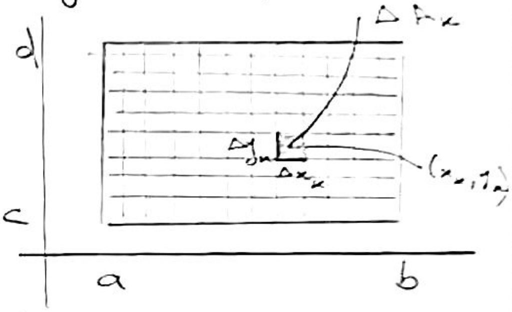
15.1 Double and Iterated Integrals over Rectangles.

We defined the definite Integral of a cont. $f(x)$ over $[a,b]$ as a limit of Riemann Sum, Now:

How to construct double Integral?

Assume $f(x,y)$ is defined on a rectangle Region R

$R: a \le x \le b,\ c \le y \le d$



- We subdivide R into n small rectangles with ~~base~~ width Δx & height Δy.
- Each small rectangle has area:

  $\Delta A = \Delta x \Delta y$
- These small n rectangles form a partition P
- The number n gets large as Δx & Δy get smaller.
- If we order the areas $\Delta A_1, \Delta A_2, \dots, \Delta A_n$, and in each ΔA_k we chouse a point (x_k, y_k) and evaluate $f(x_k, y_k)$, then:

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The Reimann Sum over R is:

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k$$

- As $\Delta x \longrightarrow 0$ & $\Delta y \longrightarrow 0$, the norm of the Partition $\|P\| = \max\{\Delta x, \Delta y\} \longrightarrow 0$ for any rectangle.

Hence $n \longrightarrow \infty$

Therefore:

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k$$

$$= \lim_{\|P\|} \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k$$

$$= \iint_R f(x,y)\,\underline{dA}$$

$$= \int_c^d \int_a^b f(x,y)\,dx\,dy$$

يعني المسح ب xy بيطلع لنا
لنفسه ابعاد z

<u>Remark</u>: If $f(x,y)$ is positive over a rectangle Region R then the double Integral is the Volume of the 3-dimensional solid over the $\underline{xy\text{-plane}}$. bounded below by R & above by the surface $z = f(x,y)$.

$$\text{Volume} = \iint_R f(x,y)\,dA \quad \longleftarrow$$

Iterated
or repeated Integral

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Example: Find the volume under the plane $Z = 4 - x - y$ over the rectangle region $R: 0 \le x \le 2, \ 0 \le y \le 1$

$$V = \int_0^1 \int_0^2 (4 - x - y)\,dx\,dy = \int_0^1 \left[4x - \frac{x^2}{2} - yx \Big|_0^2 \right] dy$$

$$= \int_0^1 [8 - 2 - 2y]\,dy = 6y - y^2 \Big|_0^1 = 5$$

or:

$$V = \int_0^2 \int_0^1 (4 - x - y)\,dy\,dx = \int_0^2 \left[4y - xy - \frac{y^2}{2} \right] \Big|_0^1 dx$$

$$= \int_0^2 \left[4 - x - \tfrac{1}{2} \right] dx = 3.5x - \frac{x^2}{2} \Big|_0^2 = 7 - 2 = 5$$

Thm: Fubini's Theorem (First form)

If $f(x,y)$ is continuous throughout the rectangle $R: a \le x \le b, \ c \le y \le d$, then:

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

Example: Find the Volume bounded above by $Z = 2\sin x \cos y$ & below by $0 \le x \le \frac{\pi}{2}, \ 0 \le y \le \frac{\pi}{4}$

$$V = \int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{2}} 2\sin x \cos y\,dx\,dy = \sqrt{2}$$

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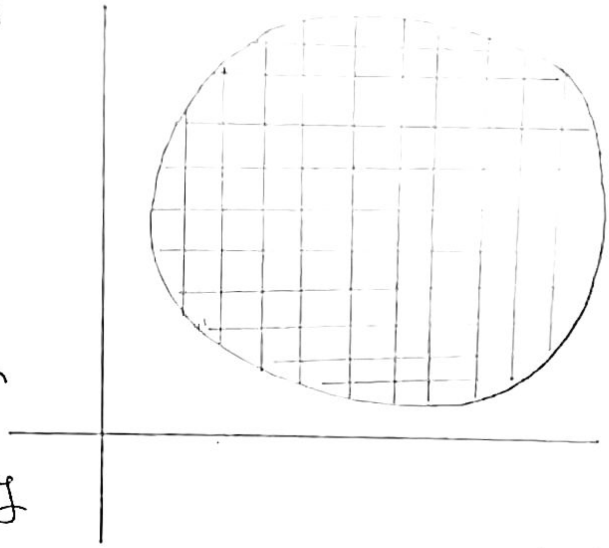
15.2 Double Integrals over General Regions:

We define a double Integral of a function $f(x,y)$ over bounded general region R similarly as we did in the previous section but we approximately cover R by a grid of small rectangles that are completely inside R.



That is

$$\iint_R f(x,y)\,dA = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k .$$

- If $z = f(x,y)$ is positive and continuous, then

$$\text{Volume} = \iint_R f(x,y)\,dA .$$

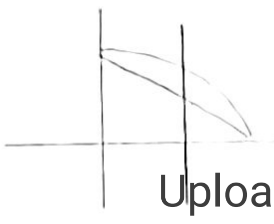
Thm: (Fubini's Theorem: Stronger form)

Let $f(x,y)$ be continuous on a region R:

1) If $R: a \le x \le b,\ g_1(x) \le y \le g_2(x)$ (s.t) g_1 & g_2 are continuous on $[a,b]$, then

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

(vertical cross section)



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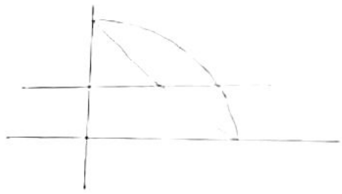
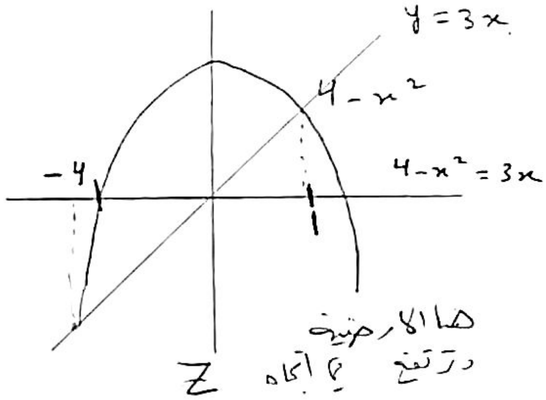
2) If $R: c \leq y \leq d$, $h_1(y) \leq x \leq h_2(y)$ with h_1 and h_2 Continuous on $[c,d]$, then:

$$\iint_R f(x,y)\, dA = \int_c^d \int_{h_1(x)}^{h_2(x)} f(x,y)\, dx\, dy.$$

(Horizontal cross section)



__Example:__ Find the Volume of the solid whose base is the region in the xy-plane that is bounded by the parabola $y = 4 - x^2$ and the line $y = 3x$ while the top of the solid is bounded by the plane $Z = x + 4$.



Vertical:

$4 - x^2 = 3x$

$x^2 + 3x - 4 = 0 \Rightarrow x = -4, 1.$

$$V = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx$$

$$= \int_{-4}^{1} \int_{3x}^{4-x^2} (x+4)\, dy\, dx = \int_{-4}^{1} \Big[(x+4)y\Big]_{3x}^{4-x^2} dx$$

$$= \int_{-4}^{1} \Big[(x+4)(4-x^2) - (x+4)(3x)\Big] dx = \dots = \frac{625}{12}$$

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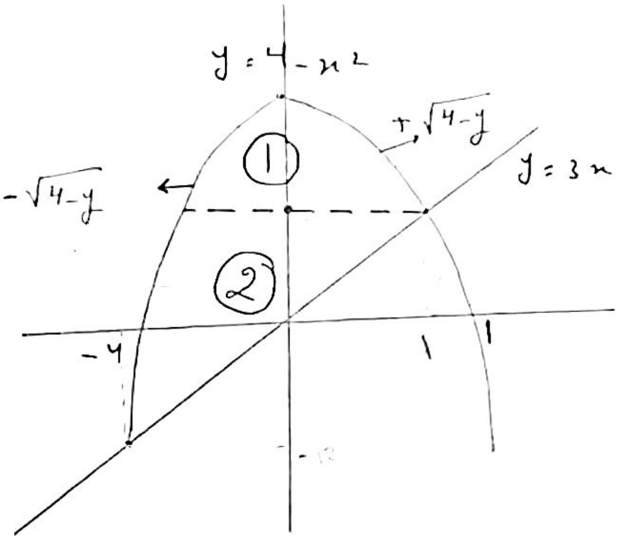
If we want to change the order of $dy\,dx$

$\Rightarrow \quad x = \frac{y}{3}$

$x = \pm\sqrt{4-y}$

we have two regions:



Then :

$$\int_{-4}^{1}\int_{3x}^{4-x^2} (x+4)\,dy\,dx = \int_{3}^{4}\int_{-\sqrt{4-y}}^{\sqrt{4-y}} (x+4)\,dx\,dy + \int_{-12}^{3}\int_{-\sqrt{4-y}}^{y/3} (x+4)\,dx\,dy$$

$$= \int_{3}^{4} \left(8\sqrt{4-y}\right) dy + \int_{-12}^{3} \frac{y^2 + 33y + 72\sqrt{4-y} - 36}{18}\, dy$$

$$= \frac{16}{3} + \frac{187}{4} = \frac{64+561}{12} = \frac{625}{12}.$$

$\left(\frac{2}{2}\right)$ (104)

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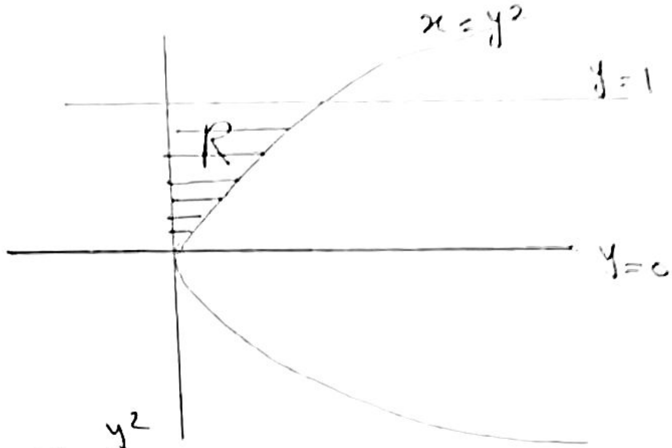
Example: Sketch the Region of Integration and evaluate the integral:



$$\int_0^1 \int_0^{y^2} 3y^3 e^{xy} \, dx \, dy$$

$$= \int_0^1 \frac{3y^3}{y} e^{xy} \Big|_0^{y^2} dy = \int_0^1 3y^2 e^{xy} \Big|_0^{y^2} dy$$

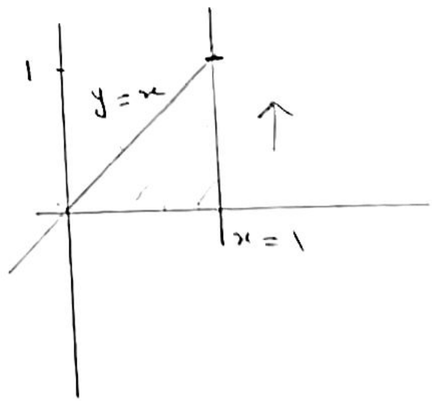
$$= \int_0^1 3y^2 \left[e^{y^3} - 1 \right] dy = \int_0^1 3y^2 e^{y^3} dy - \int_0^1 3y^2 \, dy$$

$$= e^{y^3} \Big|_0^1 - y^3 \Big|_0^1 = e - 1 - 1 = e - 2$$

Example: Evaluate $\iint_R \frac{\sin x}{x} \, dA$ where R is triangle in the xy plane bounded by x-axis & $y = x$ & $x = 1$

① $\int_0^1 \int_0^x \frac{\sin x}{x} \, dy \, dx$ (vertical) $= \int_0^1 \frac{\sin x}{x} y \Big|_0^x dx$

$$= \int_0^1 \sin x \, dx = -\cos x \Big|_0^1 = 1 - \cos 1$$



② $\int_0^1 \int_y^1 \frac{\sin x}{x} \, dx \, dy = 1 - \cos 1$

Note: It's hard here to Integrate $\frac{\sin x}{x} (dx)$

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Properties of Double Integrals:

If $f(x,y)$ & $g(x,y)$ are continuous on bounded region R, then:

1) $$\iint_R c\, f(x,y)\, dA = c \iint_R f(x,y)\, dA$$

2) $$\iint_R \big(f(x,y) \pm g(x,y)\big)\, dA = \iint_R f(x,y)\, dA \pm \iint_R g(x,y)\, dA$$

3) ⓐ $$\iint_R f(x,y)\, dA \geq 0 \quad \text{if} \quad f(x,y) \geq 0 \text{ on } R$$

ⓑ $$\iint_R f(x,y)\, dA \geq \iint_R g(x,y)\, dA \quad \text{if} \quad f(x,y) \geq g(x,y) \text{ on } R$$

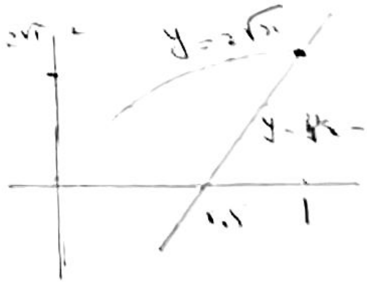
4) If $R = R_1 \cup R_2$, then: (R_1 & R_2 nonoverlapping)

$$\iint_R f(x,y)\, dA = \iint_{R_1} f(x,y)\, dA + \iint_{R_2} f(x,y)\, dA$$

Example: Find the volume of the solid that lies beneath $z = 16 - x^2 - y^2$ & above R bounded by $y = 2\sqrt{x}$ & $y = 4x - 2 \rightarrow = 0 \Rightarrow x = 0.5$

