z_o} = \frac{120 + j220}{50} = 2.4 + j4.4$$

We follow Example 11.7. At A, $y_s$=-j3 and at B, $y_s$=+j3. The required stub admittance is

$$Y_s = Y_o y_s = \frac{\pm j3}{50} = \pm j0.06 \text{ S}$$

The distance between the load and the stub is determined as follows. For A,value = 0.2308λ )

For B,

$$l_B = \frac{180 + 10 + 17}{720}\lambda = \underline{\underline{0.2875\lambda}}$$

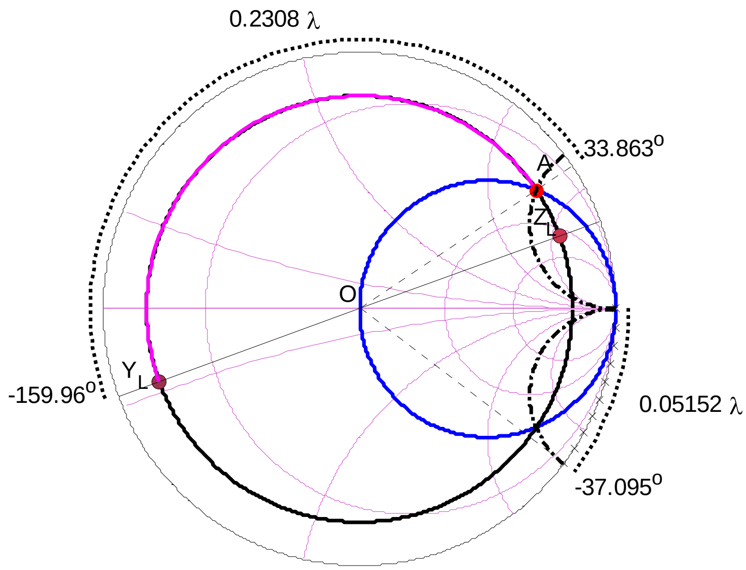
The length of the stub line is determined as follows.

$$d_A = \frac{19}{720}\lambda = \underline{\underline{0.0264\lambda}}$$

(Exact value = 0.0515 λ)

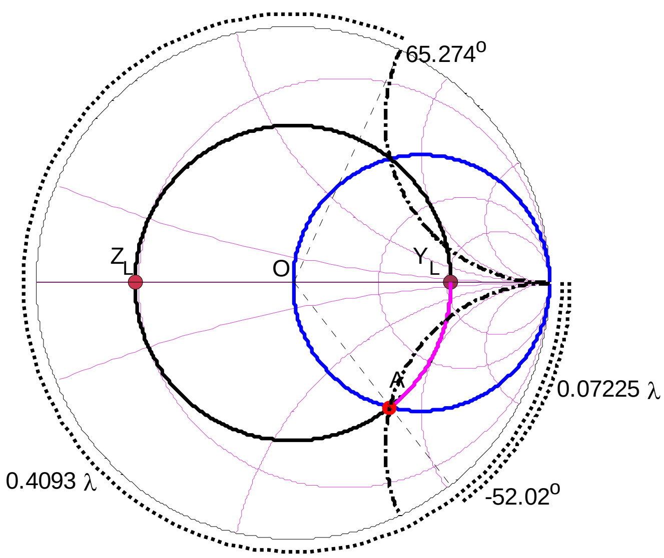
$$d_B = \frac{360 - 19}{720}\lambda = \underline{\underline{0.4736\lambda}}$$

(Exact value = 0.4485λ )


0.2308 λ
33.863°
A
Z
L
O
Y
L
-159.96°
0.05152 λ
-37.095°

**Prob. 11.61**

$$\frac{\lambda}{4} \rightarrow \frac{720^o}{4} = 180^o$$




At A, $y_{in} = 1 - j1.561$

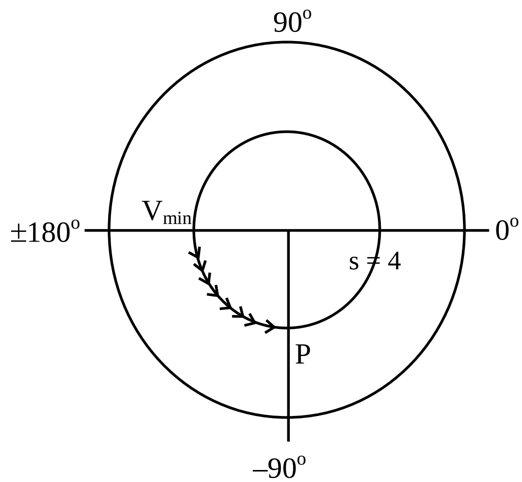
$y_{stub} = j1.5614$

Position of the stub = $\underline{\underline{0.0723\lambda}}$

Length of the stub = $\underline{\underline{0.4093\lambda}}$

**Prob. 11.62**




$$s = \frac{V_{max}}{V_{min}} = \frac{4V}{1V} = \underline{\underline{4}}$$

$$|\Gamma| = \frac{s-1}{s+1} = \frac{3}{5} = 0.6$$

$$\frac{\lambda}{2} = 25 \text{ cm} - 5 \text{ cm} = 20 \text{ cm}$$

$$\rightarrow \quad \lambda = 40 \text{ cm}$$

The load is l=5cm from $V_{min}$, i.e.

$$l = \frac{5\lambda}{40} = \frac{\lambda}{8} \quad \rightarrow \quad 90^o$$

On the s = 4 circle, move 90° from $V_{min}$ towards the load and obtain $Z_L$ = 0.46 – j0.88 at P.

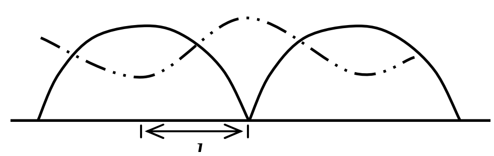
$$Z_L = Z_o z_L = 60(0.46 - j0.88) = \underline{\underline{27.6 - j52.8\ \Omega}}$$

(Exact value = 28.2353 –j52.9412 Ω)

$$\theta_\Gamma = 270^o \text{ or } -90^o$$

$$\Gamma = \underline{\underline{0.6\angle -90^o}}$$

**Prob. 11.63**

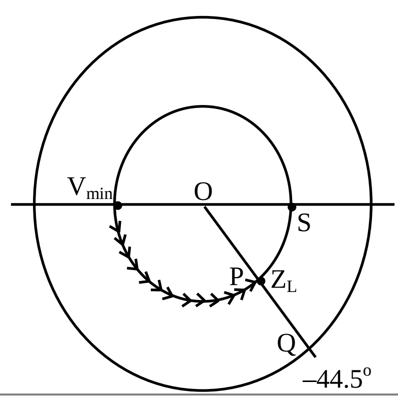



$$s = \frac{V_{max}}{V_{min}} = \frac{0.95}{0.45} = \underline{\underline{2.11}}$$

$$\frac{\lambda}{2} = 22.5 - 14 = 8.5 \quad \rightarrow \quad \lambda = 17 \text{ cm}$$

$$f = \frac{c}{\lambda} = \frac{3\times 10^8}{0.17} = \underline{\underline{1.764 \text{ GHz}}}$$

$$l = 3.2 \text{ cm} = \frac{3.2}{17}\lambda \quad \rightarrow 135.5^o$$



At P, $z_L = 1.4 - j0.8$

$$Z_L = 50(1.4 - j0.8) = \underline{\underline{70 - j40\Omega}}$$

(Exact value = 70.606-j40.496 Ω)

$$|\Gamma| = \frac{s-1}{s+1} = \frac{1.11}{3.11} = 0.357, \quad \theta_\Gamma = -44.5^o$$

$$\Gamma = \underline{\underline{0.357\angle -44.5^o}}$$

(Exact value = 0.3571∠-44.471°)

**Prob. 11.64**

$$\Gamma_s = \frac{R_g - R_o}{R_g + R_o} = \frac{0-50}{0+50} = \underline{\underline{-1}}$$

$$\Gamma_L = \frac{R_L - R_o}{R_L + R_o} = \frac{80-50}{80+50} = \underline{\underline{0.231}}$$

**Prob. 11.65**

$$\Gamma_L = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{0.5Z_o - Z_o}{1.5Z_o} = -\frac{1}{3}$$

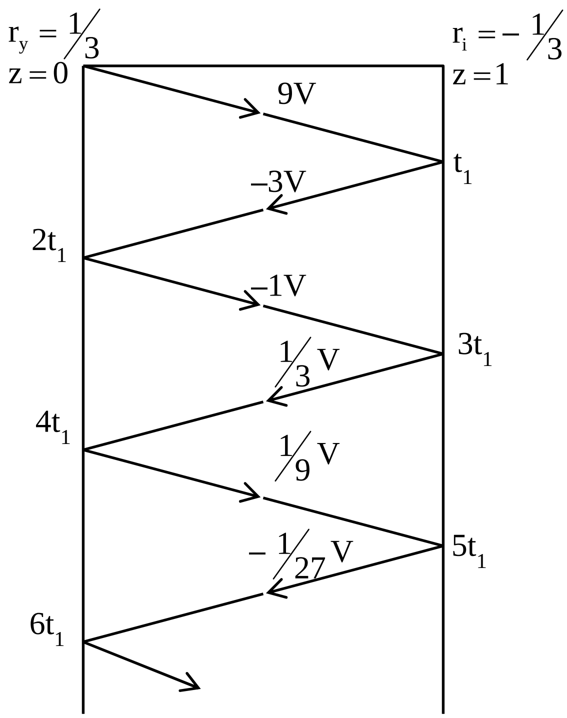
$$\Gamma_g = \frac{Z_g - Z_o}{Z_g + Z_o} = \frac{Z_o}{3Z_o} = \frac{1}{3}$$

$$t_1 = \frac{l}{u} = 2\mu s, \quad V_o = \frac{Z_o}{3Z_o}(27) = 9 \text{ V}, \quad I_o = \frac{V_o}{Z_o} = 180 \text{ mA}$$

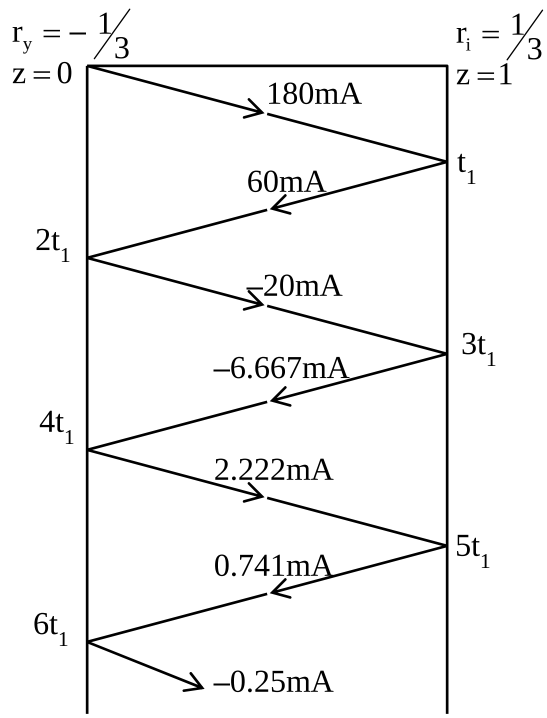
$$V_\infty = \frac{Z_L}{Z_g + Z_L} V_g = \frac{0.5}{2.5}(27) = 5.4 \text{ V}, \quad I_\infty = \frac{V_\infty}{Z_L} = 216 \text{ mA}$$

The voltage and current bounce diagrams are shown below

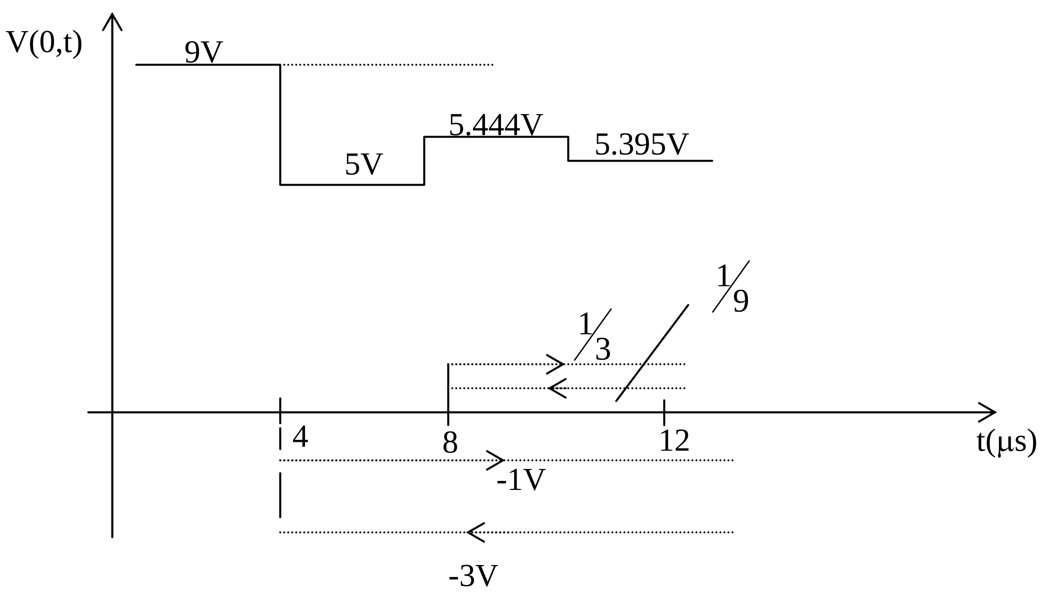


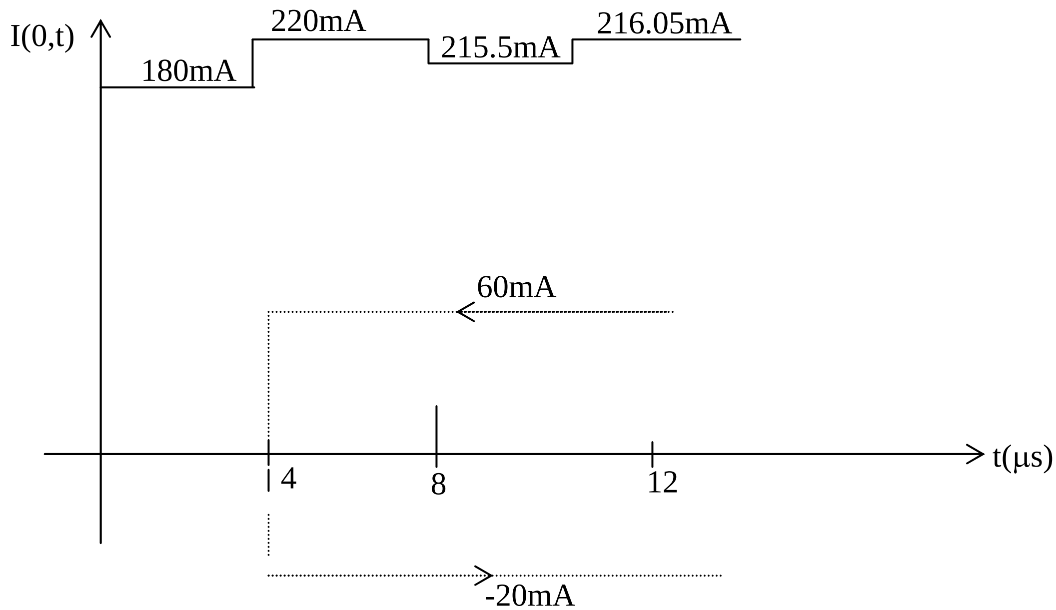

(Voltage bounce diagram)




(Current bounce diagram)

From the bounce diagrams, we obtain V(0,t) and I(0,t) as shown below:






**Prob. 11.66**

Using Thevenin equivalent at z = 0 gives

$R_g = R_s = 4Z_o = 200\Omega$

$V_g = I_s R_s = 10 \times 200 \times 10^{-3} = 2V$

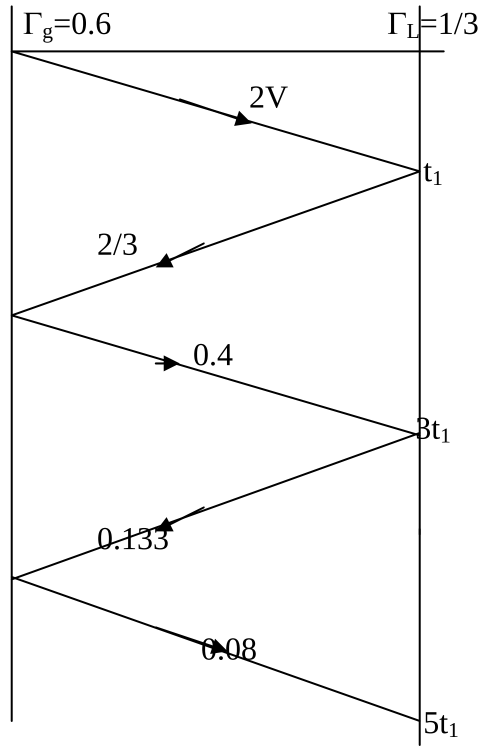
$$\Gamma_g = \frac{Z_g - Z_o}{Z_g + Z_o} = \frac{4Z_o - Z_o}{4Z_o + Z_o} = \frac{3}{5}$$

$$\Gamma_L = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{2Z_o - Z_o}{2Z_o + Z_o} = \frac{1}{3}$$

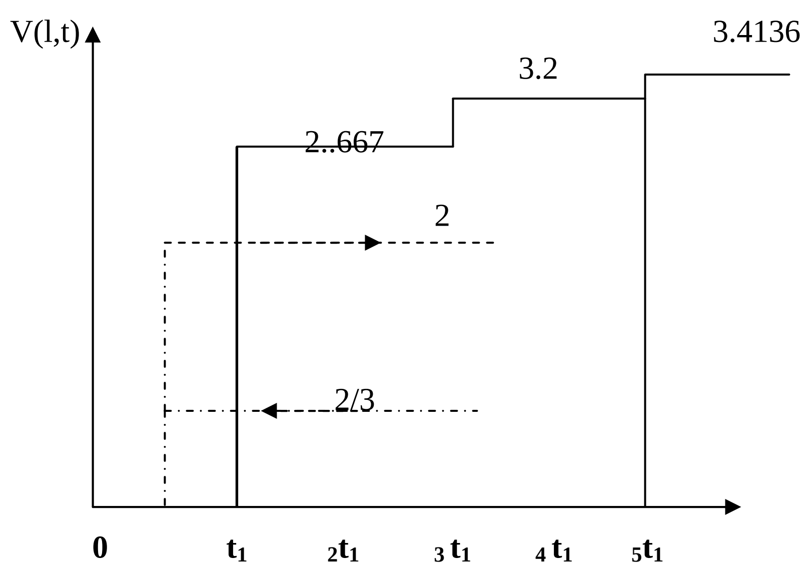
$$t_1 = \frac{\ell}{u} = \frac{10}{2 \times 10^8} = 50 \text{ ns}$$

The bounce diagram is shown below.




**T**he load voltage is sketched below.




$$I(\ell,t) = \frac{V(\ell,t)}{Z_L} = \frac{V(\ell,t)}{100}$$

To get I(I,t), we just scale down V(l,t) by 100.

**Prob. 11.67**

$$\Gamma_g = \frac{Z_g - Z_o}{Z_g + Z_o} = \frac{32 - 75}{32 + 75} = 0.4019$$

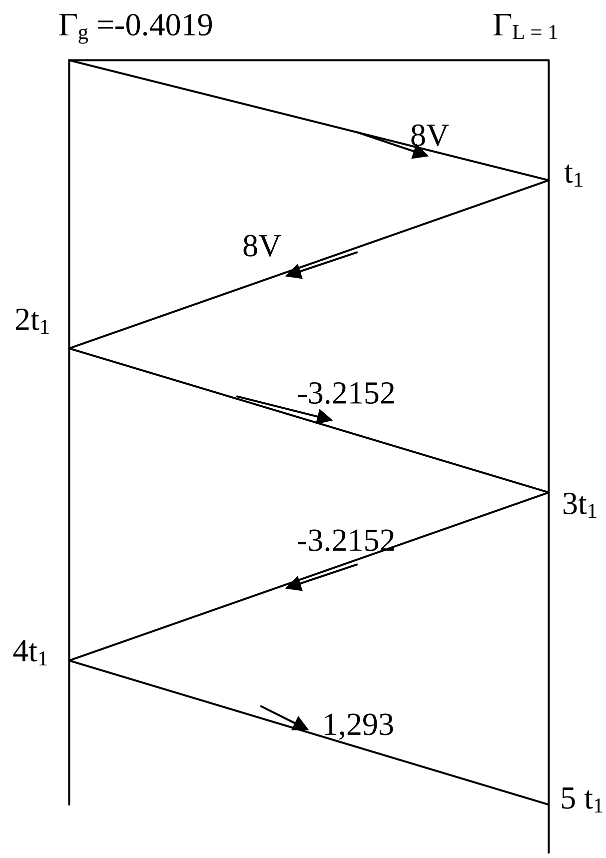
$$\Gamma_L = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{2\times10^6 - 75}{2\times10^6 + 75} \simeq 1$$

$$t_1 = \frac{\ell}{u} = \frac{50\times10^{-2}}{2\times10^8} = 2.5ns$$

The bounce diagram is shown below.

At t =20 ns = $4t_1$,

$V = 8 + 8 - 3.2152 - 3.2152 = \underline{\underline{9.57\text{ V}}}$



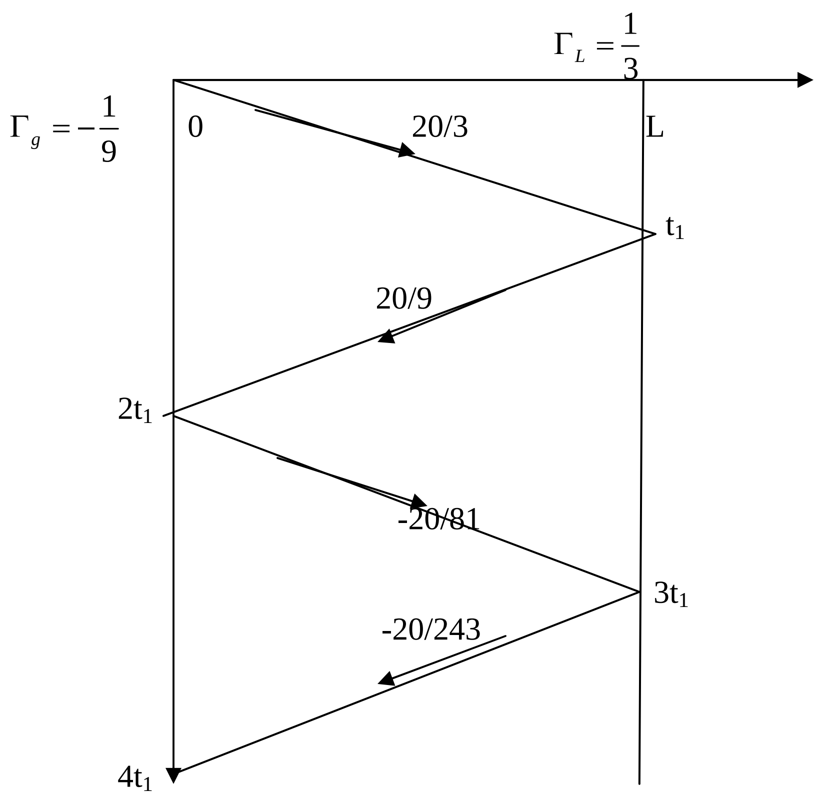
**Prob. 11.68**

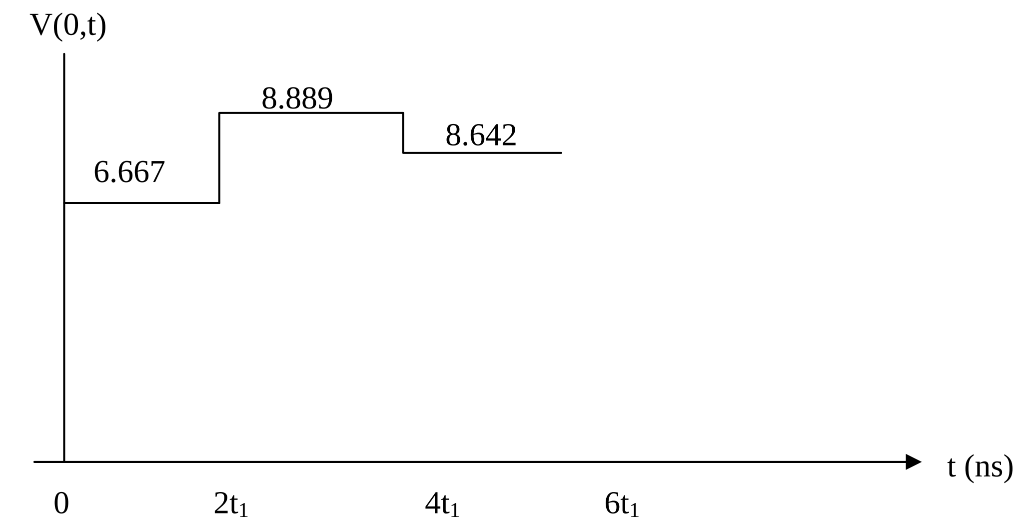
$$t_1 = \frac{l}{u} = \frac{40\times10^{-2}}{2.5\times10^{8}} = 01.6ns,$$

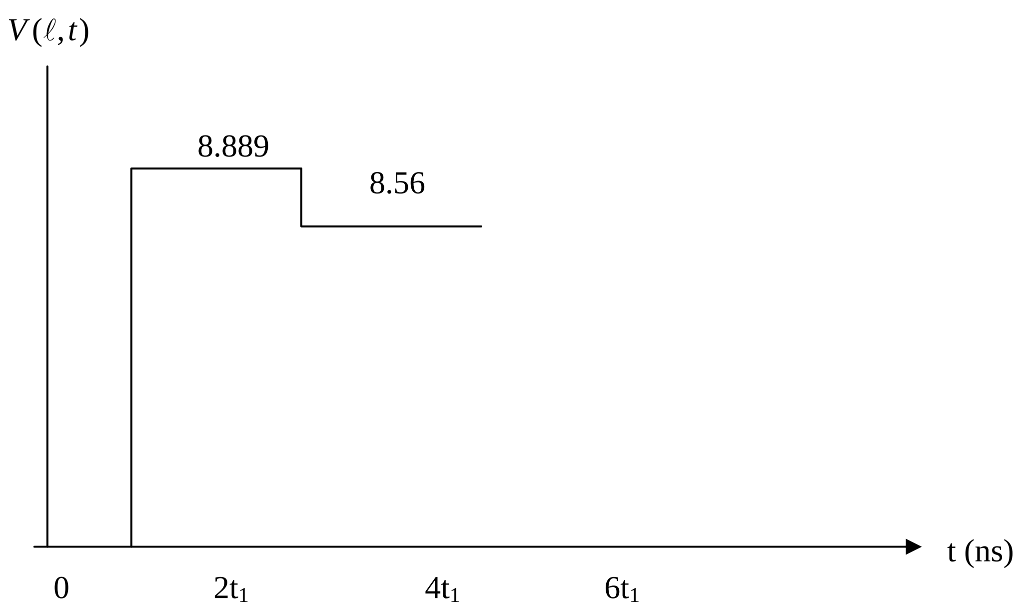
$$\Gamma_L = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{100-50}{150} = \frac{1}{3}, \qquad \Gamma_g = \frac{Z_g - Z_o}{Z_g + Z_o} = \frac{40-50}{90} = -\frac{1}{9},$$

$$V_o = \frac{Z_o V_g}{Z_o + Z_g} = \frac{50(12)}{90} = \frac{20}{3} = 6.668V$$

The bounce diagram is sketched below. From it, we obtain V(0,t) and V(L,t).






**Prob. 11.69**

$$V_o = 8V = \frac{Z_o}{Z_o + Z_g} V_g = \frac{50}{50+60} V_g \longrightarrow V_g = \frac{8 \times 110}{50} = \underline{\underline{17.6 \text{ V}}}$$

$$2t_1 = 4\mu s \longrightarrow t_1 = 2\mu s = \frac{\ell}{u}$$

$$\ell = ut_1 = 3\times 10^8 \times 2 \times 10^{-6} = \underline{\underline{600 \text{ m}}}$$

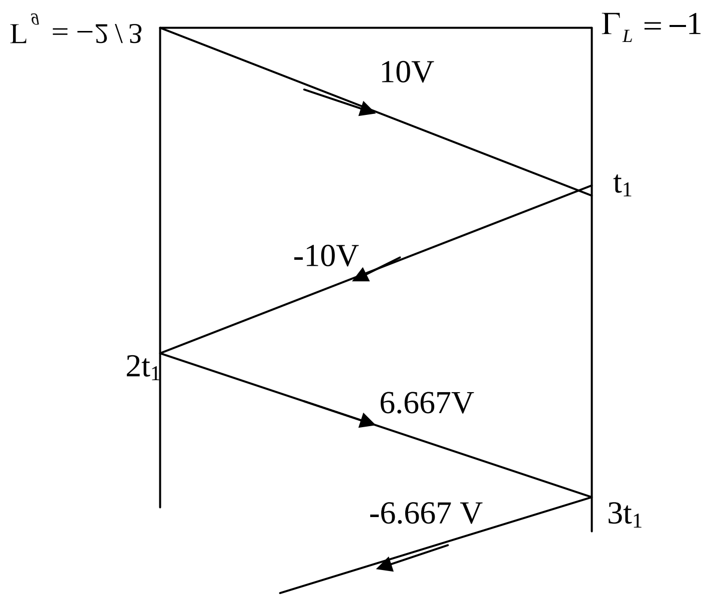
**Prob. 11.70**

$$t_1 = \frac{20}{2x10^8} = 10^{-7} = 0.1\mu s, \qquad V_o = \frac{Z_o}{Z_o + Z_g} V_g = \frac{50}{60}(12) = 10V$$

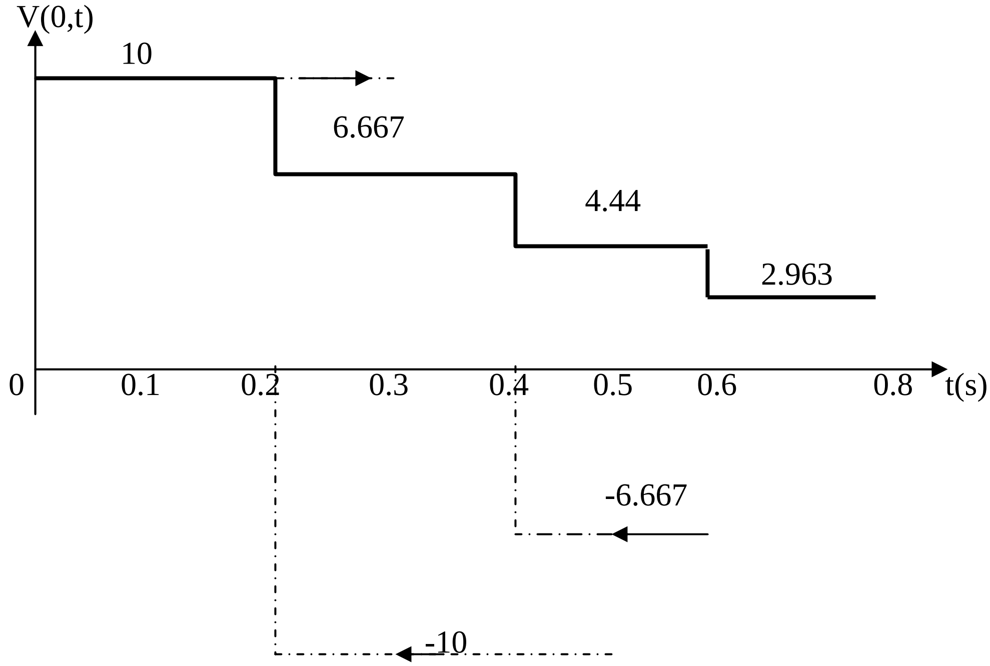
$$\Gamma_g = \frac{Z_g - Z_o}{Z_g + Z_o} = \frac{10-50}{10+50} = -2/3$$

$$\Gamma_L = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{0-50}{0+50} = -1$$

The voltage bounce diagram is shown below



From the bounce diagram, we obtain V(0,t) as shown. V(l,t) =0 due to the short circuit.




**Prob. 11.71**

The initial pulse on the line is

$$V_o = \frac{Z_o}{Z_o + Z_g} V_g \qquad (1)$$

The reflection coefficient is

$$\Gamma = \frac{Z_g - Z_o}{Z_g + Z_o} \qquad (2)$$

The first reflected wave has amplitude $\Gamma V_o$, while the second reflected wave has amplitude $\Gamma^2 V_o$, etc. For a very long time, the total voltage is

$$V_T = V_o + \Gamma V_o + \Gamma^2 V_o + ...$$

$$= V_o(1 + \Gamma + \Gamma^2 + \Gamma^3 + ...) = V_o\left(\frac{1}{1-\Gamma}\right) \qquad (3)$$

Substituting (1) and (2) into (3) gives

$$V_T = \frac{Z_o}{Z_o + Z_g} V_g \left( \frac{1}{1 - \dfrac{Z_g - Z_o}{Z_g + Z_o}} \right) = Z_o V_g \frac{1}{Z_g + Z_o - Z_g + Z_o} = \underline{\underline{\frac{V_g}{2}}}$$

**Prob.11.72**

$w = 1.5cm, \quad \text{h} = 1\text{cm}, \quad \frac{w}{h} = 1.$

(a) $\varepsilon_{\text{eff}} = \left(\frac{6 \;\; +1}{2}\right) + \frac{\varepsilon_{\text{r}} - 1}{2\sqrt{1 \;\; +12\text{h/w}}} = 1.6 \;\; + \frac{0.6}{\sqrt{1 \;+\; 12/1.5}} = \underline{\underline{1.8}}$

$$Z_0 = \frac{377}{\sqrt{1.8}\;\;(1.5 \;+1.393 \;+0.667 \;\text{In}\;(2.944))} = \frac{281}{3.613} = \underline{\underline{77.77\Omega}}$$

(b) $\alpha_c = 8.686\frac{\text{R}_\text{s}}{\text{wZ}_\text{o}}$

$$\text{R}_\text{s} = \frac{1}{\sigma_c \;\; \sigma} = \sqrt{\frac{\mu\pi\text{f}}{\sigma_\text{c}}} = \sqrt{\frac{19 \;\times\; 2.5 \;\times\; 10^9 \;\times\; 4\pi \;\times 10^{-3}}{1.1 \;\times 10^7}}$$

$= 2.995 \;\times\;\; 10^{-2}$

$$\alpha_\text{c} = \frac{8.686 \;\times 2.995 \times 10^{-2}}{1.5 \times 10^{-2} \;\; \times 77.77} = \underline{\underline{0.223\text{dB/m}}}$$

$$\text{u} = \frac{\text{c}}{\sqrt{\varepsilon_{\text{eff}}}} \;\rightarrow\; \lambda = \frac{\text{u}}{\text{f}} = \frac{\text{c}}{\text{f}\sqrt{\varepsilon_{\text{eff}}}} = \frac{3 \;\times 10^8}{2.5 \times 10^9 \;\; \sqrt{1.8}} = 8.944 \;\times 10^{-2}$$

$$\alpha_\text{d} = 27.3 \times \frac{0.8}{1.2}\frac{(2.2)}{\sqrt{1.8}} \;\; \frac{2 \;\times 10^{-2}}{8.944 \times 10^{-2}} = \frac{96.096}{14.3996}$$

$\alpha_\text{d} = \underline{\underline{6.6735 \text{ dB/m}}}$

(c) $\alpha = \alpha_\text{c} + \alpha_\text{d} = 6.8965 \text{ dB/m}$

$$\alpha\ell = 20\text{dB} \;\rightarrow\; \ell = \frac{20}{\alpha} = \frac{20}{6.8965} = \underline{\underline{2.9 \text{ m}}}$$

**Prob. 11.73**

(*a*) Let $x = w/h$. If $x < 1$,

$$50 = \frac{60}{\sqrt{4.6}} \ln\left(\frac{8}{x} + x\right)$$

$$5\sqrt{4.6} - 6\ln\left(\frac{8}{x} + x\right) = 0$$

we solve for x (e.g using Maple) and get x = 2.027 or 3.945

which contradicts our assumptiom that $x < 1$. If $x > 1$,

$$50 = \frac{120\pi}{\sqrt{4.6}\left[x + 1.393 + 0.667 \ln(x + 1.444)\right]}$$

We solve this iteratively and obtain:

$x = 1.8628, w = xh = 14.9024$ mm

For this w and h,

$$\varepsilon_{eff} = 3.4598$$

(b) $$\beta = \frac{\omega\sqrt{\varepsilon_{\text{eff}}}}{c}$$

$$\beta\ell = 45^0 = \frac{\pi}{4} = \frac{\omega\ell\sqrt{\varepsilon_{\text{eff}}}}{c}$$

$$\ell = \frac{\pi c}{4\sqrt{\varepsilon_{\text{eff}}}\, 2\pi f} = \frac{3 \times 10^8}{8 \times \sqrt{3.4598} \times 8 \times 10^9}$$

$$\underline{\underline{\ell = 0.00252 \text{ m}}}$$

**Prob. 11.74**

For w = 0.4 mm, $\frac{w}{h} = \frac{0.4\,\text{mm}}{2\text{ m}} = 0.2 \rightarrow$ narrow strip

For $\frac{w}{h} = 0.2, \quad \varepsilon_{\text{eff}} = 5.851, \quad Z_o = 91.53\Omega$

For $\frac{w}{h} = 0.4, \quad \varepsilon_{\text{eff}} = 6.072, \quad Z_o = 73.24\Omega$

Hence,

$$\underline{\underline{73.24\Omega < Z_o < 91.53\Omega}}$$

**Prob. 11.75**

$$w' = 0.5 + \frac{0.1}{3.2}\ln\left(\frac{5\times 1.2}{0.1}\right) = 0.5 + 0.1279 = 0.6279$$

$$Z_o = \frac{377}{2\pi\sqrt{2}}\ln\left\{1 + \frac{4\times 1.2}{\pi\times 0.6177}\left[\frac{8\times 1.2}{4\times 0.6279} + 3.354\right]\right\}$$

$$= 42.43\ln\{1 + 2.49(3.822 + 3.354\} = 42.43\ln(20.255)$$

$$= \underline{\underline{127.64\ \Omega}}$$

**Prob. 11.76**

Suppose we guess that w/h < 2

$$A = \frac{75}{60}\sqrt{\frac{3.3}{2}} + \frac{1.3}{3.3}\left(0.23 + \frac{0.11}{2.3}\right) = 1.117$$

$$\frac{w}{h} = \frac{8e^A}{e^{2A} - 2} = \frac{24.44}{7.337} = 3.331 \rightarrow w = 3.331h = \underline{\underline{4mm}}$$

If we guess that w/h > 2,

$$B = \frac{60\pi^2}{Z_o\sqrt{\varepsilon_r}} = \frac{60\pi^2}{75\sqrt{2.3}} = 5.206$$

$$\frac{w}{h} = \frac{2}{\pi}\left[4.266 - \ln 9.412 + \frac{1.3}{4.6}\left(\ln 4.206 + 0.39 - \frac{0.61}{2.3}\right)\right]$$

$$= 1.665 < 2$$

Thus $\frac{w}{h} = 3.331 > 2$

$$\varepsilon_{eff} = \frac{3.3}{2} + \frac{1.3}{2\sqrt{1 + \frac{12}{3.331}}} = 1.953$$

$$u = \frac{3 \times 10^8}{\sqrt{1.953}} = \underline{\underline{2.1467 \times 10^8 \text{ m/s}}}$$

**Prob. 11.77**

$$\Gamma = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{100 - 150}{250} = -0.2$$

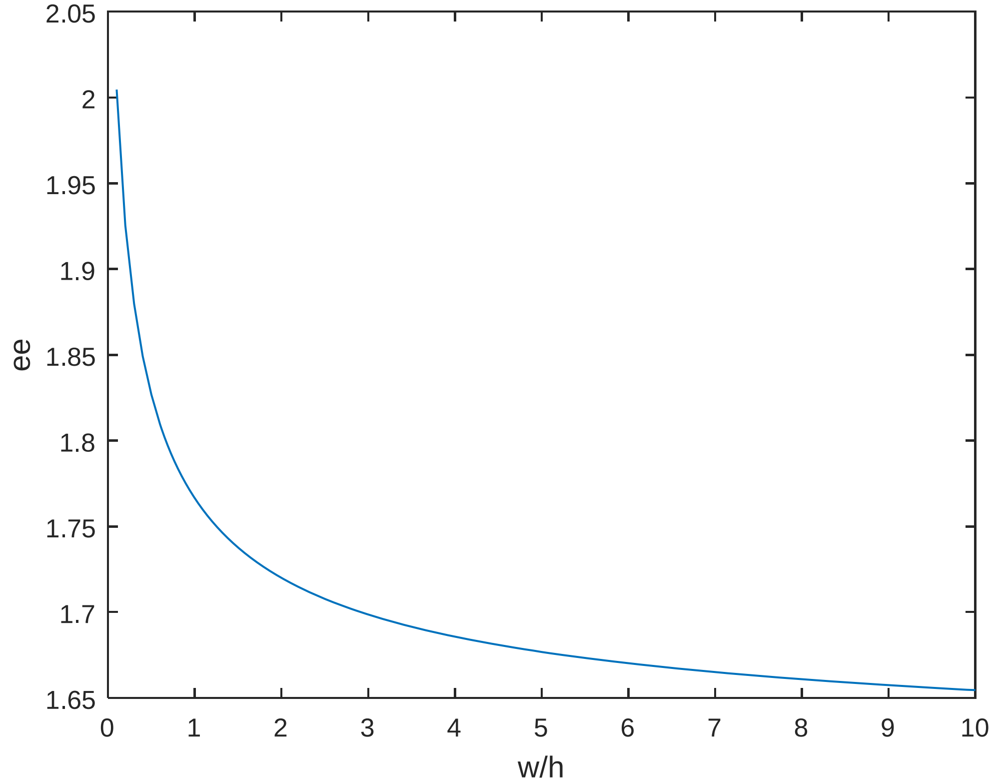
$$RL = -20 \log |\Gamma| = \underline{\underline{13.98 \text{ dB}}}$$

**Prob. 11.78**

The MATLAB code and the plot of the effective relative permittivity are presented below.

```
%    Plot effective permitivity versus x = h/w

er = 2.2
e1 = (er + 1)/2; e2 = (er -1)/2;
% x = 0.1*0.1*100
for k=1:100
    x(k)=0.1*k
    fac=x(k);
ee(k) = e1 + e2/sqrt(1 + 12*fac);
end
plot(x,ee)
xlabel('w/h')
ylabel('ee')
```


2.05
2
1.95
1.9
1.85
1.8
1.75
1.7
1.65
ee
0
1
2
3
4
5
6
7
8
9
10
w/h

# CHAPTER 12

**P. E. 12.1** (a) For $TE_{10}$, $f_c$ = 3 GHz,

$$\sqrt{1-(f_c/f)^2} = \sqrt{1-(3/15)^2} = \sqrt{0.96}, \quad \beta_o = \omega/u_o = 4\pi f/c$$

$$\beta = \frac{4\pi f}{c}\sqrt{0.96} = \frac{4\pi\times15\times10^9}{3\times10^8}\sqrt{0.96} = \underline{\underline{615.6}} \text{ rad/m}$$

$$u = \frac{\omega}{\beta} = \frac{2\pi\times15\times10^9}{615.6} = \underline{\underline{1.531\times10^8}} \text{ m/s}$$

$$\eta' = \sqrt{\frac{\mu}{\varepsilon}} = 60\pi, \quad \eta_{TE} = \frac{60\pi}{\sqrt{0.96}} = \underline{\underline{192.4\Omega}}$$

(b)For $TM_{11}$, $f_c = 3\sqrt{7.25}$ GHz, $\sqrt{1-(f_c/f)^2} = 0.8426$

$$\beta = \frac{4\pi f}{c}(0.8426) = \frac{4\pi\times15\times10^9(0.8426)}{3\times10^8} = \underline{\underline{529.4}} \text{ rad/m}$$

$$u = \frac{\omega}{\beta} = \frac{2\pi\times15\times10^9}{529.4} = \underline{\underline{1.78\times10^8}} \text{ m/s}$$

$$\eta_{TM} = 60\pi(0.8426) = \underline{\underline{158.8\Omega}}$$

**P. E. 12.2** (a) Since $E_z \neq 0$, this is a TM mode

$$E_{zs} = E_o \sin(m\pi x/a)\sin(n\pi y/b)e^{-j\beta z}$$

$E_o = 20$, $\frac{m\pi}{a} = 40\pi$ ⟶ m=2, $\frac{n\pi}{b} = 50\pi$ ⟶ n=1

i.e. $\underline{\underline{TM_{21}\text{ mode}}}$.

