Trigonometric Integrals

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Exp

$$\int \sec^2 x \tan x \, dx$$

$u = \tan x$

$du = \sec^2 x \, dx$

$$\int u \, du$$

$$= \frac{u^2}{2} + C$$

$$= \frac{\tan^2 x}{2} + C$$

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$$\int 4 \tan^3 x \, dx$$

$n = 3$ odd

$1 + \tan^2 x = \sec^2 x$

$\boxed{\tan^2 x = \sec^2 x - 1}$

$$\int 4 \tan^2 x \tan x \, dx$$

$$\int 4 (\sec^2 x - 1) \tan x \, dx$$

$u = \tan x$

$du = \sec^2 x \, dx$

$$4 \int \sec^2 x \tan x \, dx - 4 \int \tan x \, dx$$

33

$$\cancel{4}^{2} \frac{\tan^2 x}{\cancel{2}} + 4\int \frac{-\sin x}{\cos x}\,dx$$

$$= 2\tan^2 x + 4\int \frac{-\sin x}{\cos x}\,dx$$

$$= 2\tan^2 x + 4\ln|\cos x| + C$$

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(67) $\int x \sin^2 x \, dx$

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$\cos 2x = 1 - 2\sin^2 x$$
$$= 2\cos^2 x - 1$$

$$\int x\left(\frac{1-\cos 2x}{2}\right)dx$$

$$\int x\left(\frac{1}{2} - \frac{\cos 2x}{2}\right)dx$$

$$\int \frac{x}{2}\,dx - \int \frac{x}{2}\cos 2x\,dx$$

$\frac{x}{2}$	$\cos 2x$
$+$	
$\frac{1}{2}$	$\frac{1}{2}\sin 2x$
$-$	
0	$\frac{-1}{4}\cos 2x$

$$\frac{1}{2}\frac{x^2}{2} - \left[\frac{x}{4}\sin 2x + \frac{1}{8}\cos 2x\right]$$

$$\frac{x^2}{4} - \frac{x}{4}\sin 2x - \frac{1}{8}\cos 2x + C$$

$\frac{x}{4} - \frac{1}{4}\sin 2x - \frac{1}{8}\cos 2x$

(5) $n=3$ odd

$$\int \sin^3 x \, dx$$

$\sin^2 x + \cos^2 x = 1$

$\sin^2 x = 1 - \cos^2 x$

$$\int \sin^2 x \, \sin x \, dx$$

$u = \cos x$

$du = -\sin x \, dx$

$$\int (1-\cos^2 x)\sin x \, dx$$

$$-\int (1-u^2)\, du$$

$$\int (u^2-1)\, du = \frac{u^3}{3} - u + C$$

$$= \frac{\cos^3 x}{3} - \cos x + C$$

(22) $\int_0^{\frac{\pi}{2}} \sin^{2} 2\theta \, \cos^{3} 2\theta \, d\theta$

زوجي — even, odd

$\Downarrow$ odd

Q. How to integrate $\int \sin^m x \cos^n x \, dx$?

look for odd #

$$\int_0^{\frac{\pi}{2}} \sin^2 2\theta \, \cos^2 2\theta \, \cos 2\theta \, d\theta$$

$\sin^2 2\theta \to u^2$

$u = \sin 2\theta$

$du = 2\cos 2\theta \, d\theta$

$\frac{du}{2} = \cos 2\theta \, d\theta$

$$\int^{\frac{\pi}{2}} \sin^2 2\theta \,(1-\sin^2 2\theta)\cos 2\theta \, d\theta$$

