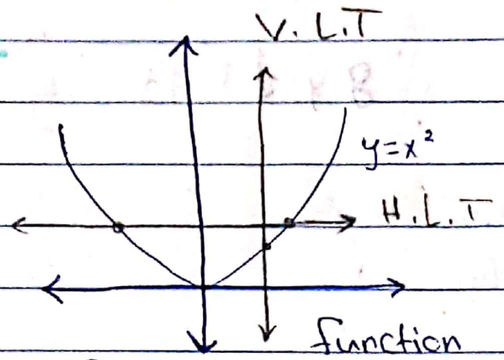
V.L.T ⇒ Vertical line test
H.L.T ⇒ Horizontal line test

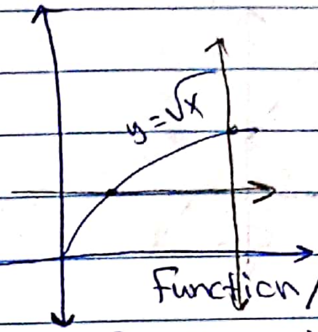
7.1 Inverse function:

Exp



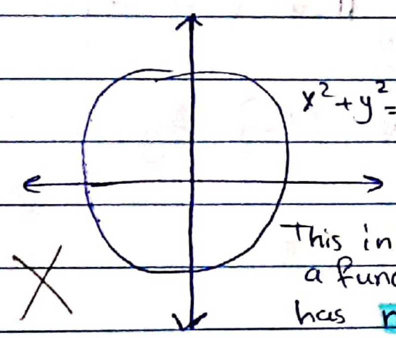
function has no inverse

$D = \mathbb{R}$

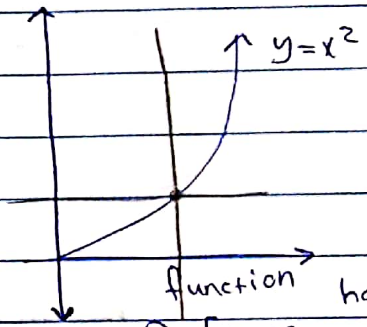


Function / has inverse

$D = [0, \infty)$



This in not a function has no inverse



function has inverse.

$D = [0, \infty$

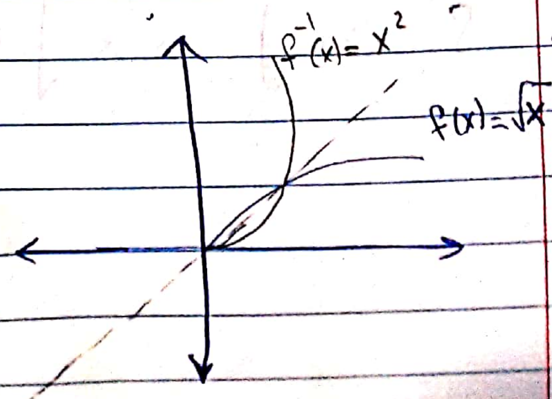
Def: The function $y=f(x)$ is "one to one" on Domain $\underline{\underline{D}}$ if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ for all $x_1, x_2 \in D$

Remark: Only 1-1 function have inverse.

Qustion: How to find $f^{-1}(x)$? . $[f(x)]^{-1} = \dfrac{1}{f(x)}$

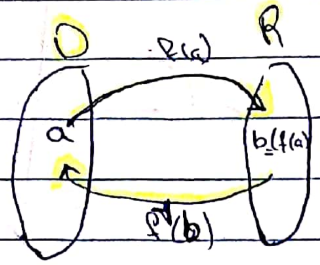
$\Rightarrow f^{-1}(x) \neq \dfrac{1}{f(x)}$

f inverse of x



7.1 → Inverse Functions:

Assume



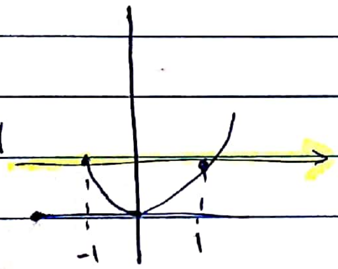
- If f is 1-1 $\Rightarrow$ $f^{-1}(b)$

  $D(f) = R(f^{-1})$

  and $D(f^{-1}) = R(f)$

- f is 1-1 on D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2 \in D$

not 1-1



- other words: f is 1-1 if f cross each HLT at most once.

- Only 1-1 function have inverse.

- the graphs of f and its inverse f^{-1} are symmetries about $x=y$.

Suppose f is 1-1 How to find it inverse f^{-1} ?

1. solve for x
2. replace x by y and replace y by x.
3. replace y by $f^{-1}(x)$.

