Discussion $\boxed{6.3}$

$\boxed{4}$ Find length of the curve

$x = \dfrac{y^{3/2}}{3} - y^{\frac{1}{2}}$ from $y=1$ to $y=9$

$\dfrac{dx}{dy} = \frac{3}{2}\,\frac{1}{3}\,y^{\frac{1}{2}} - \frac{1}{2}\,y^{-\frac{1}{2}} = \frac{1}{2}\left(\sqrt{y} - \frac{1}{\sqrt{y}}\right)$ cont. on $[1, 9]$

$\left(\dfrac{dx}{dy}\right)^2 = \frac{1}{4}\left(\sqrt{y} - \frac{1}{\sqrt{y}}\right)^2 = \frac{1}{4}\left(y - 2 + \frac{1}{y}\right)$

$L = \displaystyle\int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy = \int_1^9 \sqrt{1 + \frac{1}{4}\left(y - 2 + \frac{1}{y}\right)}\,dy$

$= \displaystyle\int_1^9 \sqrt{1 - \frac{1}{2} + \frac{1}{4}\left(y + \frac{1}{y}\right)}\,dy = \int_1^9 \sqrt{\frac{1}{2} + \frac{1}{4}\left(y + \frac{1}{y}\right)}\,dy$

$= \displaystyle\int_1^9 \sqrt{\frac{1}{4}\left(y + 2 + \frac{1}{y}\right)}\,dy = \frac{1}{2}\int_1^9 \sqrt{\left(\sqrt{y} + \frac{1}{\sqrt{y}}\right)^2}\,dy$

$= \displaystyle\frac{1}{2}\int_1^9 \left(y^{\frac{1}{2}} + y^{-\frac{1}{2}}\right)dy = \frac{1}{2}\left(\frac{2}{3}\,y^{\frac{3}{2}} + 2y^{\frac{1}{2}}\right)\Big|_1^9$

$= \displaystyle\frac{1}{3}\sqrt{y^3} + \sqrt{y}\,\Big|_1^9 = \left(\frac{1}{3}\sqrt{9\times9\times9} + \sqrt{9}\right) - \left(\frac{1}{3}\sqrt{1} + \sqrt{1}\right)$

$= \left(\cancel{\frac{1}{3}}(9)\cancel{(3)} + 3\right) - \left(\frac{1}{3} + 1\right) = 9 + 3 - \frac{1}{3} - 1$

$= 11 - \frac{1}{3} = \dfrac{32}{3}$

$\boxed{19}$. Find a curve through the point (1,1) whose length integral is

$$L = \int_1^4 \sqrt{1 + \frac{1}{4x}}\, dx$$

- How many such curves are there? Give reasons.

- Comparing with $L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx \Rightarrow$

$$(f'(x))^2 = \frac{1}{4x} \Rightarrow f'(x) = \frac{1}{2\sqrt{x}}$$

since $|x| = x$ in this case

$$f(x) = \int \frac{1}{2\sqrt{x}}\, dx$$

$$f(x) = \sqrt{x} + c$$

Since the curve $f(x)$ passes through $(1,1) \Rightarrow$

$$f(1) = 1$$

$$\sqrt{1} + c = 1 \Rightarrow c = 0$$

- This is the only curve passing through (1,1)