$4x - 2 = 2\sqrt{x} \xrightarrow{x=1} \Rightarrow x \in [0.5, 1]$

horizontal cross ...



$$V = \int_0^2 \int_{\frac{y^2}{4}}^{\frac{y+2}{4}} (16 - x^2 - y^2)\, dx\, dy = \cdots = \frac{20803}{1680}$$

≈ 12.3827

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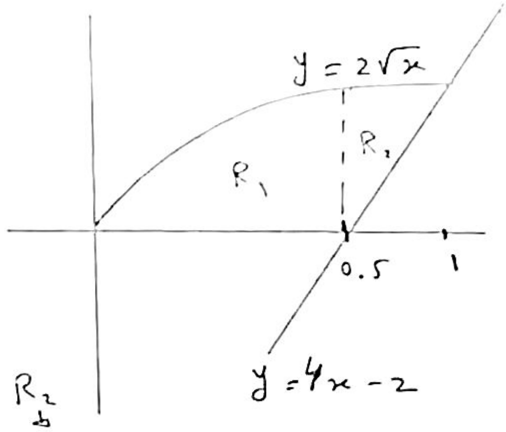
OR:

$$\text{volume} = \iint_R (16-x^2-y^2)\, dA \qquad R = R_1 \cup R_2$$



$$= \underbrace{\int_0^{0.5}\int_0^{2\sqrt{x}} (16-x^2-y^2)\, dy\, dx}_{R_1} + \underbrace{\int_{0.5}^{1}\int_{4x-2}^{2\sqrt{x}} (16-x^2-y^2)\, dy\, dx}_{R_2}$$

vertical

$$= \int_0^{0.5} -\frac{\sqrt{x}\,(6x^2+8x-96)}{3}\, dx + \int_{0.5}^{1} -\frac{\left(-76x^3+\sqrt{x}\,(6x^2+8x-96)+102x^2+144x-88\right)}{3}$$

$$= \frac{23991642}{3284995} + \frac{52916316}{10417961} \approx 12.3827.$$

Example: Find

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dx\, dy}{(x^2+1)(y^2+1)} = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \frac{1\, dx}{x^2+1}\right] \frac{dy}{y^2+1} \quad (*)$$

$$(*) \int_0^{\infty} \frac{1}{x^2+1}\, dx = \lim_{b\to\infty} \int_0^b \frac{1}{x^2+1}\, dx = \lim_{b\to\infty} \tan^{-1}\Big|_0^b = \frac{\pi}{2}$$

$$\Rightarrow \int_{-\infty}^{\infty} 2\left(\frac{\pi}{2}\right) \frac{dy}{y^2+1} = \pi \int_{-\infty}^{\infty} \frac{dy}{y^2+1} = \pi\,(2)\int_0^{\infty} \frac{dy}{(y^2+1)} = \boxed{\pi^2}$$

$$\left(\frac{2}{2}\right)(106)$$

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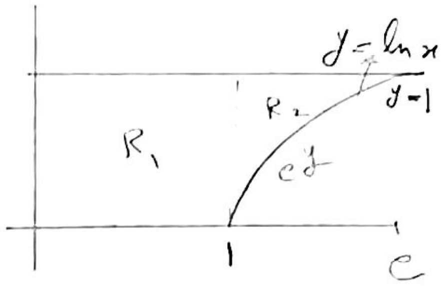
(16) Using horizontal & vertical cross section

Write $\iint_A dA$



$y = 0$, $x = 0$, $y = 1$, $y = \ln x$

(a) Vertical cross section

$$\int_0^1 \int_0^1 dy\, dx + \int_1^e \int_{\ln x}^1 dy\, dx$$

(b) horizontal cross section: $\displaystyle\int_0^1 \int_0^{e^y} dx\, dy$

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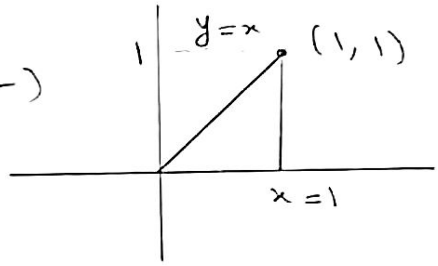
(49) $\displaystyle\int_0^1 \int_y^1 x^2 e^{xy}\, dx\, dy$ (Reverse order)

NO

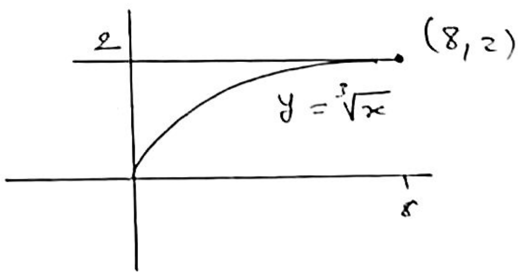


$$= \int_0^1 \int_0^x x^2 e^{xy}\, dy\, dx = \frac{e-2}{2}$$

---

(54) $\displaystyle\int_0^8 \int_{\sqrt[3]{x}}^2 \frac{1}{y^4+1}\, dy\, dx$ (Reverse order)

NO



$$= \int_0^2 \int_0^{y^3} \frac{1}{y^4+1}\, dx\, dy = \frac{\ln 17}{4}$$

$\left(\frac{1}{3}\right)(107)$

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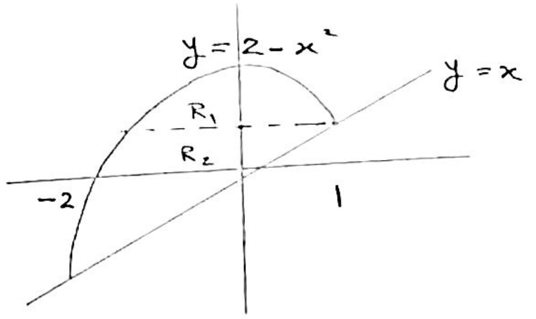
(58) Find the Volume of the solid that is bounded above by the Cylinder $z = x^2$ and below by the region enclosed by the parabola $y = 2 - x^2$ & $y = x$ in xy plane.

$2 - x^2 = x \Rightarrow x^2 + x - 2 = 0$

$\Rightarrow x = -2$ & 1



$$V = \int_{-2}^{1}\int_{x}^{2-x^2} x^2 \, dy \, dx = \frac{63}{20}$$

$$\approx 3.15$$

OR: $y = 2 - x^2 \Rightarrow x = \pm\sqrt{2-y}$

$$V = \int_{1}^{2}\int_{-\sqrt{2-y}}^{\sqrt{2-y}} x^2 \, dx \, dy + \int_{-2}^{1}\int_{-\sqrt{2-y}}^{y} x^2 \, dx \, dy$$

$$= \int_{1}^{2} -\frac{2\sqrt{2-y}\,(y-2)}{3} \, dy + \int_{-2}^{1} \frac{y^3}{3} - \frac{\sqrt{2-y}\,(y-2)}{3} \, dy$$

$$= \frac{4}{15} + \frac{173}{60} \approx 3.15$$

$\left(\frac{2}{3}\right)(107)$

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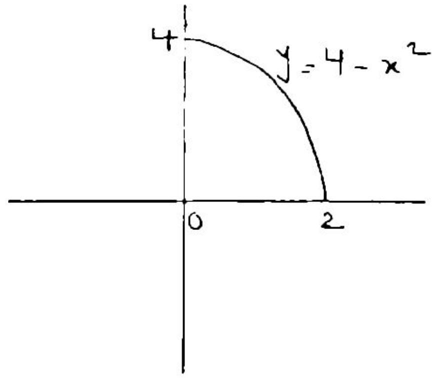
15.2

62 Find the Volume of the solid cut from the first octant by the surface $Z = 4 - x^2 - y$.

In the xy plane, we assume $Z = 0$



$$\Rightarrow \quad 4 - x^2 - y = 0$$

$$\Rightarrow \quad y = 4 - x^2.$$

$$V = \int_0^2 \int_0^{4-x^2} (4 - x^2 - y)\, dy\, dx = \frac{128}{15}$$

OR:

$$V = \int_0^4 \int_0^{\sqrt{4-y}} (4 - x^2 - y)\, dx\, dy = \frac{128}{15}$$

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15.3 Area by Double Integration:

Recall: The Riemann sum in the definition of double Integral is:

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\, \Delta A_k .$$

- If we let $f(x,y) = 1$, then the Riemann Sum becomes,

$$S_n = \sum_{k=1}^{n} \Delta A_k$$

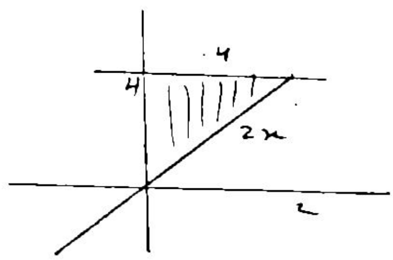
- Def: The area of a closed, bounded plane region R is

$$A = \lim_{n \to \infty} \sum_{k=1}^{n} \Delta A_k = \iint_R dA .$$

Example: Find the area of the Region R bounded by:

1) $x = 0$, $y = 2x$, $y = 4$

$$A = \iint_R dA = \int_0^2 \int_{y=2x}^{y=4} dy\, dA$$



$$= \int_0^4 \int_{x=0}^{x=\frac{y}{2}} dx\, dy = 4$$

(or 1 check)

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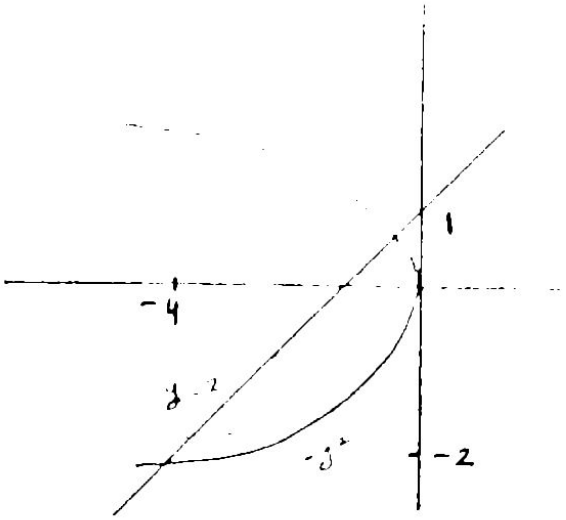
2) $x = -y^2 \ , y = x + 2$.



$$y = -y^2 + 2$$

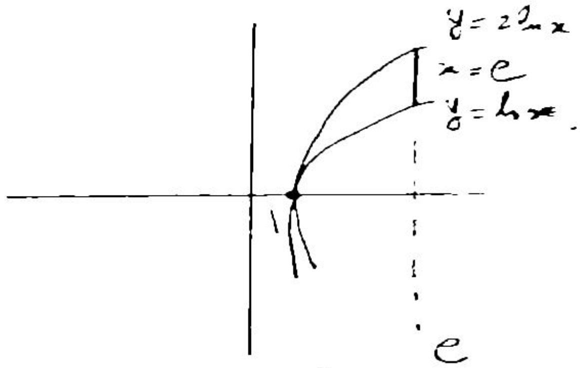
$$y^2 + y - 2 = 0$$

$$\therefore \quad y = -2 \ \& \ y = 1$$

$$A = \iint_A dA = \int_{-2}^{1} \int_{y-2}^{-y^2} dx\, dy = \frac{9}{2}.$$

---

3) $y = \ln x \ , y = 2\ln x \ , x = e$



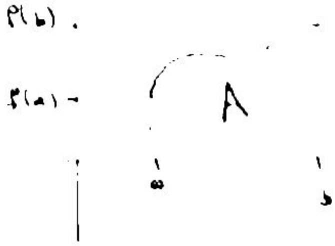
$$A = \int_1^e \int_{\ln x}^{2\ln x} dy\, dx$$

$$= \int_1^e y \Big|_{\ln x}^{2\ln x} dx = \int_1^e \overset{\text{بالاجزاء}}{\ln x\, dx} = x\ln x - x \Big]_1^e = \boxed{1}$$

<u>Average Value</u> : On bounded Region R

$$\text{Average value of } f \text{ over } R = \frac{1}{\text{area of } R} \iint_R f\, dA$$

In <u>1D</u> : $av(f) = \dfrac{1}{(b-a)} \displaystyle\int_a^b f(x)\,dx$



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Example: Find the average value of $f(x,y) = \frac{1}{xy}$ over the square $\ln 2 \le x \le 2\ln 2$ & $\ln 2 \le y \le 2\ln 2$

$$A = \int_{\ln 2}^{2\ln 2} \int_{\ln 2}^{2\ln 2} dx\, dy = \ln 2 \ln 2 = (\ln 2)^2$$

$$av(f) = \frac{1}{(\ln 2)^2} \int_{\ln 2}^{2\ln 2} \int_{\ln 2}^{2\ln 2} f(x,y)\, dx\, dy$$

$$= \frac{1}{(\ln 2)^2} \int_{\ln 2}^{2\ln 2} \int_{\ln 2}^{2\ln 2} \frac{1}{xy}\, dx\, dy$$

$$= \frac{1}{(\ln 2)^2} \int_{\ln 2}^{2\ln 2} \frac{1}{y} \ln x \Big|_{\ln 2}^{2\ln 2} dy$$

$$= \frac{1}{(\ln 2)^2} \int_{\ln 2}^{2\ln 2} \frac{1}{y} \Big(\ln(\ln 4) - \ln(\ln 2) \Big)\, dy$$

$$\ln\left(\frac{\ln 4}{\ln 2}\right) \cdot \ln\left(\frac{2\ln 2}{\ln 2}\right) = \ln 2$$

$$= \frac{1}{\ln 2} \int_{\ln 2}^{2\ln 2} \frac{1}{y}\, dy$$

$$= \frac{1}{\ln 2} (\ln y) \Big|_{\ln 2}^{2\ln 2} = \frac{1}{\ln 2} \cdot \ln 2 = \boxed{1}$$

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(4) sketch the region bounded by given lines and curves then express the region's area as a double Integral.