Exp: find $f^{-1}(x)$ for ① $f(x) = x^2$ $x \geq 0$

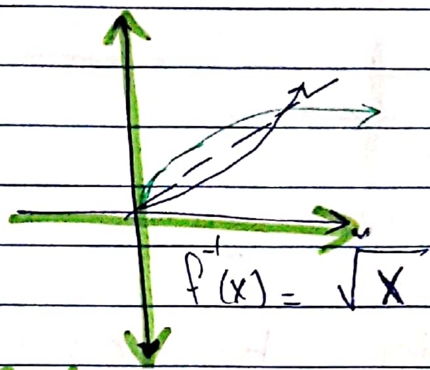
1. $y = x^2$

$\sqrt{y} = |x| = x$

2. $x = \sqrt{y}$

$y = \sqrt{x}$

3. $f^{-1}(x) = \sqrt{x}$



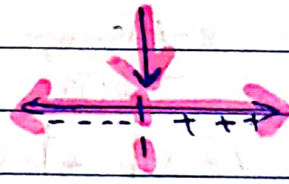
* $(f \circ f^{-1}) = x$
* $(f^{-1} \circ f) = x$

important Note

Exp:

$f(x) = x^2 - 2x$ $x \leq 1$

$y = (x-1)^2 - 1 \rightarrow \sqrt{y+1} = |x-1|$



$\sqrt{y+1} = 1 - x$

$-1 + \sqrt{y+1} = -x$

1. $x = 1 - \sqrt{y+1}$
2. $y = 1 - \sqrt{x+1}$
3. $f^{-1}(x) = 1 - \sqrt{x+1}$

* $D(f) = (-\infty, 1] = R(f^{-1})$
* $R(f) = D(f^{-1}) = [-1, \infty)$
* $(f \circ f^{-1})(x) = f(f^{-1}(x))$

$= f(1 - \sqrt{x+1})$

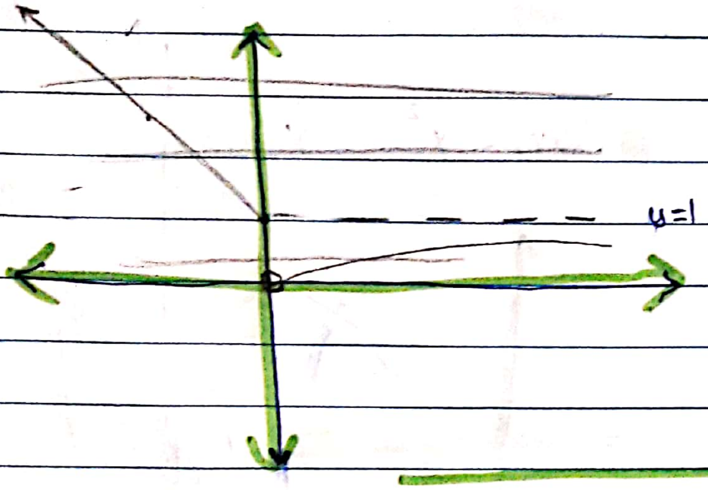
$= [1 - \sqrt{x+1}]^2 - 2[1 - \sqrt{x+1}]$

$= x$

Exp: Use $\boxed{\text{HLT}}$ to check if

$$f(x) = \begin{cases} 1 - \frac{x}{2}, & x \le 0 \\ \frac{x}{x+2}, & x > 0 \end{cases}$$

is 1-1 ?!



The HLT cross the function in one point so it is a 1-1 function.

Qustion: How to find derivative of f^{-1}?

$$\frac{df^{-1}}{dx} \text{ or } (f^{-1})'$$

Theorem: Assum $f: D \to R$ is 1-1 and diff and f' never zero. Then f^{-1}. $f^{-1}: R \to D$ is diff.

$$\left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\frac{df}{dx}\right|_{x=a=f^{-1}(b)}}$$

$b = f(a)$

$f^{-1}(b) = f^{-1}(f(a)) = a$

$$(f^{-1})'(b) = \frac{1}{f'(a)}$$

Esmaeh Jaded

Exp Given $f'(x) = 3x^2$

Find $\left.\frac{df^{-1}}{dx}\right|_{x=f(\sqrt{2})}$

$$\left.\frac{df^{-1}}{dx}\right|_{x=f(\sqrt{2})} = \frac{1}{\left.\frac{df}{dx}\right|_{x=f^{-1}(f(\sqrt{2}))}} = \frac{1}{f'(\sqrt{2})} = \frac{1}{6}$$

Exp: $f(x) = 2x + e^x$

Find $\frac{df^{-1}}{dx}(1)$?