(b) $$f_c = \frac{u'}{2}\sqrt{(m/a)^2+(n/b)^2} = \frac{3\times10^8}{2}\sqrt{40^2+50^2} = 1.5\sqrt{41} \text{ GHz}$$

$$\beta = \omega\sqrt{\mu\varepsilon}\sqrt{1-(f_c/f)^2} = \frac{2\pi f}{c}\sqrt{f^2-f_c^2} = \frac{2\pi\times10^9}{3\times10^8}\sqrt{225-92.25} = \underline{\underline{241.3 \text{ rad/m.}}}$$

(c)

$$E_{xs} = \frac{-j\beta}{h^2}(40\pi)20\cos 40\pi x \sin 50\pi y e^{-j\beta z}$$

$$E_{ys} = \frac{-j\beta}{h^2}(50\pi)20\sin 40\pi x \cos 50\pi y e^{-j\beta z}$$

$$\frac{E_y}{E_x} = 1.25 \tan 40\pi x \cot 50\pi y$$

**P. E. 12.3** If $TE_{13}$ mode is assumed, $f_c$ and $\beta$ remain the same.

$f_c$ = 28.57 GHz, $\beta$ = 1718.81 rad/m, $\gamma = j\beta$

$$\eta_{TE13} = \frac{377/2}{\sqrt{1-(28.57/50)^2}} = 229.69\ \Omega$$

For m=1, n=3, the field components are:

$E_z = 0$

$$H_z = H_o \cos(\pi x/a)\cos(3\pi y/b)\cos(\omega t - \beta z)$$

$$E_x = -\frac{\omega\mu}{h^2}\left(\frac{3\pi}{b}\right) H_o \cos(\pi x/a)\sin(3\pi y/b)\sin(\omega t - \beta z)$$

$$E_y = \frac{\omega\mu}{h^2}\left(\frac{\pi}{a}\right) H_o \sin(\pi x/a)\cos(3\pi y/b)\sin(\omega t - \beta z)$$

$$H_x = -\frac{\beta}{h^2}\left(\frac{\pi}{a}\right) H_o \sin(\pi x/a)\cos(3\pi y/b)\sin(\omega t - \beta z)$$

$$H_y = -\frac{\beta}{h^2}\left(\frac{3\pi}{a}\right) H_o \cos(\pi x/a)\sin(3\pi y/b)\sin(\omega t - \beta z)$$

Given that $H_{ox} = 2 = -\frac{\beta}{h^2}(\pi/a)H_o$,

$$H_{oy} = -\frac{\beta}{h^2}(3\pi/b)H_o = 6a/b = 6(1.5)/8 = 11.25$$

$$H_{oz} = H_o = -\frac{2h^2 a}{\beta\pi} = \frac{-2\times 14.51\pi^2\times 10^4\times 1.5\times 10^{-2}}{1718.81\pi} = -7.96$$

$$E_{oy} = \frac{\omega\mu}{h^2}\left(\frac{\pi}{a}\right)H_o = -\frac{2\omega\mu}{\beta} = 2\eta_{TE} = -459.4$$

$$E_{ox} = -E_{oy}\frac{3a}{b} = 459.4(4.5/0.8) = 2584.1$$

$$E_x = 2584.1\cos(\pi x/a)\sin(3\pi y/b)\sin(\omega t - \beta z) \text{ V/m},$$

$$E_y = -459.4\sin(\pi x/a)\cos(3\pi y/b)\sin(\omega t - \beta z) \text{ V/m},$$

$E_z = 0$,

$$H_y = 11.25\cos(\pi x/a)\sin(3\pi y/b)\sin(\omega t - \beta z) \quad \text{A/m},$$

$$H_z = -7.96\cos(\pi x/a)\cos(3\pi y/b)\cos(\omega t - \beta z) \quad \text{A/m}$$

**P. E. 12.4**

$$f_{c11} = \frac{u'}{2}\sqrt{\frac{1}{a^2}+\frac{1}{b^2}} = \frac{3\times10^8\times10^2}{2}\sqrt{1/8.636^2 + 1/4.318^2} = 3.883 \quad \text{GHz}$$

$$u_p = \frac{3\times10^8}{\sqrt{1-(3.883/4)^2}} = \underline{\underline{12.5\times10^8}} \quad \text{m/s},$$

$$u_g = \frac{9\times10^{16}}{12.5\times10^8} = \underline{\underline{7.2\times10^7}} \quad \text{m/s}$$

**P. E. 12.5** The dominant mode becomes $TE_{01}$ mode

$$f_{c01} = \frac{c}{2b} = 3.75 \quad \text{GHz}, \quad \eta_{TE} = 406.7\Omega$$

From Example 12.2,

$$E_x = -E_o\sin(3\pi y/b)\sin(\omega t - \beta z), \quad \text{where } E_o = \frac{\omega\mu b}{\pi}H_o.$$

$$\mathcal{P}_{ave} = \int_{x=0}^{a}\int_{y=0}^{b}\frac{|E_{xs}|^2}{2\eta}dxdy = \frac{E_o^{\,2}\,ab}{4\eta}$$

Hence $E_o$ = 63.77 V/m as in Example 12.5.

$$H_o = \frac{\pi E_o}{\omega\mu b} = \frac{\pi\times63.77}{2\pi\times10^{10}\times4\pi\times10^{-7}\times4\times10^{-2}} = \underline{\underline{63.34}} \quad \text{mA/m}$$

**P. E. 12.6** (a) For m=1, n=0, $f_c$ = u'/(2a)

$$\frac{\sigma}{\omega\varepsilon} = \frac{10^{-15}}{2\pi\times9\times10^9\times2.6\times10^{-9}/(36\pi)} = \frac{10^{-15}}{1.3} << 1$$

Hence,

$$u' \cong \frac{1}{\sqrt{\mu\varepsilon}} = c/\sqrt{2.6}, \quad f_c = \frac{3\times10^8}{2\times2.4\times10^{-2}\sqrt{2.6}} = 2.2149 \text{ GHz}$$

$$\alpha_d = \frac{\sigma\eta'}{2\sqrt{1-(f_c/f)^2}} = \frac{10^{-15}\times377/\sqrt{2.6}}{2\sqrt{1-(2.2149/9)^2}} = 1.205\times10^{-13} \text{ Np/m}$$

For n = 0, m=1,

$$\alpha_c = \frac{2R_s}{b\eta'\sqrt{1-(f_c/f)^2}}[\frac{1}{2}+\frac{b}{a}(f_c/f)^2]$$

$$= \frac{2\sqrt{2.6}\sqrt{\pi\times9\times10^9\times1.1\times10^7\times4\pi\times10^{-7}}}{377\times1.5\times10^{-2}\times1.1\times10^7\sqrt{1-(2.2149/9)^2}}[0.5+(2.4/1.5)(2.2148/9)^2] = \underline{\underline{2\times10^{-2}}} \text{ Np/m}$$

(b)Since $\alpha_c >> \alpha_d, \alpha = \alpha_c + \alpha_d \cong \alpha_c = 2x10^{-2}$

loss = $\alpha l = 2\times10^{-2}\times0.4 = 0.8\times10^{-2}$ Np = <u>0.06945 dB</u>

**P. E. 12.7** For $TE_{11}$ , m = 1 = n,

$$H_{zs} = H_o\cos(\pi x/a)\cos(\pi y/b)e^{-\gamma z}$$

$$E_{xs} = \frac{j\omega}{h^2}(\pi/b)H_o\cos(\pi x/a)\sin(\pi y/b)e^{-\gamma z}$$

$$E_{ys} = -\frac{j\omega\mu}{h^2}(\pi/a)H_o\sin(\pi x/a)\cos(\pi y/b)e^{-\gamma z}$$

$$H_{xs} = \frac{j\beta}{h^2}(\pi/a)H_o\sin(\pi x/a)\cos(\pi y/b)e^{-\gamma z}$$

$$H_{ys} = \frac{j\beta}{h^2}(\pi/b)H_o\cos(\pi x/a)\sin(\pi y/b)e^{-\gamma z}$$

$$E_{zs} = 0$$

For the electric field lines,

$$\frac{dy}{dx} = \frac{E_y}{E_x} = (a/b)\tan(\pi x/a)\cot(\pi y/b)$$

For the magnetic field lines

$$\frac{dy}{dx} = \frac{H_y}{H_x} = -(a/b)\cot(\pi x/a)\tan(\pi y/b)$$

Notice that $(\frac{E_y}{E_x})(\frac{H_y}{H_x}) = -1$

showing that the electric and magnetic field lines are mutually orthogonal. The field lines are as shown in Fig. 12.14.

**P. E. 12.8**

$$u' = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{\sqrt{\varepsilon_r}}$$

$$f_{TE101} = \frac{1.5\times10^{10}}{\sqrt{3}}\sqrt{1/25+0+1/100} = \underline{\underline{1.936}} \text{ GHz}$$

$$Q_{TE101} = \frac{1}{61\delta}, \text{ where}$$

$$\delta = \frac{1}{\sqrt{\pi f_{101}\mu\sigma_c}} = \frac{1}{\sqrt{\pi\times1.936\times10^9\times4\pi\times10^{-7}\times5.8\times10^7}} = 1.5\times10^{-6}$$

$$Q_{TE101} = \frac{10^6}{61\times1.5} = \underline{\underline{10,929}}$$

**P. E. 12.9**

(a) By Snell's law, $n_1 \sin\theta_1 = n_2 \sin\theta_2$. Thus
$\theta_2 = 90^o$ ⟶ $\sin\theta_2 = 1$

$\sin\theta_1 = n_2/n_1$, $\theta_1 = \sin^{-1} n_2/n_1 = \sin^{-1} 1.465/1.48 = \underline{\underline{81.83^o}}$

(b) NA = $\sqrt{n_1^2 - n_2^2} = \sqrt{1.48^2 - 1.465^2} = \underline{\underline{0.21}}$

**P. E. 12.10**

$\alpha l = 10 \log P(0)/P(l) = 0.2 \text{ X } 10 = 2$

$P(0)/P(l) = 10^{0.2}$, i.e. $P(l) = P(0)\, 10^{-0.2} = 0.631\, P(0)$

i.e. $\underline{\underline{63.1\ \%}}$

**Prob. 12.1**

(a) For $TE_{10}$mode, $f_c = \frac{u'}{2a} = \frac{3\times10^8}{2\times6\times10^{-2}} = \underline{\underline{2.5\text{ GHz}}}$

$(b)\ f = 3f_c = 7.5$ GHz

$$f_{cmn} = \frac{u^1}{2}\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}} \quad = \frac{3\times10^8\times10^2}{2}\sqrt{\left(\frac{m}{6}\right)^2+\left(\frac{n}{4}\right)^2}$$

$$=15\sqrt{\left(\frac{m}{6}\right)^2+\left(\frac{n}{4}\right)^2}\text{ GHz}$$

$f_{c20} = 15\times\frac{2}{6} = 5$ GHz

$f_{c01} = 3.75$ GHz, $\quad f_{c02} = 7.5$ GHz

$f_{c10} = 2.5$ GHz, $\quad f_{c20} = 5.0$ GHz

$f_{c21} = 6.25$ GHz, $\quad f_{c30} = 7.5$ GHz

$$f_{12} = 15\sqrt{\left(\frac{1}{6}\right)^2+\left(\frac{2}{4}\right)^2} = 7.91\text{ GHz}$$

$$f_{11} = 15\sqrt{\left(\frac{1}{6}\right)^2+\left(\frac{1}{4}\right)^2} = 4.507\text{ GHz}$$

The following modes are transmitted

$TE_{01}, TE_{02}, TE_{10}, TE_{11}, TE_{20}, TE_{21}, TE_{30}$

$TM_{11}, TM_{21}$

<u>i.e. 7 TE modes and 2 TM modes</u>

**Prob.12.2**

$$f_{c10} = \frac{u'}{2a},\ f_{c11} = \frac{u'}{2}\sqrt{\frac{1}{a^2}+\frac{1}{a^2}} = \frac{u'}{2a}\sqrt{2}$$

Since the guide can only propagate $TE_{10}$ mode,

$$f_{c10} < f < f_{c11} \quad \rightarrow \quad \frac{u'}{2a} < f < \frac{u'}{2a}\sqrt{2} \quad \rightarrow \quad u' < 2af < u'\sqrt{2}$$

$$\frac{u'}{f} < 2a < \frac{u'}{f}\sqrt{2} \quad \rightarrow \quad \lambda < 2a < \lambda\sqrt{2}$$

$$\underline{\underline{\frac{\lambda}{2} < a < \frac{\lambda}{\sqrt{2}}}}$$

**Prob. 12.3**

$$f_c = \frac{u'}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} = \frac{3\times10^8}{2\sqrt{2.25}\times10^{-2}}\left[\left(\frac{m}{2.28}\right)^2 + \left(\frac{n}{1.01}\right)^2\right]^{1/2}$$

$$= \frac{15}{\sqrt{2.25}}\left[\left(\frac{m}{2.28}\right)^2 + \left(\frac{n}{1.01}\right)^2\right]^{1/2} \text{ GHz}$$

Using this formula, we obtain the cutoff frequencies for the given modes as shown below.

| Mode | $f_c$ (GHz) |
|---|---|
| $TE_{01}$ | 9.901 |
| $TE_{10}$ | 4.386 |
| $TE_{11}$ | 10.829 |
| $TE_{02}$ | 19.802 |
| $TE_{22}$ | 21.658 |
| $TM_{11}$ | 10.829 |
| $TM_{12}$ | 20.282 |
| $TM_{21}$ | 13.228 |

**Prob. 12.4**

(a)

For $TE_{10}$ mode,

$$f_c = \frac{u'}{2}\sqrt{\left(\frac{1}{a}\right)^2} = \frac{3\times10^8}{2\times2.4\times10^{-2}} = \underline{\underline{6.25 \text{ GHz}}}$$

For $TE_{01}$ mode,

$$f_c = \frac{u'}{2}\sqrt{\left(\frac{1}{b}\right)^2} = \frac{3\times10^8}{2\times1.2\times10^{-2}} = \underline{\underline{12.5 \text{ GHz}}}$$

For $TE_{20}$ mode,

$$f_c = \frac{u'}{2}\sqrt{\left(\frac{2}{a}\right)^2} = 2\times6.25 = \underline{\underline{12.5 \text{ GHz}}}$$

For $TE_{02}$ mode,

$$f_c = \frac{u'}{2}\sqrt{\left(\frac{2}{b}\right)^2} = 2\times12.5 = \underline{\underline{25 \text{ GHz}}}$$

(b) Since f = 12 GHz, only $TE_{10}$ mode will propagate.

**Prob. 12.5** a/b = 3 ⟶ a = 3b

$$f_{c10} = \frac{u'}{2a} \longrightarrow a = \frac{u'}{2f_{c10}} = \frac{3\times10^8}{2\times18\times10^9} \text{ m} = 0.833\text{cm}$$

A design could be a = 9mm, b = 3mm.

**Prob. 12.6** For the dominant mode,

$$f_c = \frac{c}{2a} = \frac{3\times10^8}{2\times8} = 18.75 \text{ MHz}$$

(a) It will not pass the AM signal, (b) it will pass the FM signal.

**Prob. 12.7** (a) For $TE_{10}$ mode, $f_c = \frac{u'}{2a}$

Or $a = \frac{u'}{2f_c} = \frac{3\times10^8}{2\times5\times10^9} = 3 \text{ cm}$

For $TE_{01}$ mode, $f_c = \frac{u'}{2b}$

Or $b = \frac{u'}{2f_c} = \frac{3\times10^8}{2\times12\times10^9} = 1.25 \text{ cm}$

(b) Since a > b, 1/a < 1/b, the next higher modes are calculated as shown below.

| Mode | $f_c$ (GHz) |
|---|---|
| $TE_{10}$ | 5 |
| *$TE_{20}$ | 10 |
| $TE_{30}$ | 15 |
| $TE_{40}$ | 20 |
| *$TE_{01}$ | 12 |
| $TE_{02}$ | 24 |
| *$TE_{11}$ | 13 |
| $TE_{21}$ | 15.62 |

The next three higher modes are starred ones, i.e. $TE_{20}$, $TE_{01}$, $TE_{11}$

(c) $u' = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{\sqrt{2.25}} = 2\times10^8$ m/s

For $TE_{11}$ modes,

$$f_c = \frac{3\times10^8}{2\times10^{-2}\sqrt{2.25}}\sqrt{\frac{1}{3^2}+\frac{1}{1.25^2}} = \underline{\underline{8.67}}\text{ GHz}$$

**Prob. 12.8**

$$\text{Let } F_{12} = \sqrt{1-\left(\frac{f_{c12}}{f}\right)^2} = \sqrt{1-\left(\frac{25}{40}\right)^2} = 0.7806$$

$$\lambda' = \frac{c}{f} = \frac{3\times10^8}{40\times10^9} = 0.0075\text{ m} = 7.5\times10^{-3}\text{ m}$$

$$\lambda_{12} = \frac{\lambda'}{F_{12}} = \frac{7.5\times10^{-3}\text{ m}}{0.7806} = \underline{\underline{9.608\times10^{-3}\text{ m}}}$$

$$u_{12} = \frac{u'}{F_{12}} = \frac{3\times10^8}{0.7806} = \underline{\underline{3.843\times10^8\text{ m/s}}}$$

$$\beta_{12} = \frac{2\pi}{\lambda_{12}} = \frac{2\pi}{9.608\times10^{-3}} = \underline{\underline{653.95\text{ rad/m}}}$$

$$\eta_{TE12} = \frac{\eta'}{F_{12}} = \frac{120\pi}{0.7806} = \underline{\underline{482.95\Omega}}$$

**Prob. 12.9**

$$f_c = \frac{u'}{2}\sqrt{\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2}$$

For $TE_{10}$ mode, m=1, n=0,

$$f_c = \frac{u'}{2a} = \frac{c}{2a} = \frac{3\times10^8}{2\times5\times10^{-2}} = 3GHz$$

$$\beta = \omega\sqrt{\mu\varepsilon}\sqrt{1-\left(\frac{f_c}{f}\right)^2} = 2\pi\times12.5\times10^9\sqrt{4\pi\times10^{-7}\times\frac{10^{-9}}{36\pi}}\sqrt{1-\left(\frac{3}{12.5}\right)^2} = \frac{785.4}{3}(0.9708)$$

$$\underline{\underline{\beta = 254.15\text{ rad/m}}}$$

$$u = \frac{\omega}{\beta} = \frac{2\pi\times12.5\times10^9}{254.15} = \underline{\underline{3.09\times10^8\text{ m/s}}}$$

$$\eta_{TE} = \frac{\eta'}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}} = \frac{120\pi}{0.9708} = \underline{\underline{388.3\ \Omega}}$$

**Prob. 12.10**

$$f_c = \frac{u'}{2a} = \frac{c}{2a} = \frac{3\times10^8}{2\times4\times10^{-2}} = \underline{\underline{3.75\text{ GHz}}}$$

$$\beta = \omega\sqrt{\mu\varepsilon}\sqrt{1-\left(\frac{f_c}{f}\right)^2} = \frac{2\pi\times24\times10^9}{3\times10^8}\sqrt{1-\left(\frac{3.75}{24}\right)^2} = \frac{480\pi}{3}(0.9877)$$

$$\underline{\underline{\beta \quad = 496.48\text{ rad/m}}}$$

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{496.48} = \underline{\underline{0.0127\text{ m}}}$$

**Prob. 12.11**

$$u = \frac{\omega}{\beta} = \frac{u'}{\sqrt{1-(f_c/f)^2}} = \frac{3\times10^8}{\sqrt{1-(6.5/7.2)^2}} = 6.975\times10^8\text{ m/s}$$

$$u_g = \frac{9\times10^{16}}{u} = 1.2903\times10^8\text{ m/s}$$

$$t = \frac{2l}{u_g} = \frac{300}{1.2903\times10^8} = \underline{\underline{2.325}}\ \mu\text{s}$$

**Prob. 12.12**

$$f_c = \frac{u'}{2a} = \frac{c}{2a} = \frac{3\times10^8}{2\times2.4\times10^{-2}} = \underline{\underline{6.25\text{ GHz}}}$$

$$f = 1.25 f_c = \underline{\underline{7.813\text{ GHz}}}$$

$$\eta = \frac{\eta'}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}} = \frac{377}{\sqrt{1-\left(\frac{1}{1.25}\right)^2}} = \frac{377}{0.6} = \underline{\underline{628.32\ \Omega}}$$

**Prob. 12.13**

$$(a)\ f_{c10} = \frac{u'}{2a} = \frac{c}{2a\sqrt{\varepsilon_r}}$$

$$= \frac{3\times10^8}{2\times1.067\times10^{-2}\sqrt{6.8}}$$

$$= \frac{30}{2\times1.067\sqrt{6.8}}\text{ GHz}$$

$$\underline{\underline{= 5.391\text{ GHz}}}$$

(b)

$$F = \sqrt{1-\left(\frac{f_c}{f}\right)^2}$$

$$= \sqrt{1-\left(\frac{5.391}{6}\right)^2}$$

$$= 0.439$$

$$u_\rho = \frac{u'}{F} = \frac{c}{F\sqrt{\varepsilon_r}} = \frac{3\times10^8}{0.439\times\sqrt{6.8}} = \underline{\underline{2.62\times10^8 \text{ m/s}}}$$

(c)

$$\lambda = \frac{\lambda'}{F} = \frac{u'/f}{F} = \frac{c}{fF\sqrt{\varepsilon_r}} = \frac{3\times10^8}{0.439\times6\times10^9\times\sqrt{6.8}}$$

$$= \frac{10^{-1}}{2\times0.439\sqrt{6.8}} = 0.04368 \text{ m} = \underline{\underline{4.368 \text{ cm}}}$$

**Prob.12.14**
In evanescent mode,

$$k^2 = \omega^2\mu\varepsilon < \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

$$\beta = 0, \qquad \gamma = \alpha = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2} = \sqrt{4\pi^2\mu\varepsilon f_c^2 - \omega^2\mu\varepsilon}$$

$$\alpha = \sqrt{\mu\varepsilon}\sqrt{4\pi^2 f_c^2 - 4\pi^2 f^2} = \underline{\underline{2\pi\sqrt{\mu\varepsilon}\, f_c\sqrt{1-\left(\frac{f}{f_c}\right)^2}}}$$

**Prob. 12.15**
$E_z \neq 0$. This must be $TM_{23}$ mode (m=2, n=3). Since a= 2b,

$$f_c = \frac{c}{4b}\sqrt{m^2+4n^2} = \frac{3\times10^8}{4\times3\times10^{-2}}\sqrt{4+36} = 15.81 \text{ GHz}, \quad f = \frac{\omega}{2\pi} = \frac{10^{12}}{2\pi} = 159.2 \text{ GHz}$$

$$\eta_{\text{TM}} = 377\sqrt{1-(15.81/159.2)^2} = \underline{\underline{375.1\ \Omega}}$$

$$\boldsymbol{\mathcal{P}}_{\text{ave}} = \frac{|E_{xs}|^2 + |E_{ys}|^2}{2\eta_{TM}}\boldsymbol{a}_z$$

$$= \frac{\beta^2 E_o{}^2}{2h^4\eta_{TM}}\left[(2\pi/a)^2\cos^2(2\pi x/a)\sin^2(3\pi y/b)+(3\pi/b)^2\sin^2(2\pi x/a)\cos^2(3\pi y/b)\right]\boldsymbol{a}_z$$

$$P_{ave} = \int \mathscr{P}_{\text{ave}}.d\boldsymbol{S} = \int_{x=0}^{a}\int_{y=0}^{b} \mathscr{P}_{\text{ave}} dxdy\boldsymbol{a}_z$$

$$= \frac{\beta^2 E_o{}^2}{2h^4\eta_{TM}}\frac{ab}{4}\left[\frac{4\pi^2}{a^2}+\frac{9\pi^2}{b^2}\right] = \frac{\beta^2 E_o{}^2 ab}{8h^2\eta_{TM}}$$

But

$$\beta = \frac{\omega}{c}\sqrt{1-(f_c/f)^2} = \frac{10^{12}}{3\times10^8}\sqrt{1-(15.81/159.2)^2} = 3.317\times10^3$$

$$h^2 = \frac{4\pi^2}{a^2}+\frac{9\pi^2}{b^2} = \frac{10\pi^2}{b^2} = 1.097\times10^5$$

$$P_{ave} = \frac{(3.317)^2\times10^6\times5^2\times18\times10^{-4}}{8\times(1.098\times10^5)\times375.1} = \underline{\underline{1.5 \text{ mW}}}$$

**Prob. 12.16** (a) Since m=2 and n=1, we have <u>$TE_{21}$ mode</u>

(b) $\beta = \beta'\sqrt{1-(f_c/f)^2} = \omega\sqrt{\mu_o\varepsilon_o}\sqrt{1-(\omega_c/\omega)^2}$

$\beta c = \sqrt{\omega^2-\omega^2_c}$ ⟶ $\omega_c{}^2 = \sqrt{\omega^2-\beta^2c^2}$

$$f_c = \frac{\omega_c}{2\pi} = \sqrt{f^2-\frac{\beta^2c^2}{4\pi^2}} = \sqrt{36\times10^{18}-\frac{144\times9\times10^{16}}{4\pi^2}} = \underline{\underline{5.973}} \text{ GHz}$$

(c ) $\eta_{TE} = \dfrac{\eta}{\sqrt{1-(f_c/f)^2}} = \dfrac{377}{\sqrt{1-(5.973/6)^2}} = \underline{\underline{3978\Omega}}$

(d)For TE mode,

$$E_y = \frac{\omega\mu}{h^2}(m\pi/a)H_o\sin(m\pi x/a)\cos(n\pi y/b)\sin(\omega t-\beta z)$$

$$H_x = \frac{-\beta}{h^2}(m\pi/a)H_o\sin(m\pi x/a)\cos(n\pi y/b)\sin(\omega t-\beta z)$$

$\beta = 12$, m = 2, n =1

$$E_{oy} = \frac{\omega\mu}{h^2}(m\pi / a)H_o\,, \quad H_{ox} = \frac{\beta}{h^2}(m\pi / a)H_o$$

$$\eta_{TE} = \frac{E_{oy}}{H_{ox}} = \frac{\omega\mu}{\beta} = \frac{2\pi\times 6\times 10^9\times 4\pi\times 10^{-7}}{12} = 4\pi^2\times 100$$

$$H_{ox} = \frac{E_{oy}}{\eta_{TE}} = \frac{5}{4\pi^2\times 100} = 1.267 \text{ mA/m}$$

$$\underline{\underline{H_x = -1.267\sin(m\pi x / a)\cos(n\pi y / b)\sin(\omega t - \beta z)}} \text{ mA/m}$$

**Prob. 12.17** (a) Since m=2, n=3, the mode is $\underline{\underline{TE_{23}}}$.

(b) $\beta = \beta'\sqrt{1-(f_c / f)^2} = \frac{2\pi f}{c}\sqrt{1-(f_c / f)^2}$

But

$$f_c = \frac{u'}{2}\sqrt{(m/a)^2 + (n/b)^2} = \frac{3\times 10^8}{2\times 10^{-2}}\sqrt{(2/2.86)^2 + (3/1.016)^2} = 46.19 \text{ GHz}, \ \text{f = 50 GHz}$$

$$\beta = \frac{2\pi\times 50\times 10^9}{3\times 10^8}\sqrt{1-(46.19/50)^2} = 400.68 \text{ rad/m}$$

$$\gamma = j\beta = \underline{\underline{j400.7}} \text{ /m}$$

(c) $\eta = \frac{\eta'}{\sqrt{1-(f_c / f)^2}} = \frac{377}{\sqrt{1-(46.19/50)^2}} = \underline{\underline{985.3\Omega}}$

**Prob. 12.18** In free space,

$$\eta_1 = \frac{\eta_o}{\sqrt{1-(f_c / f)^2}}\,, \qquad f_c = \frac{c}{2a} = \frac{3\times 10^8}{2\times 5\times 10^{-2}} = 3 \text{ GHz}$$

$$\eta_1 = \frac{377}{\sqrt{1-(3/8)^2}} = 406.7\Omega$$

$$\eta_2 = \frac{\eta'_1}{\sqrt{1-(f_c / f)^2}}, \eta' = \frac{120\pi}{\sqrt{2.25}} = 80\pi, f_c = \frac{u'}{2a}, u' = \frac{c}{\sqrt{\varepsilon_r}}$$

$$f_c = \frac{3\times10^8}{2\times5\times10^{-2}\sqrt{2.25}} = 2 \text{ GHz}, \quad \eta_2 = \frac{80\pi}{\sqrt{1-(2/8)^2}} = 259.57\Omega$$

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = -0.2208$$

$$s = \frac{1+|\Gamma|}{1-|\Gamma|} = \underline{\underline{1.5667}}$$

**Prob. 12.19** Substituting $E_z = R\Phi Z$ into the wave equation,

$$\frac{\Phi Z}{\rho}\frac{d}{d\rho}(\rho R') + \frac{RZ}{\rho^2}\Phi'' + R\Phi Z'' + k^2 R\Phi Z = 0$$

Dividing by $R\Phi Z$,

$$\frac{1}{R\rho}\frac{d}{d\rho}(\rho R') + \frac{\Phi''}{\Phi\rho^2} + k^2 = -\frac{Z''}{Z} = -k_z^2$$

i.e. $\underline{\underline{Z'' - k_z^2 Z = 0}}$

$$\frac{1}{R\rho}\frac{d}{d\rho}(\rho R') + \frac{\Phi''}{\Phi\rho^2} + (k^2 + k_z^2) = 0$$

$$\frac{\rho}{R}\frac{d}{d\rho}(\rho R') + (k^2 + k_z^2)\rho^2 = -\frac{\Phi''}{\Phi} = k_\phi^2$$

or

$$\underline{\underline{\Phi'' + k_\phi^2 \Phi = 0}}$$

$\rho\frac{d}{d\rho}(\rho R') + (k_\rho^2\rho^2 - k_\phi^2)R = 0$, where $k_\rho^2 = k^2 + k_z^2$. Hence

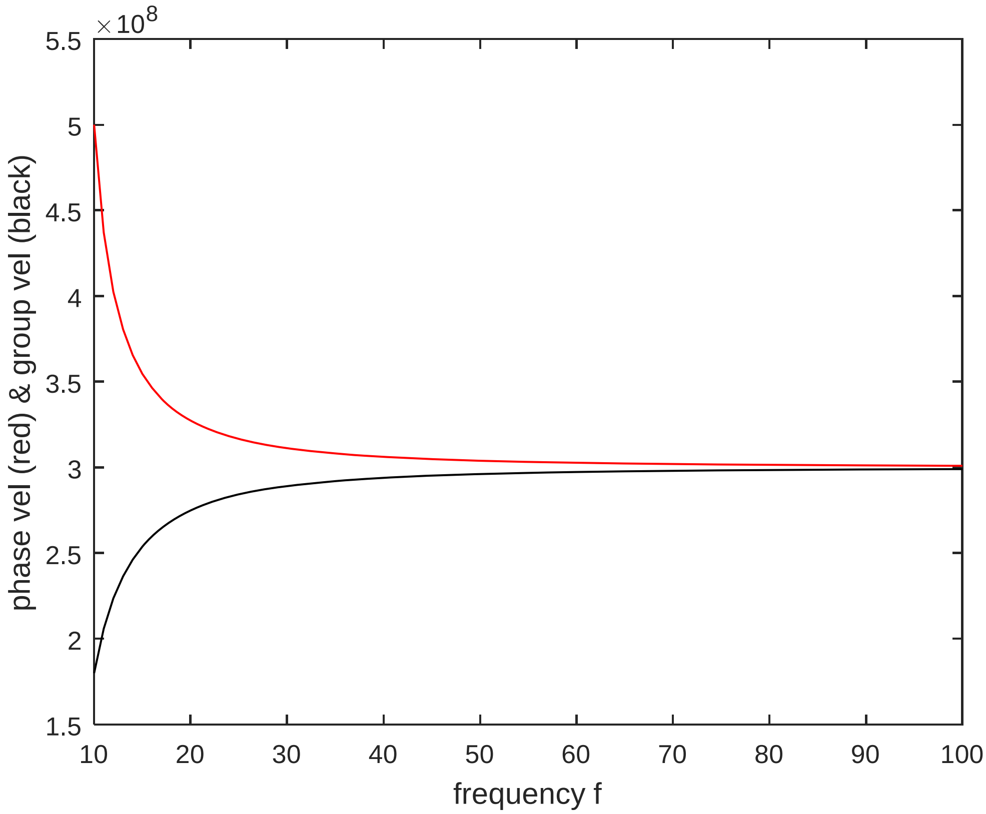
$$\underline{\underline{\rho^2 R'' + \rho R' + (k_\rho^2\rho^2 - k_\phi^2)R = 0}}$$

**Prob. 12.20**

The MATLAB code and the plot of the phase and group velocities are presented below.

```
%    Plot U_p and U_g versus frequency f  (10<f<100) in GHz

c=3*10^8;
for k=1:91
    f(k)=9+k
    fac = sqrt( 1 - (8/f(k))^2 );
    up(k) = c/fac;
    ug(k) = c*fac;
end
plot(f,up, 'r', f, ug, 'k')
xlabel('frequency f')
ylabel('phase vel (red) & group vel (black)')
```



**Prob. 12.21**

(a)

For $TE_{10}$ mode,

$$f_c = \frac{u'}{2a} = \frac{3\times10^8}{2\times7.2\times10^{-2}} = 2.083 \text{ GHz}$$

Let $F = \sqrt{1-\left(\frac{f_c}{f}\right)^2} = \sqrt{1-\left(\frac{2.083}{6.2}\right)^2} = 0.942$

$$\beta = \omega\sqrt{\mu\varepsilon}F = \frac{\omega F}{c} = \frac{2\pi\times6.2\times10^9\times0.942}{3\times10^8} = \underline{\underline{122.32 \text{ rad/m}}}$$

$$u_p = \frac{\omega}{\beta} = \frac{c}{F} = \frac{3\times10^8}{0.942} = \underline{\underline{3.185\times10^8 \text{ m/s}}}$$

$$u_g = u'F = 3\times10^8(0.942) = \underline{\underline{2.826\times10^8 \text{ m/s}}}$$

$$\eta_{TE} = \frac{\eta'}{F} = \frac{377}{0.942} = \underline{\underline{400.21\ \Omega}}$$

(b)

$$u' = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{\sqrt{\varepsilon_r}} = \frac{3\times10^8}{\sqrt{2.25}} = 2\times10^8$$

$$f_c = \frac{u'}{2a} = \frac{2\times10^8}{2\times7.2\times10^{-2}} = 1.389 \text{ GHz}$$

Let $F = \sqrt{1-\left(\frac{f_c}{f}\right)^2} = \sqrt{1-\left(\frac{1.389}{6.2}\right)^2} = 0.9746$

$$\beta = \omega\sqrt{\mu\varepsilon}F = \frac{\omega F\sqrt{\varepsilon_r}}{c} = \frac{2\pi\times6.2\times10^9\times0.9746\times1.5}{3\times10^8} = \underline{\underline{189.83 \text{ rad/m}}}$$

$$u_p = \frac{\omega}{\beta} = \frac{2\pi\times6.2\times10^9}{189.83} = \underline{\underline{2.052\times10^8 \text{ m/s}}}$$

$$u_g = u'F = 2\times10^8(0.9746) = \underline{\underline{1.949\times10^8 \text{ m/s}}}$$

$$\eta_{TE} = \frac{\eta'}{F} = \frac{377}{1.5\times0.9746} = \underline{\underline{257.88\ \Omega}}$$

**Prob. 12.22**

$$f_{c10} = \frac{u'}{2a} = \frac{\frac{1}{\sqrt{\mu\varepsilon}}}{2a} = \frac{c}{2a\sqrt{\varepsilon_r}} = \frac{3\times10^8}{2\times7.214\times10^{-2}\sqrt{2.5}} = 1.315 \text{ GHz}$$

$$\text{Let} \quad F = \sqrt{1-\left(\frac{f_c}{f}\right)^2} = \sqrt{1-\left(\frac{1.315}{4}\right)^2} = 0.9444$$

$$\beta = \omega\sqrt{\mu\varepsilon}F = \frac{\omega\sqrt{\varepsilon_r}F}{c}$$

$$u_p = \frac{\omega}{\beta} = \frac{c}{F\sqrt{\varepsilon_r}} = \frac{3\times10^8}{0.9444\times\sqrt{2.5}} = \underline{\underline{2.009\times10^8 \text{ m/s}}}$$

$$u_g = u'F = \frac{cF}{\sqrt{\varepsilon_r}} = \frac{3\times10^8\times0.9444}{\sqrt{2.5}} = \underline{\underline{1.792\times10^8 \text{ m/s}}}$$

**Prob. 12.23**

$$u_p = \frac{\omega}{\beta} = \frac{\omega\lambda}{2\pi} \quad \rightarrow \quad \omega = \beta u_p = \beta c\frac{\lambda_o^2}{\lambda^2} = \frac{c\lambda_o^2\beta^3}{4\pi^2}, \qquad (\lambda = 2\pi/\beta)$$

$$u_g = \frac{d\omega}{d\beta} = 3\left(\frac{c\lambda_o^2}{4\pi^2}\right)\beta^2 = 3c\left(\frac{\lambda_o}{\lambda}\right)^2 = \underline{\underline{3u_p}}$$