$x = y - y^2$ & $y = -x$ $\qquad x = y - y^2$

$y = -x$

$$A = \int_0^2 \int_{-y}^{y-y^2} dx\, dy$$

$$= \int_0^2 (2y - y^2)\, dy = \frac{4}{3}$$

---

(21) Find the average height of the paraboloid $z = x^2 + y^2$ over the square $0 \le x \le 2, 0 \le y \le 2$.

$$av(f) = \frac{1}{area} \int_0^2 \int_0^2 (x^2 + y^2)\, dy\, dx$$

$$area = \int_0^2 \int_0^2 dy\, dx = \int_0^2 y \Big|_0^2 dx = \int_0^2 2\, dx = 2x \Big|_0^2 = 4$$

$$\therefore av(f) = \frac{1}{4} \int_0^2 \int_0^2 (x^2 + y^2)\, dy\, dx = \frac{1}{4} \int_0^2 \left(x^2 y + \frac{y^3}{3}\right)\Big|_0^2 dx$$

$$= \frac{1}{4} \int_0^2 \left[2x^2 + \frac{8}{3}\right] dx = \frac{1}{4} \left[\frac{2x^3}{3} + \frac{8}{3}x\right]_0^2$$

$$= \frac{1}{4} \left[\frac{16}{3} + \frac{16}{3}\right] = \frac{1}{4} \left[\frac{32}{3}\right] = \boxed{\frac{8}{3}}$$

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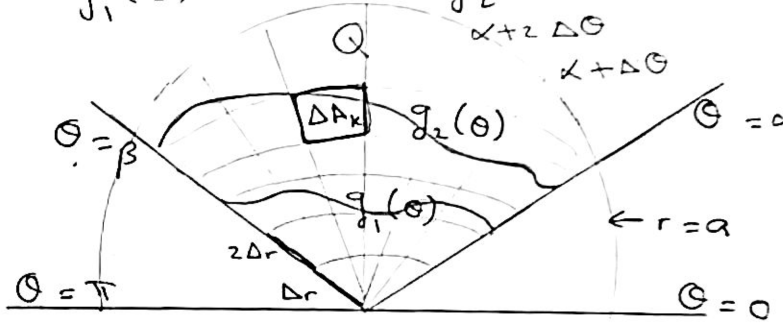
15.4 Double Integrals in Polar form:

Recall:
$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$r^2 = x^2 + y^2$$
$$\frac{y}{x} = \tan\theta.$$

Suppose that a function $f(r,\theta)$ is defined over a region R that is bounded by the rays $\theta = \alpha$ & $\theta = \beta$ and by continuous curves $r = g_1(\theta)$ & $r = g_2(\theta)$.



Suppose also $0 \le g_1(\theta) \le g_2(\theta) \le \overset{r=a}{a}$, $\forall \theta \in [\alpha, \beta]$

Then R Lies in a fanshaped region Q
$$0 \le r \le a \quad \& \quad \alpha \le \theta \le \beta.$$

We Cover Q by a grid of Circular arcs and rays "polar rectangles"

arcs have radius: $\Delta r, 2\Delta r, \ldots$, $\boxed{\Delta r = a/m}$

rays are given by: $\theta = \alpha$; $\theta = \alpha + \Delta\theta, \ldots, \alpha + \Delta\theta m' = \beta$

$\boxed{\Delta\theta = (\beta - \alpha)/m'}$

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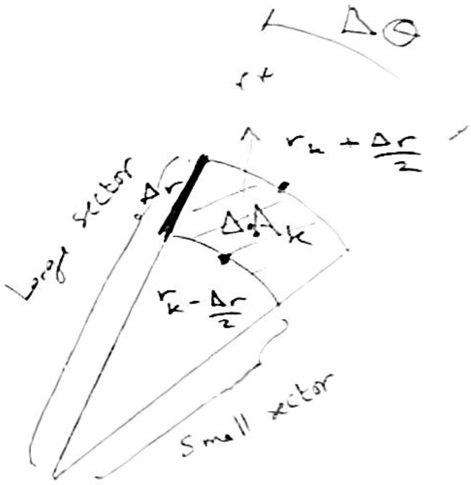
- We number the polar rectangles that lie inside R

$\Delta A_1, \Delta A_2, \ldots, \Delta A_n$, Let $(r_k, \theta_k) \in$ polar rectangle k

Then: $S_n = \sum_{k=1}^{n} f(r_k, \theta_k)\, \Delta A_k$

Now Let Δr & $\Delta\theta \longrightarrow 0$, then:

$$\lim_{n\to\infty} S_n = \iint_R f(r,\theta)\, dA.$$



<u>How to find ΔA_k ??</u>

The area of a sector of a circle is

$$A = \frac{1}{2}\theta \cdot r^2$$

$\Longrightarrow$ Area of small sector:

$$A_{small} = \frac{1}{2}\left(r_k - \frac{\Delta r}{2}\right)^2 \Delta\theta$$

$$A_{large} = \frac{1}{2}\left(r_k + \frac{\Delta r}{2}\right)^2 \Delta\theta.$$

$\Longrightarrow$ ΔA_k = area of Large sector − area of small sector.

$$= \frac{\Delta\theta}{2}\left(\left(r_k + \frac{\Delta r}{2}\right)^2 - \left(r_k - \frac{\Delta r}{2}\right)^2\right)$$

$$= \frac{\Delta\theta}{2}(2 r_k \Delta r) = r_k \Delta r \Delta\theta$$

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Then $S_n = \sum_{k=1}^{n} f(r_k, \theta_k)\, r_k \,\Delta r\, \Delta\theta$.

as $n \longrightarrow \infty$ & $\Delta r \longrightarrow 0$ & $\Delta\theta \longrightarrow 0$, then

$$\lim_{n\to\infty} S_n = \iint_R f(r,\theta)\, \underline{r}\, dr\, d\theta.$$

A version of Fubini's Theorem:

$$\iint_R f(r,\theta)\, dA = \int_{\alpha}^{\beta} \int_{r=g_1(\theta)}^{r=g_2(\theta)} f(r,\theta)\, \underline{\underline{r}}\, dr\, d\theta$$

Remark: 1) If f is positive and Continuous, then:

$$\iint_R f(r,\theta)\, dA = \text{Volume in polar Coordinates.}$$

2) If $f(r,\theta) = 1$, then $\iint_R f(r,\theta)\, dA$ = area in Polar Coordinates.

$\Longleftrightarrow$ $A = \iint_R r\, dr\, d\theta$, where R closed & bounded.

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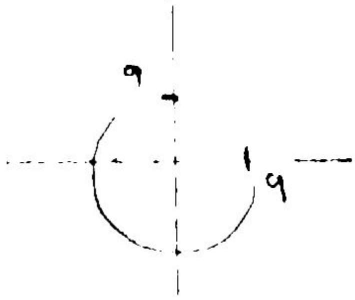
outline

Example: 1 Describe the given region in polar Coordinates:

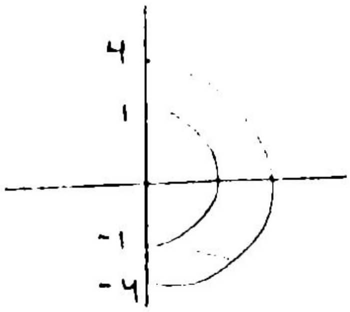
1) $x^2 + y^2 = 9^2$

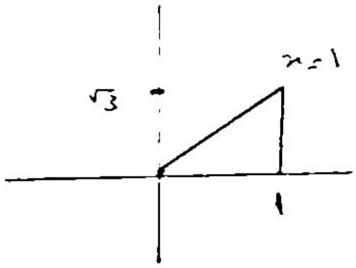
$r^2 = 81$

$\Rightarrow \quad r = 9 \quad \& \quad \frac{\pi}{2} \le \theta \le 2\pi$



2) $1 \le r \le 4 \quad \& \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$





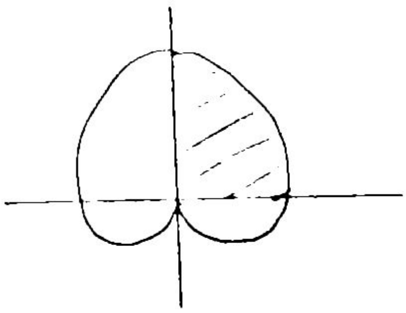
3) $x = 1 \Rightarrow r\cos\theta = 1$

then $r = \sec\theta. \Rightarrow 0 \le r \le \sec\theta$

& $\tan\theta = \sqrt{3} \Rightarrow \theta = \frac{\pi}{3} \Rightarrow 0 \le \theta \le \frac{\pi}{3}$

Example 31: Find the area of the region cut from the first quadrant by the Cardioid $r = 1 + \sin\theta$.

$$A = \int_0^{\frac{\pi}{2}}\int_0^{1+\sin\theta} r\,dr\,d\theta = \int_0^{\frac{\pi}{2}} \left[\frac{r^2}{2}\Big|_0^{1+\sin\theta}\right] d\theta$$



$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (1+\sin\theta)^2\, d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} \left(1 + 2\sin\theta + \sin^2\theta\right) d\theta$$

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$$= \frac{1}{2}\left[\int_0^{\frac{\pi}{2}} 1 + 2\sin\theta + \frac{1-\cos 2\theta}{2}\right] d\theta$$

$$= \frac{1}{2}\left[\theta + -2\cos\theta + \frac{1}{2}\left(\theta - \frac{\sin 2\theta}{2}\right)\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[\frac{\pi}{2} + \frac{\pi}{4} + 2\right] = \frac{3\pi}{8} + 1$$

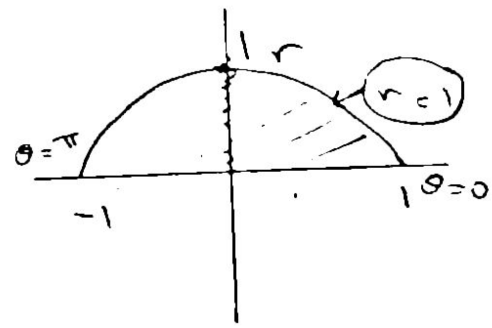
<u>Remark:</u> Changing Cartesian integrals into Polar Integrals:

$$\iint_R f(x,y)\,dx\,dy = \iint_Q f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

<u>Why Polar Integrals are important?</u>

<u>Example:</u> Find $\iint_R e^{x^2+y^2}\,dy\,dx$ where R is the semicircular region R bounded by the x-axis & $y=\sqrt{1-x^2}$.

$$\iint_R e^{x^2+y^2}\,dy\,dx = \int_0^{\pi}\int_0^{1} e^{r^2}\,r\,dr\,d\theta$$



$$= \int_0^{\pi}\left[\frac{1}{2}e^{r^2}\Big|_0^1\right]d\theta = \int_0^{\pi}\frac{1}{2}(e-1)\,d\theta$$

$$= \frac{1}{2}(e-1)\,\theta\Big|_0^{\pi} = \frac{\pi}{2}(e-1).$$

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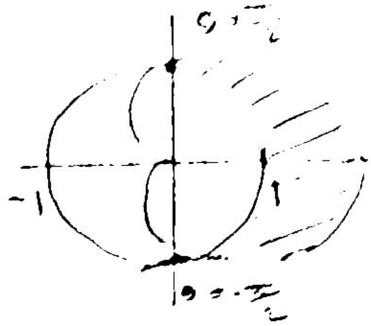
Example: Find the limits of Integration for integrating $f(r,\theta)$ over the region R that lies inside the Cardioid $r = 1 + \cos\theta$ and outside the Circle $r = 1$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{1}^{1+\cos\theta} f(r,\theta)\, r\, dr\, d\theta$$



in the above example ↑

(39) If the Top of R ↑ is the plane $Z = x$ [$f(x,y)$]. Find the Volume.

$$V = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{1}^{1+\cos\theta} \underline{r \cos\theta}\; r\, dr\, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{1}^{1+\cos\theta} r^2 \cos\theta\, dr\, d\theta$$

$$= 2 \int_{0}^{\frac{\pi}{2}} \int_{1}^{1+\cos\theta} r^2 \cos\theta\, dr\, d\theta$$

$$= \frac{2}{3} \int_{0}^{\frac{\pi}{2}} \left(3\cos^2\theta + 3\cos^3\theta + \cos^4\theta \right) d\theta$$

$$= \frac{2}{3} \left[\frac{15\theta}{8} + \sin 2\theta + 3\sin\theta - \sin^3\theta + \frac{\sin 4\theta}{32} \right]_0^{\frac{\pi}{2}}$$

$$= \frac{4}{3} + \frac{5\pi}{8}.$$

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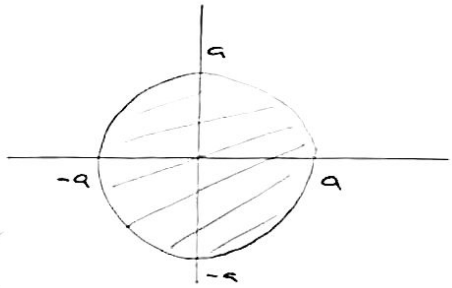
The average value of f over R is:

$$av(f) = \frac{1}{area(R)} \iint_R f(r,\theta)\, r\, dr\, d\theta.$$

Example: 33 Find the average height of the hemispherical ~~sphere~~ surface $z = \sqrt{a^2 - x^2 - y^2}$ above the disk $x^2 + y^2 \le a^2$ in the xy-plane

$$\text{Area}(R) = \pi r^2 = a^2 \pi.$$



$$\cancel{av(f) = \frac{1}{a^2\pi} \int_{-\pi/2}^{\pi/2} \int_{?}^{a} \sqrt{a^2 - r^2}\; r\, dr\, d\theta}$$