$1 = 2a + e^a$

$\boxed{a=0}$

$$\frac{df^{-1}}{dx}(\underset{b}{1}) = \frac{1}{f'(\underbrace{f^{-1}(1)}_{a})} = \frac{1}{f'(a)} = \frac{1}{2+e^a}$$

$b = f(a) = 1$

$$= \frac{1}{2+1} = \frac{1}{3}$$

$f'(x) = 2 + e^x$

Exp: Find $\left.\dfrac{d f^{-1}}{dx}\right|_{x=0}$ for $f(x) = x^2 - 4x - 5$, $x > 2$

$$\left.\frac{df^{-1}}{dx}\right|_{x=0} = \frac{1}{f'(a)}$$

$f(a) = 0$

$a^2 - 4a - 5 = 0$

$(a-5)(a+1) = 0$

$a = 5, \underset{\notin\, x>2}{-1}$

$a = 5$

$f'(x) = 2x - 4 = 2*5 - 4 = 6$

$$\frac{df^{-1}}{dx} = \frac{1}{f'(a)} = \frac{1}{6}$$

Remark: The graph of f and f^{-1} are symmetric about $y = x$

Exp $f(x) = x^2$ $x \geq 0$

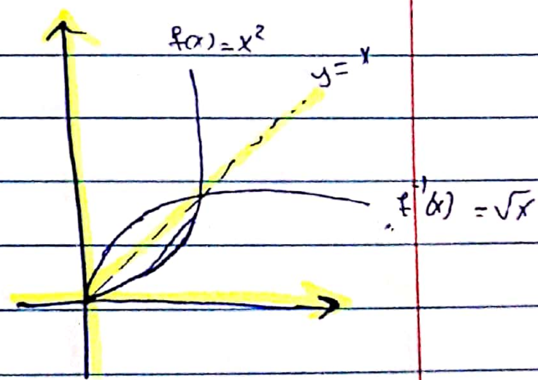
1. $\sqrt{y} = \sqrt{x^2}$

$\sqrt{y} = |x|$

$x = \sqrt{y}$

2. $y = \sqrt{x}$

3. $f^{-1}(x) = \sqrt{x}$



$$\left.\frac{df^{-1}}{dx}\right|_{\underset{b}{x=4}} = \frac{1}{2\sqrt{4}}$$

$$= \frac{1}{4}$$

$b = f(a)$

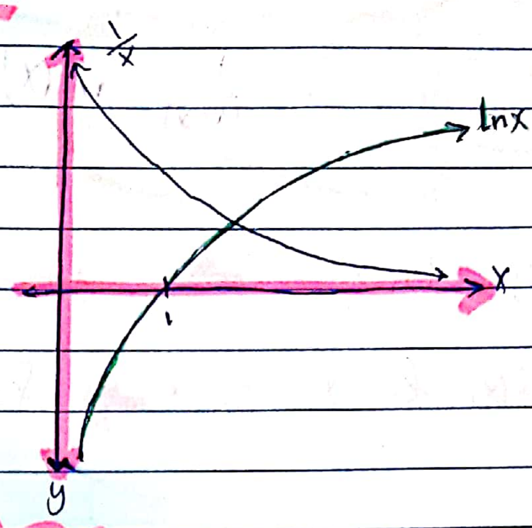
$4 = a^2$

$a = 2$

$\Rightarrow = \dfrac{1}{f'(a)} = \dfrac{1}{f'(2)}$

$$= \frac{1}{4}$$

$\boxed{7.2}$ Natural Function :



$y = \ln x$

$D = (0, \infty)$

$R = \mathbb{R}$

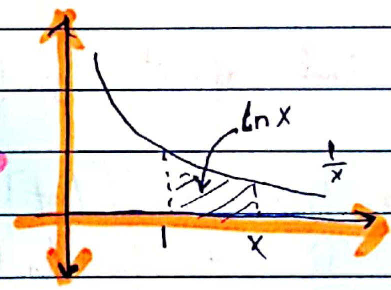
$$(\ln x)' = \frac{1}{x}$$

$$\ln x = \int_0^x \frac{1}{t} \cdot dt \quad , t$$

If $x > 1$ then $\ln x > 0$

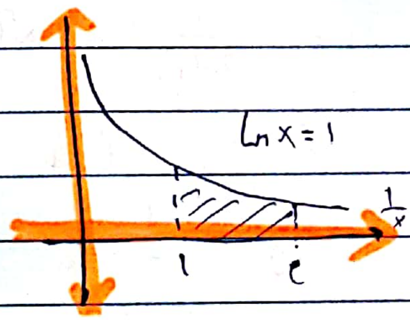
If $0 < x < 1$ then $\ln x < 0$

If $x > 1$ then $\ln x$ is the area below $\frac{1}{t}$ given by $\ln x = \int_1^x \frac{1}{t} \, dt$