**Prob. 12.24**

$$u_g = \frac{1}{4}c = u'\sqrt{1-\left(\frac{f_c}{f_1}\right)^2} \qquad (1)$$

$$u_g = \frac{1}{3}c = u'\sqrt{1-\left(\frac{f_c}{f_2}\right)^2} \qquad (2)$$

Dividing (1) by (2),

$$\frac{1/4}{1/3} = \frac{\sqrt{1-\left(\frac{f_c}{f_1}\right)^2}}{\sqrt{1-\left(\frac{f_c}{f_2}\right)^2}} \quad \rightarrow \quad \left(\frac{3}{4}\right)^2 = 0.5625 = \frac{1-\left(\frac{f_c}{f_1}\right)^2}{1-\left(\frac{f_c}{f_2}\right)^2}$$

$$1-\left(\frac{f_c}{f_1}\right)^2 = 0.5625\left[1-\left(\frac{f_c}{f_2}\right)^2\right]$$

Assuming $f_c$ is in GHz,

$$1-\frac{f_c^2}{144} = 0.5625 - \frac{0.5625 f_c^2}{225} \quad \rightarrow \quad f_c^2 = 98.44 \quad \rightarrow \quad \underline{\underline{f_c = 9.9216 \text{ GHz}}}$$

From (1),

$$u' = \frac{0.25c}{\sqrt{1-\left(\frac{f_c}{f_1}\right)^2}} = \frac{0.25c}{\sqrt{1-\left(\frac{9.9216}{12}\right)^2}} = 0.4444c$$

But $u' = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{\sqrt{\varepsilon_r}}$

$$\frac{c}{\sqrt{\varepsilon_r}} = 0.4444c \quad \rightarrow \quad \varepsilon_r = \left(\frac{1}{0.4444}\right)^2 = \underline{\underline{5.0625}}$$

**Prob. 12.25**

$$f_c = \frac{u'}{2a}$$

$$u_g = u'\sqrt{1-\left(\frac{f_c}{f}\right)^2} \quad \longrightarrow \quad \left(\frac{f_c}{f}\right)^2 = 1-\left(\frac{u_g}{u'}\right)^2 = 1-\left(\frac{1.8\times10^8}{\frac{3\times10^8}{\sqrt{2.2}}}\right)^2 = 0.208$$

$$f_c = \sqrt{0.208}\,f = 2.0523 \text{ GHz}$$

$$a = \frac{u'}{2f_c} = \frac{3\times10^8}{2\sqrt{2.2}\times2.053\times10^9} = \underline{\underline{4.927\text{ cm}}}$$

**Prob. 12.26**

Let $F = \sqrt{1-(f_c/f)^2} = \sqrt{1-(16/24)^2} = 0.7453$

$u' = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{3\times10^8}{\sqrt{2.25}} = 2\times10^8, \qquad u_p = \frac{u'}{F}, \qquad u_g = u'F = 2\times10^8\times0.7453 = \underline{\underline{1.491\times10^8}}$ m/s

$$\eta_{TE} = \eta'/F = \frac{377}{1.5x0.7453} = \underline{\underline{337.2\Omega}}$$

**Prob. 12.27**

For the $TE_{10}$ mode,

$$H_{zs} = H_o\cos\left(\frac{\pi x}{a}\right)e^{-j\beta z}$$

$$H_{xs} = \frac{j\beta a}{\pi}H_o\sin\left(\frac{\pi x}{a}\right)e^{-j\beta z}$$

$$E_{ys} = -\frac{j\omega\mu a}{\pi}H_o\sin\left(\frac{\pi x}{a}\right)e^{-j\beta z}$$

$$E_{xs} = 0 = E_{zs} = H_{ys}$$

$$\mathbf{E}_s\times\mathbf{H}_s^* = \begin{vmatrix} 0 & E_{ys} & 0 \\ H_{xs}^* & 0 & H_{zs}^* \end{vmatrix} = E_{ys}H_{zs}^*\mathbf{a}_x - E_{ys}H_{xs}^*\mathbf{a}_z$$

$$= -\frac{j\omega\mu a}{\pi}H_o^2\cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi x}{a}\right)\mathbf{a}_x + \frac{\omega\mu\beta a^2}{\pi^2}H_o^2\sin^2\left(\frac{\pi x}{a}\right)\mathbf{a}_z$$

$$\mathcal{P}_{ave} = \frac{1}{2}\text{Re}\left[\mathbf{E}_s\times\mathbf{H}_s^*\right] = \underline{\underline{\frac{\omega\mu\beta a^2}{2\pi^2}H_o^2\sin^2\left(\frac{\pi x}{a}\right)\mathbf{a}_z}}$$

**Prob. 12.28**

$$\mathbf{P}_{\text{ave}} = \frac{|E_{xs}|^2+|E_{ys}|^2}{2\eta}\mathbf{a}_z = \underline{\underline{\frac{\omega^2\mu^2\pi^2}{2\eta b^2h^4}H_o{}^2\sin^2\pi y/b\,\mathbf{a}_z}}$$

where $\eta = \eta_{TE10}$ .

$$\text{P}_{\text{ave}} = \int \mathbf{P}_{\text{ave}}.d\mathbf{S} = \frac{\omega^2\mu^2\pi^2}{2\eta b^2h^4}H_o{}^2\int_{x=0}^{a}\int_{y=0}^{b}\sin^2\pi y/b\,dxdy$$

$$P_{ave} = \frac{\omega^2 \mu^2 \pi^2}{2\eta b^2 h^4} H_o^2 ab/2$$

But $h^2 = (m\pi/a)^2 + (n\pi/b)^2 = \dfrac{\pi^2}{b^2}$,

$$P_{ave} = \frac{\omega^2 \mu^2 ab^3 H_o^2}{4\pi^2 \eta}$$

**Prob. 12.29**

$$R_s = \sqrt{\frac{\pi\mu f}{\sigma_c}} = \sqrt{\frac{\pi \times 12\times 10^9 \times 4\pi \times 10^{-7}}{5.8\times 10^7}} = 2.858\times 10^{-2}$$

$$f_{c10} = \frac{u'}{2a} = \frac{3\times 10^8}{2\sqrt{2.6}\times 2\times 10^{-2}} = 4.651 \text{ GHz}$$

$$f_{c11} = \frac{u'}{2}\left[\frac{1}{a^2} + \frac{1}{b^2}\right]^{1/2} = 10.4 \text{ GHz}$$

$$\eta' = \sqrt{\frac{\mu}{\varepsilon}} = \frac{377}{\sqrt{2.6}} = 233.81\Omega$$

(a) For $TE_{10}$ mode, eq.(12.57) gives

$$\alpha_d + j\beta_d = \sqrt{-\omega^2 \mu\varepsilon + k_x^2 + k_y^2 + j\omega\mu\sigma_d}$$

$$= \sqrt{-\omega^2/u^2 + \frac{\pi^2}{a^2} + j\omega\mu\sigma_d}$$

$$= \sqrt{-\left(\frac{2\pi\times 12\times 10^9}{3\times 10^8}\right)^2 (2.6) + \frac{\pi^2}{(2\times 10^{-2})^2} + j2\pi\times 12\times 10^9 \times 4\pi\times 10^{-7}\times 10^{-4}}$$

$= 0.012682 + j373.57$

$\alpha_d = 0.012682$ Np/m

$$\alpha_c = \frac{2R_s}{b\eta'\sqrt{1-(f_c/f)^2}}\left[\frac{1}{2} + \frac{b}{a}\left(\frac{f_c}{f}\right)^2\right]$$

$$= \frac{2x2.858x10^{-2}}{10^{-2}(233.81)\sqrt{1-(4.651\ /12)^2}}\left[\frac{1}{2}+\frac{1}{2}(\frac{4.651}{12})^2\right] = \underline{\underline{0.0153}}\ \text{Np/m}$$

(b) For $TE_{11}$ mode,

$$\alpha_d + j\beta_d = \sqrt{-\omega^2/u^2 + 1/a^2 + 1/b^2 + j\omega\mu\sigma_d}$$

$$= \sqrt{-139556.21 + \frac{\pi^2}{(10^{-2})^2} + j9.4748} = 0.02344 + j202.14$$

$\underline{\underline{\alpha_d = 0.02344}}$ Np/m

$$\alpha_c = \frac{2R_s}{b\eta'\sqrt{1-(f_c/f)^2}}\left[\frac{(b/a)^3+1}{(b/a)^2+1}\right] = \frac{2\times 2.858\times 10^{-2}}{10^{-2}(233.81)\sqrt{1-(10.4/12)^2}}\left[\frac{(1/8)+1}{(1/4)\ \ +1}\right]$$

$\underline{\underline{\alpha_c = 0.0441}}$ Np/m

**Prob. 12.30** $\quad \varepsilon_c = \varepsilon' - j\varepsilon'' = \varepsilon - j\dfrac{\sigma}{\omega}$

Comparing this with

$$\varepsilon_c = 16\varepsilon_o(1 - j10^{-4}) = 16\varepsilon_o - j16\varepsilon_o \times 10^{-4}$$

$$\varepsilon = 16\varepsilon_o, \qquad \frac{\sigma}{\omega} = 16\varepsilon_o x10^{-4}$$

For $TM_{21}$ mode,

$$f_c = \frac{u'}{2}\left[\frac{m^2}{a^2}+\frac{n^2}{b^2}\right]^{1/2} = 2.0963\ \text{GHz}, \quad f = 1.1f_c = 2.3059\ \text{GHz}$$

$$\sigma = 16\varepsilon_o\omega\times 10^{-4} = 16\times 2\pi\times 2.3059\times 10^9\times\frac{10^{-9}}{36\pi}\times 10^{-4} = 2.0525\times 10^{-4}$$

$$\eta' = \sqrt{\frac{\mu}{\varepsilon}} = 30\pi\ \Omega$$

$$\alpha_d = \frac{\sigma\eta'}{2\sqrt{1-(f_c/f)^2}} = \frac{4.1\times 10^{-4}\times 30\pi}{2\sqrt{1-1/1.12}} = \underline{\underline{0.0231}}\ \text{Np/m}$$

$$E_o e^{-\alpha_d z} = 0.8E_o \longrightarrow z = \frac{1}{\alpha_d}\ln(1/0.8) = \underline{\underline{9.66\ \text{m}}}$$

**Prob. 12.31**

For $TM_{21}$ mode,

$$\alpha_c = \frac{2R_s}{b\eta'\sqrt{1-(f_c/f)^2}}$$

$$R_s = \frac{1}{\sigma_c \delta} = \sqrt{\frac{\pi f \mu}{\sigma_c}} = \sqrt{\frac{\pi \times 2.3059 \times 10^9 \times 4\pi \times 10^{-7}}{1.5 \times 10^7}} = 0.0246\ \Omega$$

$$\alpha_c = \frac{2 \times 0.0246}{4\pi \times 10^{-2} \times 30\pi \times 0.4166} = 0.0314\ \text{Np/m}$$

$$E_o e^{-(\alpha_c+\alpha_d)z} = 0.7E_o \longrightarrow z = \frac{1}{\alpha_c+\alpha_d}\ln(1/0.7) = \underline{\underline{6.5445\ \text{m}}}$$

**Prob. 12.32**

$$f_{c10} = \frac{u'}{2a} = \frac{3\times 10^8}{2\times 6\times 10^{-2}} = 2.5\ \text{GHz}$$

$$\eta_{TE} = \frac{\eta'}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}} = \frac{377}{\sqrt{1-\left(\frac{2.5}{4}\right)^2}} = 483\ \Omega$$

From Example 12.5,

$$P_{ave} = \frac{E_o^2 ab}{2\eta} = \frac{(2.2)^2 \times 10^6 \times 6 \times 3 \times 10^{-4}}{2 \times 483} = \underline{\underline{9.0196\ \text{mW}}}$$

**Prob. 12.33**

For $TE_{10}$ mode,

$$f_c = \frac{u'}{2a} = \frac{3\times 10^8}{2\sqrt{2.11}\times 4.8\times 10^{-2}} = 2.151\ \text{GHz}$$

(a) loss tangent $= \dfrac{\sigma}{\omega\varepsilon} = d$

$$\sigma = d\omega\varepsilon = 3\times 10^{-4}\times 2\pi\times 4\times 10^9 \times 2.11\times \frac{10^{-9}}{36\pi} = 1.4086\times 10^{-4}$$

$$\eta' = \frac{120\pi}{\sqrt{2.11}} = 259.53\ \Omega$$

$$\alpha_d = \frac{\sigma\eta'}{2\sqrt{1-(f_c/f)^2}} = \frac{1.4067\times10^{-4}\times259.53}{2\sqrt{1-(2.151/4)^2}} = \underline{\underline{2.165\times10^{-2}\ \text{Np/m}}}$$

(b) $$R_s = \sqrt{\frac{\mu f \pi}{\sigma_c}} = \sqrt{\frac{\pi\times4\times10^9\times4\pi\times10^{-7}}{4.1\times10^7}} = 1.9625\times10^{-2}\ \Omega$$

$$\alpha_c = \frac{2R_s}{b\eta'\sqrt{1-(f_c/f)^2}}\left[\frac{1}{2}+\frac{b}{a}(f_c/f)^2\right] = \frac{3.925\times10^{-2}(0.5+0.5\times0.2892)}{2.4\times10^{-2}\times259.53\times0.8431}$$

$$= \underline{\underline{4.818\times10^{-3}\ \text{Np/m}}}$$

**Prob.12.34**

$$\alpha_c = \frac{2R_s}{b\eta'\sqrt{1-\left(\frac{f_c}{f}\right)^2}}\left[\frac{1}{2}+\frac{b}{a}\left(\frac{f_c}{f}\right)^2\right] = \frac{2\sqrt{\frac{\pi f\mu}{\sigma_c}}}{b\eta'\sqrt{1-\left(\frac{f_c}{f}\right)^2}}\left[\frac{1}{2}+\frac{1}{2}\left(\frac{f_c}{f}\right)^2\right]$$

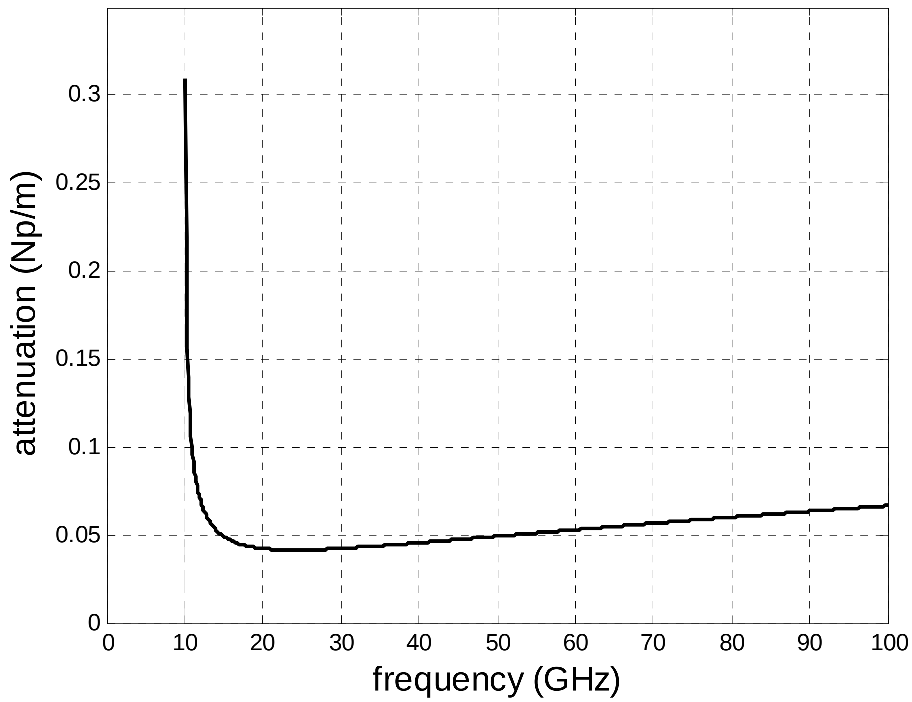
$$= \frac{2\sqrt{4\pi\times10^{-7}\times\pi}\sqrt{f}\times\frac{1}{2}}{0.5\times10^{-2}\times(120\pi/\sqrt{2.25})\sqrt{5.8\times10^7}\sqrt{1-\left(\frac{f_c}{f}\right)^2}}\left[1+\left(\frac{f_c}{f}\right)^2\right]$$

$$= \frac{10^{-5}\sqrt{f}}{30\sqrt{(5.8/2.25)}\sqrt{1-\left(\frac{f_c}{f}\right)^2}}\left[1+\left(\frac{f_c}{f}\right)^2\right]$$

The MATLAB code is shown below

```
k=10^(-5)/(30*sqrt(5.8/2.25));
fc=10^10;
for  n=1:1000
    f(n)=fc*(n/100+1);
    fn=f(n);
    num=sqrt(fn)*(1 +(fc/fn)^2 );
    den=sqrt(1- (fc/fn)^2 );
    alpha(n) =k*num/den;
end
plot(f/10^9,alpha)
xlabel('frequency (GHz)')
ylabel('attenuation')
grid
```

The plot of attenuation versus frequency is shown below.




**Prob. 12.35**

The cutoff frequency of the dominant mode is

$$f_{c10} = \frac{u}{2a} = \frac{3\times10^8}{4.576\times10^{-2}} = 6.56 \text{ GHz}$$

The surface resistance is

$$R_s = \sqrt{\frac{\pi f \mu}{\sigma_c}} = \sqrt{\frac{\pi\times8.4\times10^9\times4\pi\times10^{-7}}{5.8\times10^7}} = 23.91\times10^{-3}\ \Omega$$

For $TE_{10}$ mode,

$$\alpha_c = \frac{2R_s}{b\eta'\sqrt{1-\left(\frac{f_c}{f}\right)^2}}\left[0.5+\frac{b}{a}\left(\frac{f_c}{f}\right)^2\right]$$

$$\frac{f_c}{f} = \frac{6.56}{8.4} = 0.781, \qquad \eta' = \eta_o = 377$$

$$\alpha_c = \frac{2\times 23.91\times 10^{-3}}{1.016\times 10^{-2}\times 377\sqrt{1-0.781^2}}\left[0.5+\frac{1.016}{2.286}(0.781)^2\right]$$

$$= \frac{47.82\times 10^{-3}(0.5+0.2711)}{3.83\times 0.6245} = 15.42\times 10^{-3} \text{ Np/m}$$

$$=15.42\times 10^{-3}\times 8.686 \text{ dB/m} = \underline{\underline{0.1339 \text{ dB/m}}}$$

**Prob. 12.36**

(a) $f_{c10} = \frac{u'}{2a} = \frac{3\times 10^8}{2\times 3.8\times 10^{-2}} = 3.947 \text{ GHz}$

$$u_g = u'\sqrt{1-\left(\frac{f_c}{f}\right)^2} = 3\times 10^8\sqrt{1-(0.3947)^2} = \underline{\underline{2.756\times 10^8 \text{ m/s}}}$$

(b) $\alpha = \alpha_d + \alpha_c$

$\alpha_d = 0$ since the guide is air-filled.

$$R_s = \sqrt{\frac{\pi f \mu}{\sigma_c}} = \sqrt{\frac{\pi\times 10^{10}\times 4\pi\times 10^{-7}}{5.8\times 10^7}} = 2.609\times 10^{-2}\,\Omega$$

$$\alpha_c = \frac{2R_s}{b\eta'\sqrt{1-\left(\frac{f_c}{f}\right)^2}}\left[0.5+\frac{b}{a}\left(\frac{f_c}{f}\right)^2\right]$$

$$= \frac{2\times 2.609\times 10^{-2}}{1.6\times 10^{-2}(377)\sqrt{1-(0.3947)^2}}\left[0.5+\frac{1.6}{3.8}(0.3947)^2\right] = \frac{5.218\times 0.5656}{554.23}$$

$$\underline{\underline{= 5.325\times 10^{-3} \text{ Np/m}}}$$

$$\alpha_c(dB) = 8.686\times 5.325\times 10^{-3} = 0.04626 \text{ dB/m}$$

**Prob.12.37**

$$f_{c10} = \frac{u'}{2a} = \frac{c}{2a\sqrt{\varepsilon_r \mu_r}} = \frac{3\times 10^8}{2\times 2.5\times 10^{-2}\sqrt{2.26}} = 3.991 \text{ GHz}$$

$$\beta' = \frac{\omega}{u'} = \frac{2\pi f\sqrt{\varepsilon_r}}{c}$$

$$F = \sqrt{1-\left(\frac{f_c}{f}\right)^2} = \sqrt{1-\left(\frac{3.991}{7.5}\right)^2} = 0.8467$$

$$\beta = \beta' F = \frac{2\pi\times 7.5\times 10^9\sqrt{2.26}}{3\times 10^8}0.8467 = \underline{\underline{199.94 \text{ rad/m}}}$$

$$\alpha_d = \frac{\sigma\eta'}{2F} = \frac{\sigma\eta_o}{2F\sqrt{\varepsilon_r}} = \frac{10^{-4}(377)}{2\times 0.8467\sqrt{2.26}} = \underline{\underline{1.481\times 10^{-2}\ \text{Np/m}}}$$

$$\alpha_c = \frac{2R_s}{b\eta'\sqrt{1-\left(\frac{f_c}{f}\right)^2}}\left[0.5+\frac{b}{a}\left(\frac{f_c}{f}\right)^2\right]$$

$$R_s = \sqrt{\frac{\pi f\mu}{\sigma_c}} = \sqrt{\frac{\pi\times 7.5\times 10^9\times 4\pi\times 10^{-7}}{1.1\times 10^7}} = 0.0519\ \Omega$$

$$\alpha_c = \frac{2\times 0.0519\left[0.5+\frac{1.5}{2.5}\left(\frac{3.991}{7.5}\right)^2\right]}{1.5\times 10^{-2}\times\frac{377}{\sqrt{2.66}}\times 0.8467} = \frac{0.1038\times 0.6698}{3.1848}$$

$$\underline{\underline{= 0.02183\ \text{Np/m}}}$$

$$u_p = \frac{u'}{F} = \frac{c}{F\sqrt{\varepsilon_r}} = \frac{3\times 10^8}{0.8467\sqrt{2.26}} = \underline{\underline{2.357\times 10^8\ \text{m/s}}}$$

$$u_g = u'F = \frac{3\times 10^8\times 0.8467}{\sqrt{2.26}} = \underline{\underline{1.689\times 10^8\ \text{m/s}}}$$

$$\lambda_c = \frac{u'}{f_c} = \frac{c}{f_c\sqrt{\varepsilon_r}} = \frac{3\times 10^8}{3.991\times 10^9\sqrt{2.26}} = 0.05\ \text{m} = \underline{\underline{5\ \text{cm}}}(=2a,\ \text{as expected})$$

**Prob. 12.38** (a) For $TE_{10}$ mode,

$$f_c = \frac{u'}{2a}, \qquad u' = \frac{c}{\sqrt{2.11}}$$

$$f_c = \frac{3\times 10^8}{\sqrt{2.11}(2\times 2.25\times 10^{-2})} = \underline{\underline{4.589\ \text{GHz}}}$$

(b)
$$\alpha_{cTE10} = \frac{2R_s}{b\eta'\sqrt{1-(f_c/f)^2}}\left[\frac{1}{2}+\frac{b}{a}(f_c/f)^2\right]$$

$$R_s = \sqrt{\frac{\pi f\mu}{\sigma_c}} = \sqrt{\frac{\pi\times 5\times 10^9\times 4\pi\times 10^{-7}}{1.37\times 10^7}} = 3.796\times 10^{-2}\ \Omega$$

$$\eta' = \frac{377}{\sqrt{2.11}} = 259.54\ \Omega$$

$$\alpha_c = \frac{2\times 3.796\times 10^{-2}[0.5+\frac{1.5}{2.25}(4.589/5)^2]}{1.5\times 10^{-4}(259.54)\sqrt{1-(4.589/5)^2}} = \underline{\underline{0.05217}} \text{ Np/m}$$

**Prob. 12.39** For $TE_{10}$ mode,

$$\alpha_c = \frac{2R_s}{b\eta'\sqrt{1-(f_c/f)^2}}\left[\frac{1}{2}+\frac{b}{a}(\frac{f_c}{f})^2\right]$$

But a = b, $R_s = \frac{1}{\sigma_c\delta} = \sqrt{\frac{\pi f\mu}{\sigma_c}}$

$$\alpha_c = \frac{2\sqrt{\frac{\pi f\mu}{\sigma_c}}}{a\eta'\sqrt{1-(f_c/f)^2}}\left[\frac{1}{2}+(\frac{f_c}{f})^2\right] = \frac{k\sqrt{f}\left[\frac{1}{2}+(\frac{f_c}{f})^2\right]}{\sqrt{1-(f_c/f)^2}}$$

where k is a constant.

$$\frac{d\alpha_c}{df} = \frac{k[1-(\frac{f_c}{f})^2]^{1/2}[\frac{1}{4}f^{-1/2}-\frac{3}{2}f_c^2 f^{-5/2}]-\frac{k}{2}[\frac{1}{2}f^{1/2}+f_c^2 f^{-3/2}](2f_c^2 f^{-3})[1-(\frac{f_c}{f})^2]^{-1/2}}{1-(f_c/f)^2}$$

For minimum value, $\frac{d\alpha_c}{df} = 0$. This leads to $\underline{\underline{f = 2.962 f_c}}$.

**Prob. 12.40** For the TE mode to z,

$$E_{zs} = 0, H_{zs} = H_o\cos(m\pi x/a)\cos(n\pi y/b)\sin(p\pi z/c)$$

$$E_{ys} = -\frac{\gamma}{h^2}\frac{\partial E_{zs}}{\partial y}+\frac{j\omega\mu}{h^2}\frac{\partial H_{zs}}{\partial x} = \underline{\underline{-\frac{j\omega\mu}{h^2}(m\pi/a)H_o\sin(m\pi x/a)\cos(n\pi y/b)\sin(p\pi z/c)}}$$

as required.

$$E_{xs} = -\frac{\gamma}{h^2}\frac{\partial E_{zs}}{\partial x}-\frac{j\omega\mu}{h^2}\frac{\partial H_{zs}}{\partial y} = \frac{j\omega\mu}{h^2}(n\pi/b)H_o\cos(m\pi x/a)\sin(n\pi y/b)\sin(p\pi z/c)$$

From Maxwell's equation,

$$-j\omega\mu \boldsymbol{H}_s = \nabla \times \boldsymbol{E}_s = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_{xs} & E_{ys} & 0 \end{vmatrix}$$

$$H_{xs} = \frac{1}{j\omega\mu}\frac{\partial E_{ys}}{\partial z} = -\frac{1}{h^2}(m\pi/a)(p\pi/c)H_o \sin(m\pi x/a)\cos(n\pi y/b)\cos(p\pi z/c)$$

**Prob. 12.41** Maxwell's equation can be written as

$$H_{xs} = \frac{j\omega\varepsilon}{h^2}\frac{\partial E_{zs}}{\partial y} - \frac{\gamma}{h^2}\frac{\partial H_{zs}}{\partial x}$$

For a rectangular cavity,

$$h^2 = k_x^{\ 2} + k_y^{\ 2} = (m\pi/a)^2 + (n\pi/b)^2$$

For TM mode, $H_{zs} = 0$ and

$$E_{zs} = E_o \sin(m\pi x/a)\sin(n\pi y/b)\cos(p\pi z/c)$$

Thus

$$H_{xs} = \frac{j\omega\varepsilon}{h^2}\frac{\partial E_{zs}}{\partial y} = \frac{j\omega\varepsilon}{h^2}(n\pi/b)E_o \sin(m\pi x/a)\cos(n\pi y/b)\cos(p\pi z/c)$$

as required.

$$H_{xs} = -\frac{j\omega\varepsilon}{h^2}\frac{\partial E_{zs}}{\partial x} - \frac{\gamma}{h^2}\frac{\partial H_{zs}}{\partial y}$$

$$= -\frac{j\omega\varepsilon}{h^2}(m\pi/a)E_o \cos(m\pi x/a)\sin(n\pi y/b)\cos(p\pi z/c)$$

From Maxwell's equation,

$$j\omega\varepsilon \boldsymbol{E}_s = \nabla \times \boldsymbol{H}_s = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_{xs} & H_{ys} & 0 \end{vmatrix}$$

$$E_{ys} = \frac{1}{j\omega\varepsilon}\frac{\partial H_{xs}}{\partial z} = \frac{1}{h^2}(n\pi/b)(p\pi/c)E_o \sin(m\pi x/a)\cos(n\pi y/b)\cos(p\pi z/c)$$

**Prob. 12.42**

$$f_r = \frac{u'}{2}\sqrt{(m/a)^2 + (n/b)^2 + (p/c)^2}$$

where for TM mode to z, m = 1, 2, 3,…, n=1, 2, 3, …., p = 0, 1, 2, ….

and for TE mode to z, m = 0,1, 2, 3,…, n=0,1, 2, 3, …., p = 1, 2, 3, … , $(m+n) \neq 0$ .

(a) If a < b < c, 1/a > 1/b > 1/c,

The lowest TM mode is $TM_{110}$ with $f_r = \frac{u'}{2}\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$

The lowest TE mode is $TE_{011}$ with $f_r = \frac{u'}{2}\sqrt{\frac{1}{b^2} + \frac{1}{c^2}} < \frac{u'}{2}\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$

Hence the dominant mode is $TE_{011}$.

(b) If a > b > c, 1/a < 1/b < 1/c,

The lowest TM mode is $TM_{110}$ with $f_r = \frac{u'}{2}\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$

The lowest TE mode is $TE_{101}$ with $f_r = \frac{u'}{2}\sqrt{\frac{1}{a^2} + \frac{1}{c^2}} > \frac{u'}{2}\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$

Hence the dominant mode is $TM_{110}$.

(c) If a = c > b, 1/a = 1/c < 1/b,

The lowest TM mode is $TM_{110}$ with $f_r = \frac{u'}{2}\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$

The lowest TE mode is $TE_{101}$ with $f_r = \frac{u'}{2}\sqrt{\frac{1}{a^2} + \frac{1}{c^2}} < \frac{u'}{2}\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$

Hence the dominant mode is $TE_{101}$.

**Prob. 12.43**

(*a*) $$u' = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{\sqrt{\varepsilon_r}} = \frac{3\times10^8}{\sqrt{4.6}}$$

$$f_c = \frac{u'}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{p}{c}\right)^2}$$

For the dominant mode, m = 1, n=0, p=1

$$f_c = \frac{u'}{2}\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{c}\right)^2} = \frac{3\times10^8}{2\sqrt{4.6}}\sqrt{\frac{1}{9\times10^{-4}} + \frac{1}{36\times10^{-4}}} = \frac{3\times10^{10}}{2(2.1447)}(0.37267) = \underline{\underline{2.606 \text{ GHz}}}$$

(*b*) $$Q = \frac{(a^2+c^2)abc}{\delta\left[2b(a^3+c^3)+ac(a^2+c^2)\right]}$$

$$\delta = \frac{1}{\sqrt{\pi f_{r101}\sigma\mu_o}} = \frac{1}{\sqrt{\pi\times2.606\times10^9\times1.57\times10^7\times4\pi\times10^{-7}}} = 2.49\times10^{-6} \text{ m}$$

$$Q = \frac{(9+36)(72)\times10^{-2}}{\delta\left[8(27+216)+18(9+36)\right]} = \frac{32.42}{2.49\times10^{-6}(2754)} = \underline{\underline{4727.7}}$$

**Prob. 12.44**

(*a*)

$$f_r = \frac{u'}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{p}{c}\right)^2}$$

$$f_{rTE101} = 1.5\times10^{10}\sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{1}{4}\right)^2} = 1.5\times10^{10}\sqrt{0.1736} = \underline{\underline{6.25 \text{ GHz}}}$$

$$f_{rTE011} = 1.5\times10^{10}\sqrt{\frac{1}{6.25} + \frac{1}{16}} = 1.5\times10^{10}\sqrt{0.2225} = \underline{\underline{7.075 \text{ GHz}}}$$

$$f_{rTE110} = 1.5\times10^{10}\sqrt{\frac{1}{9} + \frac{1}{6.25}} = 1.5\times10^{10}\sqrt{0.2711} = \underline{\underline{7.81 \text{ GHz}}}$$

**Prob. 12.45**

$$u' = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{\sqrt{2.5}} = \frac{3\times10^8}{\sqrt{2.5}} = 1.897\times10^8$$

$$f_c = \frac{u'}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{p}{c}\right)^2} = \frac{1.897\times10^8\times10^2}{2}\sqrt{\left(\frac{m}{1}\right)^2 + \left(\frac{n}{2}\right)^2 + \left(\frac{p}{3}\right)^2}$$

$$= 9.485\sqrt{m^2 + 0.25n^2 + 0.111p^2} \quad \text{GHz}$$

$f_{r101} = 9.485\sqrt{1+0+0.111}$ = 10 GHz

$f_{r011} = 9.485\sqrt{0+0.25+0.111}$ = 5.701 GHz

$f_{r012} = 9.485\sqrt{0+0.25+0.444}$ = 7.906 GHz

$f_{r013} = 9.485\sqrt{0+0.25+0.999}$ = 10.61 GHz

$f_{r021} = 9.485\sqrt{0+1+0.111}$ = 10 GHz

Thus, the first five resonant frequencies are:

5.701 GHz($TE_{011}$)

7.906 GHz ($TE_{012}$)

10 GHz ($TE_{101}$ and $TE_{021}$)

10.61 GHz ($TE_{013}$ or $TM_{110}$)

11.07 GHz ($TE_{111}$ or $TM_{111}$)

**Prob. 12.46**

$$Q = \frac{(a^2+c^2)abc}{\delta\left[2b(a^3+c^3)+ac(a^2+c^2)\right]}$$

When a = b = c,

$$Q = \frac{2a^2a^3}{\delta\left[2a\times2a^3+a^2\times2a^2\right]} = \frac{2a^5}{6\delta a^4} = \frac{a}{3\delta}$$

**Prob. 12.47**

(a) Since a > b < c, the dominant mode is $TE_{101}$

$$f_{r101} = \frac{u'}{2}\sqrt{\frac{1}{a^2}+0+\frac{1}{c^2}} = \frac{3\times10^8\times10^2}{2}\sqrt{\frac{1}{2^2}+\frac{1}{1^2}} = \underline{16.77 \text{ GHz}}$$

(b) $$Q_{TE101} = \frac{(a^2+c^2)abc}{\delta\left[2b(a^3+c^3)+ac(a^2+c^2)\right]}$$

$$= \frac{(400+100)20\times 8\times 10\times 10^{-3}}{\delta[16(8000+1000)+200(400+100)]} = \frac{3.279\times 10^{-3}}{\delta}$$

But $$\delta = \frac{1}{\sqrt{\pi f_{r101}\mu_o\sigma}} = \frac{1}{\sqrt{\pi 16.77\times 10^9\times 4\pi\times 10^{-7}\times 6.1\times 10^7}} = \frac{10^{-4}}{200.961}\ \text{m}$$

$$Q_{TE101} = 3.279\times 10^{-3}\frac{200.961}{10^{-4}} = \underline{\underline{6589.51}}$$

**Prob. 12.48**

$$f_r = \frac{c}{2a}\sqrt{m^2+n^2+p^2}$$

The lowest possible modes are $TE_{101}$, $TE_{011}$, and $TM_{110}$. Hence

$$f_r = \frac{c}{2a}\sqrt{2} \longrightarrow a = \frac{c}{f_r\sqrt{2}} = \frac{3\times 10^8}{\sqrt{2}\times 3\times 10^9} = 7.071\ \text{cm}$$

<u>a = b = c = 7.071 cm</u>

**Prob. 12.49**

(a) a = b = c

$$f_r = \frac{u'}{2a}\sqrt{m^2+n^2+p^2}$$

For the dominant mode $TE_{101}$,

$$f_r = \frac{u'}{2a}\sqrt{1+1} = \frac{c}{2a}\sqrt{2}$$

$$a = \frac{c\sqrt{2}}{2f_r} = \frac{3\times 10^8\sqrt{2}}{2\times 5.6\times 10^9} = 0.03788\ \text{m}$$

$$\underline{\underline{a = b = c = 3.788\ \text{cm}}}$$

(b)

For $\varepsilon_r = 2.05$, $\quad u' = \frac{c}{\sqrt{\varepsilon_r}}$

$$a = \frac{c\sqrt{2}}{2f_r\sqrt{\varepsilon_r}} = \frac{0.03788}{\sqrt{2.05}} = 0.02646$$

$$\underline{\underline{a = b = c = 2.646\ \text{cm}}}$$

**Prob. 12.50**

(a)
This is a TM mode to z. From Maxwell's equations,

$$\nabla \times \boldsymbol{E}_s = -j\omega\mu \boldsymbol{H}_s$$

$$\boldsymbol{H}_s = -\frac{1}{j\omega\mu}\nabla \times \boldsymbol{E}_s = \frac{j}{\omega\mu}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & E_{zs}(x,y) \end{vmatrix} \quad = \frac{j}{\omega\mu}\left(\frac{\partial E_{zs}}{\partial y}\boldsymbol{a}_x - \frac{\partial E_{zs}}{\partial x}\boldsymbol{a}_y\right)$$

But

$$E_{zs} = 200\sin 30\pi x \sin 30\pi y, \quad \frac{1}{\omega\mu} = \frac{1}{6\times 10^9 \times 4\pi \times 10^{-7}} = \frac{10^{-2}}{24\pi}$$

$$\boldsymbol{H}_s = \frac{j10^{-2}}{24\pi}\times 200 \times 30\pi\left\{\sin 30\pi x \cos 30\pi y \boldsymbol{a}_x - \cos 30\pi x \sin 30\pi y \boldsymbol{a}_y\right\}$$

$$\mathbf{H} = \text{Re}\,(\mathbf{H}_s\, e^{j\omega t})$$

$$\underline{\underline{\boldsymbol{H} = 2.5\left\{-\sin 30\pi x \cos 30\pi y \boldsymbol{a}_x + \cos 30\pi x \sin 30\pi y \boldsymbol{a}_y\right\}\sin 6\times 10^9 \pi t}} \quad \text{A/m}$$

(*b*)

$$\boldsymbol{E} = E_z \boldsymbol{a}_z, \qquad \boldsymbol{H} = H_x \boldsymbol{a}_x + H_y \boldsymbol{a}_y$$

$$\boldsymbol{E} \cdot \boldsymbol{H} = 0$$

**Prob. 12.51**

(a) $a = b = c \quad \longrightarrow \quad f_{r101} = \dfrac{3\times 10^8}{a\sqrt{2}} = 12\times 10^9$

$$a = \frac{3\times 10^8}{\sqrt{2}\times 12\times 10^9} = \underline{\underline{1.77 \text{ cm}}}$$

(b) $Q_{TE101} = \dfrac{a}{3\delta} = \dfrac{a\sqrt{\pi f_{r101}\mu\sigma}}{3}$

$$= \frac{1.77\times 10^{-2}\sqrt{\pi \times 12\times 10^9 \times 4\pi \times 10^{-7}\times 5.8\times 10^7}}{3} = \underline{\underline{9767.61}}$$

**Prob. 12.52**

$$f_r = \frac{u'}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{p}{c}\right)^2}$$

$$f_{r101} = \frac{3\times10^8}{2}\sqrt{\frac{1}{(10.2)^2}+\frac{1}{(3.6)^2}} = 44.186 \text{ MHz}$$

$$f_{r011} = 150\sqrt{\frac{1}{(8.7)^2}+\frac{1}{(3.6)^2}} \text{ MHz} = 45.093 \text{ MHz}$$

$$f_{r111} = 150\sqrt{\frac{1}{(10.2)^2}+\frac{1}{(8.7)^2}+\frac{1}{(3.6)^2}} \text{ MHz} = 47.43 \text{ MHz}$$

$$f_{r110} = 150\sqrt{\frac{1}{(10.2)^2}+\frac{1}{(8.7)^2}} \text{ MHz} = 22.66 \text{ MHz}$$

$$f_{r102} = 150\sqrt{\frac{1}{(10.2)^2}+\frac{4}{(3.6)^2}} \text{ MHz} = 84.62 \text{ MHz}$$

$$f_{r201} = 150\sqrt{\frac{4}{(10.2)^2}+\frac{1}{(3.6)^2}} \text{ MHz} = 51 \text{ MHz}$$

Thus, the resonant frequences below 50 MHz are

$\underline{\underline{f_{r110}, f_{r101,} f_{r011}, \text{ and } f_{r111}}}$

**Prob. 12.53**

$$n = c/u_m = \frac{3\times10^8}{2.1\times10^8} = \underline{\underline{1.4286}}$$

**Prob. 12.54**

$$NA = \sqrt{n_1^2 - n_2^2} = \sqrt{1.51^2 - 1.45^2} = \sqrt{0.1776} = 0.421$$

**Prob. 12.55**

(a) $NA = \sqrt{n_1{}^2 - n_2{}^2} = \sqrt{1.62^2 - 1.604^2} = \underline{\underline{0.2271}}$

(b) $NA = \sin\theta_a = 0.2271$ or $\theta_a = \sin^{-1} 0.2271 = \underline{\underline{13.13^o}}$

(c) $V = \frac{\pi d}{\lambda} NA = \frac{\pi\times50\times10^{-6}\times0.2271}{1300\times10^{-9}} = 27.441$

$N = V^2/2$ $\underline{\underline{6 \text{ modes}}}$

**Prob. 12.56**

$$V = \frac{\pi d}{\lambda}\sqrt{n_1^2 - n_2^3} = \frac{\pi \times 2 \times 5 \times 10^{-6}}{1300 \times 10^{-9}}\sqrt{1.48^2 - 1.46^2} = 5.86$$

$$N = \frac{V^2}{2} = 17.17 \text{ or } \underline{\underline{17 \text{ modes}}}$$

**Prob. 12.57**

(a) NA = sin $\theta_a$ = $\sqrt{n_1^2 - n_2^2}$ = $\sqrt{1.53^2 - 1.45^2}$ = 0.4883

$\theta_a$ = $\sin^{-1}$ 0.4883 = $\underline{\underline{29.23^o}}$

(b) P(l)/P(0) = $10^{-\alpha l/10}$ = $10^{-0.4X5/10}$ = 0.631

i.e. $\underline{\underline{63.1\ \%}}$

**Prob. 12.58**

$$P(\ell) = P(0)10^{-\alpha\ell/10} = 10 \times 10^{-0.5 \times 0.85/10} = \underline{\underline{9.0678 \text{ mW}}}$$

**Prob. 12.59**

As shown in Eq. (10.35), $\log_{10}$ $P_1/P_2$ = 0.434 ln $P_1/P_2$ ,

1 Np = 20 $\log_{10}$ e = 8.686 dB or 1 Np/km = 8.686 dB/km,

or 1Np/m = 8686 dB/km. Thus,

$$\alpha_{12} = \underline{\underline{8686\alpha_{10}}}$$

**Prob. 12.60**

$$\alpha\ell = 10\log_{10}\frac{P_{in}}{P_{out}} = 10\log_{10}\frac{1.2 \times 10^{-3}}{1 \times 10^{-6}} = 30.792$$

$$\alpha = 0.4 \text{ dB/km} = \frac{0.4}{8.686} \text{ Np/km}$$

$$\ell = \frac{30.792}{\alpha} = \frac{30.392 \text{ dB}}{0.4 \text{ dB/km}} = \underline{\underline{76.98 \text{ km}}}$$

**Prob. 12.61**

P(0) = P(l) $10^{\alpha \text{ l}/10}$ = 0.2 x $10^{0.4 \text{ x } 30/10}$ mW = $\underline{\underline{3.1698 \text{ mW}}}$

**Prob. 12.62** See text.

**CHAPTER 13**

**P. E. 13.1**

(a) For this case, r is at near field.

$$H_{\phi s} = \frac{I_o dl \sin\theta}{4\pi}\left(\frac{j\beta}{r} + \frac{1}{r^2}\right)e^{-j\beta r}, \quad \beta r = \frac{2\pi}{\lambda}.\frac{\lambda}{5} = 72^o$$

$$\lambda = \frac{2\pi c}{\omega} = \frac{2\pi \times 3\times 10^8}{10^8} = 6\pi, \qquad \beta = \frac{2\pi}{\lambda} = \frac{1}{3}$$

$$H_{\phi s} = \frac{(0.25)\dfrac{6\pi}{100}\sin 30^o}{4\pi}\left(\frac{j1/3}{6\pi/5} + \frac{1}{(6\pi/5)^2}\right)e^{-j72^o} = 0.2119\angle -20.511^o \text{ mA/m}$$

$$\boldsymbol{H} = \text{Im}\left(H_{\phi s}e^{j\omega t}\,\boldsymbol{a}_\phi\right) \qquad \text{Im is used since } I = I_o \sin\omega t$$

$$= \underline{\underline{0.2119\sin(10^8 - 20.5^o)\boldsymbol{a}_\phi \text{ mA/m}}}$$

(b) For this case, r is at far field. $\beta = \frac{2\pi}{\lambda}\times 200\lambda = 0^o$

$$H_{\phi s} = \frac{j(0.25)(\dfrac{2\pi}{\lambda})\dfrac{\lambda}{100}Sin60^o e^{-j0^o}}{4\pi(6\pi\times 200)} = 0.2871e^{j90^o} \ \mu\text{A/m}$$

$$\boldsymbol{H} = \text{Im}\,(H_{\phi s}\boldsymbol{a}_\phi e^{j\omega t}) \quad = \underline{\underline{0.2871\sin(10^8 + 90^o)\boldsymbol{a}_\phi}} \ \mu\text{A/m}$$

**P. E. 13.2**

(a) $l = \frac{\lambda}{4} = \underline{\underline{1.5m}}$,

(b) $I_o$ = <u>83.3mA</u>

(c) $R_{rad} = 36.56\,\Omega$, $P_{rad} = \frac{1}{2}(0.0833)^2\,36.56$

= <u>126.8 mW</u>.

