

1.5 Solutions of Systems of Linear Equations

A solution of a linear equation with two variables x and y is the point (x, y) that will make the equation true.

Example: For the equation $4x + 3y = 11$

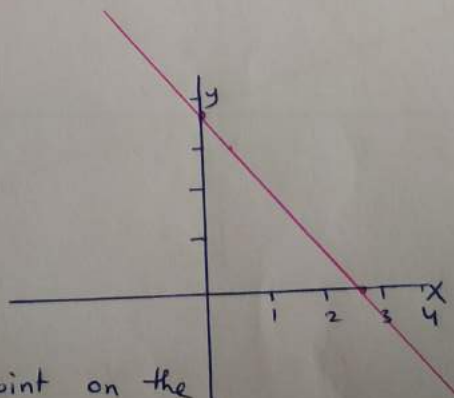
Is $(0, \frac{11}{3})$ a solution? $4(0) + 3(\frac{11}{3}) \stackrel{??}{=} 11$
 $0 + 11 \stackrel{??}{=} 11$
 $11 = 11 \quad \checkmark \quad \text{it is a solution}$

Is $(\frac{11}{4}, 0)$ a solution? $4(\frac{11}{4}) + 3(0) \stackrel{??}{=} 11$
 $11 + 0 \stackrel{??}{=} 11$
 $11 = 11 \quad \checkmark \quad \text{it is a solution}$

Is $(\frac{2}{4}, 3)$ a solution? $4(\frac{2}{4}) + 3(3) \stackrel{??}{=} 11$
 $2 + 9 \stackrel{??}{=} 11$
 $11 = 11 \quad \checkmark$

Graph the equation

x	y
0	$\frac{11}{3}$
$\frac{11}{4}$	0



In fact any point on the graph of the line is a solution.

System of equations (more than one equation):

A solution that satisfies more than equation is called a simultaneous solution and it could be found using three methods:

- 1) Graphically.
- 2) By substitution
- 3) By Elimination: the coefficient of the variables to be eliminated should be the same in both equations.

Example: Find a simultaneous solution for the following system of equations.

$$\begin{cases} 4x + 3y = 11 \\ 2x - 5y = -1 \end{cases}$$

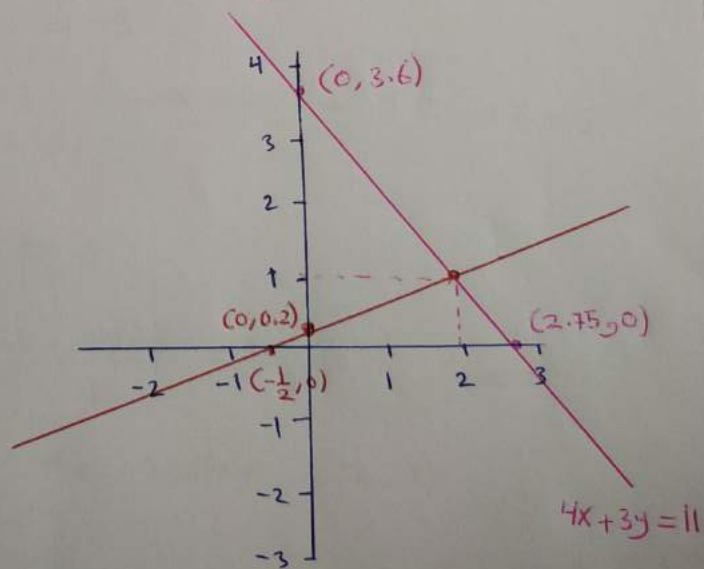
1) Graphically:

$$4x + 3y = 11$$

x	y
0	$\frac{11}{3} \approx 3.6$
$2.75 \approx \frac{11}{4}$	0

$$2x - 5y = -1$$

x	y
0	$\frac{1}{5} \approx 0.2$
$-\frac{1}{2}$	0



The solution (2, 1)

2) By Substitution:
$$\begin{cases} 4x + 3y = 11 & \text{--- (1)} \\ 2x - 5y = -1 & \text{--- (2)} \end{cases}$$

In eq. (1) $4x + 3y = 11 \rightarrow 3y = 11 - 4x$

$$\rightarrow y = \frac{11 - 4x}{3} \text{ --- (3)}$$

Substitute eq. (3) in eq. (2)

$$2x - 5y = -1$$

$$2x - 5\left(\frac{11 - 4x}{3}\right) = -1 \quad \text{multiply by 3}$$

$$3 \cdot (2x) - 3 \left(5 \left(\frac{11 - 4x}{3} \right) \right) = 3 \cdot (-1)$$

$$6x - 5(11 - 4x) = -3$$

$$6x - 55 + 20x = -3$$

$$\begin{array}{rcl} 26x & - & 55 = -3 \\ +55 & +55 & \rightarrow \frac{26x}{26} = \frac{52}{26} \rightarrow x = 2 \end{array}$$