[illegible]

$$= \frac{4}{a^2\pi} \int_0^{\pi/2} \int_0^a \sqrt{a^2 - r^2}\; r\, dr\, d\theta$$

$u = a^2 - r^2$
$du = -2r\,dr$
when $r = 0 \Rightarrow u = a^2$
$r = a \Rightarrow u = 0$

تكعيب

$$= \frac{4}{3\pi a^2} \int_0^{\pi/2} a^3\, d\theta = \frac{2a}{3}$$

$(\frac{1}{2})(118)$

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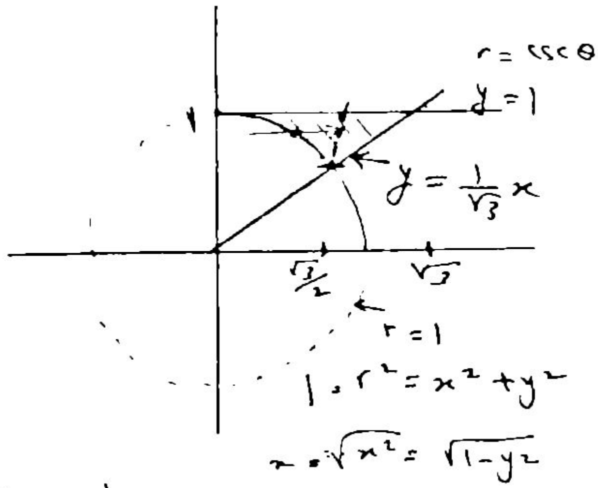
15.4 (24) Convert from polar to Cartesian:

$$\int_{\pi/6}^{\pi/2} \int_{1}^{\csc\theta} r^2 \cos\theta \, dr \, d\theta$$



$r = 1$

$r = \csc\theta \Rightarrow r\sin\theta = 1 = y.$

$\theta = \frac{\pi}{6} \Rightarrow \tan\frac{\pi}{6} = \frac{y}{x} = \frac{1}{\sqrt{3}} \Rightarrow y = \frac{1}{\sqrt{3}} x$

$\theta = \frac{\pi}{2} \Rightarrow \tan\frac{\pi}{2} = \frac{y}{x}$ (No need)

1) Horizontal Cross section:

$$\int_{1/2}^{1} \int_{\sqrt{1-y^2}}^{\sqrt{3}y} x \, dx \, dy$$

نقطة تقاطعهم

$$\begin{bmatrix} y = \frac{1}{\sqrt{3}} x \\ x^2 + y^2 = 1 \end{bmatrix}$$

$\Rightarrow x = \frac{\sqrt{3}}{2} \Rightarrow y = \frac{1}{2}$

or

2) Vertical Cross Section:

$$\int_{0}^{\sqrt{3}/2} \int_{\sqrt{1-x^2}}^{1} x \, dy \, dx + \int_{\sqrt{3}/2}^{\sqrt{3}} \int_{\frac{1}{\sqrt{3}}x}^{1} x \, dy \, dx$$

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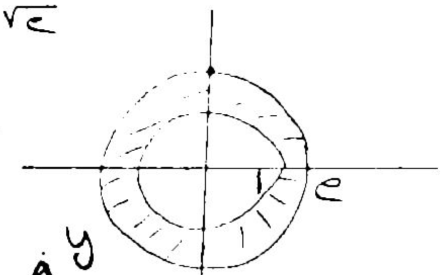
(37) Convert to a polar Integral:

$$f(x,y) = \frac{\ln(x^2+y^2)}{\sqrt{x^2+y^2}}, \quad 1 \le x^2+y^2 \le e$$

$$1 \le r^2 \le e$$

$$1 \le r \le \sqrt{e}$$

$$\int_0^{2\pi}\int_1^{\sqrt{e}} \frac{\ln r^2}{r}\, r\, dr\, d\theta = 2\pi(2-\sqrt{e}).$$



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(41) Find $I = \int_0^\infty e^{-x^2}\,dx$

$$I^2 = \left(\int_0^\infty e^{-x^2}\,dx\right)\left(\int_0^\infty e^{-y^2}\,dy\right) = \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$$

positive

$0 \le x < \infty$
$0 \le y < \infty$

$$= \int_0^{\frac{\pi}{2}}\int_0^\infty e^{-r^2}\, r\, dr\, d\theta = \int_0^{\frac{\pi}{2}}\left[\lim_{b\to\infty}\int_0^b r e^{-r^2}\,dr\right]d\theta$$

$$= \int_0^{\frac{\pi}{2}} -\frac{1}{2}\lim_{b\to\infty}\left(e^{-b^2}-1\right)d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} d\theta = \frac{\pi}{4}$$

$$\therefore I = \frac{\sqrt{\pi}}{2}.$$

---

(42) Evaluate $\displaystyle\int_0^\infty\int_0^\infty \frac{1}{(1+x^2+y^2)^2}\,dx\,dy.$

$$= \int_0^{\frac{\pi}{2}}\int_0^\infty \frac{r}{(1+r^2)^2}\,dr\,d\theta = \int_0^{\frac{\pi}{2}}\left[\lim_{b\to\infty}\int_0^b \frac{r}{(1+r^2)^2}\,dr\right]d\theta$$

$$= \int_0^{\frac{\pi}{2}}\left[\lim_{b\to\infty} \left.\frac{-1}{2(1+r^2)}\right|_0^b\right]d\theta = \ldots \quad \text{check!}$$

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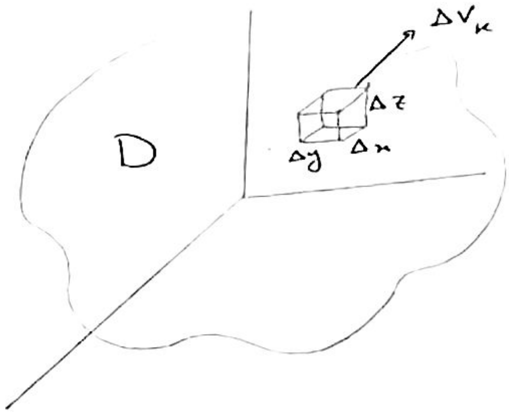
15.5 Triple Integrals in Rectangular Coordinates:

Let D be a closed and bounded region in space.

$$S_n = \sum_{k=1}^{n} F(x_k, y_k, z_k)\, \Delta V_k$$



$$\lim_{n\to\infty} S_n = \iiint_D F(x,y,z)\, dx\, dy\, dz$$

- If $F(x,y,z) = 1$, then:

$$S_n = \sum_{k=1}^{n} \Delta V_k$$

$$\lim_{n\to\infty} S_n = \iiint_D dV$$

Def: The volume of a closed, bounded region D in space is

$$V = \iiint_D dV = \iiint_D dx\, dy\, dz$$

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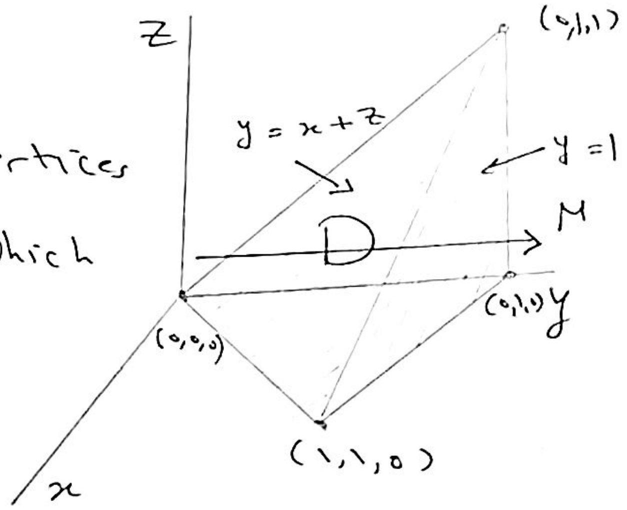
Example: Set up the limits of Integration for evaluating the triple integral of a function $f(x,y,z)$ over the tetrahedron D with vertices $(0,0,0)$, $(1,1,0)$, $(0,1,0)$, $(0,1,1)$. Use the order $dy\,dz\,dx$

1) We draw M // y-axis.



[1] M enters D at the plane with vertices $(0,0,0)$ & $(1,1,0)$, & $(0,1,1)$ which is

$$y = x + z$$

How to find equation of a plane?

Equation : $a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$

$a\vec{i} + b\vec{j} + c\vec{k} \perp$ plane. & $(x_0, y_0, z_0) \in$ plane.

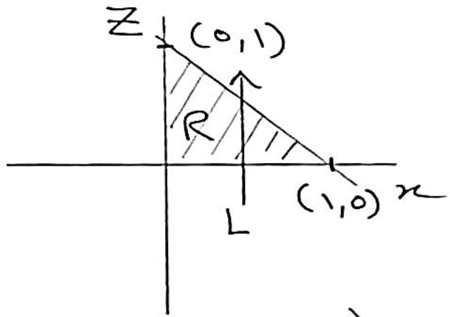
Find $\vec{V_1}$ & $\vec{V_2}$ $\Rightarrow$ $\vec{V_1} \times \vec{V_2} = a\vec{i} + b\vec{j} + c\vec{k}$

$\vec{i}+\vec{j}$ $\quad$ $\vec{j}+\vec{k}$ $\quad$ $\vec{i}$ $\quad -\vec{j}$ $\quad + \vec{k}$

[2] M leaves D at the plane with vertices : $(1,1,0)$, $(0,1,0)$, $(0,1,1)$ which is $y = 1$

2) Now assume $y=0$, to find the limits of Z in (vertical projection in xz plane) the xz plane: Draw L // z axis: then L enters R at $z=0$

& leaves at $\underbrace{z = 1-x}_{\text{equation of a Line}}$



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3) Need to find Now x limits: $x=0$ to $x=1$

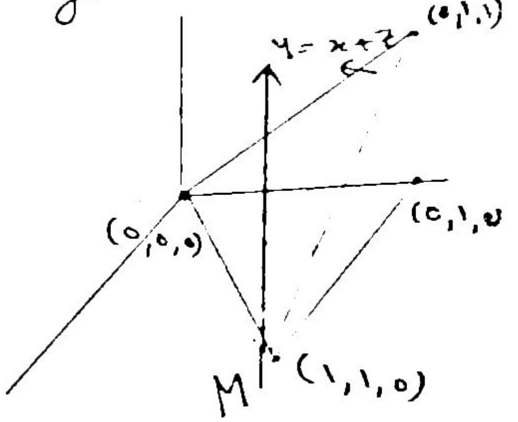
Then:
$$\int_0^1 \int_0^{1-x} \int_{x+z}^{1} F(x,y,z)\, dy\, dz\, dx$$

Example: Integrate $F(x,y,z)=1$, over the tetrahedron D in the previous example in the order $dz\,dy\,dx$ and then Integrate in the order $dy\,dz\,dx$

1) M // z-axis:

M enters at the plane with vertices $(0,0,0)$, $(1,1,0)$ & $(0,1,0)$ which is $z=0$

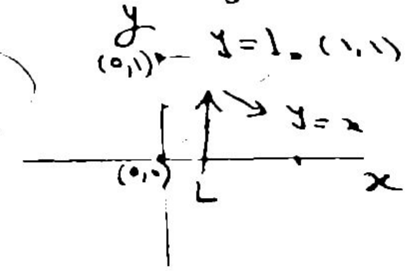
& leaves at the plane $z=y-x$



2) Assume $z=0$ to find vertical projection in xy-plane

L // y-axis, then L enters at $y=x$

& leaves at $y=1$



3) x changes from 0 to 1
when $y=0$

$$\rightarrow \int_0^1 \int_x^1 \int_0^{y-x} F(x,y,z)\, dz\, dy\, dx$$

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When $F(x,y,z)=1$, then $V = \iiint dx\,dy\,dz$

so $V = \int_0^1\int_x^1\int_0^{y-x} dz\,dy\,dx$

$$= \int_0^1\int_x^1 z\Big|_0^{y-x} dy\,dx = \int_0^1\int_x^1 (y-x)\,dy\,dx$$

$$= \int_0^1 \left(\frac{y^2}{2} - xy\right)\Big|_x^1 dx = \int_0^1\left(\frac{1}{2} - x + \frac{1}{2}x^2\right)dx$$

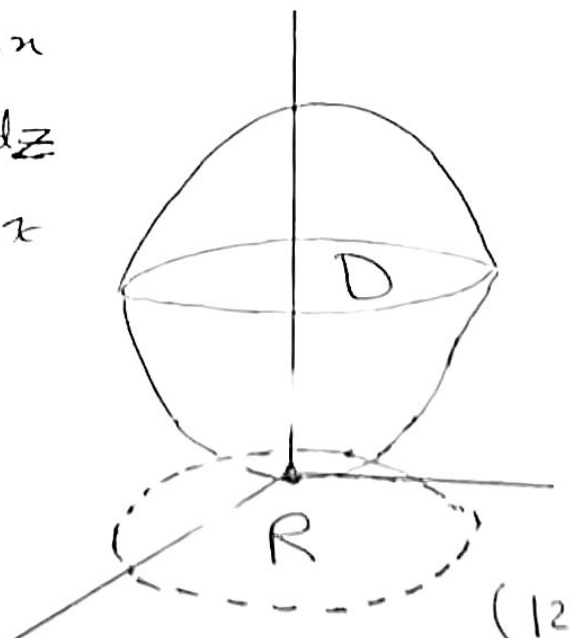
$$= \boxed{\frac{1}{6}}$$

Similarly: $V = \int_0^1\int_0^{1-x}\int_{x+z}^1 dy\,dz\,dx = \boxed{\frac{1}{6}}$