(d) $Z_L = 36.5 + j21.25$,

$$\Gamma = \frac{36.5 + j21.25 - 75}{36.5 + j21.25 + 75} = 0.3874\angle 140.3^o$$

$$s = \frac{1+0.3874}{1-0.3874} = \underline{\underline{2.265}}$$

**P. E. 13.3**

$$D = \frac{4\pi U_{\max}}{P_{rad}}$$

(a) For the Hertzian monopole

$$U(\theta,\phi) = \sin^2\theta\,,\ 0<\theta<\pi/2, \qquad 0<\phi<2\pi,\ U_{\max} = 1$$

$$P_{rad} = \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} \sin^2 \sin\theta\, d\theta\, d\phi = \frac{4\pi}{3}$$

$$D = \frac{4\pi(1)}{4\pi/3} = \underline{\underline{3}}$$

(b) For the $\frac{\lambda}{4}$ monopole,

$$U(\theta,\phi) = \frac{\cos^2(\frac{\pi}{2}\cos\theta)}{\sin^2\theta},\ U_{\max} = 1$$

$$P_{rad} = \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} \frac{\cos^2(\frac{\pi}{2}\cos\theta)}{\sin^2\theta}\sin\theta\, d\theta\, d\phi = 2\pi(0.609)$$

$$D = \frac{4\pi(1)}{2\pi(0.609)} = \underline{\underline{3.28}}$$

**P. E. 13.4**

(a) $P_{rad} = \eta_r P_{in} = 0.95(0.4)$

$$D = \frac{4\pi U_{\max}}{P_{rad}} = \frac{4\pi(0.5)}{0.4\times 0.95} = \underline{\underline{16.53}}$$

(b) $D = \frac{4\pi(0.5)}{0.3} = \underline{\underline{20.94}}$

**P. E. 13. 5**

$$P_{rad} = \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} \sin\theta \sin\theta \, d\theta \, d\phi = \frac{\pi^2}{2}, \quad U_{max} = 1$$

$$D = \frac{4\pi(1)}{\pi^2/2} = \underline{\underline{2.546}}$$

**P. E. 13. 6**

(a) $f(\theta) = |\cos\theta| \cos\left[\frac{1}{2}\left(\beta d \cos\theta + \alpha\right)\right]$

where $\alpha = \pi$, $\beta d = \frac{2\pi}{\lambda} \cdot \frac{\lambda}{2} = \pi$

$$f(\theta) = |\cos\theta| \cos\left[\frac{1}{2}(\pi\cos\theta + \pi)\right]$$

$\Downarrow$ unit pattern $\qquad$ $\Downarrow$ group pattern

For the group pattern, we have nulls at

$$\pi/2(\cos\theta + 1) = \pm\pi/2 \longrightarrow \theta = \pm\pi/2$$

and maxima at

$$\pi/2(\cos\theta + 1) = 0, \pi \longrightarrow \cos\theta = -1, 1 \longrightarrow \theta = 0, \pi$$

Thus the group pattern and the resultant patterns are as shown in Fig.13.15(a)

(b) $f(\theta) = |\cos\theta| \cos\left[\frac{1}{2}\left(\beta d \cos\theta + \alpha\right)\right]$

where $\alpha = -\pi/2$, $\beta d = \pi/2$

$$f(\theta) = |\cos\theta| \cos\left[\frac{1}{2}\left(\frac{\pi}{2}\cos\theta - \pi/2\right)\right]$$

$\Downarrow$ unit pattern $\qquad$ $\Downarrow$ group pattern

For the group pattern, the nulls are at

$$\pi/4(\cos\theta - 1) = -\pi/2 \longrightarrow \theta = 180^o$$

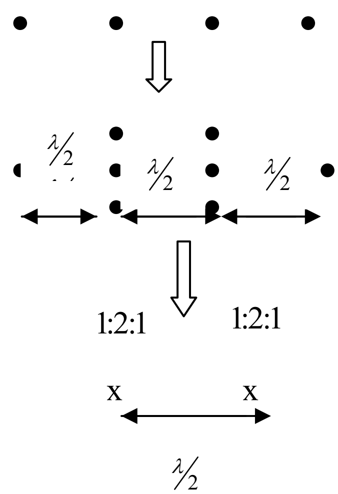
and maxima at

$\cos\theta - 1 = 0 \longrightarrow \theta = 0$

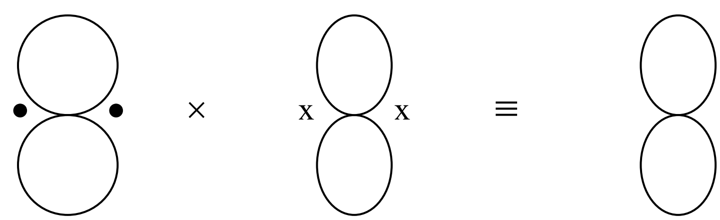
Thus the group pattern and the resultant patterns are as shown in Fig.13.15(b)

**P. E. 13.7**

(a)




Thus, we take a pair at a time and multiply the patterns as shown below.



(b) The group pattern is the normalized array factor, i.e.

$$(AF)_n = \frac{1}{\sum}\left|1 + Ne^{i\psi} + \frac{N(N-1)}{2!}e^{i2\psi} + \frac{N(N-1)(N-2)}{3!}e^{i3\psi} + \dots\dots\dots + e^{i(N-1)\psi}\right|$$

where $\sum = \sum_{i-1}^{N-1}\binom{N}{i} = 1 + N + \frac{N-1}{2!} + \frac{N(N-1)(N-2)}{3!} + \dots\dots\dots$

$$= (1+1)^{N-1} = 2^{N-1}$$

$$(AF)_n = \frac{1}{2^{N-1}}\left|1 + e^{j\psi}\right|^{N-1} = \frac{1}{2^{N-1}}\left|e^{j\psi/2}\right|\left|e^{-j\psi/2} + e^{j\psi/2}\right|^{N-1}$$

$$= \frac{1}{2^{N-1}}\left|2\cos\frac{\psi}{2}\right|^{N-1} = \left|\cos\frac{\psi}{2}\right|^{N-1}$$

**P. E. 13.8**

$$A_e = \frac{\lambda^2}{4\pi} G_d \,, \quad \lambda = \frac{c}{f} = \frac{3\times10^8}{10^8} = 3 \text{ m}$$

For the Hertzian dipole,

$$G_d = 1.5 \sin^2\theta$$

$$A_e = \frac{\lambda^2}{4\pi}(1.5\sin^2\theta)$$

$$A_{e,\max} = \frac{1.5\lambda^2}{4\pi} = \frac{1.5\times 9}{4\pi} = \underline{\underline{1.074 \text{ m}^2}}$$

By definition,

$$P_r = A_e P_{ave} \longrightarrow P_{ave} = \frac{P_r}{A_e} = \frac{3\times10^{-6}}{1.074}$$

$$= \underline{\underline{2.793\ \mu\text{W/m}^2}}$$

**P. E. 13.9**

(a) $$G_d = \frac{4\pi r^2 P_{ave}}{P_{rad}} = \frac{4\pi r^2 \frac{1}{2}\frac{E^2}{\eta}}{P_{rad}} = \frac{2\pi r^2 E^2}{\eta P_{rad}}$$

$$= \frac{2\pi\times400\times10^6\times144\times10^{-6}}{120\pi\times100\times10^3} = 0.0096$$

$$G = 10\log_{10} G_d = \underline{\underline{-20.18 \text{ dB}}}$$

(b) $$G = \eta_r G_d = 0.98\times0.0096 = \underline{\underline{9.408\times10^{-3}}}$$

**P. E. 13.10**

$$r = \left[\frac{\lambda^2 G_d^{\,2}\sigma}{(4\pi)^3}\frac{P_{rad}}{P_r}\right]^{1/4}$$

where $\lambda = \frac{c}{f} = \frac{3\times10^8}{6\times10^9} = 0.05m$

$$A_e = 0.7\pi a^2 = 0.7\pi(1.8)^2 = 7.125m^2$$

$$G_d = \frac{4\pi A_e}{\lambda^2} = \frac{4\pi(7.125)}{25\times10^{-4}} = 3.581\times10^4$$

$$r = \left[ \frac{25\times 10^{-4} \times (3.581)^2 \times 10^8 \times 5 \times 60 \times 10^3}{(4\pi)^3 \times 0.26 \times 10^{-3}} \right]^{1/4}$$

$$= 1168.4\text{m} = \underline{\underline{0.631 \text{ nm}}}$$

At $r = {r_{max}}/{2} = 584.2m,$

$$P = \frac{G_d P_{rad}}{4\pi r^2} = \frac{3.581\times 10^4 \times 60\times 10^3}{4\pi(584.2)^2} = \underline{\underline{501 \text{ W/m}^2}}$$

---

**Prob. 13.1**

Using vector transformation,

$A_{rs} = A_{xs} \sin\theta \cos\phi$ , $A_{\theta s} = A_{xs} \cos\theta \cos\phi$ , $A_{\phi s} = -A_{xs} \sin\phi$

$$\boldsymbol{A}_s = \frac{50e^{-j\beta r}}{r}(\sin\theta\cos\phi\boldsymbol{a}_r + \cos\theta\cos\phi\boldsymbol{a}_\theta - \sin\phi\boldsymbol{a}_\phi)$$

$$\frac{\nabla\times\boldsymbol{A}_s}{\mu} = \boldsymbol{H}_s = \frac{100\cos\theta\sin\phi}{\mu r^2\sin\theta}e^{-j\beta r}\boldsymbol{a}_r - \frac{50}{\mu r^2}(1-j\beta r)\sin\phi e^{-j\beta r}\boldsymbol{a}_\theta$$
$$-\frac{50}{\mu r^2}\cos\theta\cos\phi(1+j\beta r)e^{-j\beta r}\boldsymbol{a}_\phi$$

At far field, only $\frac{1}{r}$ term remains. Hence

$$\boldsymbol{H}_s = \frac{j50}{\mu r}\beta e^{-j\beta r}(\sin\phi\boldsymbol{a}_\theta - \cos\theta\cos\phi\boldsymbol{a}_\phi)$$

$$\boldsymbol{E}_s = -\eta\boldsymbol{a}_r\times\boldsymbol{H}_s = \frac{-j50\beta\eta e^{-j\beta r}}{\mu r}(\sin\phi\boldsymbol{a}_\phi + \cos\theta\cos\phi\boldsymbol{a}_\theta)$$

$$\boldsymbol{E} = \text{Re}\left[\boldsymbol{E}_s e^{j\omega t}\right] = \underline{\underline{\frac{50\eta\beta}{\mu r}\sin(\omega t-\beta r)(\sin\phi\boldsymbol{a}_\phi + \cos\theta\cos\phi\boldsymbol{a}_\theta)}} \text{ V/m}$$

$$\boldsymbol{H} = \text{Re}\left[\boldsymbol{H}_s e^{j\omega t}\right] = \underline{\underline{\frac{-50}{\mu r}\beta\sin(\omega t-\beta r)(\sin\phi\boldsymbol{a}_\theta - \cos\theta\cos\phi\boldsymbol{a}_\phi)}} \text{ A/m}$$

**Prob. 13.2**

(a) $\lambda = \frac{c}{f} = \frac{3\times10^8}{400\times10^6} = 0.75$ m

$r_{min} = \frac{2d^2}{\lambda} = \frac{2(0.02\lambda)^2}{\lambda} \ll r$

i.e. r is in the far field.

$H_{\phi s} = \frac{jI_o\beta dl}{4\pi r}\sin\theta e^{-j\beta r}$

$|H_{\phi s}| = \frac{I_o\beta dl}{4\pi r}\sin\theta = \frac{3\times\frac{2\pi}{\lambda}\times0.02\lambda\times\sin 90^o}{4\pi(60)} = 5\times10^{-4} = 0.5$ mA/m

$|E_{\theta s}| = \eta_o|H_{\phi s}| = \underline{\underline{0.1885 \text{ V/m}}}$

(b) $|H_{\phi s}| = 0.5$ mA/m|

(c) $R_{rad} = 80\pi^2\left(\frac{dl}{\lambda}\right)^2 = 80\pi^2\left(0.02\right)^2 = \underline{\underline{0.3158\ \Omega}}$

(d) $P_{rad} = \frac{1}{2}|I_o|^2 R_{rad} = \frac{1}{2}(9)(0.3158) = \underline{\underline{1.421 \text{ W}}}$

**Prob. 13.3**

$\lambda = \frac{c}{f} = \frac{3\times10^8}{50\times10^6} = 6$ m

$H_{\phi s} = \frac{jI_o\beta dl}{4\pi r}\sin\theta e^{-j\beta r}, \qquad E_{\theta s} = \eta H_{\phi s}$

$dl = 10cm << \lambda$

$\beta = \frac{2\pi}{\lambda} = \frac{2\pi}{6} = \pi/3$

$\frac{I_o\beta dl}{4\pi} = \frac{20\times\frac{\pi}{3}\times0.1}{4\pi} = 0.1667$

$\underline{\underline{\boldsymbol{H}_s = \frac{j0.1667}{r}\sin\theta e^{-j\pi r/3}\boldsymbol{a}_\phi}}$ A/m

$\boldsymbol{E}_s = \frac{j0.1667\times377}{r}\sin\theta e^{-j\pi r/3}\boldsymbol{a}_\theta = \underline{\underline{\frac{j62.83}{r}\sin\theta e^{-j\pi r/3}\boldsymbol{a}_\theta \text{ V/m}}}$

**Prob. 13.4**

$$\lambda = c/f, \quad \mathrm{R_{rad}} = 80\pi^2\left(\frac{dl}{\lambda}\right)^2$$

$$P_{rad} = \frac{1}{2}I_o^2R_{rad} \quad \rightarrow \quad I_o^2 = \frac{2P_{rad}}{R_{rad}}$$

$$I_o^2 = 2P_{rad}\left(\frac{\lambda}{dl}\right)^2\frac{1}{80\pi^2} = 2P_{rad}\left(\frac{c}{fdl}\right)^2\frac{1}{80\pi^2} = 2\times12\left(\frac{3\times10^8}{140\times10^6\times2\times10^{-2}}\right)^2\frac{1}{80\pi^2} = 348.93$$

$$\underline{\underline{I_o = 18.68\text{ A}}}$$

**Prob. 13.5**

(a) $$A_{zs} = \frac{e^{-j\beta r}}{4\pi r}\int_{-l/2}^{l/2} I_o(1-\frac{2|z|}{l})e^{j\beta z\cos\theta}\partial z$$

$$= \frac{e^{-j\beta r}}{4\pi r}I_o\left[\int_{-l/2}^{l/2}(1-\frac{2|z|}{l})\cos(\beta z\cos\theta)dz + j\int_{-l/2}^{l/2}(1-\frac{2|z|}{l})\sin(\beta z\cos\theta)dz\right]$$

$$= \frac{e^{-j\beta r}}{4\pi r}2I_o\int_0^{l/2}(1-\frac{2z}{l})\cos(\beta z\cos\theta)dz$$

$$= \frac{I_o e^{-j\beta r}}{2\pi r\beta^2\cos^2\theta}.\frac{2}{l}\left[1-\cos(\beta l/2\,\cos\theta)\right]$$

$$\boldsymbol{E}_s = -j\omega\mu\boldsymbol{A}_s \longrightarrow \qquad E_{\theta s} = j\omega\mu\sin\theta A_{zs} = j\beta\eta\sin\theta A_{zs}$$

$$E_{\theta s} = \frac{j\eta I_o e^{-j\beta r}}{\pi rl}\frac{\sin\theta\left[1-\cos(\beta l/2\,\cos\theta)\right]}{\beta\cos^2\theta}$$

If $\beta l/2 \ll 1$, $\cos(\beta l/2\,\cos\theta) = 1 - \dfrac{(\frac{\beta l}{2}\cos\theta)^2}{2!}$.

Hence

$$E_{\theta s} = \frac{j\eta I_o}{8\pi r}\beta l e^{-j\beta r}\sin\theta \quad , \quad H_{\phi s} = E_{\theta s}/\eta$$

$$P_{ave} = \frac{|E_{\theta s}|^2}{2\eta}, \qquad P_{rad} = \int P_{ave}dS$$

$$P_{rad} = \int_0^{2\pi}\int_0^{\pi}\frac{n}{2}\left(\frac{I_o\beta l}{8\pi}\right)^2\frac{1}{r^2}\sin^2\theta\, r^2\sin\theta\, d\theta\, d\phi$$

$$= 10\pi^2 I_o^2 \left(\frac{l}{\lambda}\right)^2 = \tfrac{1}{2} I_o^2 R_{rad}$$

or $R_{rad} = 20\pi^2 \left(\frac{l}{\lambda}\right)^2$

(b) $0.5 = 20\pi^2 \left(\frac{l}{\lambda}\right)^2 \longrightarrow \underline{\underline{l = 0.05\lambda}}$

**Prob. 13.6**

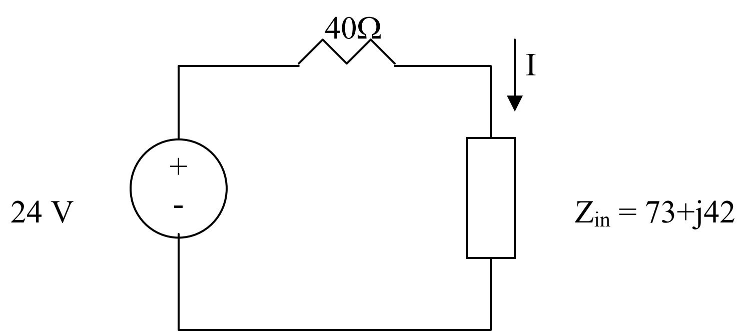
$$\mathrm{R_{rad}} = 80\pi^2 \left(\frac{dl}{\lambda}\right)^2$$

$$\lambda = c/f = \frac{3\times10^8}{1.2\times10^6} = 250m$$

$$\left(\frac{dl}{\lambda}\right)^2 = \frac{R_{rad}}{80\pi^2} = \frac{0.5}{80\pi^2} = 6.33\times10^{-4}$$

$$\frac{dl}{\lambda} = 2.516\times10^{-2} \quad \rightarrow \quad dl = 2.516\times10^{-2}\times250 = \underline{\underline{6.29\text{ m}}}$$

**Prob. 13.7**




$$I = \frac{V}{R_s + Z_{in}} = \frac{24}{40+73+j42} = 0.1866 - j0.0694$$

$$P_{rad} = \frac{1}{2}|I|^2 R_{rad}, \qquad R_{rad} = 73$$

$$P_{rad} = \frac{1}{2}(0.1991)^2\times73 = \underline{\underline{1.447\text{ W}}}$$

**Prob. 13.8**

Let us model this as a short Hertzian dipole.

$$R_{rad} = 80\pi^2\left(\frac{dl}{\lambda}\right)^2 = 80\pi^2(1/8)^2 = 12.34\Omega$$

$$P_{rad} = \frac{1}{2}I_o^2 R_{rad} = 4 \quad \longrightarrow \quad I_o = \underline{\underline{0.8052 \text{ A}}}$$

**Prob. 13.9**

Change the limits in Eq. (13.16) to $\pm l/2$ i.e.

$$A_s = \frac{\mu I_o e^{j\beta z \cos\theta}}{4\pi r}\frac{\left(j\beta\cos\theta\cos\beta t + \beta\sin\beta t\right)}{-\beta^2\cos^2\theta + \beta^2}\Bigg|_{-l/2}^{l/2}$$

$$= \frac{\mu I_o e^{j\beta r}}{2\pi r}\frac{1}{\beta\sin^2\theta}\left[\sin\frac{\beta l}{2}\cos\left(\frac{\beta l}{2}\cos\theta\right) - \cos\theta\cos\frac{\beta l}{2}\sin\left(\frac{\beta l}{2}\cos\theta\right)\right]$$

But $\boldsymbol{B} = \mu\boldsymbol{H} = \nabla\times\boldsymbol{A}$

$$H_{\phi s} = \frac{1}{\mu r}\left[\frac{\partial}{\partial r}\left(rA_\theta\right) - \frac{\partial A_r}{\partial\theta}\right],$$

where $A_o = -A_z\sin\theta, A_r = A_z\cos\theta$

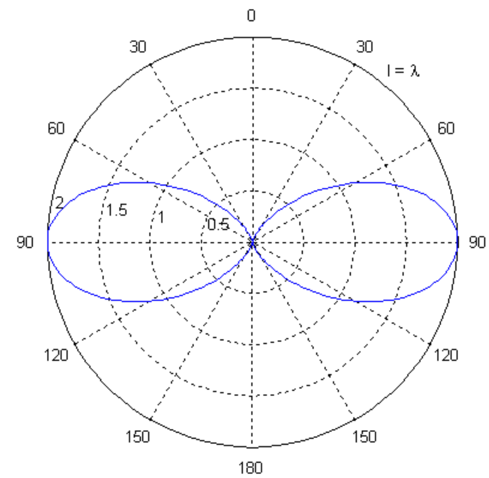
$$H_{\phi s} = \frac{I_o}{2\pi r}\frac{e^{-j\beta r}}{\beta}\left(\frac{j\beta}{\sin\theta}\right)\left[\sin\frac{\beta l}{2}\cos\left(\frac{\beta l}{2}\cos\theta\right) - \cos\theta\cos\frac{\beta l}{2}\sin\left(\frac{\beta l}{2}\cos\theta\right)\right] + \frac{I_o}{2\pi r^2}e^{-j\beta r}(\ldots\ldots)$$

For far field, only the $\frac{1}{r}$-term remains. Hence

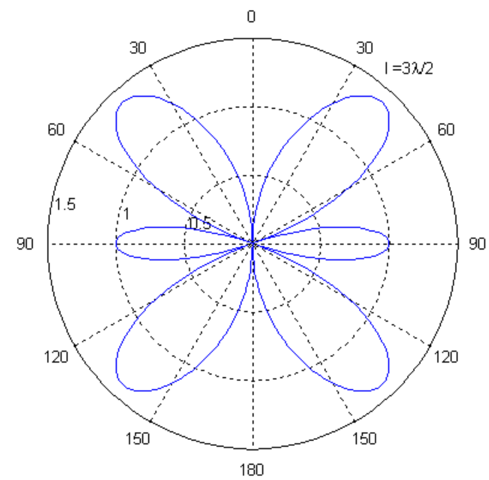
$$H_{\phi s} = \frac{jI_o}{2\pi r}e^{-j\beta r}\frac{\left[\sin\frac{\beta l}{2}\cos\left(\frac{\beta l}{2}\cos\theta\right) - \cos\theta\cos\frac{\beta l}{2}\sin\left(\frac{\beta l}{2}\cos\theta\right)\right]}{\sin\theta}$$

(b) $$f(\theta) = \frac{\cos\left(\frac{\beta l}{2}\cos\theta\right) - \cos\frac{\beta l}{2}}{\sin\theta}$$

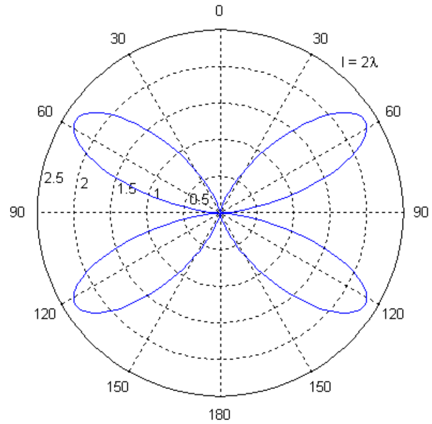
For $l = \lambda$, $$f(\theta) = \frac{\cos(\pi\cos\theta) + 1}{\sin\theta}$$




For $l = \frac{3\lambda}{2}$, $f(\theta) = \dfrac{\cos\left(\frac{3\pi}{2}\cos\theta\right)}{\sin\theta}$




For $l = 2\lambda,\ f(\theta) = \dfrac{\cos\theta\sin\left(2\pi\cos\theta\right)}{\sin\theta}$




**Prob. 13.10**

(a)

$$\lambda = \frac{c}{f} = \frac{3\times10^8}{450\times10^6} = 0.6667 \text{ m}$$

$$\ell = \frac{\lambda}{2} = \underline{\underline{0.333 \text{ m}}}$$

(b)

$$\frac{\sigma}{\omega\varepsilon} = \frac{4}{2\pi\times450\times10^6\times81\times\frac{10^{-9}}{36\pi}} = 1.975$$

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1\right]} = \frac{2\pi\times460\times10^6}{c}\sqrt{\frac{81}{2}\left[\sqrt{1+(1.975)^2}+1\right]}$$

$$= \frac{2\pi\times460\times10^6}{3\times10^8}\times11.4086 = 109.91$$

$$\lambda = \frac{2\pi}{\beta} = 0.0572$$

$$\ell = \frac{\lambda}{2} = \underline{\underline{28.58 \text{ mm}}}$$

**Prob. 13.11**

(a)

$$\lambda = \frac{c}{f} = \frac{3\times10^8}{1.150\times10^6} = 260.8 \text{ m}$$

$$\ell = \frac{\lambda}{4} = \underline{\underline{65.22 \text{ m}}}$$

(b)

$$\lambda = \frac{c}{f} = \frac{3\times10^8}{90\times10^6} = 3.333 \text{ m}$$

$$\ell = \frac{\lambda}{4} = \underline{\underline{0.8333 \text{ m}}}$$

(c )

$$\lambda = \frac{c}{f} = \frac{3\times10^8}{80\times10^6} = 3.75 \text{ m}$$

$$\ell = \frac{\lambda}{4} = \underline{\underline{0.9375 \text{ m}}}$$

(d)

$$\lambda = \frac{c}{f} = \frac{3\times10^8}{600\times10^6} = 0.5 \text{ m}$$

$$\ell = \frac{\lambda}{4} = \underline{\underline{0.125 \text{ m}}}$$

**Prob. 13.12**

$l \langle\langle \lambda$ ,hence it is a Hertzian monopole.

$$R_{rad} = 80\pi^2\left(\frac{dl}{\lambda}\right)^2 = 80\pi^2\left(\frac{1}{8}\right)^2 = \underline{\underline{12.34\ \Omega}}$$

**Prob. 13.13**

(a) $\lambda = \frac{c}{f} = \frac{3\times10^8}{10\times10^6} = 30 \text{ m}$

$$E_{\max} = \frac{\eta\pi I_o S}{r\lambda^2} \longrightarrow I_o = \frac{E_{\max} r\lambda^2}{\eta\pi S}$$

$$I_o = \frac{50\times10^{-3}\times3\times30^2}{120\pi^2\pi(0.2)^2 100} = \underline{\underline{9.071 \text{ mA}}} \quad (S=N\pi r^2)$$

(b) $R_{rad} = \frac{320\pi^4 S^2}{\lambda^4} = \frac{320\pi^4\pi^2(0.2)^4\times10^4}{30^4} = 6.077\Omega$

$$P_{rad} = \frac{1}{2}I_o{}^2 R_{rad} = \frac{1}{2}(9.071)^2\times10^{-6}\times6.077$$

$= \underline{\underline{0.25 \text{ mW}}}$

**Prob. 13.14**

$$\lambda = \frac{c}{f} = \frac{3\times10^8}{80\times10^6} = 3.75 \text{ m}$$

$$S = N\pi\rho_o^2$$

$$R_{rad} = \frac{320\pi^4 S^2}{\lambda^4} = \frac{320\pi^4 N^2 \pi^2 \rho_o^4}{\lambda^4} \longrightarrow N^2 = \frac{\lambda^4 R_{rad}}{320\pi^6 (1.2\times10^{-2})^4}$$

$$N^2 = \frac{(3.75)^4 \times 8}{320\pi^6 (1.2\times10^{-2})^4} = 248006 \longrightarrow N \simeq \underline{\underline{498}}$$

**Prob. 13.15**

(a) $$R_{rad} = \frac{320\pi^4 S^2}{\lambda^4}$$

$$S = \pi\rho_o^2 = \pi(0.4)^2 = 0.5027 \text{ m}^2$$

$$\lambda = \frac{c}{f} = \frac{3\times10^8}{6\times10^6} = 50 \text{ m}$$

$$R_{rad} = \frac{320\pi^4 (0.5027)^2}{(50)^4} = \underline{\underline{1.26 \text{ m}\Omega}}$$

(b) $$P_{rad} = \frac{1}{2} I_o{}^2 R_{rad} = \frac{1}{2}(50)^2 \times 1.26\times10^{-3} = \underline{\underline{1.575 \text{ W}}}$$

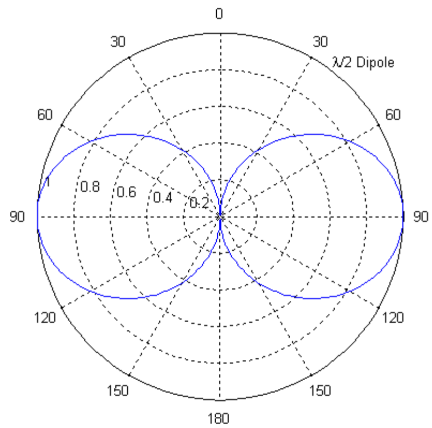
(c) $$\text{R}_{\ell} = \frac{a}{2\delta} \text{R}_{\text{dc}} = \frac{a}{2\delta}\frac{\ell}{\sigma S} = \frac{a\ell}{2\sigma\pi a^2}\sqrt{\pi \text{f} \mu\sigma} = \frac{2\pi R}{2\pi \text{a}}\sqrt{\frac{\mu f \pi}{\sigma}}$$

$$\text{R}_{\ell} = \frac{R}{a}\sqrt{\frac{\mu f\pi}{\sigma}} = \frac{0.4}{4\times10^{-3}}\sqrt{\frac{4\pi\times10^{-7}\times6\times10^6\times\pi}{5.8\times10^7}} = 63.91 \text{ m}\Omega$$

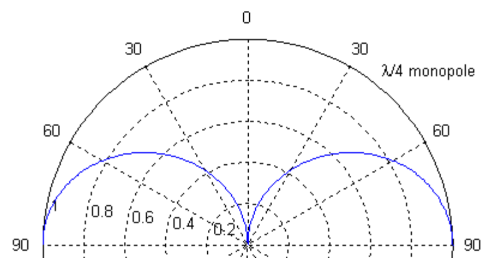
$$\eta = \frac{\text{R}_{\text{rad}}}{\text{R}_{\text{rad}} + R_\ell} = \frac{1.26}{1.26+63.91}\times100\% = \underline{\underline{1.933\%}}$$

**Prob. 13.16**

(a) $$f(\theta) = \left| \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} \right|$$




(b) The same as for $\frac{\lambda}{2}$ dipole except that the fields are zero for $\theta \rangle \frac{\pi}{2}$ as shown.




**Prob. 13.17**

Let $P_{rad1}$ and $P_{rad2}$ be the old and new radiated powers respectively.
Let $P_{ohm1}$ and $P_{ohm2}$ be the old and new ohmic powers respectively.

$$\eta_{r1} = 20\% = \frac{P_{rad1}}{P_{rad1}+P_{ohm1}} = \frac{1}{5} \quad \longrightarrow \quad 4\,P_{rad1} = P_{ohm1} \qquad (1)$$

But $\quad P_{ohm1} = \frac{1}{2} I^2 R_s \Delta z$

$$P_{ohm2} = \frac{1}{2} I^2 R_s 2\Delta z = 2P_{ohm1} \qquad (2)$$

$$P_{rad1} = \frac{1}{2} I_o^2 R_{rad} = \frac{1}{2} I_o^2 \times 80\pi^2 \left(\frac{\Delta z}{\lambda}\right)^2$$

$$P_{rad2} = \frac{1}{2} I_o^2 R_{rad} = \frac{1}{2} I_o^2 \times 80\pi^2 \left(\frac{2\Delta z}{\lambda}\right)^2 = 4P_{rad1} \qquad (3)$$

From (1) to (3),

$$\eta_{r2} = \frac{P_{rad2}}{P_{rad2}+P_{ohm2}} = \frac{4P_{rad1}}{4P_{rad1}+2P_{ohm1}} = \frac{P_{ohm1}}{3P_{ohm1}} = \underline{\underline{33.3\%}}$$

**Prob. 13.18**

$$\eta_r = \frac{P_{rad}}{P_{in}} = \frac{R_{rad}}{R_{rad} + R_\ell}$$

$$R_{rad} = 73\Omega, \qquad R_\ell = \frac{\ell}{\sigma_c S} = \frac{\ell}{\sigma_c \pi a^2}$$

$$\ell = \frac{\lambda}{2}, \qquad \lambda = c/f = \frac{3\times10^8}{6\times10^6} = 50m, \qquad \ell = 25m$$

$$R_\ell = \frac{25}{58\times10^6 \times \pi (1.2)^2 \times 10^{-6}} = \frac{25}{262.4} = 0.09528\Omega$$

$$\eta_r = \frac{73}{73 + 0.09528} = 0.9987 = \underline{\underline{99.87\%}}$$

**Prob. 13.19**

(a) Let $\boldsymbol{H}_s = \dfrac{\cos 2\theta}{\eta_o r} e^{-j\beta r} \boldsymbol{a}_H$

$$\boldsymbol{a}_E \times \boldsymbol{a}_H = \boldsymbol{a}_k \quad \longrightarrow \quad \boldsymbol{a}_\theta \times \boldsymbol{a}_H = \boldsymbol{a}_r \quad \longrightarrow \quad \boldsymbol{a}_H = \boldsymbol{a}_\phi$$

$$\boldsymbol{H}_s = \underline{\underline{\frac{\cos 2\theta}{120\pi r} e^{-j\beta r} \boldsymbol{a}_\phi}}$$

(b) $\boldsymbol{P}_{ave} = \dfrac{|E_s|^2}{2\eta} \boldsymbol{a}_r = \dfrac{\cos^2(2\theta)}{2\eta r^2} \boldsymbol{a}_r$

$$P_{rad} = \frac{1}{2\eta} \iint \frac{\cos^2 2\theta}{r^2} r^2 \sin\theta \, d\theta \, d\phi = \frac{1}{240\pi}(2\pi) \int_0^\pi \cos^2 2\theta \sin\theta \, d\theta$$

But $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$

$$P_{rad} = -\frac{1}{120} \int_0^\pi (2\cos^2\theta - 1)^2 \, d(\cos\theta)$$

$$= -\frac{1}{120} \int_0^\pi (4\cos^4\theta - 4\cos^2\theta + 1) \, d(\cos\theta)$$

$$= -\frac{1}{120} \left( \frac{4\cos^5\theta}{5} - \frac{4\cos^3\theta}{3} + \cos\theta \right) \Bigg|_0^\pi$$

$$= -\frac{1}{120} \left[ -\frac{4}{5} + \frac{4}{3} - 1 - \frac{4}{5} + \frac{4}{3} - 1 \right] = \frac{1}{120} \left( \frac{14}{15} \right)$$

$$= \underline{\underline{7.778 \text{ mW}}}$$

(c )

$$P_{rad} = -\frac{1}{120}\int_{60^o}^{120^o}(2\cos^2\theta - 1)^2 d(\cos\theta)$$

$$= -\frac{1}{120}\left(\frac{4\cos^5\theta}{5} - \frac{4\cos^3\theta}{3} + \cos\theta\right)\Bigg|_{60^o}^{120^o}$$

$$= -\frac{1}{120}[\frac{4}{5}(-\frac{1}{32}) - \frac{4}{3}(-\frac{1}{8}) - \frac{1}{2} - \frac{4}{5}(\frac{1}{32}) + \frac{4}{3}(\frac{1}{8}) - \frac{1}{2}] = \frac{1}{60}[\frac{1}{40} + \frac{1}{2} - \frac{1}{6}]$$

$$= 5.972 \text{ mW}$$

which is $\frac{5.972}{7.778} = 0.7678$ or $\underline{\underline{76.78\%}}$

**Prob. 13.20**

$$\mathscr{P}_{ave} = \frac{1}{2}\text{Re}(\boldsymbol{E}_s \times \boldsymbol{H}_s^*) = \frac{1}{2}\eta\, |H_{\phi s}|^2\, \boldsymbol{a}_r$$

$$P_{rad} = \int \mathscr{P}_{ave} \cdot d\boldsymbol{S} = \frac{1}{2}\eta\iint \frac{\beta^2 I_o^2}{16\pi^2 r^2}\sin^2\theta\cos^2\phi\; r^2\sin\theta d\theta d\phi$$

$$= \frac{\eta\beta^2 I_o^2}{32\pi^2}\int_0^{\pi}\sin^3\theta d\theta\int_0^{2\pi}\cos^2\phi d\phi = \frac{\eta\beta^2 I_o^2}{32\pi^2}\left(\frac{4}{3}\right)\frac{1}{2}\int_0^{2\pi}(1+\cos 2\phi)d\phi$$

$$= \frac{\eta\beta^2 I_o^2}{32\pi^2}\left(\frac{4}{3}\right)\pi = \frac{\eta\beta^2 I_o^2}{24\pi}$$

$$R_{rad} = \frac{2P_{rad}}{I_o^2} = \frac{\eta\beta^2}{12\pi^2}$$

Assuming free space, $\eta = 120\pi$,

$$\underline{\underline{R_{rad} = \frac{10\beta^2}{\pi}}}$$

**Prob. 13.21**

(a) $P_{rad} = \int \boldsymbol{P}_{rad} \cdot d\boldsymbol{S} = P_{ave}.2\pi r^2$ (hemisphere)

$$P_{ave} = \frac{P_{rad}}{2\pi r^2} = \frac{200\times 10^3}{2\pi(2500\times 10^6)} = 12.73\mu W / m^2$$

$$\boldsymbol{P}_{ave} = \underline{\underline{12.73\boldsymbol{a}_r\ \mu\text{W/m}^2}}\ .$$

(b) $P_{ave} = \frac{(E_{\max})^2}{2\eta}$

$$E_{\max} = \sqrt{2\eta P_{ave}} = \sqrt{240\pi \times 1273 \times 10^{-6}}$$

$$= \underline{\underline{0.098 \text{ V/m}}}$$

**Prob. 13.22**

$$U(\theta,\phi) = r^2 P_{ave} = k\sin^2\theta\sin^3\phi$$

$$\mathrm{G}_d(\theta,\phi) = \frac{U(\theta,\phi)}{U_{ave}}$$

$$U_{ave} = \frac{k}{4\pi}\iint U(\theta,\phi)\sin\theta d\theta d\phi = \frac{k}{4\pi}\int_0^\pi \sin^3\phi d\phi \int_0^\pi \sin^3\theta d\theta = \frac{k}{4\pi}\left[\int_0^\pi (1-\cos^2\phi)d(-\cos\phi)\right]^2$$

$$= \frac{k}{4\pi}\left[\frac{4}{3}\right]^2 = \frac{4k}{9\pi}$$

$$\mathrm{G}_d(\theta,\phi) = \frac{9\pi}{4k}k\sin^2\theta\sin^3\phi = \underline{\underline{7.069\sin^2\theta\sin^3\phi}}$$

$$D = G_{d,\max} = \underline{\underline{7.069}}$$

**Prob. 13.23**

From Prob. 13.11, set $\ell = \frac{3\lambda}{2}$, $\beta\ell = \frac{2\pi}{\lambda}\frac{3\lambda}{2} = 3\pi$

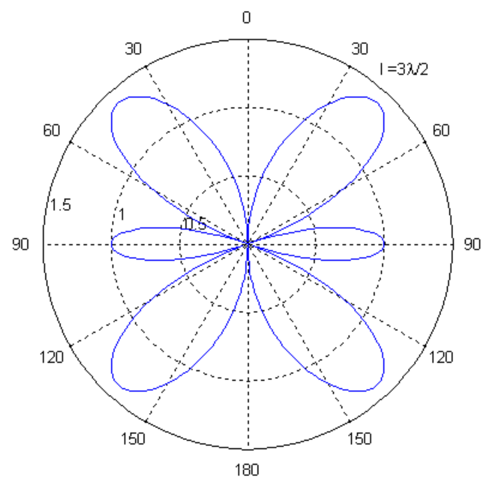
$$H_{\phi s} = \frac{jI_o e^{-\beta r}}{2\pi r}\frac{\left[\cos\left(\frac{2\pi}{\lambda}\frac{1}{2}\frac{3\lambda}{2}\cos\theta\right) - \cos\left(\frac{2\pi}{\lambda}\frac{1}{2}\frac{3\lambda}{2}\right)\right]}{\sin\theta}$$

$$= \frac{jI_o e^{-\beta r}}{2\pi r}\frac{\left[\cos\left(\frac{3\pi}{2}\cos\theta\right) - \cos\left(\frac{3\pi}{2}\right)\right]}{\sin\theta} = \frac{jI_o e^{-\beta r}}{2\pi r}\frac{\cos\left(1.5\pi\cos\theta\right)}{\sin\theta}$$

Hence, the normalized radiated field pattern is

$$f(\theta) = \frac{\cos\left(1.5\pi\cos\theta\right)}{\sin\theta}$$

which is plotted below.




