4.7: The Moment generating function Technique.

exp: $X_1 \sim N(\mu_1, \sigma_1^2)$

$X_2 \sim N(\mu_2, \sigma_2^2)$

X_1 and X_2 indep.

Find dist. of $Y = X_1 - X_2$

sol.

$$X_1 \sim N(\mu_1, \sigma_1^2) \Rightarrow M_1(t) = E(e^{tX_1}) = e^{\mu_1 t + \frac{1}{2}\sigma_1^2 t^2}$$

$$X_2 \sim N(\mu_2, \sigma^2) \Rightarrow M_2(t) = E(e^{tX_2}) = e^{\mu_2 t + \frac{1}{2}\sigma_2^2 t^2}$$

$$Y = X_1 - X_2 \Rightarrow M_Y(t) = E(e^{tY}) = E(e^{t(X_1 - X_2)})$$

$$= E(e^{tX_1} \cdot e^{-tX_2})$$

X_1, X_2 indep. $$= E(e^{tX_1})\, E(e^{-tX_2})$$

$$= M_1(t) \cdot M_2(-t)$$

$$= e^{\mu_1 t + \frac{1}{2}\sigma_1^2 t^2} \cdot e^{\mu_2(-t) + \frac{1}{2}\sigma_2^2(-t)^2}$$

$$\boxed{M_Y(t) = e^{(\mu_1 - \mu_2)t + \frac{1}{2}(\sigma_1^2 + \sigma_2^2)t^2}}.$$

$$Y \sim N(\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2).$$

Thm 1: $X_1, \dots, X_n$ independent $N(\mu_1, \sigma_1^2), \dots, N(\mu_n, \sigma_n^2)$ respectively

Define $Y = \sum_{i=1}^{n} k_i X_i = k_1 X_1 + \dots + k_n X_n$, where $k_1, \dots, k_n$ real constant

Then $Y \sim N\left(\sum_{i=1}^{n} k_i \mu_i , \sum_{i=1}^{n} k_i^2 \sigma_i^2\right)$.

proof:

$$M_Y(t) = E(e^{tY}) = E\left(e^{t(k_1 X_1 + \dots + k_n X_n)}\right)$$

$$= E\left(e^{t k_1 X_1} \cdot e^{t X_2 k_2} \cdot \dots \cdot e^{t k_n X_n}\right)$$

$$\because X_1, \dots, X_n \text{ indep.} \quad = E(e^{t k_1 X_1}) \cdot \dots \cdot E(e^{t k_n X_n}).$$

$$= e^{\mu_1 (k_1 t) + \frac{1}{2} \sigma_1^2 (k_1 t)^2} \cdot \dots \cdot e^{\mu_n (k_n t) + \frac{1}{2} \sigma_n^2 (k_n t)^2}$$

$$= e^{(\mu_1 k_1 + \dots + \mu_n k_n) t + \frac{1}{2}(\sigma_1^2 k_1^2 + \dots + \sigma_n^2 k_n^2) t^2}$$

$$\Rightarrow Y \sim N\left(\sum_{i=1}^{n} k_i \mu_i , \sum_{i=1}^{n} k_i^2 \sigma_i^2\right)$$

Thm 2: $X_1, \dots, X_n$ independent random variables with m.g.f's $M_1(t), \dots, M_n(t)$ respectively. Define $Y = \sum_{i=1}^{n} a_i X_i$. Then $M_Y(t) = M_1(a_1 t) \dots M_n(a_n t)$

$$= \prod_{i=1}^{n} M_i(a_i t).$$

proof:

$$M_Y(t) = E(e^{tY}) = E\left(e^{t(a_1 X_1 + \dots + a_n X_n)}\right)$$

$$= E\left(e^{t a_1 X_1} \cdot e^{t a_2 X_2} \cdot \dots \cdot e^{t a_n X_n}\right)$$

$$= E(e^{t a_1 X_1}) \dots E(e^{t a_n X_n}).$$

$$= M_1(a_1 t) \dots M_n(a_n t)$$

$$= \prod_{i=1}^{n} M_i(a_i t).$$

Corollary : $X_1, \ldots, X_n$ Random sample with m.g.f $M(t)$, then

a. $Y = \sum_{i=1}^{n} X_i \Rightarrow M_Y(t) = (M(t))^n$.