Example: Find the Volume of the Region D enclosed by the surfaces $z = x^2+3y^2$ & $z = 8-x^2-y^2$

1) Using the order $dz\,dy\,dx$

2) Using the order $dx\,dy\,dz$
$dx\,dy\,dz$



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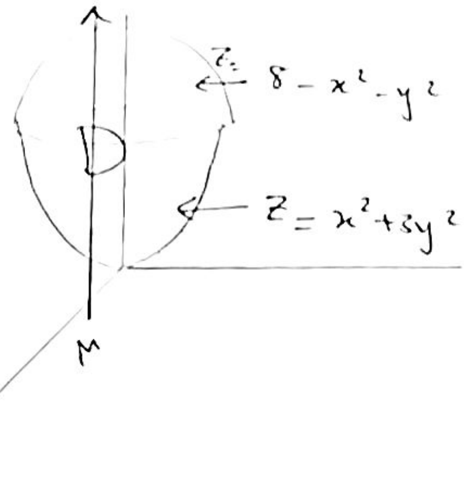
1) Limits of z:

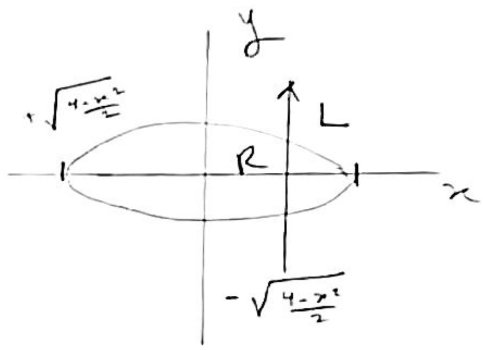
$M \parallel z$-axis: then M enters D at $z = x^2 + 3y^2$ & leaves at $z = 8 - x^2 - y^2$



2) Limits of y: Find the vertical projection when $z = 0$ in the xy-plane.

$\Rightarrow \quad 8 - x^2 - y^2 = x^2 + 3y^2$

$\Leftrightarrow \quad \boxed{x^2 + 2y^2 = 4}$ ellipse.



$L \parallel y$-axis: $\quad y = \pm\sqrt{\frac{4 - x^2}{2}}$

L enters R at $y = -\sqrt{\frac{4 - x^2}{2}}$ & leaves at $y = \sqrt{\frac{4 - x^2}{2}}$

3) limits of x: when $y = 0$, then $x = \pm 2$

$$\therefore V = \int_{-2}^{2} \int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} \int_{x^2+3y^2}^{8-x^2-y^2} dz\, dy\, dx$$

$$= 8\pi\sqrt{2}.$$

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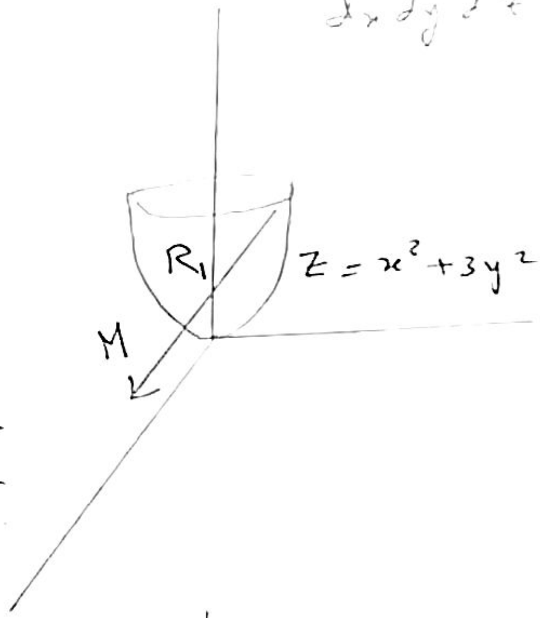
$dx\,dy\,dz$

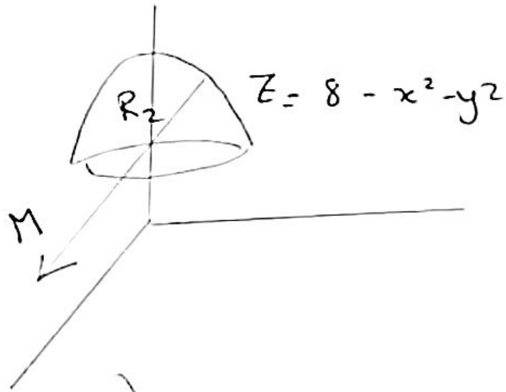
2) (i) M // x-axis $\underline{\underline{(R_1)}}$ $x = \pm\sqrt{-3y^2+z}$

M enters D at $-\sqrt{z-3y^2}$ & leaves at $+\sqrt{z-3y^2}$



(ii) M // x-axis $\underline{\underline{(R_2)}}$: $x = \pm\sqrt{8-y^2-z}$

M enters D at $-\sqrt{8-y^2-z}$ & leaves at $\sqrt{8-y^2-z}$

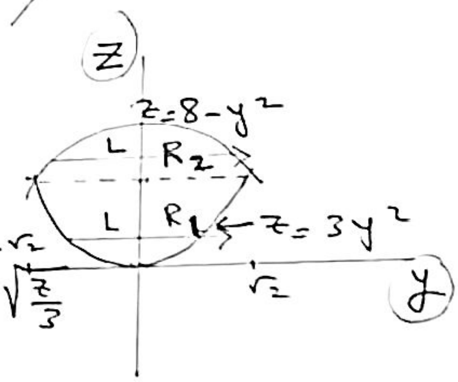


Now: assume $x=0$ to find the vertical projection in yz plane.

$\therefore$

① L // y-axis $\underline{\underline{(R_1)}}$: $y = \pm\sqrt{\frac{z}{3}}$.

$\therefore$ L enters R at $y = -\sqrt{\frac{z}{3}}$ & leaves at $y = \sqrt{\frac{z}{3}}$



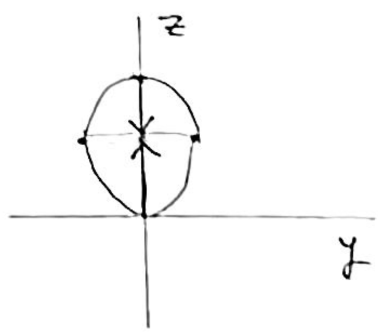
② L // y-axis in $\underline{\underline{(R_2)}}$: $y = \pm\sqrt{8-z}$

$\therefore$ L enters R at $y = -\sqrt{8-z}$ & leaves at $y = \sqrt{8-z}$

Now: $3y^2 = 8-y^2 \Rightarrow 4y^2 = 8$

$\therefore$ $y = \pm\sqrt{2}$, then:

z starts at 0 & ends at $3(\sqrt{2})^2 = \boxed{6}$ in $\boxed{R_1}$



& z starts at 6 & ends at 8 in $\boxed{R_2}$

Therefore,

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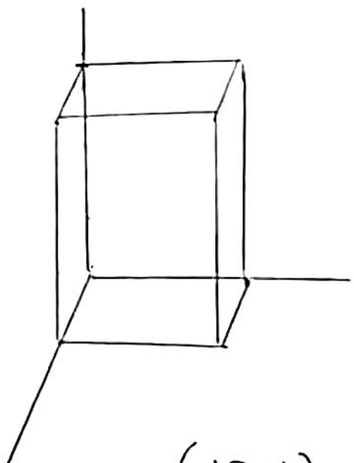
$$\therefore V = \int_0^6 \int_{-\sqrt{z/3}}^{\sqrt{z/3}} \int_{-\sqrt{z-3y^2}}^{\sqrt{z-3y^2}} dx\, dy\, dz$$

$$+ \int_6^8 \int_{-\sqrt{8-z}}^{\sqrt{8-z}} \int_{-\sqrt{8-y^2-z}}^{\sqrt{8-y^2-z}} dx\, dy\, dz = 8\pi\sqrt{2}.$$

The Average Value of a Function in space:

$$av(f) = \frac{1}{\text{Volume}(D)} \iiint_D F\, dv$$

Example: Find the average value of $f(x,y,z) = xyz$ throughout the rectangular Region D in the first octant bounded by the Coordinates planes & the planes $x = 1$, $y = 2$, $z = 3$



$$V(D) = \int_0^3 \int_0^2 \int_0^1 dx\, dy\, dz = 6$$

$$av(f) = \frac{1}{6} \int_0^3 \int_0^2 \int_0^1 xyz\, dx\, dy\, dz = \boxed{\frac{3}{4}}$$

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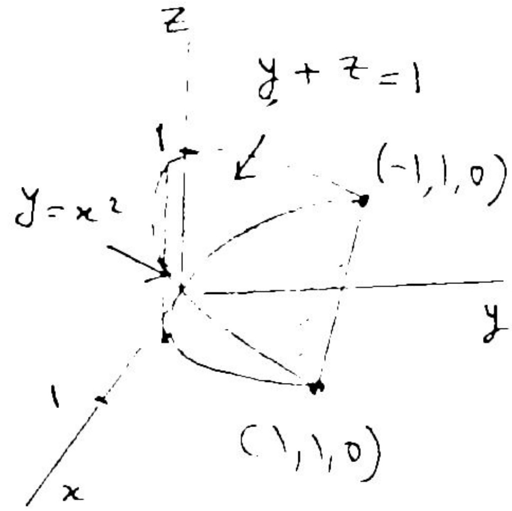
(21) $\displaystyle\int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y} dz\, dy\, dx$



$z = 1-y = 1-x^2$ نعوض

(a) $dy\, dz\, dx$:

$$\int_{-1}^{1}\int_{0}^{1-x^2}\int_{x^2}^{1-z} dy\, dz\, dx$$

تعوض لأنه

حسب $- - = x$

$x = \sqrt{y}$

$x = \pm\sqrt{1-z}$

(b) $dy\, dx\, dz$: $\displaystyle\int_{0}^{1}\int_{-\sqrt{1-z}}^{\sqrt{1-z}}\int_{x^2}^{1-z} dy\, dx\, dz$

(c) $dx\, dy\, dz$: $\displaystyle\int_{0}^{1}\int_{0}^{1-z}\int_{-\sqrt{y}}^{\sqrt{y}} dx\, dy\, dz$

(d) $dx\, dz\, dy$: $\displaystyle\int_{0}^{1}\int_{0}^{1-y}\int_{-\sqrt{y}}^{\sqrt{y}} dx\, dz\, dy$

(e) $dz\, dx\, dy$: $\displaystyle\int_{0}^{1}\int_{-\sqrt{y}}^{\sqrt{y}}\int_{0}^{1-y} dz\, dx\, dy$

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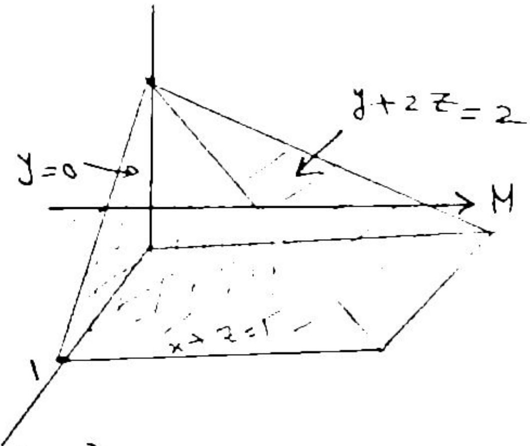
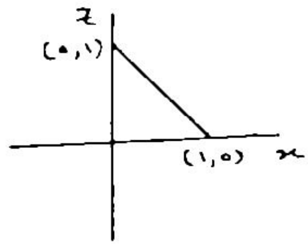
(24) The Region in the first octant bounded by the coordinate planes and the planes $x+z=1$, $y+2z=2$. Find the Volume? $dy\,dz\,dx$



1) y-limits: M // y-axis:

$y=0$ to $y=2-2z$

2) z-limits: L // z-axis. (xz-plane)



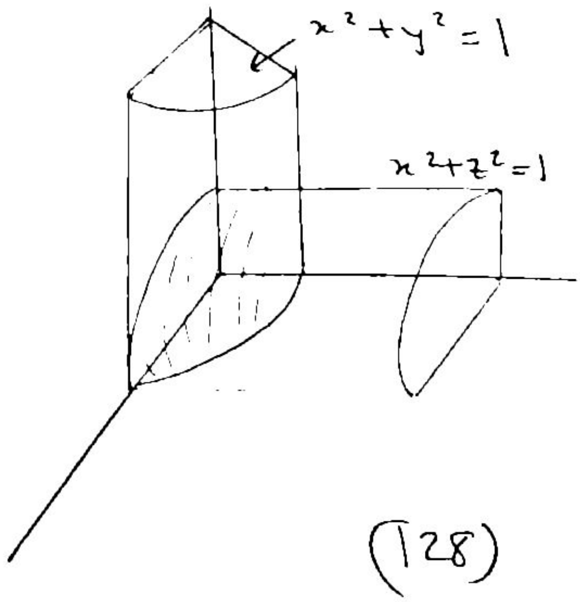
$z=0$ to $z=1-x$

3) x-limits: $x=0$ to $x=1$

$$V=\int_0^1\int_0^{1-x}\int_0^{2-2z} 1\,dy\,dz\,dx = \boxed{\frac{2}{3}}$$

(29) The Region common to the Interiors of the Cylinders $x^2+y^2=1$ and $x^2+z^2=1$, one eighth of which is shown in the figure, Find V?

$$V=8\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2}} 1\,dz\,dy\,dx$$



$$=\boxed{\frac{16}{3}}$$

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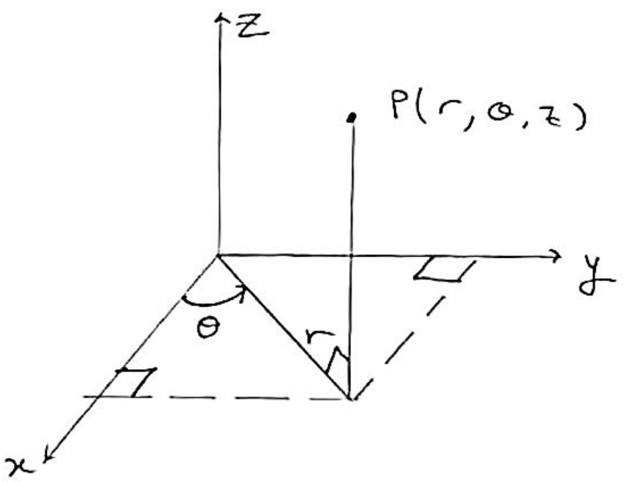
15.7 Triple Integrals in Cylindrical and spherical Coordinates:

Def: Cylindrical Coordinates represent a point P in space by ordered triples (r, θ, z) in which

1. r & θ are polar Coordinates for the vertical projection P on the xy-plane
2. z is the rectangular vertical Coordinate.