**Prob. 13.24**
The MATLAB code is shown below

```
N=20;
del= 2*pi/N;
sum=0;
for k=1:N
  theta = del*k;
  term = (1 – cos(theta))/theta;
  sum = sum + term;
end
int = del*sum
```

When the program is run, it gives the value of 2.4335. The accuracy may be increased by increasing N.

**Prob. 13.25**

(a) $E_{\theta s} = \dfrac{j\eta I_o \beta dl}{4\pi r}\sin\theta e^{-j\beta r}$

$$R_{rad} = 80\pi^2\left(\frac{dl}{\lambda}\right)^2$$

$$G_d = \frac{4\pi r^2 P_{ave}}{P_{rad}} = \frac{4\pi r^2 \cdot \frac{1}{2\eta}\left|E_{\theta s}\right|^2}{\frac{1}{2}I_o^2 R_{rad}}$$

$$= \frac{4\pi r^2}{I_o^2}\cdot\frac{1}{80\pi^2}\left(\frac{\lambda}{dl}\right)^2\cdot\frac{1}{\eta}\frac{\eta^2 I_o^2\beta^2(dl)^2\sin^2\theta}{16\pi^2 r^2}$$

$G_d = \underline{\underline{1.5\sin^2\theta}}$

(b) $D = G_{d,\max} = \underline{\underline{1.5}}$

(c) $A_e = \frac{\lambda^2}{4\pi} G_d = \underline{\underline{\frac{1.5\lambda^2 \sin^2\theta}{4\pi}}}$

(d) $R_{rad} = 80\pi^2 \left(\frac{1}{16}\right)^2 = \underline{\underline{3.084\ \Omega}}$

**Prob. 13.26**

$$G_d(\theta,\phi) = \frac{4\pi U(\theta,\phi)}{P_{rad}} = \frac{4\pi f^2(\theta)}{\iint f^2(\theta) d\Omega}$$

$$f(\theta) = \sin\theta$$

$$G_d(\theta,\phi) == \frac{4\pi \sin^2\theta}{\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \sin^3\theta d\theta} = \frac{4\pi\sin^2\theta}{2\pi(4/3)} = \underline{\underline{1.5\sin^2\theta}}$$

$$D = G_{d,\max} = \underline{\underline{1.5}}$$

**Prob. 13.27**

(a)

$$\lambda = \frac{c}{f} = \frac{3\times10^8}{1.2\times10^6} = 250$$

$$\ell = \frac{\lambda}{4} = \underline{\underline{62.5\text{ m}}}$$

(b) From eq. (13.30), $R_{rad} = \underline{\underline{36.5\ \Omega}}$

(c )

For $\lambda/4$-monopole,

$$f(\theta) = \frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta}, \qquad 0 < \theta < \pi/2$$

$$G_d(\theta,\phi) = \frac{4\pi f^2(\theta)}{\int f^2(\theta) d\Omega} = \frac{\dfrac{4\pi\cos^2(\frac{\pi}{2}\cos\theta)}{\sin^2\theta}}{\int_0^{2\pi}\int_0^{\pi/2} \dfrac{\cos^2(\frac{\pi}{2}\cos\theta)}{\sin\theta} d\theta d\phi}$$

$$= \frac{4\pi\cos^2(\frac{\pi}{2}\cos\theta)}{\sin^2\theta}\frac{1}{2\pi(0.6094)} = \frac{3.282\cos^2(\frac{\pi}{2}\cos\theta)}{\sin^2\theta}$$

$$D = G_{d,\max} = \underline{\underline{3.282}}$$

**Prob. 13.28**

(a) $U_{max} = 1$

$$U_{ave} = \frac{P_{rad}}{4\pi} = \frac{\int U d\Omega}{4\pi}$$

$$= \frac{1}{4\pi}\int\int \sin^2 2\theta \sin\theta d\theta d\phi$$

$$= \frac{1}{4\pi}(2\pi)\int_0^{\pi}\left(2\sin\theta\cos\theta\right)^2 d\left(-\cos\theta\right)$$

$$= 2\int_0^{\pi}\left(\cos^4\theta - \cos^2\theta\right)d\left(\cos\theta\right)$$

$$= 2\left[\frac{\cos^5\theta}{5} - \frac{\cos^3\theta}{3}\right]\Bigg|_0^{\pi}$$

$$= 2\left[-\frac{2}{5} + \frac{2}{3}\right] = \frac{8}{15}$$

$$U_{ave} = \underline{\underline{0.5333}}$$

$$D = \frac{U_{\max}}{U_{ave}} = \underline{\underline{1.875}}$$

(b) $U_{max} = 4$

$$U_{ave} = \frac{1}{4\pi}\int U d\Omega = \frac{4}{4\pi}\int\int \frac{\sin\theta}{\sin^2\theta} d\theta d\phi$$

$$= \frac{1}{\pi}\int_0^{\pi} d\phi \int_{\pi/3}^{\pi/2} \cos ec\theta d\theta = \frac{\pi}{\pi}\ln\sqrt{3}$$

$U_{ave} = \underline{\underline{0.5493}}$

$$D = \frac{U_{max}}{U_{ave}} = \frac{16}{3\ln\sqrt{3}} = \underline{\underline{9.7092}}$$

(c ) $U_{max} = 2$

$$U_{ave} = \frac{1}{4\pi}\int U d\Omega = \frac{1}{4\pi}\int\int 2\sin^2\theta \sin^2\phi \sin\theta d\theta d\phi$$

$$= \frac{1}{2\pi}\int_0^{\pi} \sin^2\phi d\phi \int_0^{\pi} \left(1 - \cos^2\theta\right) d(-\cos\theta)$$

$$= \frac{1}{2\pi}.\frac{\pi}{2}\left(\frac{\cos^3\theta}{3} - \cos\theta\right)\Bigg|_0^{\pi} = \frac{1}{4}\left[-\frac{2}{3} + 2\right] = \frac{1}{3}$$

$U_{ave} = \underline{\underline{0.333}}$

$$D = \frac{U_{max}}{U_{ave}} = \underline{\underline{6}}$$

**Prob. 13.29**

(a) $$G_d(\theta,\phi)=\frac{U(\theta,\phi)}{U_{ave}}$$

$$U_{ave}=\frac{1}{4\pi}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}10\sin\theta\sin^2\phi\times\sin\theta d\theta d\phi$$

$$=\frac{10}{4\pi}\int_0^{2\pi}\sin^2\phi d\phi\int_0^{\pi}\sin^2\theta d\theta$$

$$=\frac{10}{4\pi}\frac{1}{2}\left(\theta-\frac{\sin 2\theta}{2}\right)\Bigg|_0^{2\pi}\frac{1}{2}\left(\phi-\frac{\sin 2\phi}{2}\right)\Bigg|_0^{\pi}$$

$$=\frac{10}{16\pi}(2\pi-0)(\pi-0)=\frac{5\pi}{4}$$

$$G_d(\theta,\phi)=\frac{40\sin\theta\sin^2\phi}{5\pi}=\underline{\underline{2.546\sin\theta\sin^2\phi}}$$

$$D=G_{d.\max}=\underline{\underline{2.546}}$$

(b) $$U_{ave}=\frac{1}{4\pi}\int_{\phi=0}^{\pi}\int_{\theta=0}^{\pi}2\sin^2\theta\sin^3\phi\times\sin\theta d\theta d\phi$$

$$=\frac{2}{4\pi}\int_0^{\pi}\sin^3\phi d\phi\int_0^{\pi}\sin^3\theta d\theta=\frac{2}{4\pi}\left[\int_0^{\pi}(1-\cos^2\phi)d(-\cos\phi)\right]^2$$

$$=\frac{1}{2\pi}\left[\left(\frac{\cos^3\phi}{3}-\cos\phi\right)\Bigg|_0^{\pi}\right]=\frac{1}{2\pi}\left(\frac{4}{3}\right)^2=\frac{16}{18\pi}$$

$$G_d(\theta,\phi)=\frac{18\pi}{16}2\sin^2\theta\sin^3\phi=\underline{\underline{2.25\pi\sin^2\theta\sin^3\phi}}$$

$$D=G_{d.\max}=\underline{\underline{7.069}}$$

(c) $$U_{ave} = \frac{1}{4\pi}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} 5(1+\sin^2\theta\sin^2\phi)\times\sin\theta d\theta d\phi$$

$$= \frac{5}{4\pi}\left[\int_0^{\pi}\sin\theta d\theta\int_0^{2\pi} d\phi + \int_0^{\pi}\sin^3\theta d\theta\int_0^{2\pi}\sin^2 d\phi\right]$$

$$= \frac{5}{4\pi}\left[2\pi(-\cos\theta)\Big|_0^{\pi} + \frac{4}{3}\left(\frac{\phi}{2} - \frac{\sin 2\phi}{4}\right)\Big|_0^{2\pi}\right]$$

$$= \frac{5}{4\pi}\left[4\pi + \frac{4}{3}\pi\right] = \frac{20}{3}$$

$$G_d(\theta,\phi) = \frac{3}{20}5(1+\sin^2\theta\sin^2\phi) = \underline{\underline{0.75(1+\sin^2\theta\sin^2\phi)}}$$

$$D = G_{d.\max} = \underline{\underline{1.5}}$$

**Prob. 13.30**

$$U_{\max} = 4$$

$$U_{ave} = \frac{1}{4\pi}\int U d\Omega = \frac{1}{4\pi}\iint 4\sin^2\theta\sin\frac{\phi}{2}\sin\theta d\theta d\phi$$

$$= \frac{1}{\pi}\int_0^{\pi}\sin^3\theta d\theta\int_0^{\pi}\sin\frac{\phi}{2}d\phi = \frac{1}{\pi}\int_0^{\pi}(1-\cos^2\theta)d(-\cos\theta)(-2\cos\frac{\phi}{2})\Big|_0^{\pi}$$

$$= \frac{1}{\pi}\left(\frac{4}{3}\right)(2) = \frac{8}{3\pi}$$

$$D = \frac{U_{\max}}{U_{ave}} = 4\times\frac{3\pi}{8} = \underline{\underline{4.712}}$$

**Prob. 13.31**

$$\boldsymbol{P}_{ave} = \frac{|E_r|^2}{2\eta}\boldsymbol{a}_r = \frac{I_o^2\sin^2\theta}{2\eta r^2}\boldsymbol{a}_r$$

$$P_{rad} = \frac{I_o^2}{2\eta}\iint\frac{\sin^2\theta}{r^2}r^2\sin\theta d\theta d\phi = \frac{I_o^2}{240\pi}(2\pi)\int_0^{2\pi}(1-\cos^2\theta)d(-\cos\theta)$$

$$= \frac{I_o^2}{120}\left(\frac{\cos^3\theta}{3} - \cos\theta\right)\Big|_0^{\pi} = \frac{I_o^2}{120}(-1/3+1-1/3+1) = \frac{I_o^2}{90}$$

$$I_o^2 = 90P_{ave} = 90\times 50\times 10^{-3} \longrightarrow I_o = \underline{\underline{2.121\text{ A}}}$$

**Prob. 13.32**

$$U(\theta,\phi) = r^2 P_{ave} = r^2 \frac{|E_\phi|^2}{2\eta} = \frac{r^2}{2\eta}\frac{1400\pi^4 I_o^2 S^2 \sin^2\theta}{r^2\lambda^4}, \qquad \text{where } \eta = 120\pi$$

$$U(\theta,\phi) = \frac{120\pi^3 I_o^2 S^2 \sin^2\theta}{2\lambda^4} = \underline{\underline{\frac{60\pi^3 I_o^2 S^2 \sin^2\theta}{\lambda^4}}}$$

**Prob. 13.33**

$$P_{ave} = \frac{E^2}{2\eta} = \frac{\alpha^2 I^2}{2\eta r^2}$$

$$R_{rad} = \frac{2P_{rad}}{I^2} = \frac{2P_{ave}}{I^2}4\pi r^2 = \frac{8\pi r^2}{I^2}\frac{\alpha^2 I^2}{2\eta r^2} = \frac{4\pi\alpha^2}{\eta} = \frac{4\pi\alpha^2}{120\pi} = \underline{\underline{\frac{\alpha^2}{30}}}$$

**Prob. 13.34**

According to eq. (13.10),

$$P_{rad} = k\int_0^\pi \sin^3\theta d\theta = \frac{4k}{3}, \qquad \text{where k is a constant.}$$

$$P_{rad}' = k\int_0^{\pi/3} \sin^3\theta d\theta = k\left[\frac{\cos^3\theta}{3} - \cos\theta\right]_0^{\pi/3} = k\left[\frac{1}{3\times 8} - \frac{1}{2} - \frac{1}{3} + 1\right] = \frac{5}{24}k$$

$$\text{Fraction} = \frac{\frac{5}{24}k}{\frac{4k}{3}} = \frac{5}{32} = \underline{\underline{0.1562}}$$

**Prob. 13.35**

This is similar to Fig. 13.10 except that the elements are z-directed.

$$\boldsymbol{E}_s = \boldsymbol{E}_{s1} + \boldsymbol{E}_{s2} = \frac{j\eta\beta I_o dl}{4\pi}\left[\sin\theta_1 \frac{e^{-j\beta r_1}}{r_1}\boldsymbol{a}_{\theta 1} + \sin\theta_2 \frac{e^{-j\beta r_2}}{r_2}\boldsymbol{a}_{\theta 2}\right]$$

where $r_1 \cong r - \frac{d}{2}\cos\theta, \qquad r_2 \cong r + \frac{d}{2}\cos\theta, \qquad \theta_1 \cong \theta_2 \cong \theta, \qquad \boldsymbol{a}_{\theta 1} \cong \boldsymbol{a}_{\theta 2} = \boldsymbol{a}_\theta$

$$\boldsymbol{E}_s = \frac{j\eta\beta I_o dl}{4\pi}\sin\theta\, \boldsymbol{a}_\theta\left[e^{j\beta d\cos\theta/2} + e^{-j\beta d\cos\theta/2}\right]$$

$$\underline{\underline{\boldsymbol{E}_s = \frac{j\eta\beta I_o dl}{2\pi}\sin\theta\ \cos(\frac{1}{2}\beta d\cos\theta)\boldsymbol{a}_\theta}}$$

**Prob. 13.36**

(a) $AF = 2\cos\left[\frac{1}{2}\left(\beta d\cos\theta + \alpha\right)\right], \quad \alpha = 0, \quad \beta d = \frac{2\pi}{\lambda}\lambda = 2\pi$

$AF = \underline{\underline{2\cos(\pi\cos\theta)}}$

(b)Nulls occur when

$\cos(\pi\cos\theta) = 0 \quad \longrightarrow \quad \pi\cos\theta = \pm\pi/2, \pm 3\pi/2, \ldots$

or

$$\theta = \underline{\underline{60^o, 120^o}}$$

(c) Maxima and minima occur when

$\frac{df}{d\theta} = 0 \quad \longrightarrow \quad \sin(\pi\cos\theta)\pi\sin\theta = 0$

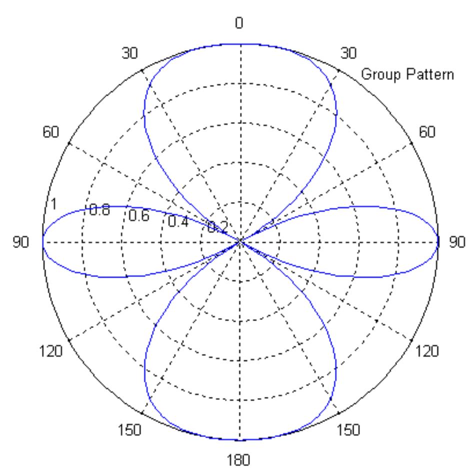
i.e. $\sin\theta = 0 \quad \longrightarrow \quad \theta = 0^o, 180^o$

$\cos\theta = 0 \quad \longrightarrow \quad \theta = 90^o$

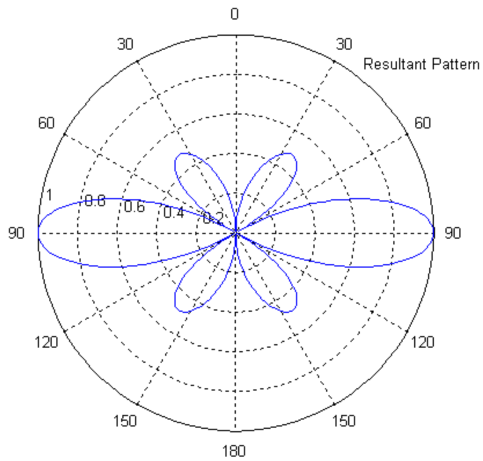
or

$$\theta = \underline{\underline{0^o, 90^o, 180^o}}$$

(d ) The group pattern is sketched below.






**Prob. 13.37**

$$f(\theta) = \cos\left[\frac{1}{2}\left(\beta d \cos\theta + \alpha\right)\right]$$

(a) $\alpha = \pi/2, \beta d = \frac{2\pi}{\lambda}.\lambda = 2\pi$

$$f(\theta) = \cos\left(\pi \cos\theta + \pi/4\right)$$

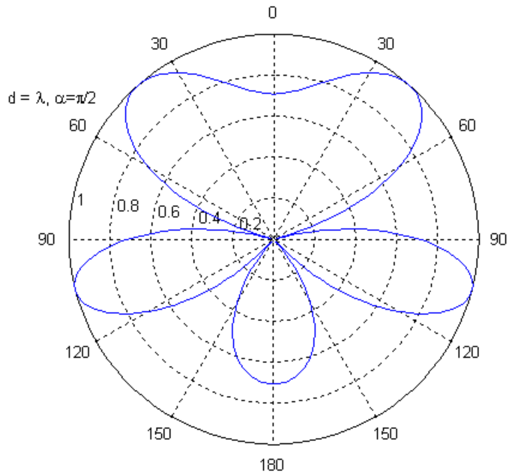
Nulls occur at $\pi \cos\theta + \pi/4 = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$ or $\theta = 75.5^o, 138.6^o$

Maxima occur at $\frac{\partial f}{\partial\theta} = 0 \longrightarrow \sin\theta = 0 \longrightarrow \theta = 0^o, 180^o$

Or $\sin\left(\pi\cos\theta + \frac{\pi}{4}\right) = 0 \longrightarrow \theta = 41.4^o, 104.5^o$

With $f_{max} = 0.71, 1$.

Hence the group pattern is sketched below.



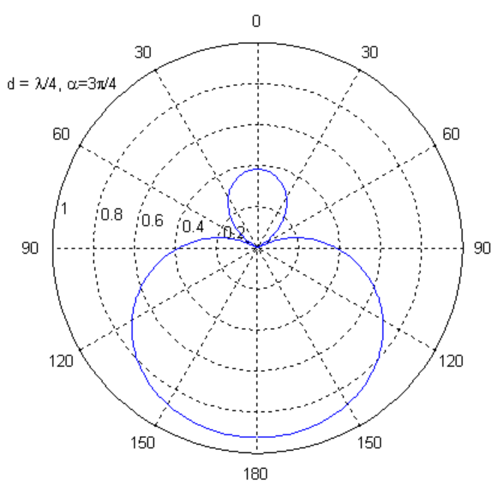

(b) $\alpha = \frac{3\pi}{4}, \beta d = \frac{2\pi}{\lambda}.\frac{\lambda}{4} = \frac{\pi}{2}$

$$f(\theta) = \left|\cos\left(\frac{\pi}{4}\cos\theta + \frac{3\pi}{8}\right)\right|$$

Nulls occur at $\frac{\pi}{4}\cos\theta + \frac{3\pi}{8} = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots \quad \longrightarrow \quad \theta = 60^o$

Minima and maxima occur at $\sin\theta\cos\left(\frac{\pi}{4}\cos\theta + \frac{3\pi}{8}\right) = 0$

i.e. $\theta = 0^o, 180^o \rightarrow f(\theta) = 0.383, 0.924$



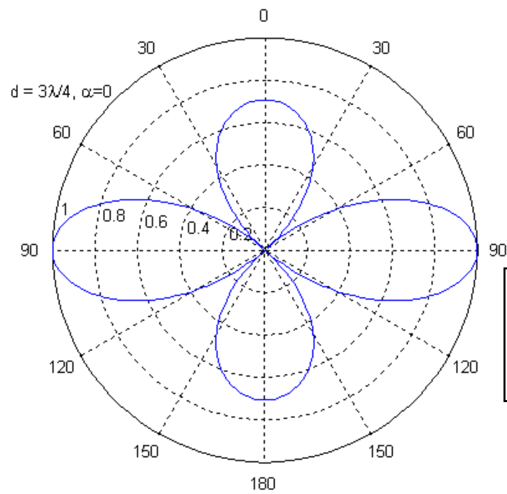
(c) $\alpha = 0, \beta d = \frac{2\pi}{\lambda}.\frac{3\lambda}{4} = \frac{3\pi}{2}$

$$f(\theta) = \left|\cos\left(\frac{3\pi}{4}\cos\theta\right)\right|$$

It has nulls at $\frac{3\pi}{4}\cos\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots \rightarrow \theta = 48.2^o, 131.8^o$

It has maxima and minima at $\frac{df}{d\theta} = 0 \rightarrow \sin\theta \sin\left(\frac{3\pi}{4}\cos\theta\right) = 0$

i.e. $\theta = 0^o, 180^o \rightarrow f(\theta) = 0.71, 1, \quad \theta = \pm 90^o, \rightarrow f(\theta) = 1$




| | |
|---|---|
| $\theta = 0^o$, | abs(f) = 1/√2=0.707 |
| $\theta = 90^o$, | abs(f) = 1, |
| $\theta = 180^o$, | abs(f) = 1/√2=0.707 |

**Prob. 13.38**

(a) For N = 2, $f(\theta) = \cos\left[\frac{1}{2}(\beta d\cos\theta + \alpha)\right]$

$\alpha = 0, d = \frac{\lambda}{4}$

$$f(\theta) = \cos\left[\frac{1}{2}\left(\frac{2\pi}{\lambda}.\frac{\lambda}{4}\cos\theta + 0\right)\right] = \cos\left(\frac{\pi}{4}\cos\theta\right)$$

Maxima and minima occur at

$$\frac{d}{d\theta}\left[\cos\left(\frac{\pi}{4}\cos\theta\right)\right] = 0$$

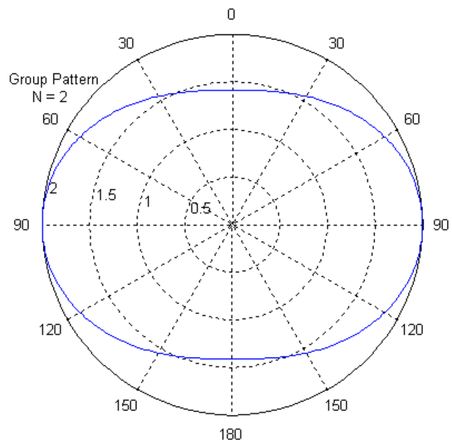
$$\sin\theta\sin\left(\frac{\pi}{4}\cos\theta\right) = 0$$

$$\sin\theta = 0 \rightarrow \theta = \pi, 0 \text{ and } f(\theta) = 0.707$$

$$\sin\left(\frac{\pi}{4}\cos\theta\right) \rightarrow \cos = 0 \rightarrow \theta = 90^o, f(\theta) = 1$$

Nulls occur as $\frac{\pi}{4} cos\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$ (No Solution)

The group pattern is sketched below.




(b) For N = 4,

$$AF = \frac{\sin 2(\beta d \cos\theta + 0)}{\sin \frac{1}{2}(\beta d \cos\theta + 0)}$$

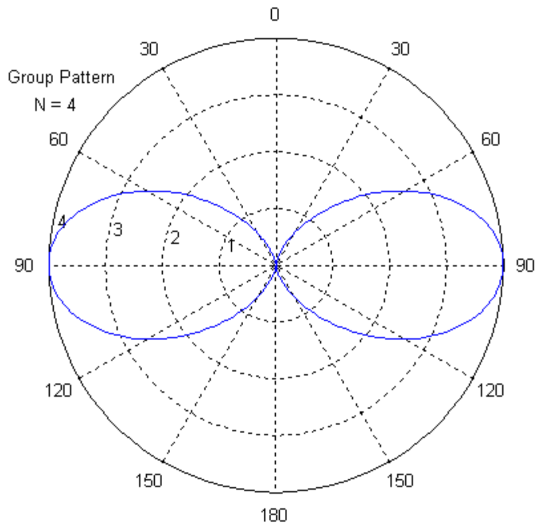
Now, $\frac{\sin 4\theta}{\sin\theta} = \frac{2 \sin 2\theta \cos 2\theta}{\sin\theta} = 4 \cos 2\theta \cos\theta$

$$AF = 4 \cos(\beta d \cos\theta) \cos\left(\frac{1}{2}\beta d \cos\theta\right)$$

$$f(\theta) = \cos\left(\frac{2\pi}{\lambda}.\frac{\lambda}{4}\cos\theta\right) \cos\left(\frac{1}{2}\frac{2\pi}{\lambda}\frac{\lambda}{4}\right) \cos\theta$$

$$= \cos\left(\frac{\pi}{2}\cos\theta\right) \cos\left(\frac{\pi}{4}\cos\theta\right)$$

The plot is shown below.

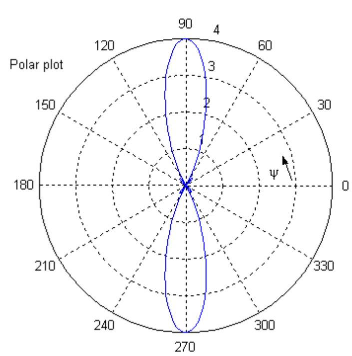


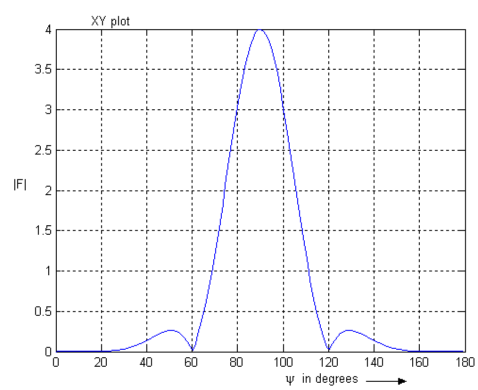

**Prob. 13.39**

The MATLAB code is shown below.

```
for n=1:180
    phi=n*pi/180;
    p(n)=n;
    sn=sin(2*pi*cos(phi));
    cn=cos(0.5*pi*cos(phi));
    sd=sin(0.5*pi*cos(phi));
    fun=sn*cn*cn/sd;
    f(n)= abs(fun);
end
polar(p,f)
```

The polar plot and the xy plot are shown below.

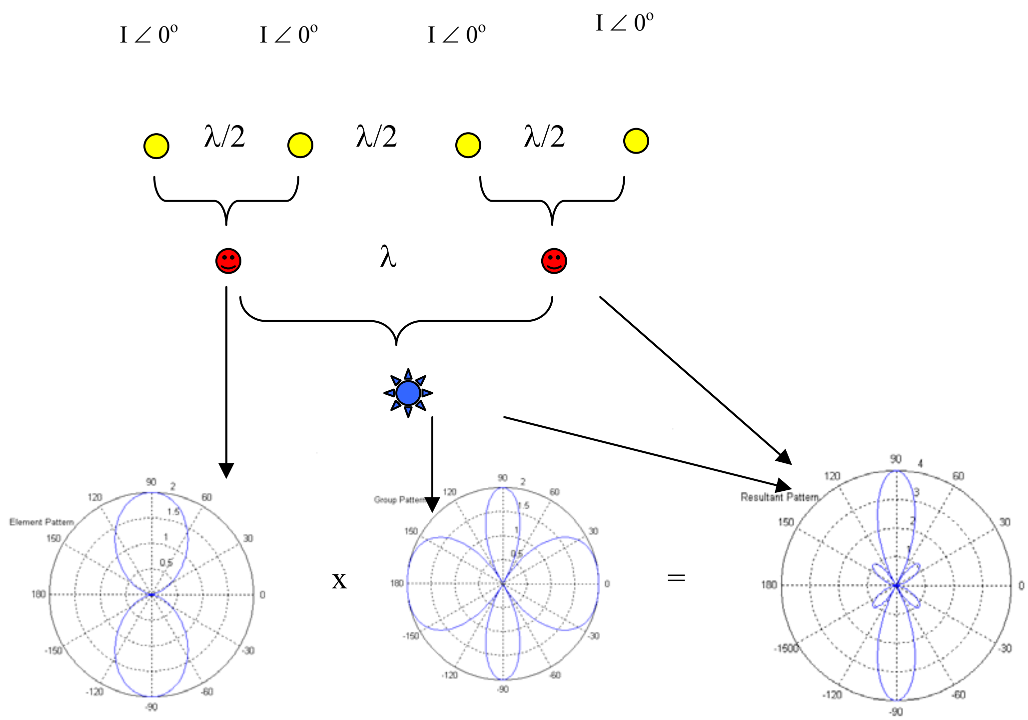


**Prob. 13.40**

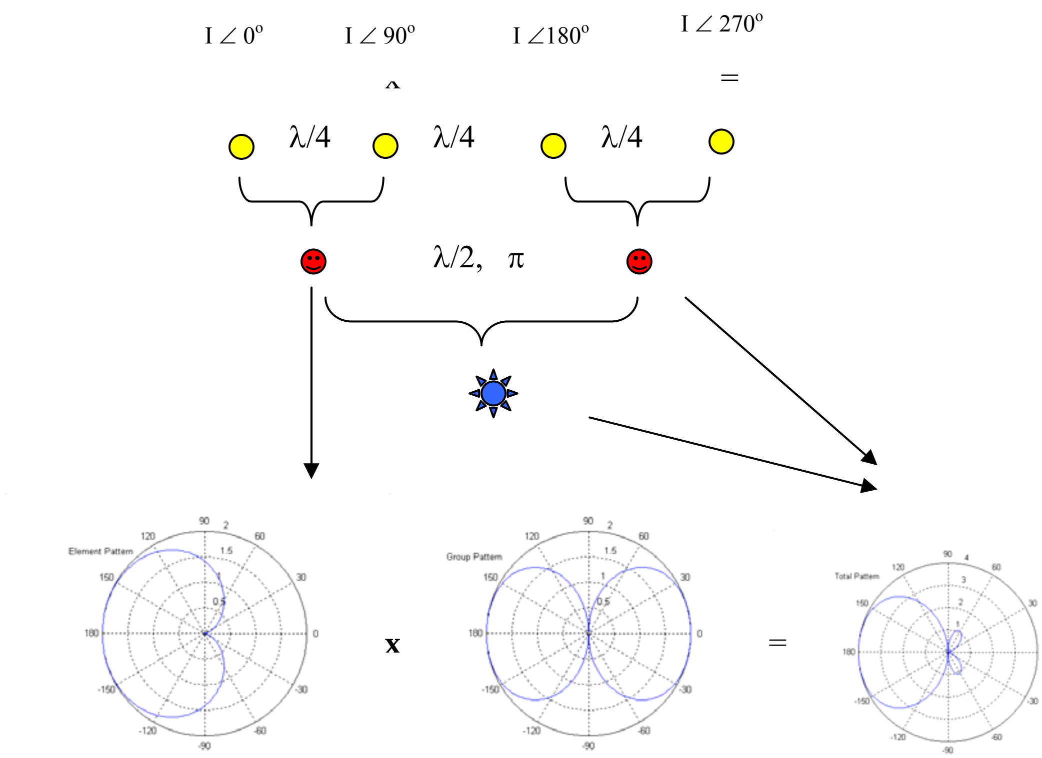
(a) The resultant pattern is obtained as follows.



(b) The array is replaced by by + $\angle 0^o$ $\frac{\lambda}{4}$ + $\angle \pi/2$

where + stands for $\cdot \longleftrightarrow \cdot$ ($\angle 0^o$, $\angle \pi$)

Thus the resultant pattern is obtained as shown.




**Prob. 13.41**

$$G_d(dB) = 20dB = 10\log_{10} G_d \longrightarrow G_d = 10^2 = 100$$

$$\lambda = \frac{c}{f} = \frac{3\times10^8}{10\times10^9} = 3\times10^{-2}$$

$$A_e = \frac{\lambda^2}{4\pi}G_d = \frac{9\times10^{-4}}{4\pi}100 = \underline{\underline{7.162\times10^{-2}\ \text{m}^2}}$$

**Prob. 13.42**

$$A_e = \frac{P_r}{P_{ave}} = \frac{P_r}{|E_r|^2 / 2\eta} = \frac{2\eta P_r}{|E_r|^2}$$

$$= \frac{2\times 120\pi \times 2\times 10^{-6}}{25\times 10^2 \times 10^{-6}} = \frac{48\pi}{250} = \underline{\underline{0.6031}} \text{ m}^2$$

**Prob. 13.43**

Friis equation states that

$$\frac{P_r}{P_t} = G_r G_t \left(\frac{\lambda}{4\pi r}\right)^2$$

$$\lambda = \frac{c}{f} = \frac{3\times 10^8}{200\times 10^6} = 1.5 \text{ m}, \qquad r = 238,857\times 1.609\times 10^3 = 3.843\times 10^8$$

$$G_t(dB) = 15dB = 10\log_{10} G_t \quad \longrightarrow \quad G_t = 10^{15/10} = 31.623$$

$$G_r = \left(\frac{4\pi r}{\lambda}\right)^2 \frac{P_r}{P_t} = \left(\frac{4\pi \times 3.843\times 10^8}{1.5}\right)^2 \frac{4\times 10^{-9}}{120\times 10^{-3}} = 34.55\times 10^{10}$$

$$G_r(dB) = 10\log_{10} G_r = 10\log_{10} 34.55\times 10^{10} = \underline{\underline{115.384 \text{ dB}}}$$

**Prob. 13.44**

Using Frii's equation,

$$P_r = G_r G_t \left[\frac{\lambda}{4\pi r}\right]^2 P_t$$

$$P_t = \left[\frac{4\pi r}{\lambda}\right]^2 \frac{P_r}{G_r G_t}$$

$$\lambda = \frac{c}{f} = \frac{3\times 10^8}{3\times 10^9} = 0.1, \qquad r = 42 \text{ km}$$

$$G_t(dB) = 10\log_{10} G_t = 25 \quad \longrightarrow \quad G_t = 10^{2.5} = 316.23$$

$$G_r(dB) = 10\log_{10} G_r = 20 \quad \longrightarrow \quad G_t = 10^2 = 100$$

$$P_t = \left(\frac{4\pi \times 42\times 10^3}{0.1}\right)^2 \frac{3\times 10^{-6}}{31623} = \underline{\underline{2.642 \text{ kW}}}$$

**Prob. 13.45**

$$G_{dt} = 10^4, G_{dr} = 10^{3.2} = 1585$$

$$\lambda = \frac{c}{f} = \frac{3\times 10^8}{15\times 10^9} = 0.02m = \frac{1}{50}$$

$$P_r = G_{dr}G_{dt}\left(\frac{\lambda}{4\pi r}\right)^2 P_t = 10^4(1585)\left(\frac{0.02}{4\pi\times 2.456741\times 10^7}\right)^2 320$$

$$= 2.129\times 10^{-11}\text{ W} = \underline{\underline{21.29\text{ pW}}}$$

**Prob. 13.46**

$$P_r = G_{dt}G_{dr}\left(\frac{\lambda}{4\pi r}\right)^2 P_t$$

$$\lambda = \frac{c}{f} = \frac{3\times 10^8}{20\times 10^6} = 15\text{ m},\qquad G_{dt} = G_{dr} = 1.64$$

$$P_t = \frac{P_r}{G_{dt}G_{dr}}\left(\frac{4\pi r}{\lambda}\right)^2 = 0.5\times 10^{-6}\left(\frac{4\pi\times 80\times 10^3}{1.64\times 15}\right)^2 = 835.025\text{ W}$$

But $\quad R_{rad} = \dfrac{2P_{rad}}{I_o^2} = 73 \quad\rightarrow\quad I_o^2 = \dfrac{2P_{rad}}{73} = \dfrac{2\times 835.025}{73} = 22.8774$

$$I_o = \underline{\underline{4.783\text{ A}}}$$

**Prob. 13.47**

$$30dB = \log\frac{P_t}{P_r} \rightarrow \frac{P_t}{P_r} = 10^3 = 1000$$

But $\quad P_r = (G_d)^2\left(\dfrac{3}{50\times 4\pi\times 12}\right)^2 P_t = P_t\left(\dfrac{G_d}{800\pi}\right)^2$

$$\left(\frac{G_d}{800\pi}\right)^2 = \frac{P_r}{P_t} = \frac{1}{1000} = \left(\frac{1}{10\sqrt{10}}\right)^2$$

or $\quad G_d = \dfrac{800\pi}{10\sqrt{10}} = 79.476$

$$G_d = 10\log 79.476 = \underline{\underline{19\text{ dB}}}$$

**Prob. 13.48**

(a) $P_i = \dfrac{|E|^2}{2\eta_o} = \dfrac{P_{rad}G_d}{4\pi r^2} \rightarrow |E_i| = \sqrt{\dfrac{240\pi P_{rad}G_d}{4\pi r^2}}$

$$|E_i| = \frac{1}{r}\sqrt{60P_{rad}G_d} = \frac{1}{120\times 10^3}\sqrt{60\times 200\times 10^3\times 3500}$$

$$= \underline{\underline{1708\ \text{V/m}}}$$

(b) $|E_s| = \sqrt{\dfrac{|E_i|^2\sigma}{4\pi r^2}} = \sqrt{\dfrac{1708^2\times 8}{4\pi\times 14400\times 10^6}} = \underline{\underline{11.36\ \mu\text{V/m}}}$

(c) $P_c = P_i\sigma = \dfrac{1708^2}{240\pi}(8) = \underline{\underline{30.95\ \text{mW}}}$

(d) $P_i = \dfrac{|E|^2}{2\eta_o} = \dfrac{(11.36)^2\times 10^{-12}}{240\pi} = 1712\times 10^{-13}\ \ \text{W/m}^2$

$$\lambda = \frac{3\times 10^8}{15\times 10^8} = 0.2m,\ A_{2r} = \frac{\lambda^2 G}{4\pi} = \frac{0.04\times 3500}{4\pi}$$

$$P_r = P_a A_{er} = 1712\times 10^{-13}\times 11.14 = 1.907\times 10^{-12}$$

or $P_r = \dfrac{(\lambda G_d)^2\sigma P_{rad}}{(4\pi)^3 r^4} = \dfrac{(0.2\times 3500)^2\times 8\times 2\times 10^5}{(4\pi)^3\times 12^4\times 10^{16}}$

$$= \underline{\underline{1.91\times 10^{-12}\ \text{W}}}$$

**Prob. 13.49**

$$P_r = \frac{(\lambda G_d)^2\sigma P_{rad}}{(4\pi)^3 r^4}$$

$$G_d(dB) = 30dB = 10\log_{10} G_d \longrightarrow G_d = 10^3$$

$$\lambda = \frac{c}{f} = \frac{3\times 10^8}{4\times 10^9} = 0.075\ \text{m}$$

$$P_r = \frac{(0.075\times 10^3)^2\times 12\times 80\times 10^3}{(4\pi)^3(10\times 10^3)^4} = \underline{\underline{272.1\ \text{pW}}}$$

**Prob. 13.50**

(a) $t_r = \dfrac{2d}{u} = \dfrac{2\times160\times10^3}{3\times10^8} = \dfrac{32}{3}\times10^{-4} = \underline{\underline{1.067\text{ ms}}}$

(b) $\lambda = c/f = \dfrac{3\times10^8}{4\times10^9} = 0.075m, \quad P_r = \dfrac{A_e\sigma G_d P_{rad}}{(4\pi r^2)^2}$

But $A_e = \dfrac{\lambda^2 G_d}{4\pi} \quad \rightarrow \quad G_d = \dfrac{4\pi A_e}{\lambda^2}$

$$P_r = 4\pi\sigma\left(\frac{A_e}{4\pi r^2\lambda}\right)^2 P_{rad} = 4\pi\times5\left(\frac{2}{0.075\times4\pi\times160^2\times10^6}\right)^2(60\times10^3) = \underline{\underline{2.59\times10^{-14}\text{ W}}}$$

(c) $$r = \left[\frac{\lambda^2 G_d^2\sigma}{(4\pi)^3}\frac{P_{rad}}{P_r}\right]^{1/4} = \left[\frac{\lambda^2\sigma}{(4\pi)^3}\frac{(4\pi)^2A_e^2}{\lambda^4}\frac{P_{rad}}{P_r}\right]^{1/4} = \left[\frac{A_e^2\sigma}{4\pi\lambda^2}\frac{P_{rad}}{P_r}\right]^{1/4}$$

$$= \left[\frac{5\times4}{4\pi(0.075)^2}\frac{60\times10^3}{8\times10^{-12}}\right]^{1/4} = \underline{\underline{38.167\text{ km}}}$$

**Prob. 13.51**

$P_r = \dfrac{kP_{rad}}{r^4} \quad \rightarrow \quad P_{rad} = \dfrac{r^4}{k}P_r$

If $R = 2r, \quad P_{rad}' = \dfrac{(2r)^4}{k}P_r = 16\dfrac{r^4}{k}P_r = 16P_{rad}$

i.e. the transmitted power must be increased $\underline{\underline{\text{16 times}}}$.