Substitute $x=2$ in eq. (3)

$$y = \frac{11 - 4x}{3} = \frac{11 - 4(2)}{3} = \frac{11 - 8}{3} = \frac{3}{3} = 1$$

The solution of the system $(2, 1)$

3) By Elimination:
$$\begin{cases} 4x + 3y = 11 & \text{--- ①} \\ 2x - 5y = -1 & \text{--- ②} \end{cases}$$

Multiply eq. ② by -2 , you get $-4x + 10y = 2$

- You have the system
$$\begin{array}{r} 4x + 3y = 11 \\ -4x + 10y = 2 \\ \hline \end{array}$$

- add the two equations:
$$\frac{13y}{13} = \frac{13}{13} \rightarrow \boxed{y=1}$$

- Substitute $y=1$ in eq. ① $4x + 3y = 11$

$$4x + 3(1) = 11 \rightarrow 4x + 3 = 11$$

$\quad \quad -3 \quad -3$

$$\rightarrow \frac{4x}{4} = \frac{8}{4} \rightarrow \boxed{x=2} \rightarrow \text{The solution } (2, 1)$$

If you want to check the solution substitute $(2, 1)$

in the second equation $2x - 5y = -1$

$$2(2) - 5(1) \stackrel{?}{=} -1$$

$$4 - 5 \stackrel{?}{=} -1$$

$$-1 = -1 \checkmark$$

Example: Solve the following system of linear equations

$$\begin{cases} 3x + 4y = 1 & \text{--- ①} \\ 2x - 3y = 12 & \text{--- ②} \end{cases}$$

By elimination: multiply eq ① by 3 and multiply eq. ② by 4

$$\begin{aligned} \text{You have } 9x + 12y &= 3 \\ 8x - 12y &= 48 \end{aligned}$$

$$\text{add the two equations: } \frac{17x}{17} = \frac{51}{17} \rightarrow \boxed{x = 3}$$

Substitute $x = 3$ in eq ②: $8x - 12y = 48$

$$8(3) - 12y = 48 \rightarrow 24 - 12y = 48$$

$\quad \quad -24 \quad \quad -24$

$$\rightarrow \frac{-12y}{-12} = \frac{24}{-12} \rightarrow \boxed{y = -2}$$

The solution is $(3, -2)$

check the solution: substitute $(3, -2)$ in eq ①

$$9x + 12y = 3 \rightarrow 9(3) + 12(-2) \stackrel{??}{=} 3$$

$$27 - 24 \stackrel{??}{=} 3$$

$$3 = 3 \quad \checkmark$$

Example: Solve the following
$$\begin{cases} 4x + 3y = 4 & \text{---①} \\ 8x + 6y = 18 & \text{---②} \end{cases}$$

This system is called **inconsistent** system (Note that they have the same slope)

$$4x + 3y = 4 \rightarrow y = \frac{4}{3} - \frac{4x}{3} \rightarrow \text{slope} = -\frac{4}{3}$$

$$8x + 6y = 18 \rightarrow y = \frac{18}{6} - \frac{8x}{6} \rightarrow \text{slope} = -\frac{8}{6} = -\frac{4}{3}$$

Multiply eq ① by -2 , You get $-8x - 6y = -8$

$$\text{You have } -8x - 6y = -8$$

$$8x + 6y = 18$$

add the two equations: $0 + 0 = 10$

$$\boxed{0 = 10}$$

There are no solutions of the system, their graphs are parallel lines.

Example: Solve the following $\begin{cases} 4x + 3y = 4 & \text{--- ①} \\ 8x + 6y = 8 & \text{--- ②} \end{cases}$

This system is called **independent** system. Note that they have the same slope and multiple of each other ($\text{eq ②} = 2(\text{eq ①})$)

Multiply eq ① by -2 , you get $-8x - 6y = -8$

$$\begin{array}{r} \text{You have } -8x - 6y = -8 \\ \phantom{\text{You have }} 8x + 6y = 8 \end{array}$$

add the equations: $0 + 0 = 0 \rightarrow \boxed{0 = 0}$

So the system has infinitely many solutions, their graphs coincide and each point on this graph represents a solution of the system.

Question: Try to solve the following system:

$$\begin{cases} 2x - 4 = 5y & \text{--- ①} \\ x + 2y = 3 & \text{--- ②} \end{cases}$$

You can rearrange eq ① , $2x - 5y = 4$