يعني Z العادية
الي بنعرفها



Equations Relating Rectangular (x, y, z) and Cylindrical (r, θ, z) Coordinates:

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

$r^2 = x^2 + y^2$

$\tan\theta = \frac{y}{x}$

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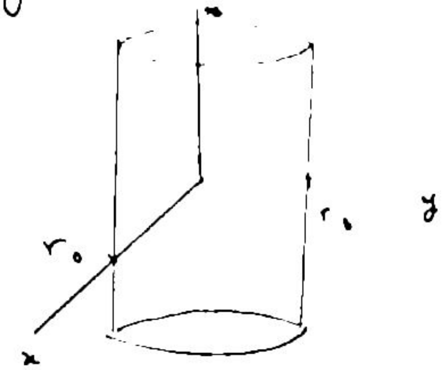
Remark: In cylindrical Coordinates:

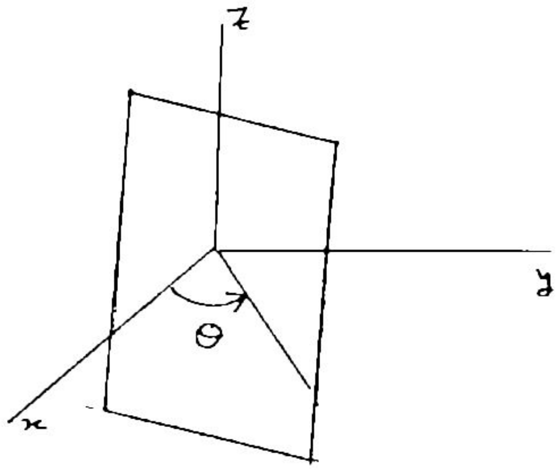
1) $r = r_0$, describes an entire cylinder about Z-axis

θ & Z vary:



2) $\theta = \theta_0$, describes the plane that Contains the Z-axis and makes angle θ with the positive x-axis

r & Z vary.



3) $Z = Z_0$, describes plane $\perp$ Z axis.

r & θ vary.

Note: $r = 0$ describe Z-axis.

$r = 4$ describes cylinder with $r = 4$ about Z axis

$\theta = \frac{\pi}{3}$ = planes Containing Z-axis

$Z = 2$ = plane $\perp$ Z-axis

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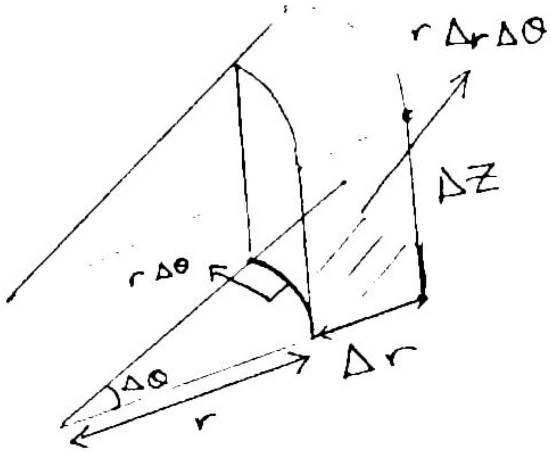
To find triple integral over region D in cylindrical Coordinates, we partition the region D into small n cylindrical wedges.



. The kth cylindrical wedge r_k, θ_k, z_k changes by

. $\Delta r_k, \Delta\theta_k, \Delta z_k$.

. The norm of the partition $\max\{\Delta r_k, \Delta\theta_k, \Delta z_k\}$.

. $\Delta A_k = r_k \Delta r_k \Delta\theta_k$.

. $\Delta V_k = \Delta Z_k \, \Delta A_k = \Delta Z_k \, r_k \Delta\theta_k \Delta r_k$

For a point in the center of the kth wedge, the Riemann sum of f on D has the form:

$$S_n = \sum_{k=1}^{n} f(r_k, \theta_k, z_k) \, \Delta z_k \, r_k \, \Delta r_k \, \Delta\theta_k$$

as norm $\to 0 \implies n \to \infty$

$$\lim_{n\to\infty} S_n = \iiint_D f \, dV = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{g_1(r,\theta)}^{g_2(r,\theta)} f \, dZ \, \underline{r \, dr \, d\theta}$$

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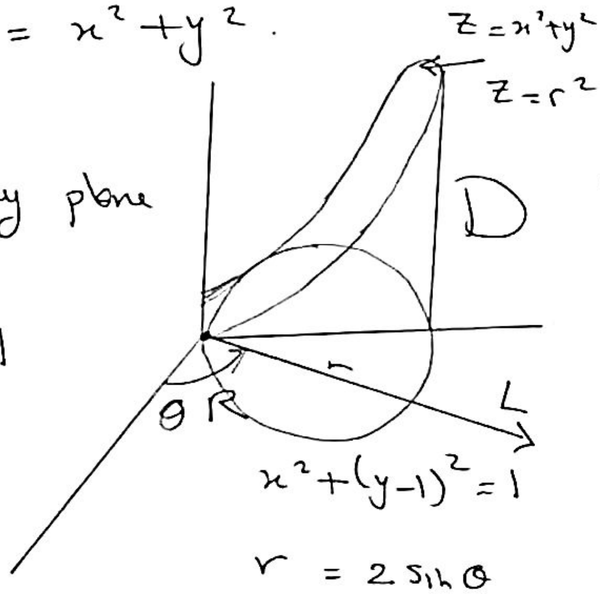
Example: Find the limits of integration in cylindrical coordinates for integrating a function $f(r,\theta,z)$ over the region D bounded below by the plane $z=0$, & by the circular cylinder $x^2+(y-1)^2=1$ and above by the paraboloid $z=x^2+y^2$.



- Base of D is the projection R in xy plane

Boundary of R is $x^2+(y-1)^2=1$

$$x^2+y^2-2y+1=1$$

$$r^2 - 2r\sin\theta = 0$$

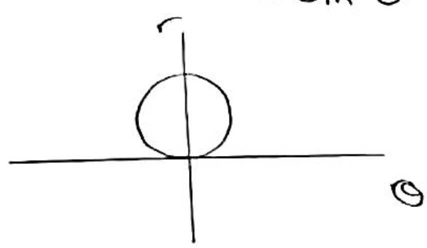
$$r(r-2\sin\theta)=0$$

$$\Longrightarrow \quad r = 2\sin\theta.$$

1) z-limits: M enters at $z=0$ & leaves at $z=x^2+y^2=r^2$

2) r-limits: L enters at $r=0$ & leaves at $r=2\sin\theta$

3) θ-limits: $\theta=0$ to $\theta=\pi$



$$\iiint_D f(r,\theta,z)\,dv = \int_0^{\pi}\int_0^{2\sin\theta}\int_0^{r^2} f(r,\theta,z)\,dz\,r\,dr\,d\theta$$

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Example (14) Convert the Integral:

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}}\int_{0}^{x} (x^2+y^2)\, dz\, dx\, dy$$ to an equivalent

integral in cylindrical Coordinates.

. z from 0 to x $\Rightarrow$ z from 0 to $r\cos\theta$

. x from 0 to $\sqrt{1-y^2}$ $\Rightarrow$ r from 0 to $x=\sqrt{1-y^2}$

$x^2+y^2=1$

$\therefore r^2=1$

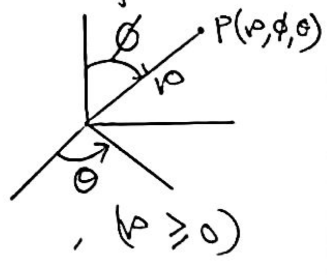
$r=1$

. y from -1 to 1 $\Rightarrow$ $\theta = \sin^{-1}(-1)$ to $\theta = \sin^{-1} 1$

$$\Rightarrow \int_{-\pi/2}^{\pi/2}\int_{0}^{1}\int_{0}^{r\cos\theta} r^2\, dz\, r\, dr\, d\theta.$$

Spherical Coordinates and Integration:

Spherical Coordinates represent a point P in space by (ρ, ϕ, θ), in which:



1. ρ is the distance from P to the origin, $(\rho \geq 0)$
2. ϕ is the angle $\overrightarrow{OP}$ makes with the positive z-axis $(0 \leq \phi \leq \pi)$
3. θ is the angle from cylindrical Coordinates. $(0 \leq \theta \leq 2\pi)$.

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Equations Relating Spherical Coordinates to Cartesian and Cylindrical Coordinates:

$$\boxed{r = \rho \sin \phi}$$

$$x = r \cos \theta = \rho \sin \phi \cos \theta = x$$

$$\boxed{z = \rho \cos \phi}$$

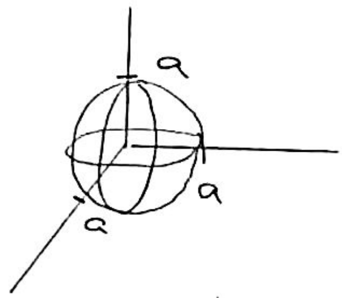
$$y = r \sin \theta = \rho \sin \phi \sin \theta = y$$

$$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{r^2+z^2}$$

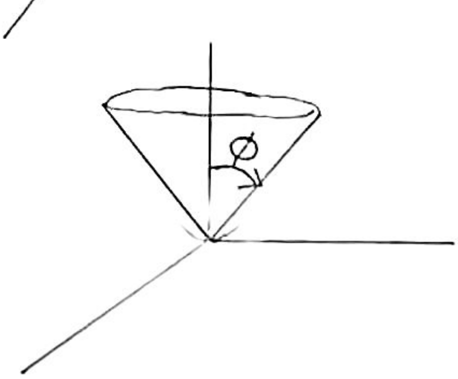
Remark: In spherical Coordinates:

1) $\rho = a$ is sphere with radius a centered at origin

ϕ & θ vary.

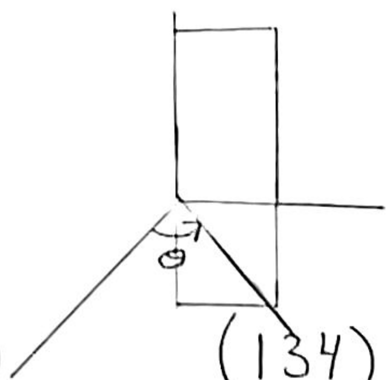


2) $\phi = \phi_0$ is Cone whose vertex at origin and whose axis is the z-axis (ρ & θ vary)



- As $\boxed{\phi = \frac{\pi}{2}}$ $\Rightarrow$ we get xy-plane
- The Cone open down for $\boxed{\phi > \frac{\pi}{2}}$

3) $\theta = \theta_0$ is half plane Contains z-axis and makes angle θ_0 with positive x-axis. (Not in negative axis)



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To find triple Integral over region D in spherical Coordinates:

- we partition D into n small spherical wedges.
- in the kth spherical wedge ρ_k, ϕ_k, θ_k change by $\Delta\rho_k, \Delta\phi_k, \Delta\theta_k$.
- $\Delta V_k = \rho_k^2 \sin\phi_k \, \Delta\rho_k \, \Delta\phi_k \, \Delta\theta_k$
- Reimann Sum:

$$S_n = \sum f(\rho_k, \phi_k, \theta_k) \, \rho_k^2 \sin\phi_k \, \Delta\rho_k \Delta\phi_k \Delta\theta_k$$

$$\Rightarrow \lim_{n\to\infty} S_n = \iiint_D f(\rho, \phi, \theta) \, dV$$

$$= \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{g_1(\phi,\theta)}^{g_2(\phi,\theta)} f(\rho, \phi, \theta) \, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