**Prob. 13.52**

$$P_{rad} = \frac{4\pi}{G_{dt}G_{dr}}\left(\frac{4\pi r_1 r_2}{\lambda}\right)^2\frac{P_r}{\sigma}$$

But $G_{dt} = 36dB = 10^{3.6} = 3981.1$

$G_{dr} = 20dB = 10^2 = 100$

$\lambda = \dfrac{c}{f} = \dfrac{3\times10^8}{5\times10^9} = 0.06$

$r_1 = 3km, \; r_2 = 5km$

$$P_{rad} = \frac{4\pi}{3981.1 \times 100}\left(\frac{4\pi \times 15 \times 10^{6}}{6 \times 10^{-2}}\right)^{2} \frac{8 \times 10^{-12}}{2.4}$$

$$= \underline{\underline{1.038 \text{ kW}}}$$

**Prob. 13.53**

(a)

$$F = \frac{\pi fL}{R} = \frac{\pi \times 300 \times 10^{6} \times 50 \times 10^{-9}}{20} = 2.356$$

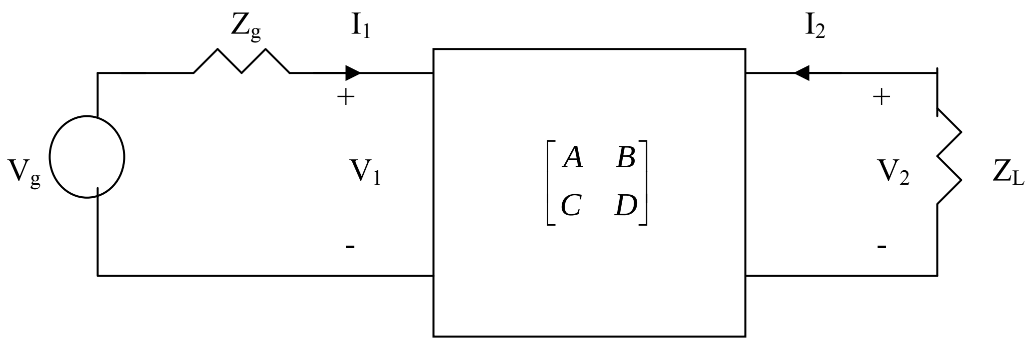
$$IL = 10\log_{10}(1+F^{2}) = 10\log_{10}(1+2.356^{2}) = \underline{\underline{8.164 \text{ dB}}}$$

(b)

$$F = \pi fRC = \pi \times 300 \times 10^{6} \times 10 \times 10^{3} \times 60 \times 10^{-12} = 180\pi = 565.5$$

$$IL = 10\log_{10}(1+F^{2}) = 10\log_{10}(1+565.5^{2}) = \underline{\underline{55.05 \text{ dB}}}$$

**Prob. 13.54**




By definition,

$$V_1 = AV_2 - BI_2 \qquad (1)$$

$$I_1 = CV_2 - DI_2 \qquad (2)$$

Let $V_2$ and $\overline{V_2}$ be respectively the load voltages when the filter circuit is present and when it is absent.

$$V_2 = -I_2 Z_L = \frac{I_1 Z_L}{CZ_L + D}$$

$$= \frac{V_g Z_L}{\left(Z_g + \frac{V_1}{I_1}\right)(CZ_L + D)} = \frac{V_g Z_L}{\left(Z_g + \frac{AV_2 - BI_2}{CV_2 - DI_2}\right)(CZ_L + D)}$$

$$= \frac{V_g Z_L}{\left(Z_g + \frac{AZ_L + B}{CZ_L + D}\right)(CZ_L + D)}$$

$$= \frac{V_g Z_L}{\left(Z_g (CZ_L + D) + AZ_L + B\right)}$$

$$\overline{V}_2 = \frac{V_g Z_L}{\left(Z_g + Z_L\right)}$$

Ratio and modulus give

$$\left|\frac{\overline{V}_2}{V_2}\right| = \frac{\left(Z_g (CZ_L + D) + AZ_L + B\right)}{Z_g + Z_L}$$

Insertion loss =

$$\text{IL} = 20\log_{10}\left|\frac{\overline{V}_2}{V_2}\right| = 20\log_{10}\left|\frac{\left(Z_g (CZ_L + D) + AZ_L + B\right)}{Z_g + Z_L}\right|$$

which is the required result

**Prob. 13.55**

(a) $R_{dc} = \frac{l}{\sigma S} = \frac{10^3}{0.96\times 10^{-4} \times 6.1\times 10^7} = \underline{\underline{17.1 \text{ m}\Omega \text{ /km}}}$

(b) $R_{ac} = \frac{l}{\delta w \sigma}$, $\quad \pi a^2 = 0.8 \times 1.2 = 0.96$ or $a = 0.5528$

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}} = \frac{1}{\sqrt{\pi \times 6\times 10^6 \times 4\pi \times 10^{-7} \times 6.1\times 10^7}} = \frac{1}{12.1\times \pi \times 10^3}$$

$$R_{ac} = \frac{1000\times 12.1\times \pi \times 10^3}{1.2\times 10^{-2} \times 6.1\times 10^7} = \underline{\underline{51.93 \ \Omega}}$$

**Prob. 13.56**

$$SE = 20\log_{10}\frac{E_i}{E_o} = 20\log_{10}\frac{6}{20\times10^{-6}} = 20\log_{10}(3\times10^5)$$

$$= \underline{\underline{109.54 \text{ dB}}}$$

# CHAPTER 14

**P. E. 14.1** The program in Fig. 14.3 was used to obtain the plot in Fig. 14.5.

**P. E. 14.2** For the exact solution,

$(D^2 + 1)\, y = 0 \longrightarrow y = A \cos x + B \sin x$

$y(0) = 0 \longrightarrow A = 0$

$y(1) = 1 \longrightarrow 1 = B \sin 1$ or $B = 1/\sin 1$

Thus, $y = \sin x/\sin 1$

For the finite difference solution,

$$y'' + y = 0 \longrightarrow \frac{y(x+\Delta) - 2y(x) + y(x-\Delta)}{\Delta^2} + y = 0$$

or

$$y(x) = \frac{y(x+\Delta) + y(x-\Delta)}{2-\Delta^2}, y(0) = 0, y(1) = 1, \Delta = 1/4$$

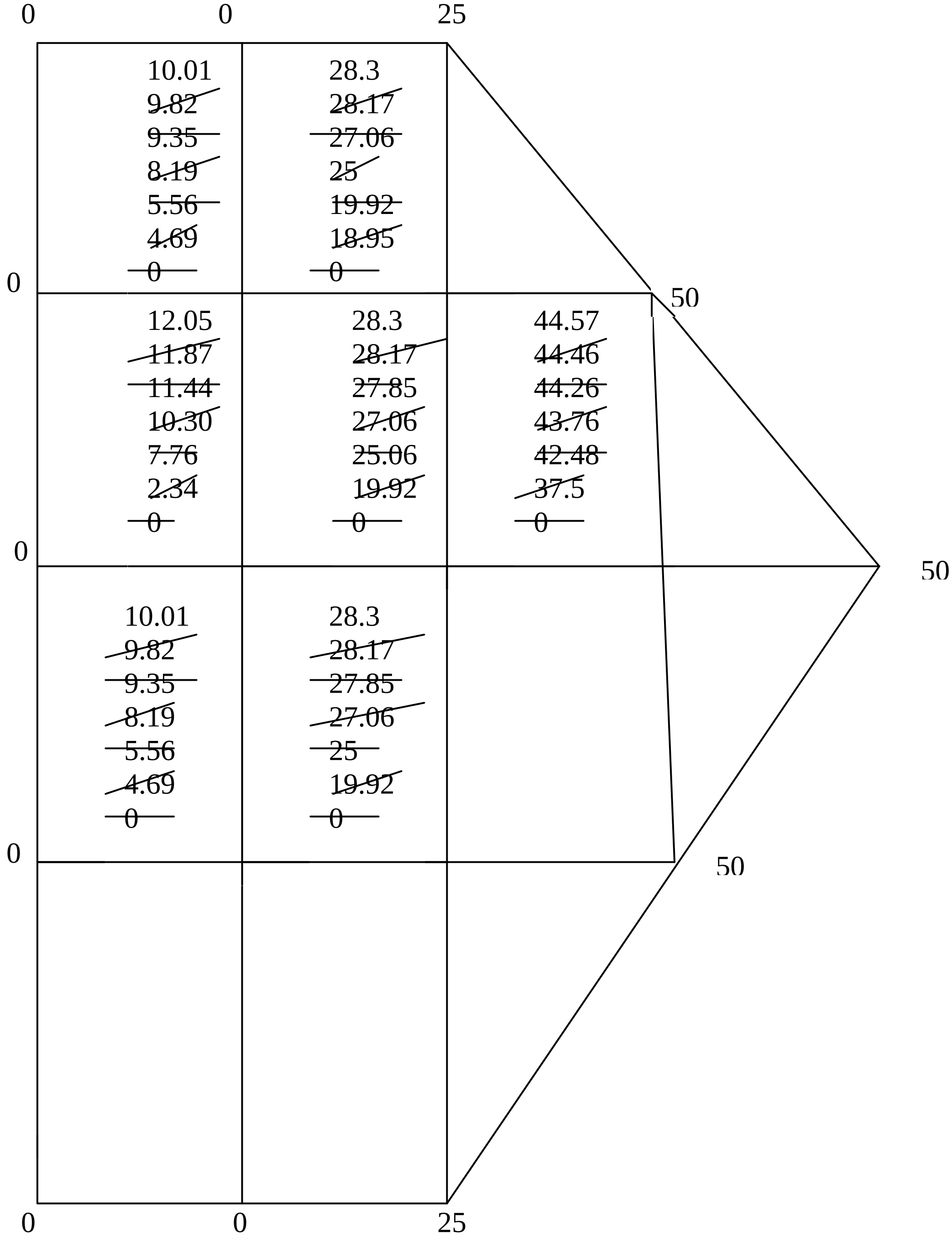
With the MATLAB program shown below, we obtain the exact result $y_e$ and FD result y.

```
y(1)=0.0;
y(5)=1.0;
del=0.25;
for  n=1:20
    for k=2:4
        y(k)=( y(k+1) +y(k-1) )/(2-del*del)
        x=(k-1)*del;
        ye=sin(x)/sin(1.0)
    end
end
```

The results are listed below.

| y(x) | N=5 | N=10 | N=15 | N=20 | Exact $y_e(x)$ |
|---|---|---|---|---|---|
| y(0.25) | 0.2498 | 0.2924 | 0.2942 | 0.2943 | 0.2940 |
| y(0.5) | 0.5242 | 0.5682 | 0.5701 | 0.5702 | 0.5697 |
| y(0.75) | 0.7867 | 0.8094 | 0.8104 | 0.8104 | 0.8101 |

**P. E. 14.3** By applying eq. (14.16) to each node as shown below, we obtain the following results after 5 iterations.




**P. E. 14.4** (a) Using the program in Fig. 14.16 with nx = 4+1=5 and ny = 8+1=9, we obtain the potential at center as

V(3,5) = 23.796 V

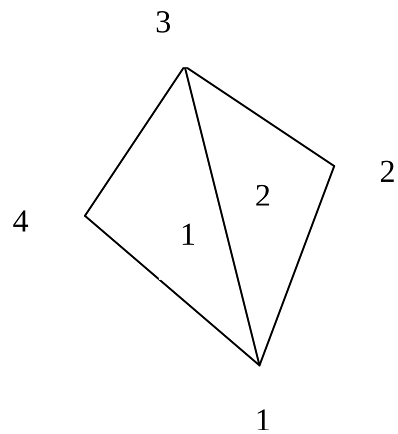
 

(b) Using the same program with nx = 12+1=13 and ny = 24+1=25, the potential at the center is

$$V(7,13) = \underline{\underline{23.883\text{ V}}}$$

**P. E. 14.5** By combining the ideas in Figs. 14.20 and 14.24, and dividing each wire into N segments, the results listed in Table 14.2 is obtained.

**P. E. 14.6**
(a)




For element 1, local 1-2-3 corresponds with global 1-3-4 so that $A_1 = 0.35$,

$P_1 = 0.8$, $P_2 = 0.6$, $P_3 = -1.4$, $Q_1 = -0.5$, $Q_2 = 0.5$, $Q_3 = 0$

$$C^{(1)} = \begin{bmatrix} 0.6357 & 0.1643 & -0.8 \\ 0.1643 & 0.4357 & -0.6 \\ -0.8 & -0.6 & 1.4 \end{bmatrix}$$

For element 2, local 1-2-3 corresponds with global 1-2-3 so that $A_2 = 0.7$,

$P_1 = 0.1$, $P_2 = 1.4$, $P_3 = -1.5$, $Q_1 = -1$, $Q_2 = 0$, $Q_3 = 1$

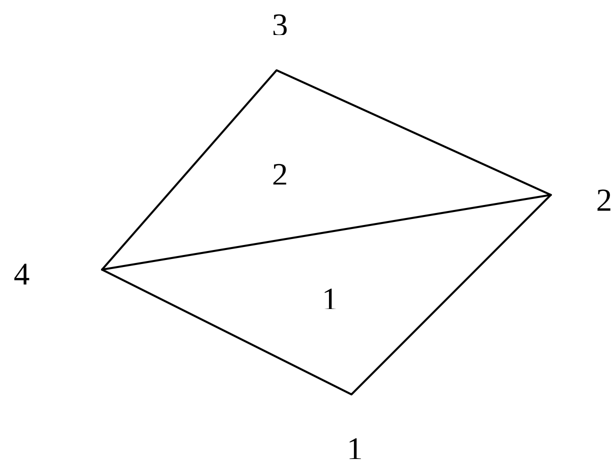
$$C^{(2)} = \begin{bmatrix} 0.3607 & 0.05 & -0.4107 \\ 0.05 & 0.7 & -0.75 \\ -0.4107 & -0.75 & 1.1607 \end{bmatrix}$$

The global coefficient matrix is given by

$$C = \begin{bmatrix} C^{(1)}_{11} + C_{11}^{(2)} & C_{12}^{(2)} & C_{12}^{(1)} + C_{13}^{(2)} & C_{13}^{(1)} \\ C_{21}^{(2)} & C_{22}^{(2)} & C_{23}^{(2)} & 0 \\ C_{21}^{(1)} + C_{31}^{(2)} & C_{32}^{(2)} & C_{22}^{(1)} + C_{33}^{(2)} & C_{23}^{(1)} \\ C_{31}^{(1)} & 0 & C_{32}^{(2)} & C_{33}^{(1)} \end{bmatrix}$$

$$= \begin{bmatrix} 0.9964 & 0.05 & -0.2464 & -0.8 \\ 0.05 & 0.7 & -0.75 & 0 \\ -0.2464 & -0.75 & 1.596 & -0.6 \\ -0.8 & 0 & -0.75 & 1.4 \end{bmatrix}$$

(b)




For element 1, local 1-2-3 corresponds with global 1-2-4 .
$P_1 =$ 0.9000 ; $P_2 =$ 0.6000 ; $P_3 =$ -1.5000
$Q_1 =$ -1.5000 ; $Q_2 =$ 0.5000; $Q_3 =$ 1;

$A_1 =$ 0.6750;

$$C^{(1)} = \begin{bmatrix} 1.1333 & -0.0778 & -1.0556 \\ -0.0778 & 0.2259 & -0.1481 \\ -1.0556 & -0.1481 & 1.2037 \end{bmatrix}$$

For element 2, local numbering 1-2-3 corresponds with global numbering 2-3-4.
$P_1 =$ 0.8000; $P_2 =$ -0.9000 ; $P_3 =$ 0.1000 ;
$Q_1 =$ -0.5000 ; $Q_2 =$ 1.5000 ; $Q_3 =$ -1 ;

$A_2 =$ 0.3750 ;

$$C^{(2)} = \begin{bmatrix} 0.5933 & -0.9800 & 0.3867 \\ -0.9800 & 2.0400 & -1.0600 \\ 0.3867 & -1.0600 & 0.6733 \end{bmatrix}$$

The global coefficient matrix is

$$C = \begin{bmatrix} C^{(1)}_{11} & C_{12}^{(1)} & 0 & C_{13}^{(1)} \\ C_{21}^{(1)} & C_{22}^{(1)} + C_{11}^{(2)} & C_{12}^{(2)} & C_{23}^{(1)} + C_{13}^{(2)} \\ 0 & C_{12}^{(2)} & C_{22}^{(2)} & C_{23}^{(2)} \\ C_{31}^{(1)} & C_{32}^{(1)} + C_{31}^{(2)} & C_{32}^{(2)} & C_{33}^{(1)} + C_{33}^{(2)} \end{bmatrix}$$

$$C = \begin{bmatrix} 1.1333 & -0.0778 & 0 & -1.0556 \\ -0.0778 & 0.8193 & -0.9800 & 0.2385 \\ 0 & -0.9800 & 2.0400 & -1.0600 \\ -1.0556 & 0.2385 & -1.0600 & 1.8770 \end{bmatrix}$$

**P. E. 14.7** We use the MATLAB program in Fig. 14.33. The input data for the region in Fig. 14.34 is as follows:

```
NE = 32;  ND =  26;  NP = 18;
NL = [ 1  2  4
     2  5  4
     2  3  5
     3  6  5
     4  5  9
     5  10  9
     5  6  10
     6  11  10
     7  8  12
      8  13  12
     8  9  13
     9  14  13
      9  10  14
     10  15  14
     10  11  15
     11  16  15
     12  13  17
      13  18  17
     13  14  18
```

```
    14  19  18
    14  15  19
    15  20  19
    15  16  20
    16  21  20
    17  18  22
    18  23  22
    18  19  23
    19   24  23
     19  20  24
     20  25  24
    20  21  25
    21  26  25];
X = [ 1.0 1.5 2.0 1.0 1.5 2.0 0.0  0.5 1.0 1.5  2.0 0.0 0.5 1.0 1.5 2.0 0.0 0.5  1.0 1.5
      2    0.0  0.5 1.0 1.5 2.0];
 Y = [ 0.0  0.0  0.0 0.5 0.5  0.5 1.0 1.0  1.0 1.0  1.0  1.5  1.5  1.5  1.5  1.5  2.0  2.0  2.0
   2.0  2.0  2.5  2.5  2.5  2.5  2.5  ];
 NDP = [ 1   2  3  6  11  16  21  26  25  24  23  22  17  12  7  8  9  4];
 VAL = [0.0  0.0  15.0  30.0  30.0  30.0  30.0  25.0  20.0  20.0  20.0 10.0  0.0  0.0 0.0
        0.0  0.0  0.0];
```

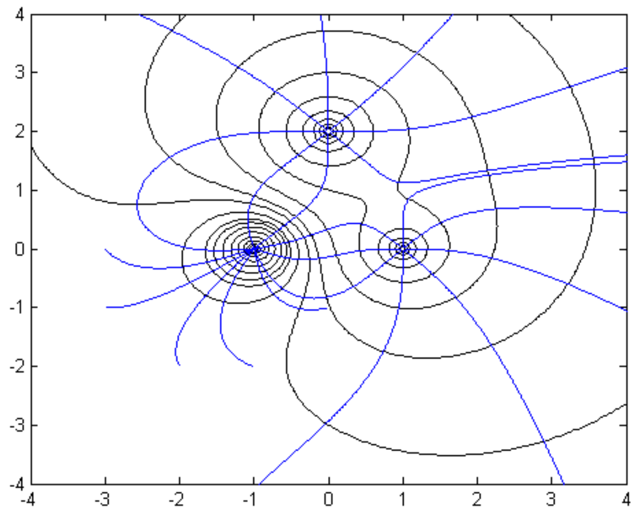
With this data, the finite element (FEM) solution is compared with the finite difference (FD) solution as shown in the table below.

| Node # | X | Y | FEM | FD |
|---|---|---|---|---|
| 5 | 1.5 | 0.5 | 11.265 | 11.25 |
| 10 | 1.5 | 1.0 | 15.06 | 15.02 |
| 13 | 0.5 | 1.5 | 4.958 | 4.705 |
| 14 | 1.0 | 1.5 | 9.788 | 9.545 |
| 15 | 1.0 | 1.5 | 18.97 | 18.84 |
| 18 | 0.5 | 2.0 | 10.04 | 9.659 |
| 19 | 1.0 | 2.0 | 15.32 | 15.85 |
| 20 | 1.5 | 2.0 | 21.05 | 20.87 |

**Prob. 14.1** (a) Using the Matlab code in Fig. 14.3, we input the data as:

>> plotit( [-1 2 1], [-1 0; 0 2; 1 0], 1, 1, 0.01, 0.01, 8, 2, 5 )

and the plot is shown below.

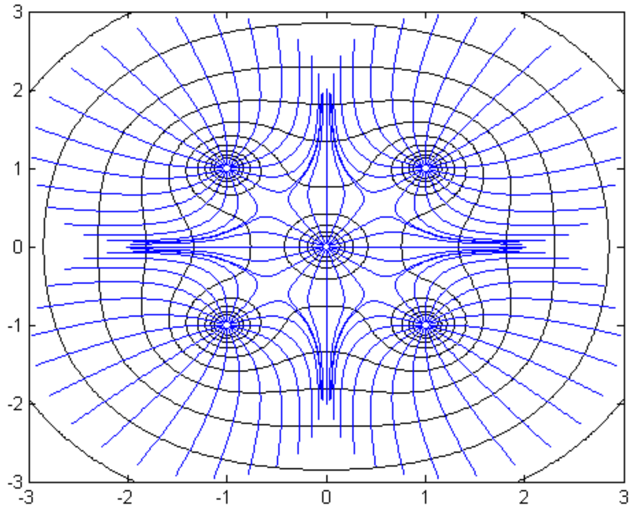



.

(b) Using the MATLAB code in Fig. 14.3, we input the required data as:

>> plotit( [1 1 1 1 1], [-1 -1; -1 1; 1 –1; 1 1; 0 0], 1, 1, 0.02, 0.01, 6, 2, 5 )

and obtain the plot shown below.




**Prob.14.2**

The exact solution is

$$V(x) = \frac{1}{6}x^3 + \frac{1}{2}x^2 + \frac{1}{3}x$$

so that V(0.5) = 0.3125. For the finite difference solution,

$$\frac{V(x+\Delta) - 2V(x) + V(x-\Delta)}{\Delta^2} = x + 1$$

which leads to

$$V(x) = \frac{1}{2}\left[V(x+\Delta) + V(x-\Delta) - \Delta^2(x+1)\right]$$

We apply this at x = 0.25, 0.5, 0.75 for 5 iterations as tabulated below.

| No. of iterations | V(0) | V(0.25) | V(0.5) | V(0.75) | V(1.0) |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 1 |
| 1 | 0 | -0.03916 | -0.0664 | 0.4121 | 1 |
| 2 | 0 | -0.07226 | 0.1232 | 0.5069 | 1 |
| 3 | 0 | 0.02254 | 0.2178 | 0.5542 | 1 |
| 4 | 0 | 0.06984 | 0.2651 | 0.5779 | 1 |
| 5 | 0 | 0.09349 | 0.2888 | 0.5897 | 1 |

From the table, V(0.5) = 0.2888 which is smaller than the exact value due to the fact that the number of iterations is not sufficiently large and also Δ= 0.25 is large.

**Prob. 14.3** (a)

$$\frac{dV}{dx} = \frac{V(x+\Delta x) - V(x - \Delta x)}{2\Delta x}$$

For $\Delta x = 0.05$ and at x = 0.15,

$$\frac{dV}{dx} = \frac{2.0134 - 1.00}{0.05 X 2} = \underline{\underline{10.117}}$$

$$\frac{d^2V}{dx^2} = \frac{V(x + \Delta x) - 2V(x) + V(x - \Delta x)}{(\Delta x)^2} = \frac{2.0134 + 1.0017 - 2x1.5056}{(0.05)^2} = \underline{\underline{1.56}}$$

(b) V = 10 sinh x, dV/dx = 10 cosh x. At x = 0.15, dV/dx = $\underline{\underline{10.113}}$

which is close to the numerical estimate.

$d^2V/dx^2$ = 10 sinh x. At x = 0.15, $d^2V/dx^2$ = $\underline{\underline{1.5056}}$

which is slightly lower than the numerical value.

**Prob. 14.4**

$$\nabla^2 V = \frac{\partial^2 V}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial V}{\partial \rho} + \frac{\partial^2 V}{\partial z^2} = 0$$

The equivalent finite difference expression is

$$\frac{V(\rho_o + \Delta\rho, z_o) - 2V(\rho_o, z_o) + V(\rho_o - \Delta\rho, z_o)}{(\Delta\rho)^2} + \frac{1}{\rho_o}\frac{V(\rho_o + \Delta\rho, z_o) - V(\rho_o - \Delta\rho, z_o)}{2\Delta\rho}$$

$$+ \frac{V(\rho_o, z_o + \Delta z) - 2V(\rho_o, z_o) + V(\rho_o, z_o - \Delta z)}{(\Delta z)^2} = 0$$

If $\Delta z = \Delta\rho = h,$ rearranging terms gives

$$V(\rho_o, z_o) = \frac{1}{4}V(\rho_o, z_o + h) + \frac{1}{4}V(\rho_o, z_o - h) + (1 + \frac{h}{2\rho_o})V(\rho + h, z_o)$$

$$+ (1 - \frac{h}{2\rho_o})V(\rho - h, z_o)$$

as expected.

**Prob. 14.5**

$$\nabla^2 V = \frac{\partial^2 V}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial V}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} = 0\,, \qquad (1)$$

$$\frac{\partial^2 V}{\partial \rho^2} = \frac{V_{m+1}{}^{n} - 2V_m{}^{n} + V_{m+1}{}^{n}}{(\Delta\rho)^2}\,, \qquad (2)$$

$$\frac{\partial^2 V}{\partial \phi^2} = \frac{V_m{}^{n+1} - 2V_m{}^{n} + V_m{}^{n-1}}{(\Delta\phi)^2}\,, \qquad (3)$$

$$\left.\frac{\partial V}{\partial \rho}\right|_{m,n} = \frac{V^{n}{}_{m+1} - V^{n}{}_{m-1}}{2\Delta\rho}\,. \qquad (4)$$

Substituting (2) to (4) into (1) gives

$$\nabla^2 V = \frac{V^{n}{}_{m+1} - V^{n}{}_{m-1}}{m\Delta\rho(2\Delta\rho)} + \frac{V_{m+1}{}^{n} - 2V_m{}^{n} + V_{m+1}{}^{n}}{(\Delta\rho)^2} + \frac{V_m{}^{n+1} - 2V_m{}^{n} + V_m{}^{n-1}}{(m\Delta\rho\Delta\phi)^2}$$

$$= \frac{1}{(\Delta\rho)^2}\left[(1-\frac{1}{2m})V_{m-1}{}^{n} - 2V_m{}^{n} + (1+\frac{1}{2m})V_{m-1}{}^{n} + \frac{1}{(m\Delta\phi)^2}(V_m{}^{n+1} - 2V_m{}^{n} + V_m{}^{n-1})\right]$$

as required.

**Prob. 14.6**

$$V_o = \frac{V_1 + V_2 + V_3 + V_4}{4} = \frac{10 - 40 + 50 + 80}{4} = \underline{\underline{25\text{V}}}$$

**Prob. 14.7**

| Iteration → | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $V_1$ | 0.0000 | 25.0000 | 35.6250 | 38.9063 | 39.7266 | 39.9316 |
| $V_2$ | 0.0000 | 26.2500 | 32.8125 | 34.4531 | 34.8633 | 34.9658 |
| $V_3$ | 0.0000 | 16.2500 | 22.8125 | 24.4531 | 24.8633 | 24.9658 |
| $V_4$ | 0.0000 | 15.6250 | 18.9063 | 19.7266 | 19.9316 | 19.9829 |

**Prob. 14.8**

$$V_a = \frac{1}{4}\left[0+100+100+V_b\right] = \frac{1}{4}(V_b + 200) \qquad (1)$$

$$V_b = \frac{1}{4}\left[0+0+V_a+V_c\right] = \frac{1}{4}(V_a + V_c) \qquad (2)$$

$$V_c = \frac{1}{4}\left[V_b+100+100+0\right] = \frac{1}{4}(200 + V_b) \qquad (3)$$

Using these relationships, we obtain the data in the table below.

| Iteration | 1st | 2nd | 3rd | 4th | 5th |
|---|---|---|---|---|---|
| $V_a$ | 50 | 53.115 | 56.641 | 57.08 | 57.135 |
| $V_b$ | 12.5 | 26.56 | 28.32 | 28.54 | 28.57 |
| $V_c$ | 53.125 | 56.64 | 57.08 | 57.135 | 57.142 |

Alternatively, we can solve (1) to (3) simultaneously.

From (1) and (3), $V_a = V_c$

From (2), $V_b = \dfrac{V_a}{2}$

Thus (1) becomes $V_a = \dfrac{1}{4}\left(\dfrac{V_a}{2} + 200\right) \rightarrow V_a = 400/7 = 51.143 = V_c$

$$V_b = \frac{V_a}{2} = 28.57$$

**Prob. 14.9**

(a) We follow Example 6.5 with a=b.

$$V = V_1 + V_2 = \underline{\underline{\frac{4V_o}{\pi}\sum_{n=odd}^{\infty}\frac{\sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi y}{a}\right)}{n\sinh(n\pi)} + \frac{4V_o}{\pi}\sum_{n=odd}^{\infty}\frac{\sin\left(\frac{n\pi y}{a}\right)\sinh\left(\frac{n\pi x}{a}\right)}{n\sinh(n\pi)}}}$$

(b) At the center of the region, finite difference gives

$$V(a/2, a/2) = \frac{1}{4}(0 + 0 + V_o + V_o) = \frac{V_o}{2} = \underline{\underline{25 \text{ V}}}$$

**Prob. 14.10**

$$k = \frac{h^2 \rho_s}{\varepsilon} = 10^{-4} \times \frac{50 \times 10^{-9}}{\frac{10^{-9}}{36\pi}} = 0.18\pi = 0.5655$$

At node 1,

$$V_1 = \frac{1}{4}[0 + V_2 + V_3 + k] \quad \longrightarrow \quad 4V_1 - V_2 - V_3 = k \qquad (1)$$

At node 2,

$$V_2 = \frac{1}{4}[0 + V_1 + V_4 + k] \quad \longrightarrow \quad 4V_2 - V_1 - V_4 = k \qquad (2)$$

At node 3,

$$V_3 = \frac{1}{4}[0 + 2V_1 + V_4 + k] \quad \longrightarrow \quad 4V_3 - 2V_1 - V_4 = k \qquad (3)$$

At nde 4,

$$V_4 = \frac{1}{4}[0 + 2V_2 + V_3 + k] \quad \longrightarrow \quad 4V_4 - 2V_2 - V_3 = k \qquad (4)$$

Putting (1) to (4) in matrix form,

$$\begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -2 & 0 & 4 & -1 \\ 0 & -2 & -1 & 4 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} 0.5655 \\ 0.5655 \\ 0.5655 \\ 0.5655 \end{bmatrix}$$

Using a calculator or MATLAB, we obtain

$$\underline{\underline{V_1 = V_2 = 0.3231 \text{ V}, \quad V_3 = V_4 = 0.4039 \text{ V}}}$$

**Prob. 14.11**

(a)

$$\underbrace{\begin{bmatrix} -4 & 1 & 0 & 1 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & 1 & -4 & 0 & 0 & 1 \\ 1 & 0 & 0 & -4 & 1 & 0 \\ 0 & 1 & 0 & 1 & -4 & 1 \\ 0 & 0 & 1 & 0 & 1 & -4 \end{bmatrix}}_{[A]} \begin{bmatrix} V_a \\ V_b \\ V_c \\ V_d \\ V_e \\ V_f \end{bmatrix} = \underbrace{\begin{bmatrix} -200 \\ -100 \\ -100 \\ -100 \\ 0 \\ 0 \end{bmatrix}}_{[B]}$$

(b)

$$\begin{bmatrix} -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -4 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -4 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & -4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ V_7 \\ V_8 \end{bmatrix} = \begin{bmatrix} -30 \\ -15 \\ -30 \\ -7.5 \\ 0 \\ -7.5 \\ 0 \\ 0 \end{bmatrix}$$

[A] [B]

**Prob. 14.12** (a) Matrix [A] remains the same. To each term of matrix [B], we add

$-h^2 \rho_v / \varepsilon$ .

(b) Let $\Delta x = \Delta y = h = 0.25$ so that nx = 5= ny.

$$\frac{\rho_v}{\varepsilon} = \frac{x(y-1)10^{-9}}{10^{-9}/36\pi} = 36\pi x(y-1)$$

Modify the program in Fig. 14.16 as follows.

```
    H=0.25;
  for   I=1:nx –1
    for   J=1: ny-1
    X = H*I;
    Y=H*J;
    RO = 36.0*pi*X*(Y-1);
    V(I,J) = 0.25*( V(I+1,J) + V(I-1,J) + V(I,J+1) + V(I,J-1) + H*H*RO );
      end
   end
```

This is the major change. However, in addition to this, we must set

```
v1 = 0.0;
v2 = 10.0;
v3 = 20.0;
v4 = -10.0;
nx = 5;
ny = 5;
```

The results are:

$V_a$ = 4.6095 $V_b$= 9.9440 $V_c$= 11.6577
$V_d$ = -1.5061 $V_e$ =3.5090 $V_f$= 6.6867
$V_g$= -3.2592 $V_h$ = 0.2366 $V_i$ = 3.3472

**Prob. 14.13**

$$V_1 = \frac{1}{4}(0+0+V_2+V_4) = \frac{1}{4}(V_2+V_4)$$

$$V_2 = \frac{1}{4}(0+50+V_1+V_3) = \frac{1}{4}(50+V_1+V_3)$$

$$V_3 = \frac{1}{4}(0+100+50+V_2) = \frac{1}{4}(150+V_2)$$

$$V_4 = \frac{1}{4}(0+50+V_1+V_5) = \frac{1}{4}(50+V_1+V_5)$$

$$V_5 = \frac{1}{4}(0+0+V_4+V_6) = \frac{1}{4}(V_4+V_6)$$

$$V_6 = \frac{1}{4}(0+50+V_5+V_7) = \frac{1}{4}(50+V_5+V_7)$$

$$V_7 = \frac{1}{4}(0+100+V_6+50) = \frac{1}{4}(150+V_6)$$

Initially set all free potentials equal to zero. Apply the seven formulas above iteratively and obtain the results shown below.

| n | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $V_1$ | 0 | 6.25 | 9.77 | 10.63 | 10.97 |
| $V_2$ | 12.5 | 24.22 | 25.83 | 26.15 | 26.25 |
| $V_3$ | 40.625 | 43.55 | 43.96 | 44.04 | 44.06 |
| $V_4$ | 12.5 | 14.84 | 16.70 | 17.73 | 17.97 |
| $V_5$ | 3.12 | 7.03 | 10.29 | 10.93 | 11.05 |
| $V_6$ | 13.281 | 24.46 | 25.98 | 26.23 | 26.28 |
| $V_7$ | 40.82 | 43.62 | 43.99 | 44.06 | 44.07 |

**Prob. 14.14**

$$\frac{1}{c^2}\frac{\Phi^{j+1}_{m,n} + \Phi^{j-1}_{m,n} - 2\Phi^{j}_{m,n}}{(\Delta t)^2} = \frac{\Phi^{j}_{m+1,n} + \Phi^{j}_{m-1,n} - 2\Phi^{j}_{m,n}}{(\Delta x)^2}$$

$$+\frac{\Phi^{j}_{m,n+1} + \Phi^{j}_{m,n-1} - 2\Phi^{j}_{m,n}}{(\Delta z)^2}$$

If $h = \Delta x = \Delta z$, then after rearranging we obtain

$$\Phi^{j+1}{}_{m,n} = 2\Phi^{j}{}_{m,n} - \Phi^{j-1}{}_{m,n} + \alpha(\Phi^{j}{}_{m+1,n} + \Phi^{j}{}_{m-1,n} - 2\Phi^{j}{}_{m,n})$$
$$+\alpha(\Phi^{j}{}_{m,n+1} + \Phi^{j}{}_{m,n-1} - 2\Phi^{j}{}_{m,n})$$

where $\alpha = (c\Delta t / h)^2$.