$$\int_0^{\pi/2} \sin^2 2\theta \,(1-\sin^2 2\theta)\, \cos 2\theta \, d\theta$$

$\frac{du}{2} = \cos 2\theta \, d\theta$

when $\theta = 0 \Rightarrow$

$u = \sin 0 = 0$

when $\theta = \frac{\pi}{2} \Rightarrow$

$u = \sin \pi = 0$

$$\int_0^0 u^2 (1-u^2) \frac{du}{2}$$

0

Exp

$$\int \sin^3 x \cos^3 x \, dx$$

odd odd

$u = \cos x$

$du = -\sin x \, dx$

$-du = \sin x \, dx$

$$\int \sin^2 x \sin x \cos^3 x \, dx$$

u^3

$$\int (1-\cos^2 x) \cos^3 x \sin x \, dx$$

$$-\int (1-u^2)\, u^3 \, du$$

$$\int (u^2-1)\, u^3 \, du = \int (u^5 - u^3)\, du$$

$$= \frac{u^6}{6} - \frac{u^4}{4} + C$$

$- - 6$

$$= \frac{\cos^6 x}{6} - \frac{\cos^4 \cancel{x}}{4} + C$$

Exp $\int 16 \sin^2 x \cos^2 x \, dx$

even

n and m

even

$\Downarrow$

use

$$\int \overset{4}{\cancel{16}} \left(\frac{1-\cos 2x}{\cancel{2}}\right)\left(\frac{1+\cos 2x}{\cancel{2}}\right) dx$$

$\cos^2 x = \frac{1+\cos 2x}{2}$

$\sin^2 x = \frac{1-\cos 2x}{2}$

$$4 \int (1-\cos 2x)(1+\cos 2x) \, dx$$

$x^2 - y^2 = (x-y)(x+y)$

$x = 1$

$y = \cos 2x$

$$4 \int (1-\cos^2 2x) \, dx$$

even

$$\int 4 \, dx - 4 \int \cos^2 2x \, dx$$

$\cos^2 x = \frac{1+\cos 2x}{2}$

$\cos^2 (2x) = \frac{1+\cos 4x}{2}$

$$4x - \overset{2}{\cancel{4}} \int \left(\frac{1+\cos 4x}{\cancel{2}}\right) dx$$

$$4x - 2\int(1+\cos 4x)\,dx$$

$$4x - 2\left(x + \frac{\sin 4x}{4}\right) + c$$

$$4x - 2x - \frac{1}{2}\sin 4x + c$$

$$\boxed{2x - \frac{1}{2}\sin 4x + c}$$

$\sin 2x = 2\sin x\cos x$

$\sin^2 2x = 4\sin^2 x\cos^2 x$

$$\int 16\sin^2 x\cos^2 x\,dx$$

$$4\boxed{\int 4\sin^2 x\cos^2 x\,dx}$$

$$4\int \sin^2 2x\,dx$$

$\sin^2 2x = \dfrac{1-\cos 4x}{2}$

$$\overset{2}{\cancel{4}}\int \frac{1-\cos 4x}{\cancel{2}}\,dx$$

$$2\int(1-\cos 4x)\,dx$$

$$2\left(x - \frac{\sin 4x}{4}\right) + C$$

$$\boxed{2x - \frac{1}{2}\sin 4x + C}$$

$$\int \sin mx \ \sin nx \ dx = \int \frac{1}{2}\left[\cos(m-n)x - \cos(m+n)x\right] dx$$

$$\int \cos mx \ \cos nx \ dx = \int \frac{1}{2}\left[\cos(m-n)x + \cos(m+n)x\right] dx$$

$$\int \sin mx \ \cos nx \ dx = \int \frac{1}{2}\left[\sin(m-n)x + \sin(m+n)x\right] dx$$

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$$\int \sin \overset{m=3}{3x} \ \cos \overset{n=2}{2x} \ dx = \int \frac{1}{2}\left[\sin(3-2)x + \sin(3+2)x\right] dx$$

$$= \int \frac{1}{2}\left[\sin x + \sin 5x\right] dx$$

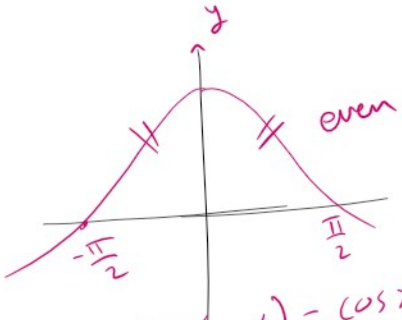
$$= \frac{1}{2}\left[-\cos x - \frac{\cos 5x}{5}\right] + C$$

$$= -\frac{1}{2}\cos x - \frac{1}{10}\cos 5x + C$$

Exp

$$\int \cos 3x \cos 4x \, dx = \frac{1}{2}\int \left[\cos(3-4)x + \cos(3+4)x\right] dx$$