Example: Find spherical Coordinate to $x^2+y^2+(z-1)^2=1$

$$x^2+y^2+(z-1)^2 = 1$$

$$\Rightarrow \rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta + (\rho\cos\phi - 1)^2 = 1$$

$$\Rightarrow \rho^2\sin^2\phi + \rho^2\cos^2\phi - 2\rho\cos\phi + 1 = 1$$

$$\Rightarrow \rho^2 - 2\rho\cos\phi = 0 \Rightarrow \rho^2 = 2\rho\cos\phi$$

$$\rho = 2\cos\phi, \quad \rho > 0$$

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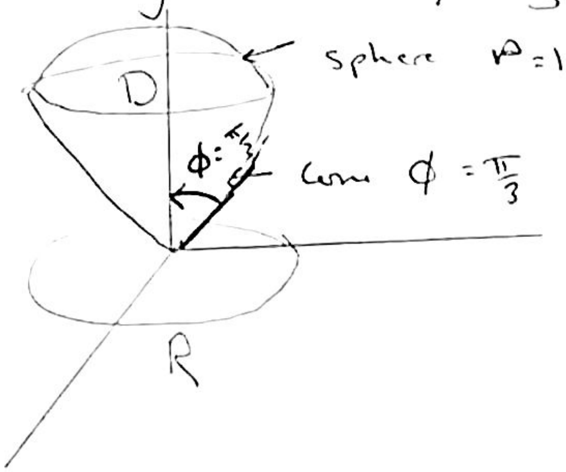
Example: Find the Volume of the ice cream Cone D cut from the solid sphere $\rho \le 1$ by the Cone $\phi = \frac{\pi}{3}$

$$V = \int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_0^{1} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

$$= \boxed{\frac{\pi}{3}}$$



Remark: Coordinate Conversion formulas:

1) from Cylindrical (r, θ, z) to rectangular (x, y, z):

$$r\cos\theta = x \quad , \quad r\sin\theta = y \quad , \quad z = z$$

2) from spherical (ρ, ϕ, θ) to rectangular (x, y, z):

$$\rho\sin\phi\cos\theta = x$$
$$\rho\sin\phi\sin\theta = y$$
$$\rho\cos\phi = z$$

3) from spherical (ρ, ϕ, θ) to cylindrical (r, θ, z):

$$\rho\sin\phi = r$$
$$\rho\cos\phi = z$$
$$\theta = \theta$$

$$dv = dx\,dy\,dz = dz\,r\,dr\,d\theta = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

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(12) Let D be the region bounded below by the cone

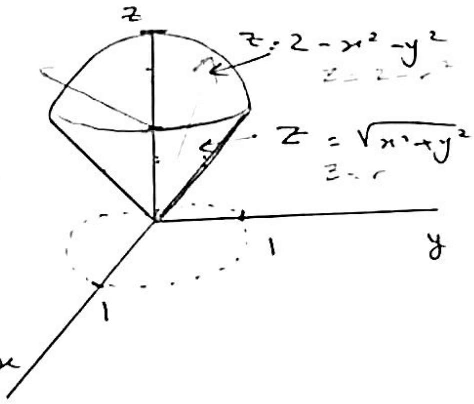
$z = \sqrt{x^2+y^2}$ and above by the paraboloid $z = 2 - x^2 - y^2$.

a) $dz\,dr\,d\theta$ b) $dr\,dz\,d\theta$ c) $d\theta\,dz\,dr$

a) $V = \int_0^{2\pi}\int_0^{1}\int_r^{2-r^2} dz\, r\,dr\, d\theta$

نقطة تقاطع المجسمين $r = 1 \Leftarrow$

$\xrightarrow{r}$



b) $V = \int_0^{2\pi}\int_0^{1}\int_0^{z} r\,dr\,dz\,d\theta + \int_0^{2\pi}\int_1^{2}\int_0^{\sqrt{2-z}} r\,dr\,dz\,d\theta$

c) $V = \int_0^{1}\int_r^{2-r^2}\int_0^{2\pi} r\,d\theta\,dz\,dr$

لإيجاد r نعمل ray [illegible]

$x = \sqrt{1-y^2}$

$x^2 = 1 - y^2$

$x^2 + y^2 = 1$

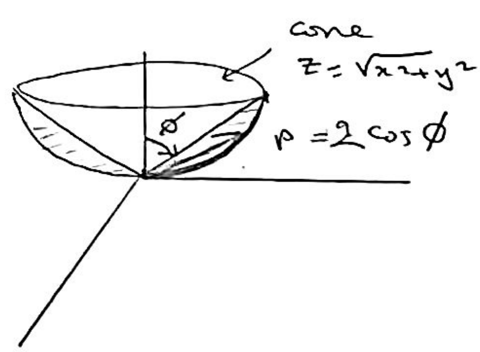
$r^2 = 1$

(14) $\int_{-1}^{1}\int_0^{\sqrt{1-y^2}}\int_0^{x} (x^2+y^2)\,dz\,dx\,dy = \int_{-\pi/2}^{\pi/2}\int_0^{1}\int_0^{r\cos\theta} r^3\,dz\,dr\,d\theta$

---

(37) Find the Volume.

$V = \int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^{2\cos\phi} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$



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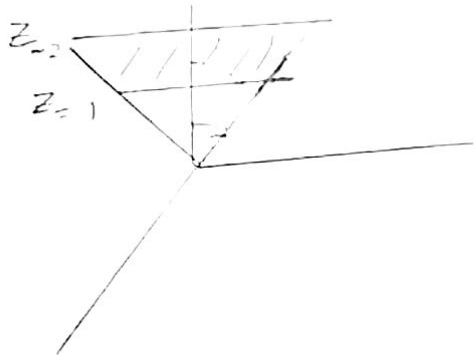
(52) Find the Volume of the Solid enclosed by the Cone $z = \sqrt{x^2+y^2}$ between $z = 1$ & $z = 2$

$$V = 4\int_0^{\cancel{\frac{\pi}{2}}}\int_0^{\frac{\pi}{4}}\int_{\sec\phi}^{2\sec\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

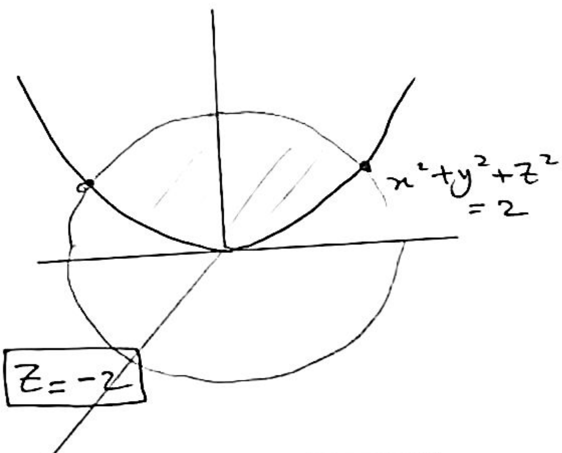


2π و 0 بدل
$4\times(\frac{\pi}{2}$ و 0 ~ بين$)$

$z = \rho\cos\phi \Rightarrow$ $z = 1 \Rightarrow \rho = \sec\phi \Rightarrow$
$z = 2 \Rightarrow \rho = 2\sec\phi$

---

(62) Find the volume of the region bounded ~~by~~ above by the sphere $x^2+y^2+z^2 = 2$ & below by the paraboloid $z = \overset{(*)}{x^2+y^2} =$



They Intersect when:

$x^2+y^2+z^2 = 2$ & $x^2+y^2 = z$

$\Rightarrow z^2 + z - 2 = 0 \Rightarrow \boxed{z = 1}$ or $\boxed{z = -2}$

Now since $\overset{\text{from }(*)}{z \geq 0} \Rightarrow \boxed{z = 1} \Rightarrow x^2+y^2 = 1 \Rightarrow \boxed{r^2 = 1}$

$$\Rightarrow V = \int_0^{2\pi}\int_0^1\int_{r^2}^{\sqrt{2-r^2}} dz \, r \, dr \, d\theta = \frac{\pi(8\sqrt{2}-7)}{6}$$

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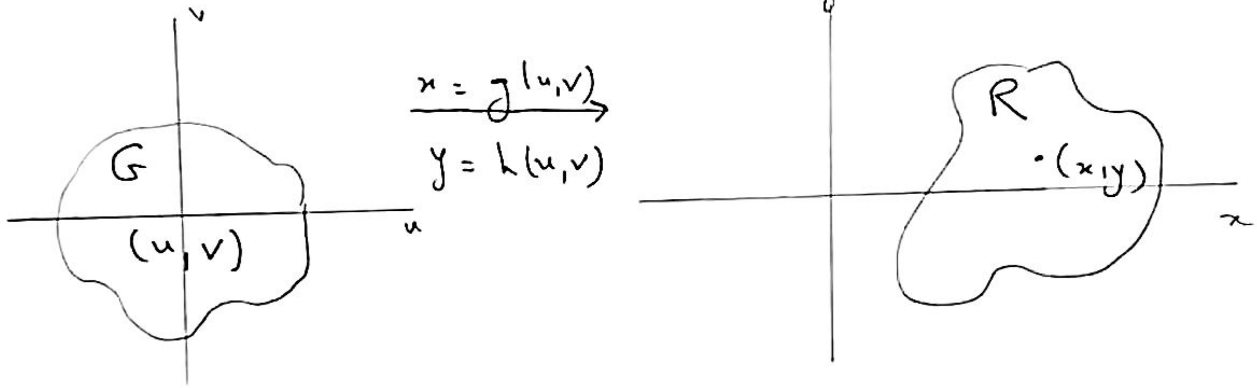
15.8 Substitution in Multiple Integrals:

1) Substitution in Double Integrals:

Suppose that a region G in uv-plane is transformed (1-1) into a region R in the xy-plane through the equations $\quad x = g(u,v) \quad , \quad y = h(u,v)$



- If $f(x,y)$ is defined on R, then $f(x,y)$ is defined on the region G by $f(g(u,v), h(u,v))$
- If g, h and f have continuous partial derivatives, then:

$$\iint_R f(x,y)\,dx\,dy = \iint_G f(g(u,v), h(u,v))\,|J(u,v)|\,du\,dv$$

where $J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = x_u y_v - x_v y_u$

$J(u,v)$ is called the <u>Jacobian</u> of x & y.

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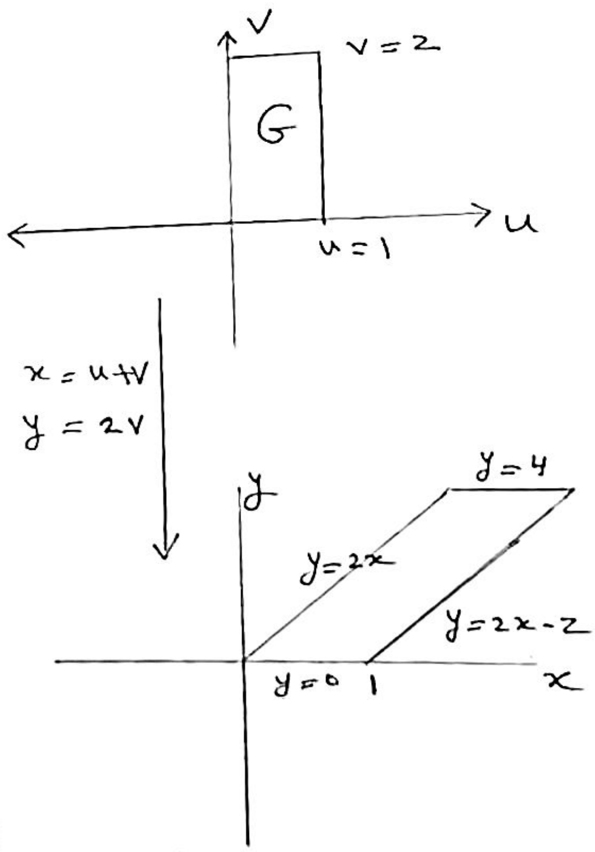
Example: Evaluate $\displaystyle\int_0^4 \int_{x=\frac{y}{2}}^{x=\frac{y}{2}+1} \frac{2x-y}{2}\, dx\, dy$

by applying the transformation $u = \dfrac{2x-y}{2}$ & $v = \dfrac{y}{2}$

and Integrate w.r.t u and v

- $u = x - \dfrac{y}{2} = x - v \iff \begin{cases} x = u + v \\ y = 2v \end{cases}$

$\Rightarrow J(u,v) = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2$



when $y = 0 \Rightarrow v = 0$

$y = 4 \Rightarrow v = 2$

when $x = \dfrac{y}{2} \Rightarrow u = 0$

$x = u + v \Rightarrow v = u + v$

$x = \dfrac{y}{2} + 1 \Rightarrow u = 1$

$$\Rightarrow \int_0^4 \int_{\frac{y}{2}}^{\frac{y}{2}+1} \frac{2x-y}{2}\, dx\, dy$$

$$= \int_0^2 \int_0^1 u\,(2)\, du\, dv = \int_0^2 u^2 \Big|_0^1 dv = \int_0^2 dv = \boxed{2}$$

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Example: Evaluate $\int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2\,dy\,dx$

Let $\boxed{u = x+y}$ ... (*), $v = y - 2x$

$\Rightarrow \quad x = u - y \quad , \quad y = v + 2x$

$\Rightarrow \quad x = u - (v + 2x) = u - v - 2x$

$\Rightarrow \quad 3x = u - v$

then $\quad x = \frac{u}{3} - \frac{v}{3} \quad \cdots (1)$

Now: $\quad y = v + 2x = v + 2\left(\frac{u}{3} - \frac{v}{3}\right) =$

$\Rightarrow \quad y = \frac{2u}{3} + \frac{v}{3} \quad \cdots (2).$

$$J(u,v) = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{vmatrix} = \frac{3}{9} = \boxed{\frac{1}{3}}$$

when $\quad y = 0 \Rightarrow v = -2u \quad$ from (2)

$\quad y = 1 - x \Rightarrow u = 1 \quad$ from (*)

when $\quad x = 0 \Rightarrow u = v \quad$ from (1)

$\quad x = 1 \Rightarrow 3 = u - v \Rightarrow v = u - 3 \quad$ from (1)