**Prob. 14.15**

$$\frac{\partial^2 V}{\partial x^2} = \frac{\partial^2 V}{\partial t^2} \quad \longrightarrow \quad \frac{V(x+\Delta x,t) - 2V(x,t) + V(x-\Delta x,t)}{(\Delta x)^2} =$$
$$\frac{V(x,t+\Delta t) - 2V(x,t) + V(x,t-\Delta t)}{(\Delta t)^2}$$

$$V(x,t+\Delta t) = \left(\frac{\Delta t}{\Delta x}\right)^2 \left[V(x+\Delta x,t) - 2V(x,t) + V(x-\Delta x,t\right] + 2V(x,t) - V(x,t-\Delta t)$$

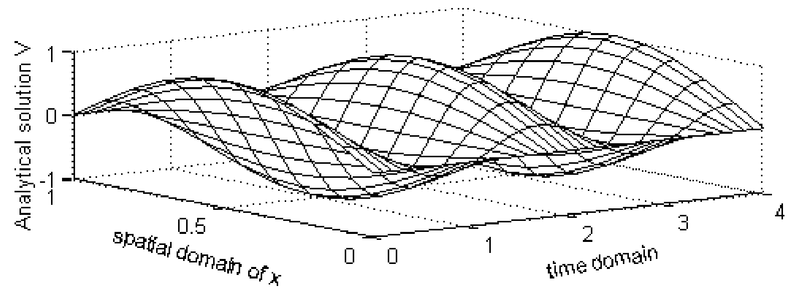
or

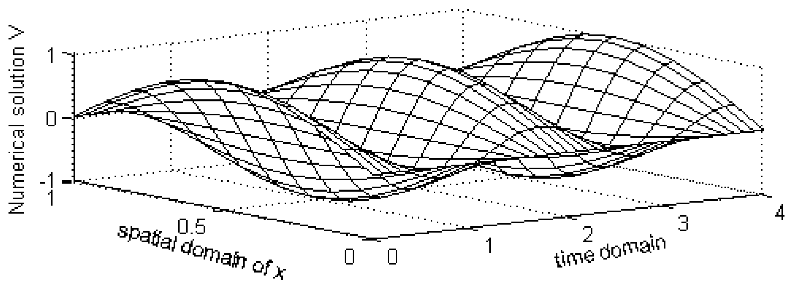
$$V(i,\, j+1) = \alpha\left[V(i+1,\, j) + v(i-1,\, j)\right] + 2(1-\alpha)V(i,\, j) - V(i,\, j-1)$$

where $\alpha = \left(\frac{\Delta t}{\Delta x}\right)^2$. Applying the finite difference formula derived above, the following programs was developed.

```
xd=0:.1:1;td=0:.1:4;
[t,x]=meshgrid(td,xd);
Va=sin(pi*x).*cos(pi*t);%Analytical result
subplot(211) ;mesh(td,xd,Va);colormap([0 0 0])
% Numerical result
N=length(xd);M=length(td);
v(:,1)=sin(pi*xd');
v(2:N-1,2)=(v(1:N-2,1)+v(3:N,1))/2;
for k=2:M-1
    v(2:N-1,k+1)=-v(2:N-1,k-1)+v(1:N-2,k)+v(3:N,k);
end
subplot(212);mesh(td,xd,v);colormap([0 0 0])
```

The results of the finite difference algorithm agree perfectly with the exact solution as shown below.



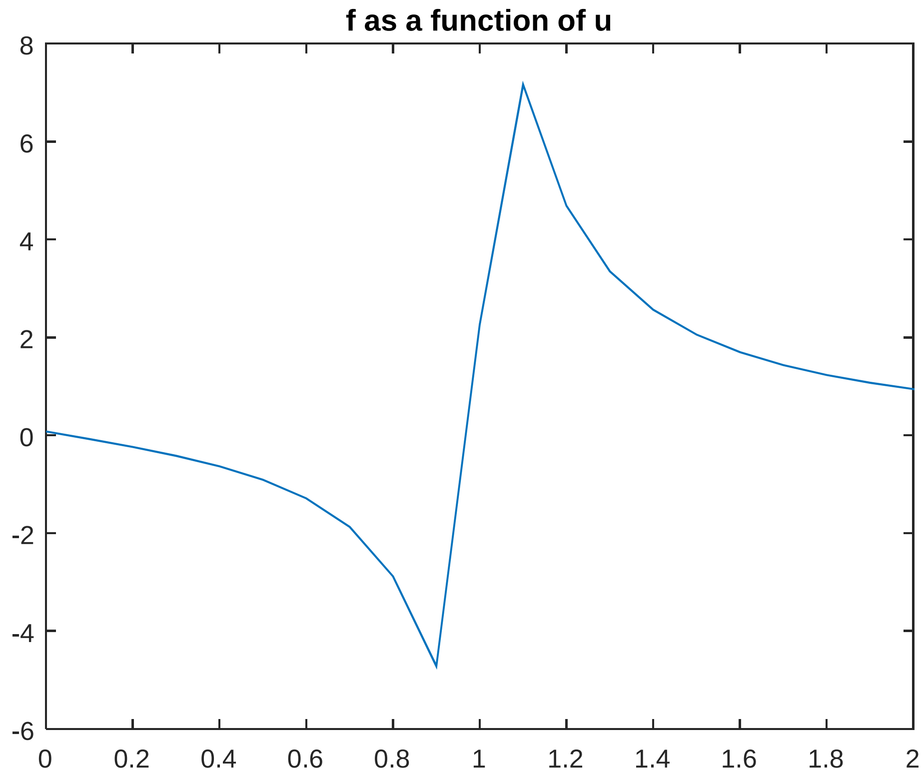




**Prob. 14.16**

The MATLAB code and the plot of F(u) are presented below.

```
% Integration using MATLAB

N=40;
del=pi/N;
for k=1:21
    u(k)=0.1*(k-1);
    sum=0.0;
   for n=1:N
       theta=del*n;
       num= u(k)- cos(theta)
    den=( 1 +u(k)^2 - 2*u(k)*cos(theta) )^1.5;
    term = num/den;
    sum=sum + term;
   end
   f(k) = sum*del;
end
plot(u,f)
title('f as a function of u')
```




**Prob. 14.17** Combining the ideas in the programs in Figs. 14.20 and 14.24, we develop a MATLAB code which gives

N = 20 ⟶ C = 19.4 pF/m

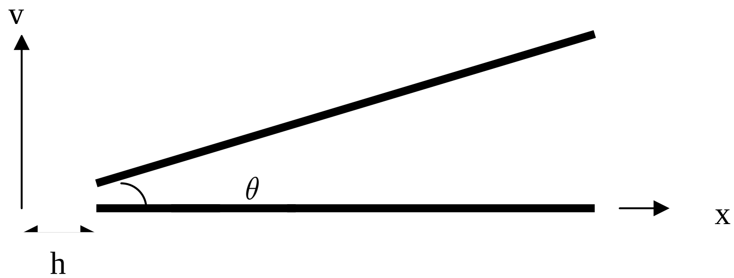
N = 40 ⟶ C = 13.55 pF/m

N = 100 ⟶ C = 12.77 pF/m

For the exact value, d/2a = 50/10 = 5

$$C = \frac{\pi\varepsilon}{\cosh^{-1}\dfrac{d}{2a}} = \frac{\pi \times 10^{-9}/36\pi}{\cosh^{-1}5} = \underline{\underline{12.12}}\ \text{pF/m}$$

**Prob. 14.18**




To find C, take the following steps:

(1)Divide each line into N equal segments. Number the segments in the lower conductor as 1, 2, …, N and segments in the upper conductor as N+1, N+2, …, 2N,

(2) Determine the coordinate ($x_k$, $y_k$) for the center of each segment.

For the lower conductor, $y_k = 0$, k=1, …, N, $x_k = h + \Delta (k-1/2)$, k = 1,2,… N

For the upper conductor, $y_k = [h + \Delta (k-1/2)] \sin\theta$, k=N+1, N+2, …,2 N,

$x_k = [h + \Delta (k-1/2)] \cos\theta$, k = N+1,N+2,… 2N

where h is determined from the gap g as

$$h = \frac{g}{2\sin\theta/2}$$

(3)Calculate the matrices [V] and [A] with the following elements

$$V_k = \begin{cases} V_o, k = 1,\ldots,N \\ -V_o, k = N+1,\ldots 2N \end{cases}$$

$$A_{ij} = \begin{cases} \dfrac{\Delta}{4\pi\varepsilon R_{ij}}, i \neq j \\ 2\ln\Delta/a, i = j \end{cases}$$

where $R_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$

(4) Invert matrix [A] and find $[\rho] = [A]^{-1} [V]$.

(5) Find the charge Q on one conductor

$$Q = \sum \rho_k \Delta = \Delta \sum_{k=1}^{N} \rho_k$$

(6) Find C = |Q|/2$V_o$

Taking N= 10, $V_o$ = 1.0, a program was developed to obtain the following result.

| $\theta$ | C (in pF) |
|---|---|
| 10 | 8.5483 |
| 20 | 9.0677 |
| 30 | 8.893 |
| 40 | 8.606 |
| 50 | 13.004 |
| 60 | 8.5505 |
| 70 | 9.3711 |
| 80 | 8.7762 |
| 90 | 8.665 |
| 100 | 8.665 |
| 110 | 10.179 |
| 120 | 8.544 |
| 130 | 9.892 |
| 140 | 8.7449 |
| 150 | 9.5106 |
| 160 | 8.5488 |
| 170 | 11.32 |
| 180 | 8.6278 |

**Prob. 14.19** We may modify the program in Fig. 14.24 and obtain the result in the table below. $Z_o \cong \underline{\underline{100\ \Omega}}$.

| N | $Z_o$, in Ω |
|---|---|
| 10 | 97.2351 |
| 20 | 97.8277 |
| 30 | 98.0515 |
| 40 | 98.1739 |
| 50 | 98.2524 |

**Prob. 14.20**

We make use of the formulas in Problem 14.19.

$$V_i = \sum_{j=1}^{2N} A_{ij} \rho_i$$

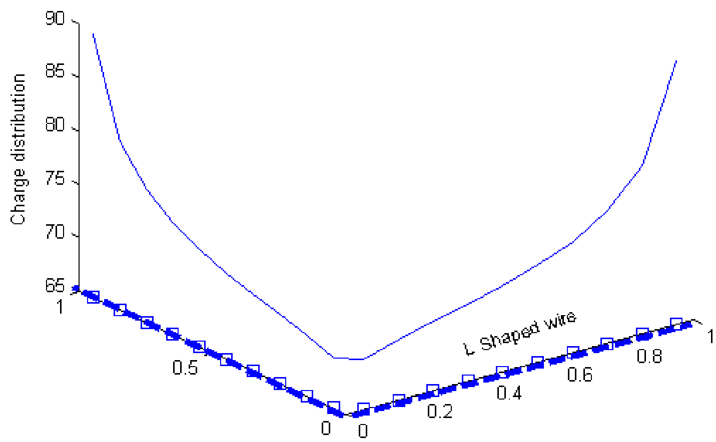
where N is the number of divisions on each arm of the conductor.

The MATLAB code is as follows:

```
aa=0.001;
vo=10;
eo = (10^(-9))/(36*pi);
L=2.0;
N=10;  %no.of divisions on each arm
NT=N*2;
delta=L/(NT);
x=zeros(NT,1);
y=zeros(NT,1);
%Second calculate the elements of the coefficient matrix
for i=1:N-1
    y(i)=0;
    x(i)=delta*(i-0.5)
end
for i=N+1:NT
    x(i)=0;
    y(i)=delta*(i-N-0.5);
end
for  i=1:NT
    for j=1:NT
        if (i ~=j)
            R=sqrt( (x(i)-x(j))^2 + (y(i)-y(j))^2 )
        A(i,j)=-delta*R;
      else
        A(i,j)=-delta*(log(delta)-1.5);
       end
    end
end
%Determine the matrix of constant vector B and find rho
B=2*pi*eo*vo*ones(NT,1);
rho=inv(A)*B;
```

The result is presented below.




| Segment | x | y | ρ in pC/m |
|---|---|---|---|
| **1** | 0.9500 | | 89.6711 |
| **2** | 0.8500 | 0 | 80.7171 |
| **3** | 0.7500 | 0 | 77.3794 |
| **4** | 0.6500 | 0 | 75.4209 |
| **5** | 0.5500 | 0 | 74.0605 |
| **6** | 0.4500 | 0 | 73.0192 |
| **7** | 0.3500 | 0 | 72.1641 |
| **8** | 0.2500 | 0 | 71.4150 |
| **9** | 0.1500 | 0 | 70.6816 |
| **10** | 0.0500 | 0 | 69.6949 |
| **11** | 0 | 0 | 69.6949 |
| **12** | 0 | 0.0500 | 70.6816 |
| **13** | 0 | 0.1500 | 71.4150 |
| **14** | 0 | 0.2500 | 72.1641 |
| **15** | 0 | 0.3500 | 73.0192 |
| **16** | 0 | 0.4500 | 74.0605 |
| **17** | 0 | 0.5500 | 75.4209 |
| **18** | 0 | 0.6500 | 77.3794 |
| **19** | 0 | 0.7500 | 80.7171 |
| **20** | 0 | 0.8500 | 89.6711 |

**Prob. 14.21**(a) Exact solution yields

$C = 2\pi\varepsilon / \ln(\Delta / a) = 8.02607 \times 10^{-11}$ F/m and $\underline{\underline{Z_o = 41.559\Omega}}$

where a = 1cm and Δ = 2cm. The numerical solution is shown below.

| N | C (pF/m) | $Z_o (\Omega)$ |
|---|---|---|
| 10 | 82.386 | 40.486 |
| 20 | 80.966 | 41.197 |
| 40 | 80.438 | 41.467 |
| 100 | 80.025 | 41.562 |

(b)For this case, the numerical solution is shown below.

| N | C (pF/m) | $Z_o (\Omega)$ |
|---|---|---|
| 10 | 109.51 | 30.458 |
| 20 | 108.71 | 30.681 |
| 40 | 108.27 | 30.807 |
| 100 | 107.93 | 30.905 |

**Prob. 14.22 We** modify the MATLAB code in Fig. 14.24 (for Example 14.5) by changing the input data and matrices [A] and [B]. We let

$x_i = h + \Delta (i-1/2)$, $i = 1,2,\dots N$, $\Delta = L/N$

$y_i = h/2$, $j = 1,2,\dots N$, $z_k = t/2$, $k = 1,2,\dots N$

and calculate

$$R_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2}$$

We obtain matrices [A] and [B]. Inverting [A] gives

$[q] = [A]^{-1} [B]$, $[\rho_v] = [q]/(ht\Delta)$, $C = \dfrac{\sum_{i=1}^{N} q_i}{10}$

The computed values of $[\rho_v]$ and C are shown below.

| I | $\rho_{vi}(\times 10^{-6})C/m^3$ |
|---|---|
| 1, 20 | 0.5104 |
| 2, 19 | 0.4524 |
| 3, 18 | 0.4324 |
| 4, 17 | 0.4215 |
| 5, 16 | 0.4144 |
| 6, 15 | 0.4096 |
| 7, 14 | 0.4063 |
| 8, 13 | 0.4041 |
| 9, 12 | 0.4027 |
| 10,11 | 0.4020 |

C = 17.02 pF

**Prob. 14.23** From the given figure, we obtain

$$\alpha_1 = \frac{A_1}{A} = \frac{1}{2A}\begin{vmatrix} 1 & x & y \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = \frac{1}{2A}[(x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y]$$

as expected. The same applies for $\alpha_2$ and $\alpha_3$.

**Prob. 14.24**

**(a)** $P_1 = 1.5, P_2 = 0.5, P_3 = -2, Q_1 = -1, Q_2 = 1.5, Q_3 = -0.5$

$$A = \frac{1}{2}(P_2Q_3 - P_3Q_2) = 1.375$$

$$C_{ij} = \frac{1}{4A}[P_iP_j + Q_iQ_j]$$

$$C = \begin{bmatrix} 0.5909 & -0.1364 & -0.4545 \\ -0.1364 & 0.4545 & -0.3182 \\ -0.4545 & -0.3182 & 0.7727 \end{bmatrix}$$

(b)

$P_1 = -4, P_2 = 4, P_3 = 0, Q_1 = 0, Q_2 = -3, Q_3 = 3$

$$A = \frac{1}{2}(P_2Q_3 - P_3Q_2) = 6$$

$$C = \begin{bmatrix} 0.6667 & -0.6667 & 0 \\ -0.6667 & 1.042 & -0.375 \\ 0 & -0.375 & 0.375 \end{bmatrix}$$

**Prob. 14.25** (a)

$$2A = \begin{vmatrix} 1 & 1/2 & 1/2 \\ 1 & 3 & 1/2 \\ 1 & 2 & 2 \end{vmatrix} = 15/4$$

$$\alpha_1 = \frac{4}{15}[(6-1)+(-1\frac{1}{2})x+(-1)y] = \frac{4}{15}(5-1.5x-y)$$

$$\alpha_2 = \frac{4}{15}[(1-1)+\frac{3}{2}x-\frac{3}{2}y] = \frac{4}{15}(1.5x-1.5y)$$

$$\alpha_3 = \frac{4}{15}[(1/4-3/2)+0x+\frac{5}{2}y] = \frac{4}{15}(-1.25+2.5y)$$

$V = \alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3$

Substituting V=80, $V_1$ = 100, $V_2$ = 50, $V_3$ = 30, $\alpha_1$, $\alpha_2$, and $\alpha_3$ leads to

20 = 7.5x + 10y + 3.75

Along side 12, y=1/2 so that

20 = 15x/2 + 5 + 15/4 $\longrightarrow$ x=3/2, i.e (1.5, 0.5)

Along side 13, x =y

20 = 15x/2 + 10x + 15/4 $\longrightarrow$ x=13/4, i.e. (13/14, 13/14)
Along side 23, y = -3x/2 + 5

20 = 15x/2 – 15 + 50 + 15/4 $\longrightarrow$ x=-5/2 (not possible)

Hence intersection occurs at

(1.5, 0.5) along 12 and (0.9286, 0.9286) along 13

(b) At (2,1),

$$\alpha_1 = \frac{4}{15}, \quad \alpha_2 = \frac{6}{15}, \quad \alpha_3 = \frac{5}{15}$$

$V(2,1) = \alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3 =$ (400 + 300 + 150)/15 = 56.67 V

**Prob. 14.26**

$$2A = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 1 & 4 \end{vmatrix} = 9$$

$$\alpha_1 = \frac{1}{9}[(0-0)+(4-0)x+(0-1)y] = \frac{1}{9}(4x-y)$$

$$\alpha_2 = \frac{1}{9}[(0-0)+(0+1)x+(2-0)y] = \frac{1}{9}(x+2y)$$

$$\alpha_3 = \frac{1}{9}[(8+1)+(-1-4)x+(1-2)y] = \frac{1}{9}(9-5x-y)$$

$$V_e = \alpha_1 V_{e1} + \alpha_2 V_{e2} + \alpha_{31} V_{e3}$$

V(1,2) = 8(4-2)/9 + 12(1+4)/9 + 10(9-5-1)/9 = 96/9 = 10.667 V

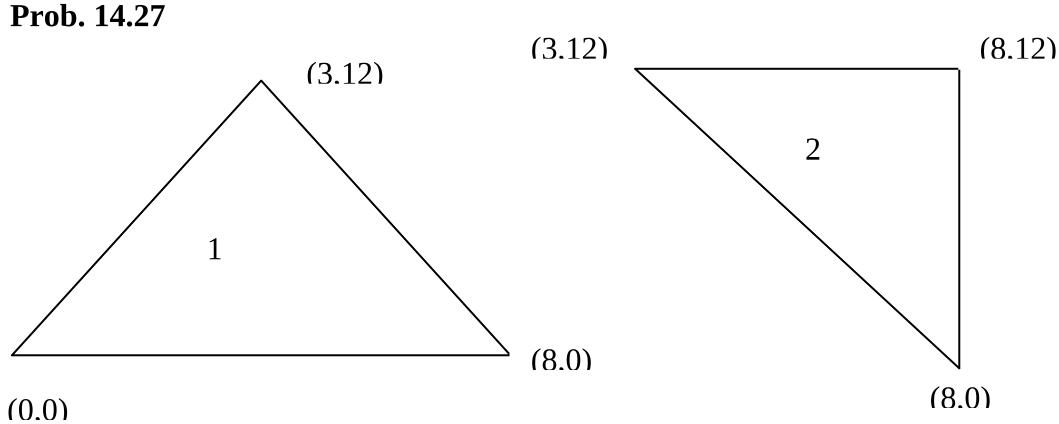
At the center $\alpha_1 = \alpha_2 = \alpha_3 = 1/3$ so that

V(center) = (8 + 12 + 10)/3 = 10

Or at the center, (x, y) = (0 + 1 + 2, 0 + 4 –1)/3 = (1,1)

V(1,1) = 8(3)/9 + 12(3)/9 + 10(3)/9 = 10 V

**Prob. 14.27**




For element 1, local numbering 1-2-3 corresponds to global numbering 4-2-1.

$P_1 = 12$, $P_2 = 0$, $P_3 = -12$, $Q_1 = -3$, $Q_2 = 8$, $Q_3 = -5$,

A = (0 + 12 x 8)/2 = 48

$$C_{ij} = \frac{1}{4x48}[P_j P_i + Q_j Q_j]$$

$$C^{(1)} = \begin{bmatrix} 0.7969 & -0.125 & -0.6719 \\ -0.125 & 0.3333 & -0.2083 \\ -0.6719 & -0.2083 & 0.8802 \end{bmatrix}$$

For element 2, local numbering 1-2-3 corresponds to global numbering 2-4-3.

$P_1$ = -12, $P_2$ = 0, $P_3$ = 12, $Q_1$ = 0, $Q_2$ = -5, $Q_3$ = 5,

A = (0 + 60)/2 = 30

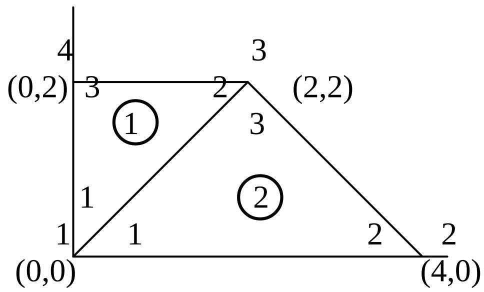
$$C_{ij} = \frac{1}{4x48}[P_j P_i + Q_j Q_j]$$

$$C^{(2)} = \begin{bmatrix} 1.2 & 0 & -1.2 \\ 0 & 0.208 & -0.208 \\ -1.2 & -0.208 & 1.408 \end{bmatrix}$$

$$C = \begin{bmatrix} C^{(1)}{}_{33} & C_{23}{}^{(1)} & 0 & C_{31}{}^{(1)} \\ C_{23}{}^{(1)} & C_{22}{}^{(1)} + C_{11}{}^{(2)} & C_{13}{}^{(2)} & C_{21}{}^{(1)} + C_{12}{}^{(2)} \\ 0 & C_{31}{}^{(2)} & C_{33}{}^{(2)} & C_{32}{}^{(2)} \\ C_{13}{}^{(1)} & C_{21}{}^{(1)} + C_{21}{}^{(2)} & C_{23}{}^{(2)} & C_{22}{}^{(2)} + C_{11}{}^{(1)} \end{bmatrix}$$

$$= \begin{bmatrix} 0.8802 & -0.2083 & 0 & -0.6719 \\ -0.2083 & 1.533 & -1.2 & -0.125 \\ 0 & -1.2 & 1.4083 & -0.2083 \\ -0.6719 & -0.125 & -0.2083 & 1.0052 \end{bmatrix}$$

**Prob. 14.28**




For element 1,

$$P_1 = 0, P_2 = 2, P_3 = -2, Q_1 = -2, Q_2 = 0, Q_3 = 2$$

$$A = \frac{1}{2}(4-0) = 2, \qquad 4A = 8$$

$$C^{(1)} = \frac{1}{8}\begin{bmatrix} 4 & 0 & -4 \\ 0 & 4 & -4 \\ -4 & -4 & 8 \end{bmatrix} = \begin{bmatrix} 0.5 & 0 & -0.5 \\ 0 & 0.5 & -0.5 \\ -0.5 & -0.5 & 1 \end{bmatrix}$$

For element 2,

$$P_1 = -2, P_2 = 2, P_3 = 0, Q_1 = -2, Q_2 = -2, Q_3 = 4$$

$$A = \frac{1}{2}(8-0) = 4, \qquad 4A = 16$$

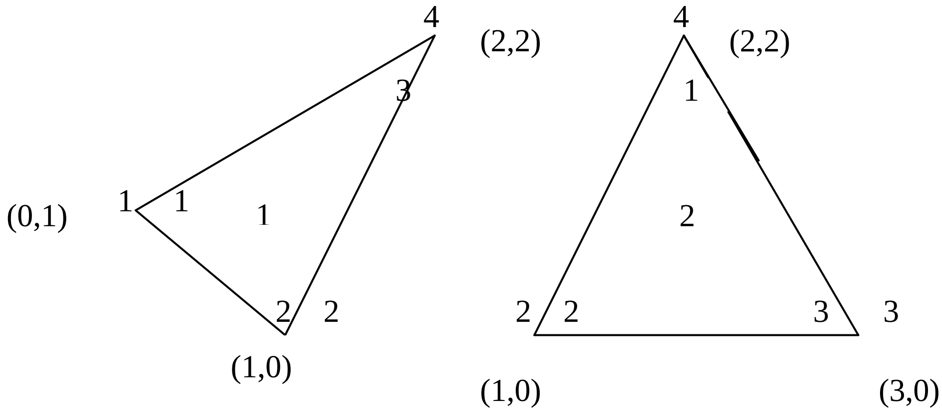
$$C^{(2)} = \frac{1}{16}\begin{bmatrix} 8 & 0 & -8 \\ 0 & 8 & -8 \\ -8 & -8 & 16 \end{bmatrix} = \begin{bmatrix} 0.5 & 0 & -0.5 \\ 0 & 0.5 & -0.5 \\ -0.5 & -0.5 & 1 \end{bmatrix}$$

The global coefficient matrix is

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{21} & C_{22} & C_{23} & C_{24} \\ C_{31} & C_{32} & C_{33} & C_{34} \\ C_{41} & C_{42} & C_{43} & C_{44} \end{bmatrix} = \begin{bmatrix} C_{11}^{(1)} + C_{11}^{(2)} & C_{12}^{(2)} & C_{12}^{(1)} + C_{13}^{(2)} & C_{13}^{(1)} \\ C_{21}^{(2)} & C_{22}^{(2)} & C_{23}^{(2)} & 0 \\ C_{21}^{(1)} + C_{31}^{(2)} & C_{32}^{(2)} & C_{22}^{(1)} + C_{33}^{(2)} & C_{23}^{(1)} \\ C_{31}^{(1)} & 0 & C_{32}^{(1)} & C_{33}^{(1)} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -0.5 & -0.5 \\ 0 & 0.5 & -0.5 & 0 \\ -0.5 & -0.5 & 1.5 & -0.5 \\ -0.5 & 0 & -0.5 & 1 \end{bmatrix}$$

**Prob. 14.29**




For element 1, local numbering 1-2-3 corresponds to global numbering 1-2-4.

$P_1 = -2$, $P_2 = 1$, $P_3 = 1$, $Q_1 = 1$, $Q_2 = -2$, $Q_3 = 1$,

$A = (P_2 Q_3 - P_3 Q_2)/2 = 3/2$, i.e. $4A = 6$

$$C_{ij} = \frac{1}{4A}[P_j P_i + Q_j Q_j]$$

$$C^{(1)} = \frac{1}{6}\begin{bmatrix} 5 & -4 & -1 \\ -4 & 5 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

For element 2, local numbering 1-2-3 corresponds to global numbering 4-2-3.

$P_1 = 0$, $P_2 = -2$, $P_3 = 2$, $Q_1 = 2$, $Q_2 = -1$, $Q_3 = -1$,

$A = 2$, $4A = 8$

$$C^{(2)} = \frac{1}{8}\begin{bmatrix} 4 & -2 & -2 \\ -2 & 5 & -3 \\ -2 & -3 & 5 \end{bmatrix}$$

The global coefficient matrix is

$$C = \begin{bmatrix} C^{(1)}_{11} & C_{12}^{(1)} & 0 & C_{13}^{(1)} \\ C_{12}^{(1)} & C_{22}^{(1)} + C_{22}^{(2)} & C_{23}^{(2)} & C_{23}^{(1)} + C_{21}^{(2)} \\ 0 & C_{23}^{(2)} & C_{33}^{(2)} & C_{31}^{(2)} \\ C_{13}^{(1)} & C_{23}^{(1)} + C_{21}^{(2)} & C_{31}^{(2)} & C_{33}^{(1)} + C_{11}^{(2)} \end{bmatrix}$$

$$= \begin{bmatrix} 0.8333 & -0.667 & 0 & -0.1667 \\ -0.6667 & 1.4583 & -0.375 & -0.4167 \\ 0 & -0.375 & 0.625 & -0.25 \\ -0.1667 & -0.4167 & -0.25 & 0.833 \end{bmatrix}$$

**Prob. 14.30** We can do it by hand as in Example 14.6. However, it is easier to prepare an input file and use the program in Fig. 14.54. The MATLAB input data is

```
NE = 2;
ND = 4;
NP = 2;
NL = [1  2  4
      4    2   3];
X = [ 0.0    1.0   3.0   2.0];
Y = [ 1.0  0.0  0.0  2.0];
NDP= [ 1  3 ];
VAL = [ 10.0   30.0]
```

The result is $V = \begin{bmatrix} 10 \\ 18 \\ 30 \\ 20 \end{bmatrix}$

From this,

$V_2 = 18$ V, $V_4 = 20$ V

**Prob. 14.31**

Compare your finite element solution to the exact or finite difference solution:

$\underline{V_5 = 25 \text{ V}}$

**Prob. 14.32** As in P. E. 14.7, we use the program in Fig. 14.33. The input data based on Fig. 14.64 is as follows.

```
NE =50;   ND= 36;   NP= 20;
NL = [1     8     7
      1     2     8
      2     9     8
      2     3     9
      3     10    9
      3     4     10
      4     11    10
      4     5     11
      5     12    11
      5     6     12
      7     14    13
      7     8     14
      8     15    14
      8     9     15
      9     16    15
      9     10    16
      10    17    16
      10    11    17
      11    18    17
      11    12    18
      13    20    19
      13    14    20
      14    21    20
      14    15    21
      15    22    21
      15    16    22
      16    23    22
      16    17    23
      17    24    23
      17    18    24
      19    26    25
      19    20    26
      20    27    26
      20    21    27
```

```
        21      28      27
        21      22      28
        22      29      28
        22      23      29
        23      30      29
        23      24      30
        25      32      31
        25      26      32
        26      33      32
        26      27      33
        27      34      33
        27      28      34
        28      35      34
        28      29      35
        29      36      35
        29      30      36];
X = [0.0  0.2  0.4  0.6  0.8  1.0  0.0  0.2  0.4  0.6  0.8  1.0  0.0  0.2  0.4  0.6 0.8  1.0
     0.0  0.2  0.4  0.6  0.8  1.0  0.0  0.2  0.4  0.6  0.8  1.0  0.2  0.4  0.6  0.8  1.0];
Y = [0.0  0.0   0.0  0.0  0.0  0.0  0.2  0.2  0.2  0.2  0.2  0.2  0.4  0.4  0.4  0.4  0.4
0.4 0.6  0.6  0.6  0.6  0.6  0.6  0.8  0.8  0.8  0.8  0.8  0.8  1.0 1.0  1.0  1.0  1.0 1.0];
NDP = [ 1   2  3  4  5  6  12  18  24  30  36  35  34  33  32  31  25  19  13  7];
VAL = [ 0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  50.0  100.0  100.0  100.0
     100.0  50.0  0.0  0.0  0.0  0.0];
```

With this data, the potentials at the free nodes are compared with the exact values as shown below.

| Node no. | FEM Solution | Exact Solution |
|---|---|---|
| 8 | 4.546 | 4.366 |
| 9 | 7.197 | 7.017 |
| 10 | 7.197 | 7.017 |
| 11 | 4.546 | 4.366 |
| 14 | 10.98 | 10.60 |
| 15 | 17.05 | 16.84 |
| 16 | 17.05 | 16.84 |
| 17 | 10.98 | 10.60 |
| 20 | 22.35 | 21.78 |
| 21 | 32.95 | 33.16 |
| 22 | 32.95 | 33.16 |
| 23 | 22.35 | 21.78 |
| 26 | 45.45 | 45.63 |
| 27 | 59.49 | 60.60 |
| 28 | 59.49 | 60.60 |
| 29 | 45.45 | 45.63 |

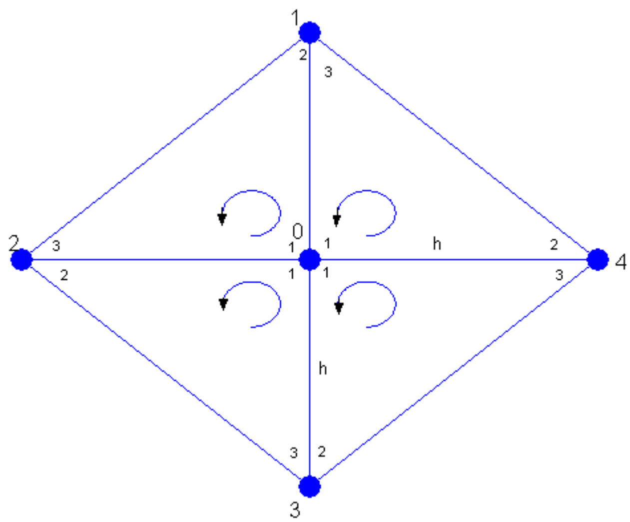
**Prob. 14.33** We use exactly the same input data as in the previous problem except that the last few lines are replaced by the following lines.

```
VAL = [ 0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  29.4  58.8  95.1  95.1
        58.8  29.4  0.0  0.0  0.0  0.0];
```

The potential at the free nodes obtained with the input data are compared with the exact solution as shown below.

| Node no. | FEM Solution | Exact Solution |
|---|---|---|
| 8 | 3.635 | 3.412 |
| 9 | 5.882 | 5.521 |
| 10 | 5.882 | 5.521 |
| 11 | 3.635 | 3.412 |
| 14 | 8.659 | 8.217 |
| 15 | 14.01 | 13.30 |
| 16 | 14.01 | 13.30 |
| 17 | 8.659 | 8.217 |
| 20 | 16.99 | 16.37 |
| 21 | 27.49 | 26.49 |
| 22 | 27.49 | 26.49 |
| 23 | 16.99 | 16.37 |
| 26 | 31.81 | 31.21 |
| 27 | 51.47 | 50.5 |
| 28 | 51.47 | 50.5 |
| 29 | 31.81 | 31.21 |

**Prob. 14.34**




For element 1, the local numbering 1-2-3 corresponds with nodes with $V_1$ , $V_2$ , and $V_3$.

$$V_o = -\frac{1}{C_{oo}}\sum_{i=1}^{4} V_i C_{io}$$

$$C_{oo} = \sum_{j=1}^{4} C_{oj}^{(e)} = \frac{1}{4h^2/2}(hh+hh)\times 2 + \frac{1}{4h^2/2}(hh+0)\times 4 = 4$$

$$C_{o1} = \frac{2\times 1}{2h^2}[P_3P_1 + Q_3Q_1] = \frac{2}{2h^2}[-hh-0] = -1$$

$$C_{o2} = \frac{2\times 1}{2h^2}[P_1P_2 + Q_1Q_2] = \frac{2}{2h^2}[-h\times 0 + h\times(-h)] = -1$$

Similarly, $C_{03}$ = -1 = $C_{04}$.   Thus

$V_o = ( V_1 + V_2 + V_3 + V_4 )/4$

which is the same result obtained using FDM.

**Prob. 4.35**

$$V_1 = \frac{1}{4}(0+0+100+V_2) = 25 + \frac{V_2}{4}$$

$$V_2 = \frac{1}{4}(0+100+V_1+V_3) = 25 + \frac{V_1+V_3}{4}$$

$$V_3 = \frac{1}{4}(0+0+100+V_2) = 25 + \frac{V_2}{4}$$

$$V_4 = \frac{1}{4}(0+0+100+V_5) = 25 + \frac{V_5}{4}$$

$$V_5 = \frac{1}{4}(0+0+100+V_4+V_6) = 25 + \frac{(V_4+V_6)}{4}$$

$$V_6 = \frac{1}{4}(0+0+100+V_5) = 25 + \frac{V_5}{4}$$

We initially set $V_1 = V_2 = V_3 = V_4 = V_5 = V_6 = 0$ and then apply above formulas iteratively. The solutions are presented in the table below.

| iteration | 1st | 2nd | 3rd | 4th | 5th |
|---|---|---|---|---|---|
| $V_1$ | 25 | 32.81 | 35.35 | 35.67 | 35.71 |
| $V_2$ | 31.25 | 41.41 | 42.68 | 42.83 | 42.85 |
| $V_3$ | 32.81 | 35.35 | 35.67 | 35.71 | 35.71 |
| $V_4$ | 25 | 23.81 | 35.35 | 35.67 | 35.71 |
| $V_5$ | 31.25 | 41.41 | 42.68 | 42.83 | 42.85 |
| $V_6$ | 32.81 | 35.35 | 35.67 | 35.71 | 35.71 |

$$V_1 = V_4 = 35.71 \text{ V}, \quad V_2 = V_5 = 42.85\text{V}, \quad V_3 = V_6 = 35.71 \text{ V}$$

Alternatively, if we take advantage of the symmetry, $V_1 = V_3 = V_4 = V_6$ and $V_2 = V_5$. We need to find solve two equations, namely,

$V_1 = 25 + V_2 \ / 4$

$V_2 = 25 + V_1 / 2$

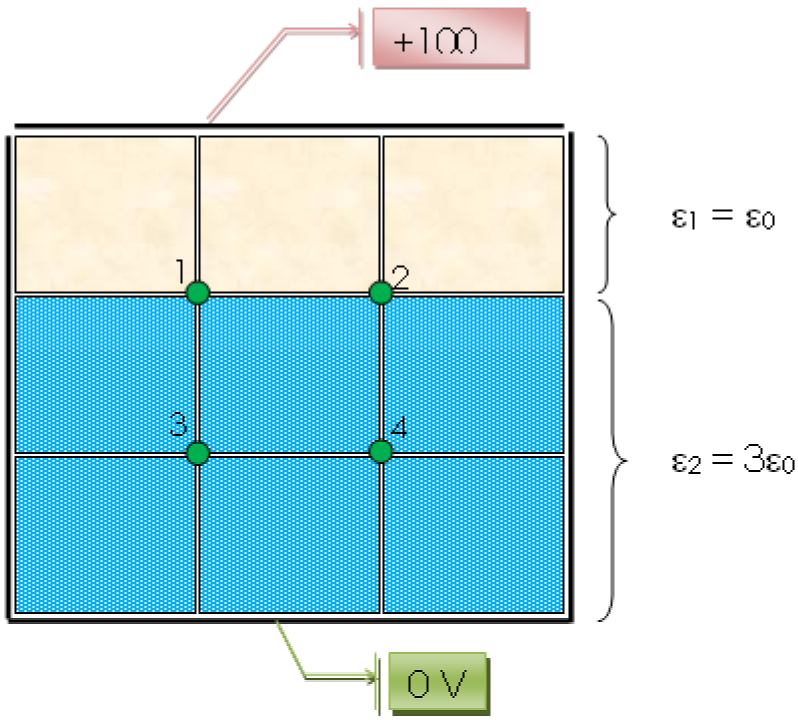
Solving these gives

$V_1 = 35.714$

$V_2 = 42.857$

Other node voltages follow.

**Prob. 14.36**




On the interface,

$$\frac{\varepsilon_1}{2(\varepsilon_1+\varepsilon_2)} = \frac{1}{8}, \qquad \frac{\varepsilon_2}{2(\varepsilon_1+\varepsilon_2)} = \frac{3}{8}$$

$$V_1 = \frac{V_2}{4} + \frac{3V_3}{8} + 12.5$$

$$V_2 = 12.5 + \frac{3V_4}{8} + \frac{V_1}{4}$$

$$V_3 = \frac{1}{4}(V_1 + V_4)$$

$$V_4 = \frac{1}{4}(V_2 + V_3)$$

Applying this iteratively, we obtain the results shown in the table below.

| No. of iterations | 0 | 1 | 2 | 3 | 4 | 5… | 100 |
|---|---|---|---|---|---|---|---|
| $V_1$ | 0 | 12.5 | 17.57 | 19.25 | 19.77 | 19.93 | 20 |
| $V_2$ | 0 | 15.62 | 18.65 | 19.58 | 19.87 | 19.96 | 20 |
| $V_3$ | 0 | 3.125 | 5.566 | 6.33 | 6.56 | 6.6634 | 6.667 |
| $V_4$ | 0 | 4.688 | 6.055 | 6.477 | 6.608 | 6.649 | 6.667 |

**Prob. 14.37**

The MATLAB code is similar to the one in Fig.14.40. When the program is run, it gives $Z_o = \underline{\underline{40.587\ \Omega}}$.