b. $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i \Rightarrow M_{\bar{X}}(t) = \left(M\left(\frac{t}{n}\right)\right)^n$

Proof :

a. Thm 2 , $a_i = 1$, $i = 1, \ldots, n$.

b. Thm 2 , $a_i = \frac{1}{n}$, $i = 1, \ldots, n$.

exp 3 :

$X_1, \ldots, X_n$, n Bernoulli trails

$X_1, \ldots, X_n$ independent $b(1, p)$.

Find dist. of $Y = \sum_{i=1}^{n} X_i$.

sol.

$X_1, \ldots, X_n$ Random sample with $M(t) = pe^t + 1 - p$.

$M_Y(t) = (M(t))^n = (pe^t + 1 - p)^n$

$Y = \sum_{i=1}^{n} X_i \sim b(n, p)$.

exp 4: X_1, X_2, X_3 Random sample from exponential dist. with mean β.

1. $Y = \sum_{i=1}^{3} X_i$

X_1, X_2, X_3 indep. with $M(t) = (1 - \beta t)^{-1}$, $t < \frac{1}{\beta}$

$M_Y(t) = (M(t))^3 = ((1 - \beta t)^{-1})^3$, $t < \frac{1}{\beta}$

$M_Y(t) = (1 - \beta t)^{-3}$, $t < \frac{1}{\beta}$

$\Rightarrow Y \sim Gamma(3, \beta)$.

2. $\bar{X} = \frac{1}{n} \sum_{i=1}^{3} X_i$.

$$M_{\bar{X}}(t) = \left(M\left(\tfrac{t}{3}\right)\right)^3 = \left(1 - \beta \tfrac{t}{3}\right)^{-3}, \quad \tfrac{t}{3} < \tfrac{1}{\beta}$$

$$= \left(1 - \tfrac{\beta}{3} t\right)^{-3}, \quad t < \tfrac{3}{\beta}$$

$$\Rightarrow \bar{X} \sim \text{Gamma}\left(3, \tfrac{\beta}{3}\right).$$

Thm 3: $X_1, \ldots, X_n$ indep., $\chi^2(r_1), \ldots, \chi^2(r_n)$, define $Y = \sum_{i=1}^{n} X_i$ then $Y \sim \chi^2(r_1 + \cdots + r_n)$.

proof: m.g.f of X_i

$$M_i(t) = (1 - \beta t)^{-\alpha}, \quad t < \tfrac{1}{\beta}.$$

$$\alpha = \tfrac{r_i}{2}, \quad \beta = 2$$

$$M_i(t) = (1 - 2t)^{-\frac{r_i}{2}}, \quad t < \tfrac{1}{2}$$

$$M_Y(t) = (1 - 2t)^{-\frac{r_1}{2}} \cdots (1 - 2t)^{-\frac{r_n}{2}}, \quad t < \tfrac{1}{2}$$

$$M_Y(t) = (1 - 2t)^{-\frac{(r_1 + \cdots + r_n)}{2}}, \quad t < \tfrac{1}{2}$$

$$\Rightarrow Y \sim \chi^2(r_1 + \cdots + r_n).$$

$\boxtimes$

Thm 4:

$X_1, \dots, X_n$ Random sample from $N(\mu, \sigma^2)$.

Define $Y = \sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma} \right)^2$ then $Y \sim \chi^2(n)$.

proof:

$X_i \sim N(\mu, \sigma^2)$ indep.

$\frac{X_i - \mu}{\sigma} \sim N(0,1)$ indep.

$\left(\frac{X_i - \mu}{\sigma} \right)^2 \sim \chi^2(1)$ indep.

$\sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma} \right)^2 \sim \chi^2(n)$ ∎