NOT Boundary

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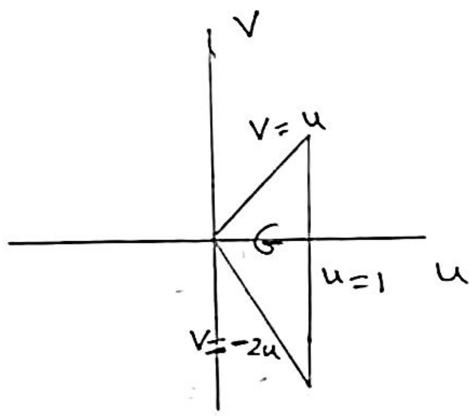
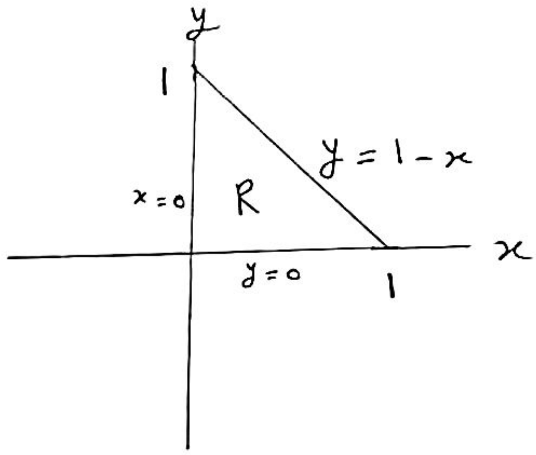
$$\int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2\,dy\,dx$$

Vertical cross section

$$= \int_0^1 \int_{-2u}^{u} \sqrt{u}\; v^2 \left(\tfrac{1}{3}\right) dv\,du = \cdots$$





$$= \frac{1}{3}\int_0^1 \sqrt{u}\;\frac{v^3}{3}\Bigg|_{-2u}^{u} du = \frac{1}{9}\int_0^1 \sqrt{u}\,(u^3 + 8u^3)\,du$$

$$= \frac{1}{9}\int_0^1 \sqrt{u}\,(9u^3)\,du = \int_0^1 u^{7/2}\,du$$

$$= \frac{u^{9/2}}{9/2}\Bigg|_0^1 = \boxed{\frac{2}{9}}$$

(We choose the Boundary of R to Identify the Boundary of G).

$\left(\frac{1}{2}\right)(142)$

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Example: Evaluate $\displaystyle\int_1^2 \int_{\frac{1}{y}}^{y} \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dx\, dy.$

Sol: Let $u = \sqrt{xy}$, $v = \sqrt{\frac{y}{x}}$.

squaring the equations:

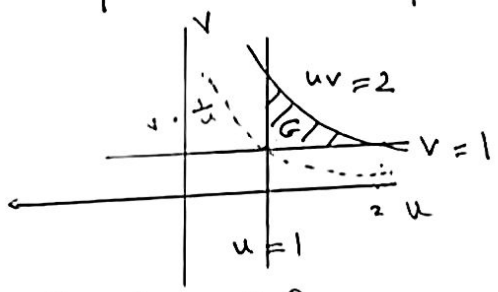
$$u^2 = xy \quad \dots (1) \quad \& \quad v^2 = \frac{y}{x}. \quad \dots (2)$$

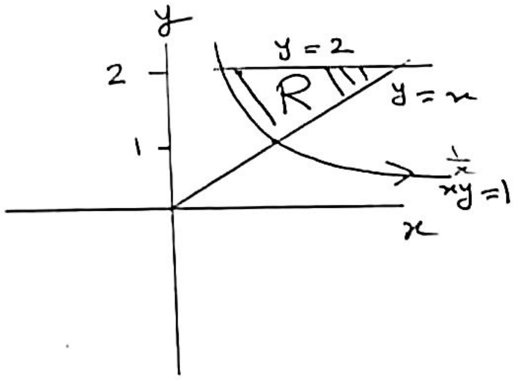
$(2) \Rightarrow y = v^2 x$. Now substitute y in (1), we have

$$u^2 = x(v^2 x) \Rightarrow \frac{u^2}{v^2} = x^2 \Rightarrow \boxed{x = \frac{u}{v}}$$

Now from (2): $y = v^2 x = v^2\left(\frac{u}{v}\right) = \boxed{uv = y.}$ $(*)$

$$J(u,v) = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{vmatrix} = \frac{2u}{v}$$





$x = \frac{1}{y}$ Boundaries of R

when $xy = 1 \Rightarrow u = 1$ from (1)

$y = x \Rightarrow v = 1$ from (2)

when $y = 2 \Rightarrow v = \frac{2}{u}$ from $(*)$

$\boxed{y = 1 \Rightarrow uv = 1,\ v = \frac{1}{u}}$

G

$$\rightarrow \int_1^2 \int_{\frac{1}{y}}^{y} \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dx\, dy = \int_1^2 \int_1^{2/u} v e^{u} \left(\frac{2u}{v}\right) dv\, du =$$

$2e(e-2)$

$\left(\frac{2}{2}\right)(142)$

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2) Substitution in Triple Integrals:

- Suppose a region G in uvw-space is transformed (1-1) to a region R in xyz-space through the equations

$$x = g(u,v,w) \;,\; y = h(u,v,w) \;,\; z = k(u,v,w).$$

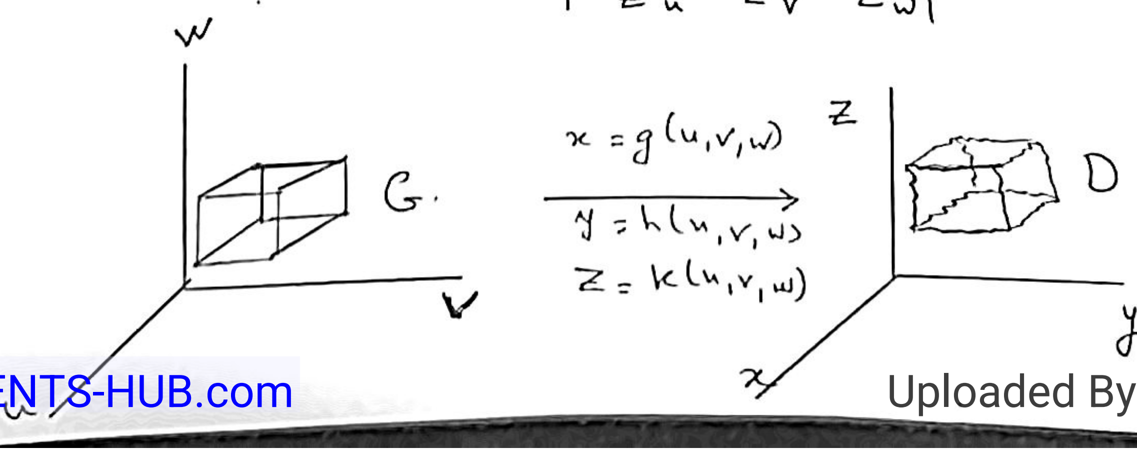
- If $f(x,y,z)$ is defined on R, then f is defined on G

$$f(x,y,z) = f\big(g(u,v,w)\,,\,h(u,v,w)\,,\,k(u,v,w)\big)$$
$$= H(u,v,w).$$

- If g, h, k & f have continuous first partial derivatives then:

$$\iiint_R f(x,y,z)\,dx\,dy\,dz = \iiint_G H(u,v,w)\,|J(u,v,w)|\,du\,dv\,dw$$

where
$$J(u,v,w) = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$



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Example: 24 Let D be the region in xyz-plane defined by the inequalities:

$$1 \le x \le 2 \quad , \quad 0 \le xy \le 2 \quad , \quad 0 \le z \le 1$$

Evaluate $\iiint_D (x^2 y + 3xyz)\, dx\, dy\, dz$

by applying the transformation

$$u = x \quad , \quad v = xy \quad , \quad w = 3z .$$

$$\Rightarrow \quad x = u \quad , \quad y = \frac{v}{x} = \frac{v}{u} \quad , \quad z = \frac{w}{3}$$

$$J(u,v,w) = \begin{vmatrix} 1 & 0 & 0 \\ -\frac{v}{u^2} & \frac{1}{u} & 0 \\ 0 & 0 & \frac{1}{3} \end{vmatrix} = \frac{1}{3u} .$$

when $1 \le x \le 2 \iff 1 \le u \le 2$

$0 \le \overset{v}{xy} \le 2 \iff 0 \le v \le 2 .$

$0 \le z \le 1 \iff 0 \le w \le 3$

$$\iiint_D (x^2 y + 3xyz)\, dx\, dy\, dz = \int_1^2 \int_0^2 \int_0^3 (uv + vw) \frac{1}{3u}\, dw\, dv\, du$$

$$= 2 + 3 \ln 2 .$$

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Note: For Cylindrical Coordinates: r, θ & z take the place of u, v, w. The transformation from $r\theta z$-space to xyz-space is given by

$$x = r\cos\theta \quad , y = r\sin\theta \quad , z = z.$$

$$J(r,\theta,z) = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = \boxed{r}$$

$$\Rightarrow \iiint_D f(x,y,z)\,dx\,dy\,dz = \iiint_G H(r,\theta,z)\,|r|\,dr\,d\theta\,dz.$$

Note: For spherical Coordinates: ρ, ϕ and θ take the place of u, v, w. The transformation from $\rho\phi\theta$-space to xyz-space is given by:

$$x = \rho\sin\phi\cos\theta \quad , y = \rho\sin\phi\sin\theta \quad , z = \rho\cos\phi$$

$$J(\rho,\phi,\theta) = \begin{vmatrix} x_\rho & x_\phi & x_\theta \\ y_\rho & y_\phi & y_\theta \\ z_\rho & z_\phi & z_\theta \end{vmatrix} = \rho^2\sin\phi.$$

$$\Rightarrow \iiint_D f(x,y,z)\,dx\,dy\,dz = \iiint_G H(\rho,\phi,\theta)\,\underline{|\rho^2\sin\phi|}\,d\rho\,d\phi\,d\theta.$$

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⑨ Let R be the region in the first quadrant of xy-plane bounded by $xy = 1$, $xy = 9$ & $y = x$ & $y = 4x$

Use the transformation

$x = \frac{u}{v}$, $y = uv$ with $u > 0$, $v > 0$ to rewrite

$\iint_R \left(\sqrt{\frac{y}{x}} + \sqrt{xy}\right) dx\, dy$ as an Integral in uv-plane

- $x = \frac{u}{v}$ & $y = uv \Rightarrow \frac{y}{x} = \frac{uv}{(u/v)} = v^2$

$\Rightarrow xy = u^2$.

$$J(u,v) = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{vmatrix} = \frac{2u}{v}.$$

when $y = x \Rightarrow uv = \frac{u}{v} \Rightarrow v^2 = 1 \Rightarrow v = 1$

$y = 4x \Rightarrow uv = \frac{4u}{v} \Rightarrow v^2 = 4 \Rightarrow v = 2$

when $xy = 1 \Rightarrow u^2 = 1 \Rightarrow u = 1$

$xy = 9 \Rightarrow u^2 = 9 \Rightarrow u = 3$

$$\Rightarrow \iint_R \left(\sqrt{\frac{y}{x}} + \sqrt{xy}\right) dx\, dy = \int_1^3 \int_1^2 (v+u)\left(\frac{2u}{v}\right) dv\, du$$

$$= 8 + \frac{52}{3}(\ln 2).$$

$(\frac{1}{3})\ (146)$

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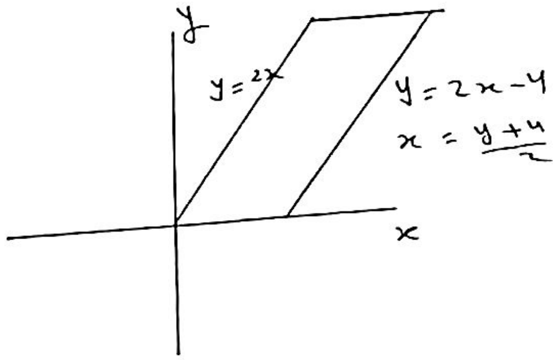
(14) Use the transformation $x = u + \frac{1}{2}v$, $y = v$

to evaluate :

$$\int_0^2 \int_{y/2}^{\frac{y+4}{2}} y^3 (2x-y)\, e^{(2x-y)^2}\, dx\, dy.$$

- $x = u + \frac{1}{2}v$, $y = v$



$$J(u,v) = \begin{vmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{vmatrix} = 1$$

Now : $x = \frac{y}{2} \Rightarrow y = 2x \Rightarrow v = 2x \Rightarrow v = 2(u + \frac{1}{2}v)$

$\Rightarrow 2u = 0 \Rightarrow \boxed{u = 0}$

$x = \frac{y+4}{2} \Rightarrow u + \frac{1}{2}v = \frac{v+4}{2} \Rightarrow$

$u + \frac{1}{2}v = \frac{1}{2}v + 2 \Rightarrow \boxed{u = 2}$

$y = 0 \Rightarrow v = 0$

$y = 2 \Rightarrow v = 2$

$$\int_0^2 \int_{y/2}^{y/2+2} y^3 (2x-y)\, e^{(2x-y)^2}\, dx\, dy = \int_0^2 \int_0^2 v^3 (2u)\, e^{4u^2}\, du\, dv$$

$$= e^{16} - 1$$

(2) (146)

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(22) Find the Volume of the ellipsoid:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

(Hint : Let $x = au$, $y = bv$, and $z = cw$)

$$J(u,v,w) = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc.$$

$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \Rightarrow u^2 + v^2 + w^2 = 1$$

$$V = \iiint_K dx\,dy\,dz = \iiint_G abc\;du\,dv\,dw$$

$$= \frac{4\pi abc}{3}$$

(spherical Region $V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi$

$\left(\frac{3}{3}\right)$ (146)

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