$$= \frac{1}{2}\int \left[\cos(-x) + \cos(7x)\right] dx$$



$\cos(-x) = \cos x$

$$= \frac{1}{2}\int \left[\cos x + \cos 7x\right] dx$$

$$= \frac{1}{2}\left(\sin x + \frac{\sin 7x}{7}\right) + C$$

$$= \frac{\sin x}{2} + \frac{\sin 7x}{14} + C$$

Estimating Square Roots

$\cos^2 x = 1 - \sin^2 x$

Exp

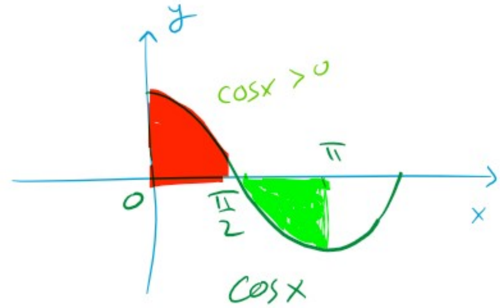
$$\int_0^{\pi} \sqrt{1 - \sin^2 x}\, dx$$

$\sqrt{\cos^2 x} = \cancel{\cos x}$

$= |\cos x|$

$$\int_0^{\pi} \sqrt{\cos^2 x}\, dx$$

$$\int_0^{\pi} |\cos x|\, dx$$

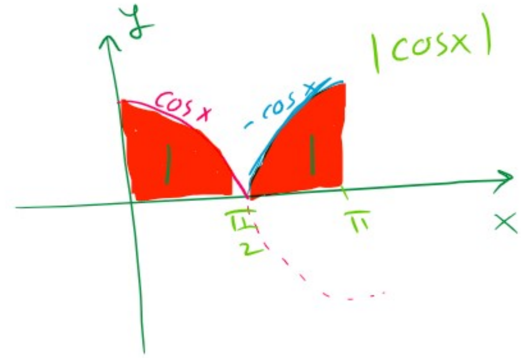


$$= 2\int^{\frac{\pi}{2}} \cos x \, dx$$

$\int^{\pi} \cos x \, dx = 0$

$$= 2 \int_0^{\frac{\pi}{2}} \cos x \, dx$$

$$\int_0^{\pi} \cos x \, dx = 0$$



$$= 2 \quad \sin x \Big|_0^{\frac{\pi}{2}}$$

$$= 2 \left[\sin \frac{\pi}{2} - \sin 0 \right]$$

$$= 2 [1 - 0] = 2$$

or

$$\int_0^{\pi} |\cos x| \, dx = \int_0^{\frac{\pi}{2}} \cos x \, dx + \int_{\frac{\pi}{2}}^{\pi} -\cos x \, dx$$

$$|\cos x| = \begin{cases} \cos x & \text{if } 0 \le x \le \frac{\pi}{2} \\ -\cos x & \text{if } \frac{\pi}{2} \le x \le \pi \end{cases}$$

$$= \sin x \Big|_0^{\frac{\pi}{2}} - \sin \Big|_{\frac{\pi}{2}}^{\pi}$$

$$= \sin \frac{\pi}{2} - \sin 0 - \left(\sin \pi - \sin \frac{\pi}{2} \right)$$

$$= 1 - 0 - (0 - 1)$$

$$= 1 + 1$$

$$= 2$$

Exp $\int \sec^4 x \, dx = \int \underbrace{\sec^2 x}_{1 + \tan^2 x} \sec^2 x \, dx$

$1+\tan^2 x$

$$= \int (1+\tan^2 x)\,\underline{\sec^2 x\, dx}$$

$u = \tan x$

$du = \underline{\sec^2 x\, dx}$

$$= \int (1+u^2)\, du$$

$$= u + \frac{u^3}{3} + C$$

$$= \tan x + \frac{1}{3}\tan^3 x + C$$

$$\int \sin^m x \cos^n x \, dx$$

✓ if m odd $\Rightarrow$ $\sin^m x = \sin^{2k+1} x = \sin^{2k} x \ \sin x$

and use $\sin^2 x = 1-\cos^2 x$

✓ if n odd $\Rightarrow$ $\cos^n x = \cos^{2k+1} x = \cos^{2k} x \ \cos x$

and use $\cos^2 x = 1-\sin^2 x$

$\Rightarrow$ use $\cos^2 x = \frac{1+\cos 2x}{}$

if m and n both even $\Rightarrow$ use $\cos^2 x = \frac{1+\cos 2x}{2}$

$\sin^2 x = \frac{1-\cos 2x}{2}$