**Prob. 14.38**

The finite difference solution is obtained by following the same steps as in Example 14.8. We obtain $\underline{\underline{Z_o = 43\ \Omega}}$

**Prob.14.39**

$$V_1 = \frac{1}{4}(V_2 + 100 + 100 + 100) = \frac{1}{4}V_2 + 75$$

$$V_2 = \frac{1}{4}(V_1 + V_4 + 2V_3)$$

$$V_3 = \frac{1}{4}(V_2 + V_5 + 200) = \frac{1}{4}(V_2 + V_5) + 50$$

$$V_4 = \frac{1}{4}(V_2 + V_7 + 2V_5)$$

$$V_5 = \frac{1}{4}(V_3 + V_4 + V_6 + V_8)$$

$$V_6 = \frac{1}{4}(V_5 + V_9 + 200) = \frac{1}{4}(V_5 + V_9) + 50$$

$$V_7 = \frac{1}{4}(V_4 + 2V_8 + 0) = \frac{1}{4}(V_4 + 2V_8)$$

$$V_8 = \frac{1}{4}(V_5 + V_7 + V_9)$$

$$V_9 = \frac{1}{4}(V_6 + V_8 + 100 + 0) = \frac{1}{4}(V_6 + V_8) + 25$$

Using these equations, we apply iterative method and obtain the results shown below.

| | 1st | 2nd | 3rd | 4th | 5th |
|---|---|---|---|---|---|
| $V_1$ | 75 | 79.687 | 87.11 | 89.91 | 92.01 |
| $V_2$ | 18.75 | 48.437 | 59.64 | 68.06 | 74.31 |
| $V_3$ | 54.69 | 65.82 | 73.87 | 79.38 | 82.89 |
| $V_4$ | 4.687 | 19.824 | 34.57 | 46.47 | 53.72 |
| $V_5$ | 14.687 | 35.14 | 49.45 | 57.24 | 61.78 |
| $V_6$ | 53.71 | 68.82 | 74.2 | 77.01 | 78.6 |

| | | | | | |
|---|---|---|---|---|---|
| $V_7$ | 1.172 | 6.958 | 18.92 | 26.08 | 30.194 |
| $V_8$ | 4.003 | 20.557 | 28.93 | 33.53 | 36.153 |
| $V_9$ | 39.43 | 47.34 | 50.78 | 52.63 | 53.69 |

**Prob. 14.40**

Applying the difference method,

$$V_1 = \frac{V_3}{4} + \frac{V_2}{2} + 25$$

$$V_2 = \frac{1}{4}(V_1 + V_4) + 50$$

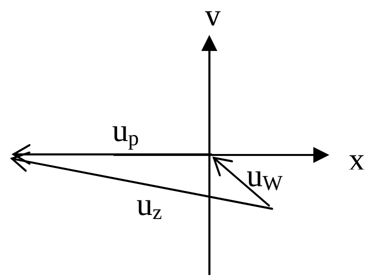
$$V_3 = \frac{1}{4}(V_1 + 2V_4)$$

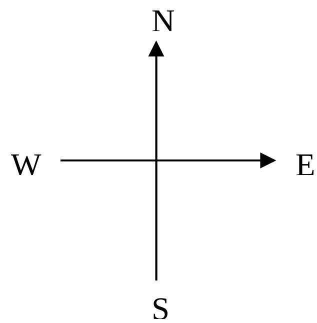
$$V_4 = \frac{1}{4}(V_2 + V_3 + V_5)$$

$$V_5 = \frac{V_4}{4} + 50$$

Applying these equations iteratively, we obtain the results below.

| Iterations | 0 | 1 | 2 | 3 | 4 | 5… | 100 |
|---|---|---|---|---|---|---|---|
| $V_1$ | 0 | 25.0 | 54.68 | 64.16 | 70.97 | 73.79 | 74.68 |
| $V_2$ | 0 | 56.25 | 67.58 | 74.96 | 78.23 | 79.54 | 80.41 |
| $V_3$ | 0 | 6.25 | 17.58 | 33.91 | 38.72 | 40.63 | 41.89 |
| $V_4$ | 0 | 15.63 | 35.74 | 41.96 | 44.86 | 45.31 | 45.95 |
| $V_5$ | 0 | 53.91 | 58.74 | 60.49 | 61.09 | 61.37 | 51.49 |




**P. E. 1.4**

Using the dot product,

$$\cos\theta_{AB} = \frac{\mathbf{A} \bullet \mathbf{B}}{AB} = \frac{-13}{\sqrt{10}\sqrt{65}} = -\sqrt{\frac{13}{50}}$$

$$\underline{\underline{\theta_{AB} = 120.66^\circ}}$$

**P. E. 1.5**

(a) $$\boldsymbol{E_F} = (\boldsymbol{E} \cdot \boldsymbol{a_F})\boldsymbol{a_F} = \frac{(\boldsymbol{E} \cdot \boldsymbol{F})\boldsymbol{F}}{|\boldsymbol{F}|^2} = \frac{-10(4,-10,5)}{141}$$

$$= \underline{\underline{-0.2837\boldsymbol{a_x} + 0.7092\boldsymbol{a_y} - 0.3546\boldsymbol{a_z}}}$$

(b) $$\boldsymbol{E} \times \boldsymbol{F} = \begin{vmatrix} \boldsymbol{a_x} & \boldsymbol{a_y} & \boldsymbol{a_z} \\ 0 & 3 & 4 \\ 4 & -10 & 5 \end{vmatrix} = (55,16,-12)$$

$$\boldsymbol{a_{E\times F}} = \underline{\underline{\pm(0.9398,0.2734,-0.205)}}$$

**P. E. 1.6** $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} = 0$ showing that $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ form the sides of a triangle.

$\boldsymbol{a} \cdot \boldsymbol{b} = 0$,
hence it is a right angle triangle.

$$\text{Area} = \frac{1}{2}|\boldsymbol{a} \times \boldsymbol{b}| = \frac{1}{2}|\boldsymbol{b} \times \boldsymbol{c}| = \frac{1}{2}|\boldsymbol{c} \times \boldsymbol{a}|$$

$$\frac{1}{2}|\boldsymbol{a} \times \boldsymbol{b}| = \frac{1}{2}\begin{vmatrix} 4 & 0 & -1 \\ 1 & 3 & 4 \end{vmatrix} = \frac{1}{2}|(3,-17,12)|$$

$$\text{Area} = \frac{1}{2}\sqrt{9 + 289 + 144} = \underline{\underline{10.51}}$$

**P. E. 1.7**

(a) $P_1P_2 = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

$= \sqrt{25+4+64} = \underline{\underline{9.644}}$

(b) $\boldsymbol{r_P} = \boldsymbol{r_{P_1}} + \lambda(\boldsymbol{r_{P_2}} - \boldsymbol{r_{P_1}})$

$= (1,2,-3) + \lambda(-5,-2,8)$

$= \underline{\underline{(1-5\lambda, 2-2\lambda, -3+8\lambda)}}.$

(c) The shortest distance is

$$\boldsymbol{d} = \boldsymbol{P_1P_3}\sin\theta = \left|\boldsymbol{P_1P_3} \times \boldsymbol{a_{P_1P_2}}\right|$$

$$= \frac{1}{\sqrt{93}}\begin{vmatrix} 6 & -3 & 5 \\ -5 & -2 & 8 \end{vmatrix}$$

$$= \frac{1}{\sqrt{93}}\left|(-14,-73,-27)\right| = \underline{\underline{8.2}}$$

---

**Prob.1.1**

$\boldsymbol{r}_{OP} = 4\boldsymbol{a}_x - 5\boldsymbol{a}_y + \boldsymbol{a}_z$

$$\boldsymbol{a}_{r_{OP}} = \frac{\boldsymbol{r}_{OP}}{|\boldsymbol{r}_{OP}|} = \frac{(4,-5,1)}{\sqrt{(16+25+1)}} = \underline{\underline{0.6172\boldsymbol{a}_x - 0.7715\boldsymbol{a}_y + 0.1543\boldsymbol{a}_z}}$$

**Prob. 1.2**

Method 1:

$r_{AB} = r_B - r_A, \quad r_{BC} = r_C - r_B, \quad r_{CA} = r_A - r_C$

$r_{AB} + r_{BC} + r_{CA} = r_B - r_A + r_C - r_B + r_A - r_C = \mathbf{0}$

Method 2

$r_{AB} = r_B - r_A = (-2,0,3) - (4,-6,2) = (-6,6,1)$

$r_{BC} = r_C - r_B = (10,1,-7) - (-2,0,3) = (12,1,-10)$

$r_{CA} = r_A - r_C = (4,-6,2) - (10,1,-7) = (-6,-7,9)$

$r_{AB} + r_{BC} + r_{CA} = (0,0,0) = \mathbf{0}$

**Prob. 1.3**

(a)

$$\boldsymbol{A}-3\boldsymbol{B}=(4,-2,6)-3(12,18,-8)=(4,-2,6)-(36,54,-24)$$
$$=\underline{\underline{(-32,-56,-30)}}$$

(b)

$$2\boldsymbol{A}+5\boldsymbol{B}=2(4,-2,6)+5(12,18,-8)=(68,86,-28)$$
$$|\boldsymbol{B}|=\sqrt{12^2+18^2+8^2}=\sqrt{532}=23.065$$
$$(2\boldsymbol{A}+5\boldsymbol{B})/|\boldsymbol{B}|=(68,86,-28)/23.065=2.948\boldsymbol{a}_x+3.728\boldsymbol{a}_y-1.214\boldsymbol{a}_z$$

**(c )**

$$\boldsymbol{a}_x\times\boldsymbol{A}=\begin{vmatrix}1 & 0 & 0\\ 4 & -2 & 6\end{vmatrix}=\underline{\underline{-6\boldsymbol{a}_y-2\boldsymbol{a}_z}}$$

**(d)**

$$\boldsymbol{B}\times\boldsymbol{a}_x=\begin{vmatrix}12 & 18 & -8\\ 1 & 0 & 0\end{vmatrix}=-8\boldsymbol{a}_y-18\boldsymbol{a}_z$$
$$(\boldsymbol{B}\times\boldsymbol{a}_x)\bullet\boldsymbol{a}_y=-8$$

**Prob. 1.4**

**(a)** $\boldsymbol{A}\cdot\boldsymbol{B}=(10,-6,8)\cdot(1,0,2)=10+16=\underline{\underline{26}}$

**(b)**
$$\boldsymbol{A}\times\boldsymbol{B}=\begin{vmatrix}10 & -6 & 8\\ 1 & 0 & 2\end{vmatrix}=(-12-0)\boldsymbol{a}_x+(8-20)\boldsymbol{a}_y+(0+6)\boldsymbol{a}_z$$
$$=\underline{\underline{-12\boldsymbol{a}_x-12\boldsymbol{a}_y+6\boldsymbol{a}_z}}$$

**(c)** $2\boldsymbol{A}-3\boldsymbol{B}=(20,-12,16)-(3,0,6)=\underline{\underline{17\boldsymbol{a}_x-12\boldsymbol{a}_y+10\boldsymbol{a}_z}}$

**Prob. 1.5**

(a) $\boldsymbol{A}-\boldsymbol{B}+\boldsymbol{C}=(-2,5,1)+(-1,0,-3)+(4,-6,10)=\underline{\underline{(1,-1,8)}}$

(b)
$$\boldsymbol{B}\times\boldsymbol{C}=\begin{vmatrix}1 & 0 & 3\\ 4 & -6 & 10\end{vmatrix}=(18,2,-6)$$
$$\boldsymbol{A}\bullet(\boldsymbol{B}\times\boldsymbol{C})=(-2,5,1)\bullet(18,2,-6)=-36+10-6=\underline{\underline{-32}}$$

(c) $\cos\theta_{AB}=\dfrac{\boldsymbol{A}\bullet\boldsymbol{B}}{AB}=\dfrac{-2+0+3}{\sqrt{4+25+1}\sqrt{1+9}}=0.05773 \quad\rightarrow\quad \theta_{AB}=\underline{\underline{86.69^o}}$

**Prob. 1.6**

(a)
$$\boldsymbol{B}\times\boldsymbol{C}=\begin{vmatrix}1 & 1 & 1\\ 0 & 1 & 2\end{vmatrix}=\boldsymbol{a}_x-2\boldsymbol{a}_y+\boldsymbol{a}_z$$
$$\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C})=(1,0,-1)\cdot(1,-2,1)=1+0-1=\underline{\underline{0}}$$

(b) $\boldsymbol{A}\times\boldsymbol{B} = \begin{vmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{vmatrix} = \boldsymbol{a}_x - 2\boldsymbol{a}_y + \boldsymbol{a}_z$

$(\boldsymbol{A}\times\boldsymbol{B})\cdot\boldsymbol{C} = (1,-2,1)\cdot(0,1,2) = 0-2+2 = \underline{\underline{0}}$

(c ) $\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C}) = \begin{vmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \end{vmatrix} = \underline{\underline{-2\boldsymbol{a}_x - 2\boldsymbol{a}_y - 2\boldsymbol{a}_z}}$

(d) $(\boldsymbol{A}\times\boldsymbol{B})\times\boldsymbol{C} = \begin{vmatrix} 1 & -2 & 1 \\ 0 & 1 & 2 \end{vmatrix} = \underline{\underline{-5\boldsymbol{a}_x - 2\boldsymbol{a}_y + \boldsymbol{a}_z}}$

**Prob.1.7**

(a) T = (4, 6, -1) and S = (10, 12, 8)

(b) $r_{TS} = r_S - r_T = (10,12,8)-(4,6,-1) = \underline{\underline{6\boldsymbol{a}_x + 6\boldsymbol{a}_y + 9\boldsymbol{a}_z}}$

(c ) $TS = |r_{TS}| = \sqrt{36+36+81} = \underline{\underline{12.37}}$

**Prob. 1.8**

(a) If **A** and **B** are parallel, **B**=k**A**, where k is a constant.

$B_x = kA_x, \quad B_y = kA_y, \quad B_z = kA_z$

For $B_z$, $\quad 3 = k(-1) \quad \rightarrow \quad k=-3$

$B_x = \alpha = kA_x = (-3)(4) = -12$

$B_y = \beta = kA_y = (-3)(2) = -6$

Hence, $\underline{\underline{\alpha = -12, \beta = -6}}$

(b) If A and B are perpendicular to each other,

$$\boldsymbol{A}\bullet\boldsymbol{B} = 0 \quad \longrightarrow \quad \underline{\underline{4\alpha + 2\beta - 3 = 0}}$$

**Prob. 1.9**

(a) $\boldsymbol{A}\times\boldsymbol{a}_y = \begin{vmatrix} 10 & 5 & -2 \\ 0 & 1 & 0 \end{vmatrix} = \underline{\underline{2\boldsymbol{a}_x + 10\boldsymbol{a}_z}}$

(b) $\boldsymbol{A}\bullet\boldsymbol{a}_z = \underline{\underline{-2}}$

(c) $\cos\theta_z = \dfrac{\boldsymbol{A}\bullet\boldsymbol{a}_z}{\sqrt{100+25+4}} = \dfrac{-2}{11.358} \quad \rightarrow \quad \theta_z = \underline{\underline{100.14^o}}$

**Prob. 1.10**

(a) $A \cdot B = \text{AB}\cos\theta_{AB}$

$A \times B = \text{AB}\sin\theta_{AB} a_n$

$(A \cdot B)^2 + |A \times B|^2 = (AB)^2 (\cos^2\theta_{AB} + \sin^2\theta_{AB}) = (\text{AB})^2$

(b) $\boldsymbol{a}_x \cdot (\boldsymbol{a}_y \times \boldsymbol{a}_z) = \boldsymbol{a}_x \cdot \boldsymbol{a}_x = 1.$ Hence,

$$\frac{\boldsymbol{a}_y \times \boldsymbol{a}_z}{\boldsymbol{a}_x \cdot \boldsymbol{a}_y \times \boldsymbol{a}_z} = \frac{\boldsymbol{a}_x}{1} = \boldsymbol{a}_x$$

$$\frac{\boldsymbol{a}_z \times \boldsymbol{a}_x}{\boldsymbol{a}_x \cdot \boldsymbol{a}_y \times \boldsymbol{a}_z} = \frac{\boldsymbol{a}_y}{1} = \boldsymbol{a}_y$$

$$\frac{\boldsymbol{a}_x \times \boldsymbol{a}_y}{\boldsymbol{a}_x \cdot \boldsymbol{a}_y \times \boldsymbol{a}_z} = \frac{\boldsymbol{a}_z}{1} = \boldsymbol{a}_z$$

**Prob. 1.11**

(a) $\boldsymbol{P} + \boldsymbol{Q} = (6, 2, 0),\ \boldsymbol{P} + \boldsymbol{Q} - \boldsymbol{R} = (7, 1, -2)$

$|\boldsymbol{P} + \boldsymbol{Q} - \boldsymbol{R}| = \sqrt{49+1+4} = \sqrt{54} = \underline{\underline{7.3485}}$

(b) $\boldsymbol{P}.\boldsymbol{Q} \times \boldsymbol{R} = \begin{vmatrix} 2 & -1 & -2 \\ 4 & 3 & 2 \\ -1 & 1 & 2 \end{vmatrix} = 2(6-2) + (8+2) - 2(4+3) = 8 + 10 - 14 = \underline{\underline{4}}$

$\boldsymbol{Q} \times \boldsymbol{R} = \begin{vmatrix} 4 & 3 & 2 \\ -1 & 1 & 2 \end{vmatrix} = (4, -10, 7)$

$\boldsymbol{P}.\boldsymbol{Q} \times \boldsymbol{R} = (2, -1, -2) \cdot (4, -10, 7) = 8 + 10 - 14 = \underline{\underline{4}}$

(c) $\boldsymbol{Q} \times \boldsymbol{P} = \begin{vmatrix} 4 & 3 & 2 \\ 2 & -1 & -2 \end{vmatrix} = (-4, 12, -10)$

$\boldsymbol{Q} \times \boldsymbol{P} \cdot \boldsymbol{R} = (-4, 12, -10) \cdot (-1, 1, 2) = 4 + 12 - 20 = \underline{\underline{-4}}$

or $\boldsymbol{Q} \times \boldsymbol{P} \cdot \boldsymbol{R} = \boldsymbol{R} \cdot \boldsymbol{Q} \times \boldsymbol{P} = \begin{vmatrix} -1 & 1 & 2 \\ 4 & 3 & 2 \\ 2 & -1 & -2 \end{vmatrix} = -(-6+2) - (-8-4) + 2(-4-6) = \underline{\underline{-4}}$

(d) $(\boldsymbol{P} \times \boldsymbol{Q}) \cdot (\boldsymbol{Q} \times \boldsymbol{R}) = (4, -12, 10) \cdot (4, -10, 7) = 16 + 120 + 70 = \underline{\underline{206}}$

(e) $(\boldsymbol{P} \times \boldsymbol{Q}) \times (\boldsymbol{Q} \times \boldsymbol{R}) = \begin{vmatrix} 4 & -12 & 10 \\ 4 & -10 & 7 \end{vmatrix} = \underline{\underline{16a_x + 12a_y + 8a_z}}$

(f) $\cos\theta_{PR} = \frac{\mathbf{P}\cdot\mathbf{R}}{|\boldsymbol{P}||\boldsymbol{R}|} = \frac{(-2-1-4)}{\sqrt{4+1+4}\sqrt{1+1+4}} = \frac{-7}{3\sqrt{6}} = -0.9526$

$\underline{\underline{\theta_{PR} = 162.3^\circ}}$

(g) $\sin\theta_{PQ} = \frac{|P\times Q|}{|P||Q|} = \frac{\sqrt{16+144+100}}{3\sqrt{16+9+4}} = \frac{\sqrt{260}}{3\sqrt{29}} = 0.998$

$\theta_{PQ} = \underline{\underline{86.45^\circ}}$

**Prob. 1.12**

$\boldsymbol{A}\cdot\boldsymbol{B} = (4,-6,1)\cdot(2,0,5) = 8-0+5 = 13$

(a) $|\boldsymbol{B}|^2 = 2^2+5^2 = 29$

$\boldsymbol{A}\cdot\boldsymbol{B} + 2|\boldsymbol{B}|^2 = 13+2\times29 = \underline{\underline{71}}$

(b)

$\boldsymbol{a}_\perp = \pm\frac{\boldsymbol{A}\times\boldsymbol{B}}{|\boldsymbol{A}\times\boldsymbol{B}|}$

Let $\boldsymbol{C} = \boldsymbol{A}\times\boldsymbol{B} = \begin{vmatrix} 4 & -6 & 1 \\ 2 & 0 & 5 \end{vmatrix} = (-30,-18,12)$

$\boldsymbol{a}_\perp = \pm\frac{\boldsymbol{C}}{|\boldsymbol{C}|} = \pm\frac{(-30,-18,12)}{\sqrt{30^2+18^2+12^2}} = \underline{\underline{\pm(-0.8111\boldsymbol{a}_x - 0.4867\boldsymbol{a}_y + 0.3244\boldsymbol{a}_z)}}$

**Prob. 1.13**

$\boldsymbol{P}\cdot\boldsymbol{Q} = (2,-6,5)\cdot(0,3,1) = 0-18+5 = \underline{-13}$

$\boldsymbol{P}\times\boldsymbol{Q} = \begin{vmatrix} 2 & -6 & 5 \\ 0 & 3 & 1 \end{vmatrix} = \underline{\underline{-21\boldsymbol{a}_x - 2\boldsymbol{a}_y + 6\boldsymbol{a}_z}}$

$\cos\theta_{PQ} = \frac{\boldsymbol{P}\cdot\boldsymbol{Q}}{PQ} = \frac{-13}{\sqrt{10}\sqrt{65}} = -0.51 \longrightarrow \theta_{PQ} = \underline{\underline{120.66^o}}$

**Prob. 1.14**

**P** and **Q** are orthogonal if the angle between them is 90$^o$. Hence

$\boldsymbol{P}\bullet\boldsymbol{Q} = PQ\cos\theta = 0$

$\boldsymbol{P}\bullet\boldsymbol{Q} = (2,4,-6)\bullet(5,2,3) = 10+8-18 = 0$

showing that they are perpendicular or orthogonal.

**Prob. 1.15**

(a) Using the fact that

$$(\boldsymbol{A}\times\boldsymbol{B})\times\boldsymbol{C} = (\boldsymbol{A}\cdot\boldsymbol{C})\boldsymbol{B} - (\boldsymbol{B}\cdot\boldsymbol{C})\boldsymbol{A},$$

we get

$$\boldsymbol{A}\times(\boldsymbol{A}\times\boldsymbol{B}) = -(\boldsymbol{A}\times\boldsymbol{B})\times\boldsymbol{A} = \underline{\underline{(\boldsymbol{B}\cdot\boldsymbol{A})\boldsymbol{A} - (\boldsymbol{A}\cdot\boldsymbol{A})\boldsymbol{B}}}$$

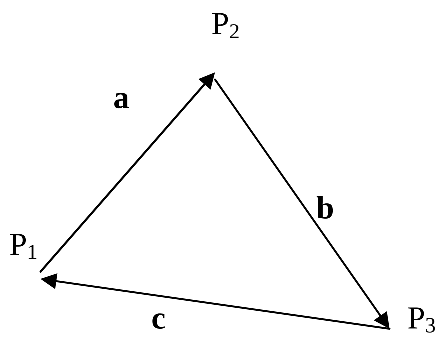
(b) $$\boldsymbol{A}\times(\boldsymbol{A}\times(\boldsymbol{A}\times\boldsymbol{B})) = \boldsymbol{A}\times\left[(\boldsymbol{A}\bullet\boldsymbol{B})\boldsymbol{A} - (\boldsymbol{A}\bullet\boldsymbol{A})\boldsymbol{B}\right]$$

$$= (\boldsymbol{A}\bullet\boldsymbol{B})\ (\boldsymbol{A}\times\boldsymbol{A}) - (\boldsymbol{A}\bullet\boldsymbol{A})\ (\boldsymbol{A}\times\boldsymbol{B})$$

$$= \underline{\underline{-A^2\ (\boldsymbol{A}\times\boldsymbol{B})}}$$

since **A**x**A** = **0**

**Prob. 1.16**




$$\boldsymbol{a} = \boldsymbol{r}_{p2} - \boldsymbol{r}_{p1} = (1,-2,4) - (5,-3,1) = (-4,1,3)$$

(a) $$\boldsymbol{b} = \boldsymbol{r}_{p3} - \boldsymbol{r}_{p2} = (3,3,5) - (1,-2,4) = (2,5,1)$$

$$\boldsymbol{c} = \boldsymbol{r}_{p1} - \boldsymbol{r}_{p3} = (5,-3,1) - (3,3,5) = (2,-6,-4)$$

Note that **a** + **b** + **c** = **0**

$\boldsymbol{a}\cdot\boldsymbol{b} = -8+5+3 = 0 \quad \longrightarrow$ perpendicular

$\boldsymbol{b}\cdot\boldsymbol{c} = 4-30-4 \neq 0$

$\boldsymbol{c}\cdot\boldsymbol{a} = -8-6-12 \neq 0$

Hence $\underline{\underline{P_2}}$ is a right angle.

(b) $$\text{Area} = \frac{1}{2}|\boldsymbol{a}\times\boldsymbol{b}| = \frac{1}{2}\begin{vmatrix} -4 & 1 & 3 \\ 2 & 5 & 1 \end{vmatrix} = \frac{1}{2}|(1-15)\boldsymbol{a}_x + (6+4)\boldsymbol{a}_y + (-20-2)\boldsymbol{a}_z|$$

$$= \frac{1}{2}|(-14,10,-22)| = \frac{1}{2}\sqrt{196+100+484} = \underline{\underline{13.96}}$$

**Prob. 1.17**

Given $\boldsymbol{r}_P = (-1,4,8), \quad \boldsymbol{r}_Q = (2,-1,3), \quad \boldsymbol{r}_R = (-1,2,3)$

(a) $|PQ| = \sqrt{9+25+25} = \underline{\underline{7.6811}}$

(b) $\boldsymbol{PR} = \underline{\underline{-2\boldsymbol{a}_y - 5\boldsymbol{a}_z}}$

(c )

$$\boldsymbol{QP} = (-1,4,8)-(2,-1,3) = (-3,5,5)$$

$$\boldsymbol{QR} = (-1,2,3)-(2,-1,3) = (-3,3,0)$$

$$\frac{\boldsymbol{QP}\cdot\boldsymbol{QR}}{|\boldsymbol{QP}||\boldsymbol{QR}|} = \frac{9+15+0}{\sqrt{59}\sqrt{18}} = 0.7365$$

$$\angle PQR = \cos^{-1}\left(0.7365\right) = \underline{\underline{42.64^o}}$$

(d) $\text{Area} = \frac{1}{2}\left|QP \times QR\right| = \frac{1}{2}\begin{vmatrix} -3 & 5 & 5 \\ -3 & 3 & 0 \end{vmatrix} = 0.5\left|(-15,-15,8)\right| = 0.5\sqrt{225+225+36} = \underline{\underline{10.677}}$

(e) Perimeter $= PQ + QR + RP = \sqrt{59} + \sqrt{18} + \sqrt{29} = \underline{\underline{17.31}}$

**Prob.1.18**

Let R be the midpoint of PQ.

$$\boldsymbol{r}_R = \frac{1}{2}\{(2,4,-1)+(12,16,9)\} = (7,10,4)$$

$$OR = \sqrt{49+100+16} = \sqrt{165} = 12.845$$

$$t = \frac{OR}{v} = \frac{12.845}{300} = \underline{\underline{42.82 \text{ ms}}}$$

**Prob. 1.19**

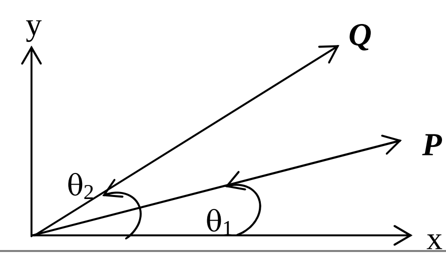
Area = Twice the area of a triangue

$$= |\boldsymbol{D}\times\boldsymbol{E}| = \begin{vmatrix} 4 & 1 & 5 \\ -1 & 2 & 3 \end{vmatrix} = |(3-10)\boldsymbol{a}_x + (-5-12)\boldsymbol{a}_y + (8+1)\boldsymbol{a}_z|$$

$$= |(-7,-19,9)| = \sqrt{49+361+81} = \underline{\underline{22.16}}$$

**Prob. 1.20**

(a) Let **P** and **Q** be as shown below:



$|\boldsymbol{P}| = \cos^2\theta_1 + \sin^2\theta_1 = 1, |\boldsymbol{Q}| = \cos^2\theta_2 + \sin^2\theta_2 = 1,$

Hence **P** and **Q** are unit vectors.

(b) $\boldsymbol{P}\cdot\boldsymbol{Q} = (1)(1)\cos(\theta_2 - \theta_1)$

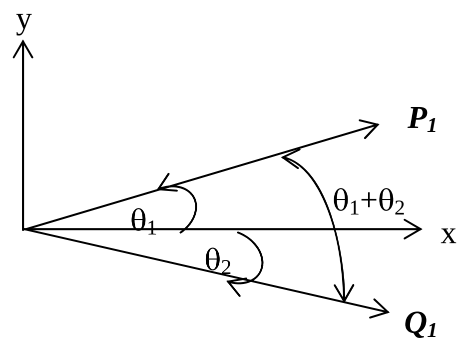
But $\boldsymbol{P}\cdot\boldsymbol{Q} = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2$. Thus,

$$\underline{\underline{\cos(\theta_2 - \theta_1) = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2}}$$

Let $\boldsymbol{P}_1 = \boldsymbol{P} = \cos\theta_1\boldsymbol{a}_x + \sin\theta_1\boldsymbol{a}_y$ and

$\boldsymbol{Q}_1 = \cos\theta_2\boldsymbol{a}_x - \sin\theta_2\boldsymbol{a}_y$.

$\boldsymbol{P_1}$ and $\boldsymbol{Q_1}$ are unit vectors as shown below:




$\boldsymbol{P}_1\cdot\boldsymbol{Q}_1 = (1)(1)\cos(\theta_1 + \theta_2)$

But $\boldsymbol{P}_1\cdot\boldsymbol{Q}_1 = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$,

$$\underline{\underline{\cos(\theta_2 + \theta_1) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2}}$$

Alternatively, we can obtain this formula from the previous one by replacing $\theta_2$ by $-\theta_2$ in $\boldsymbol{Q}$.

(c )

$$\frac{1}{2}|\boldsymbol{P}-\boldsymbol{Q}| = \frac{1}{2}|(\cos\theta_1 - \cos\theta_2)a_x + (\sin\theta_1 - \sin\theta_2)a_y|$$

$$= \frac{1}{2}\sqrt{\cos^2\theta_1 + \sin^2\theta_1 + \cos^2\theta_2 + \sin^2\theta_2 - 2\cos\theta_1\cos\theta_2 - 2\sin\theta_1\sin\theta_2}$$

$$= \frac{1}{2}\sqrt{2 - 2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)} = \frac{1}{2}\sqrt{2 - 2\cos(\theta_2 - \theta_1)}$$

Let $\theta_2 - \theta_1 = \theta$, the angle between **P** and **Q.**

$$\frac{1}{2}|\boldsymbol{P}-\boldsymbol{Q}| = \frac{1}{2}\sqrt{2 - 2\cos\theta}$$

But $\cos 2A = 1 - 2\sin^2 A$.

$$\frac{1}{2}|\boldsymbol{P}-\boldsymbol{Q}| = \frac{1}{2}\sqrt{2-2+4\sin^2\theta/2} = \sin\theta/2$$

Thus,

$$\frac{1}{2}|\boldsymbol{P}-\boldsymbol{Q}| = |\sin\frac{\theta_2-\theta_1}{2}|$$

**Prob. 1.21**

$$\boldsymbol{\omega} = \frac{\omega(1,-2,2)}{3} = (1,-2,2), \quad r = r_p - r_o = (1,3,4)-(2,-3,1) = (-1,6,3)$$

$$u = \boldsymbol{\omega}\times r = \begin{vmatrix} 1 & -2 & 2 \\ -1 & 6 & 3 \end{vmatrix} = (-18,-5,4)$$

$$u = -18a_x - 5a_y + 4a_z$$

**Prob. 1.22**

$\boldsymbol{r}_1 = (1,1,1), \qquad \boldsymbol{r}_2 = (1,0,1)-(0,1,0) = (1,-1,1)$

$$\cos\theta = \frac{\boldsymbol{r}_1\cdot\boldsymbol{r}_2}{r_1 r_2} = \frac{(1-1+1)}{\sqrt{3}\sqrt{3}} = \frac{1}{3} \quad \longrightarrow \quad \theta = 70.53^o$$

**Prob. 1.23**

(a) $$T_s = T\cdot a_s = \frac{T\cdot S}{|S|} = \frac{(2,-6,3)\cdot(1,2,1)}{\sqrt{6}} = \frac{-7}{\sqrt{6}} = -2.8577$$

(b) $$\boldsymbol{S_T} = (\boldsymbol{S}\cdot\boldsymbol{a_T})\boldsymbol{a_T} = \frac{(\boldsymbol{S}\cdot\boldsymbol{T})\boldsymbol{T}}{\boldsymbol{T}^2} = \frac{-7(2,-6,3)}{7^2}$$

$$= -0.2857\boldsymbol{a}_x + 0.8571\boldsymbol{a}_y - 0.4286\boldsymbol{a}_z$$

(c) $$\sin\theta_{TS} = \frac{|\boldsymbol{T}\times\boldsymbol{S}|}{|\boldsymbol{T}||\boldsymbol{S}|} = \begin{vmatrix} 2 & -6 & 3 \\ 1 & 2 & 1 \end{vmatrix} = \frac{|(-12,1,10)|}{7\sqrt{6}} = \frac{\sqrt{245}}{7\sqrt{6}} = 0.9129$$

$$\Rightarrow \theta_{TS} = 65.91^\circ$$

**Prob. 1.24**

Let $\boldsymbol{A} = \boldsymbol{A}_{B\parallel} + \boldsymbol{A}_{B\perp}$

$$\boldsymbol{A}_{B\parallel} = (\boldsymbol{A}\cdot\boldsymbol{a}_B)\boldsymbol{a}_B = \frac{\boldsymbol{A}\cdot\boldsymbol{B}}{\boldsymbol{B}\cdot\boldsymbol{B}}\boldsymbol{B}$$

Hence,

$$\boldsymbol{A}_{B\perp} = \boldsymbol{A} - \boldsymbol{A}_{B\parallel} = \boldsymbol{A} - \frac{\boldsymbol{A}\cdot\boldsymbol{B}}{\boldsymbol{B}\cdot\boldsymbol{B}}\boldsymbol{B}$$

**Prob. 1.25**

(a) $\boldsymbol{A}\bullet\boldsymbol{B} = 20 + 0 - 10 = \underline{\underline{10}}$

(b) $\boldsymbol{A}\times\boldsymbol{B} = \begin{vmatrix} 20 & 15 & -10 \\ 1 & 0 & 1 \end{vmatrix} = \underline{\underline{15\boldsymbol{a}_x - 30\boldsymbol{a}_y - 15\boldsymbol{a}_z}}$

(c) $\boldsymbol{A}_B = (\boldsymbol{A}\bullet\boldsymbol{a}_B)\boldsymbol{a}_B = \dfrac{(\boldsymbol{A}\bullet\boldsymbol{B})\boldsymbol{B}}{B^2} = \dfrac{10(\boldsymbol{a}_x + \boldsymbol{a}_z)}{2} = \underline{\underline{5\boldsymbol{a}_x + 5\boldsymbol{a}_z}}$

**Prob. 1.26**

$$\boldsymbol{A}\bullet\boldsymbol{a}_x = A_x = A\cos\alpha \quad \rightarrow \quad \cos\alpha = \frac{A_x}{A} = \frac{2}{\sqrt{4+16+36}} = 0.2673 \quad \rightarrow \quad \underline{\underline{\alpha = 74.5^o}}$$

$$\cos\beta = \frac{A_y}{A} = \frac{-4}{\sqrt{56}} = -0.5345 \quad \rightarrow \quad \underline{\underline{\beta = 122.31^o}}$$

$$\cos\gamma = \frac{A_z}{A} = \frac{6}{\sqrt{56}} = 0.8018 \quad \rightarrow \quad \underline{\underline{\gamma = 36.7^o}}$$

**Prob.1.27**

(a) $\boldsymbol{H}(1,3,-2) = 6\boldsymbol{a}_x + \boldsymbol{a}_y + 4\boldsymbol{a}_z$

$$\boldsymbol{a}_H = \frac{(6,1,4)}{\sqrt{36+1+16}} = \underline{\underline{0.8242\boldsymbol{a}_x + 0.1374\boldsymbol{a}_y + 0.5494\boldsymbol{a}_z}}$$

(b) $|\,\boldsymbol{H}\,| = 10 = \sqrt{4x^2y^2 + (x+z)^2 + z^4}$

or

$$\underline{\underline{100 = 4x^2y^2 + x^2 + 2xz + z^2 + z^4}}$$

**Prob. 1.28**

$\boldsymbol{R} = R\boldsymbol{a}_R, \qquad R = 4$

$$\boldsymbol{a}_R = \frac{\boldsymbol{P}\times\boldsymbol{Q}}{|\boldsymbol{P}\times\boldsymbol{Q}|}$$

$$\boldsymbol{P}\times\boldsymbol{Q} = \begin{vmatrix} 2 & -4 & 1 \\ 1 & 2 & 0 \end{vmatrix} = -2\boldsymbol{a}_x + \boldsymbol{a}_y + \mathbf{8}a_z$$

$$\boldsymbol{a}_R = \frac{-2\boldsymbol{a}_x + \boldsymbol{a}_y + \mathbf{8}a_z}{\sqrt{4+1+64}} = -0.2408\boldsymbol{a}_x + 0.1204\boldsymbol{a}_y + 0.9631\boldsymbol{a}_z$$

$$\boldsymbol{R} = R\boldsymbol{a}_R = 4(-0.2408\boldsymbol{a}_x + 0.1204\boldsymbol{a}_y + 0.9631\boldsymbol{a}_z) = \underline{\underline{-0.9631\boldsymbol{a}_x + 0.4815\boldsymbol{a}_y + 3.852\boldsymbol{a}_z}}$$

An alternate choice of $\boldsymbol{R}$ is $-0.9631\boldsymbol{a}_x + 0.4815\boldsymbol{a}_y + 3.852\boldsymbol{a}_z$

**Prob. 1.29**

(a) At (1, -2, 3), x = 1, y = -2, z = 3.

$\boldsymbol{G} = \boldsymbol{a}_x + 2\boldsymbol{a}_y + 6\boldsymbol{a}_z, \qquad \boldsymbol{H} = -6\boldsymbol{a}_x + 3\boldsymbol{a}_y - 3\boldsymbol{a}_z$

$$G = \sqrt{1+4+36} = \underline{\underline{6.403}}$$

$$H = \sqrt{36+9+9} = \underline{\underline{7.348}}$$

**(b)** $\boldsymbol{G}\cdot\boldsymbol{H} = -6+6-18 = \underline{\underline{-18}}$

**(c )**
$$\cos\theta_{GH} = \frac{\boldsymbol{G}\cdot\boldsymbol{H}}{GH} = \frac{-18}{6.403\times 7.348} = -0.3826$$

$$\underline{\underline{\theta_{GH} = 112.5^o}}$$

**Prob. 1.30**

(a) $\boldsymbol{H} = 10(2)(16)\boldsymbol{a}_x - 8(-8)\boldsymbol{a}_y + 12(4)\boldsymbol{a}_z = 320\boldsymbol{a}_x + 64\boldsymbol{a}_y + 48\boldsymbol{a}_z$

Let $\boldsymbol{F} = \boldsymbol{a}_x - \boldsymbol{a}_y$

(b)
$$\boldsymbol{H}_F = (\boldsymbol{H}\bullet\boldsymbol{a}_F)\boldsymbol{a}_F = \frac{(\boldsymbol{H}\bullet\boldsymbol{F})\boldsymbol{F}}{F^2} = \frac{(320-64)(1,-1,0)}{1+1} = 128\boldsymbol{a}_x - 128\boldsymbol{a}_y$$

**Prob. 1.31**

(a) At (1,2,3), $\boldsymbol{E}$ = (2,1,6)

$$\left|\boldsymbol{E}\right| = \sqrt{4+1+36} = \sqrt{41} = \underline{\underline{6.403}}$$

(b) At (1,2,3), $\boldsymbol{F}$ = (2,-4,6)

$$\boldsymbol{E_F} = (\boldsymbol{E} \cdot \boldsymbol{a_F})\boldsymbol{a_F} = \frac{(\boldsymbol{E} \cdot \boldsymbol{F})\boldsymbol{F}}{|\boldsymbol{F}|^2} = \frac{36}{56}(2,-4,6)$$

$$= \underline{\underline{1.286\boldsymbol{a_x} - 2.571\boldsymbol{a_y} + 3.857\boldsymbol{a_z}}}$$

(c) At (0,1,-3), $\boldsymbol{E}$ = (0,1,-3), $\boldsymbol{F}$ = (0,-1,0)

$$\boldsymbol{E} \times \boldsymbol{F} = \begin{vmatrix} 0 & 1 & -3 \\ 0 & -1 & 0 \end{vmatrix} = (-3,0,0)$$

$$\boldsymbol{a}_{\boldsymbol{E}\times\boldsymbol{F}} = \pm \frac{\boldsymbol{E} \times \boldsymbol{F}}{|\boldsymbol{E} \times \boldsymbol{F}|} = \underline{\underline{\pm\, \boldsymbol{a_x}}}$$

**Prob. 1.32**

(a) At P, x = -1, y = 2, z = 4

$$\boldsymbol{D} = 8\boldsymbol{a}_x - 4\boldsymbol{a}_y - 2\boldsymbol{a}_z, \qquad \boldsymbol{E} = -10\boldsymbol{a}_x + 24\boldsymbol{a}_y + 128\boldsymbol{a}_z$$

$$\boldsymbol{C} = \boldsymbol{D} + \boldsymbol{E} = -2\boldsymbol{a}_x + 20\boldsymbol{a}_y + 126\boldsymbol{a}_z$$

(b)

$$\boldsymbol{C} \cdot \boldsymbol{a}_x = C\cos\theta_x \longrightarrow \cos\theta_x = \frac{\boldsymbol{C} \cdot \boldsymbol{a}_x}{C} = \frac{-2}{\sqrt{2^2 + 20^2 + 126^2}} = -0.01575$$

$$\underline{\underline{\theta_x = 90.9^o}}$$