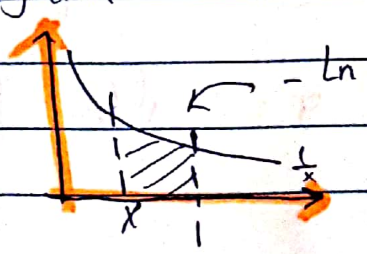
Exp : Take $x = e \approx 2.718$

$$\Rightarrow \ln e = 1 = \int_1^e \frac{1}{t} \, dt$$



If $x < 1$ then $\ln x$ is negative and given by

$$\ln x = \int_x^1 \frac{1}{t} \, dt$$



Remark Assume $u(x) > 0$ and diff for $y = \ln(u(x))$

Then $y' = \frac{1}{u(x)} u'(x)$.

Exp ① $y = \ln x \Rightarrow y' = \frac{1}{x}$

② $f(x) = \ln x^2 - \sin x \Rightarrow f'(x) = \frac{1}{x^2 - \sin x} * (2x - \cos x)$

③

→ Properties of Natural function:

Assume a and b are positive constant. Then

① $\ln(ab) = \ln a + \ln b$

② $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$

③ $\ln a^b = b \ln a$

④ $\ln \frac{1}{a} = -\ln a$

Exp: Find y' if $y = t\sqrt{\ln t}$

S1 → $y = t\,[\ln t]^{\frac{1}{2}}$

$y' = t\,\frac{1}{2}[\ln t]^{-\frac{1}{2}} \frac{1}{t} + [\ln t]^{\frac{1}{2}}$

$= \frac{1}{2\sqrt{\ln t}} + \sqrt{\ln t}$

S2 → $\ln y = \ln t + \ln(\ln t)^{\frac{1}{2}}$

$\frac{y'}{y} = \frac{1}{t} + \frac{1}{\sqrt{\ln t}} \frac{1}{2} (\ln t)^{-\frac{1}{2}} \frac{1}{t}$

$y' = y\left[\frac{1}{t} + \frac{1}{\sqrt{\ln t}} \frac{1}{2} (\ln t)^{-\frac{1}{2}} \frac{1}{t}\right]$

$y' = t\sqrt{\ln t}\left[\frac{1}{t} + \frac{1}{2t\ln t}\right]$

(2) $y = t\,(t+1)(t+2)(t+3)$

$\ln y = \ln t + \ln(t+1) + \ln(t+2) + \ln(t+3)$

$$\frac{y'}{y} = \frac{1}{t} + \frac{1}{t+1} + \frac{1}{t+2} + \frac{1}{t+3}$$

$$y' = y\left(\frac{1}{t} + \frac{1}{t+1} + \frac{1}{t+2} + \frac{1}{t+3}\right)$$

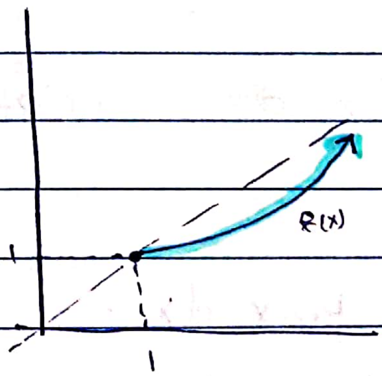
(3) $y = \ln\left[\ln(\ln x)\right]$

$$= \frac{1}{\ln(\ln x)} * \frac{1}{\ln x} * \frac{1}{x}$$

Exp. show that $f(x) = x - \ln x$, $x > 1$ is increasing.

(a) $f'(x) = 1 - \frac{1}{x} \Rightarrow f(x)\ + + + + + + \rightarrow \quad x > 1$

(b) show that $x > \ln x$



$f(1) = 1 - \ln 1$

$= 1 - 0 = 1$

and $f \uparrow$

$f(x) > 0$

$x - \ln x > 0$

$x > \ln x$

Exp: Express $\ln\sqrt{13.5}$ interms of $\ln 2$ and $\ln 3$??

$\ln (13.5)^{1/2}$

$= \frac{1}{2} \ln (13.5)$

$= \frac{1}{2} \ln \frac{27}{2}$

$= \frac{1}{2} (\ln 27 - \ln 2)$

$= \frac{1}{2} (3\ln 3 - \ln 2)$

- Remember :-

① $\int \sec^2 x \, dx = \tan x + c$

② $\int \csc^2 x \, dx = -\cot x + c$

③ $\int \sec x \tan x \, dx = \sec x + c$

④ $\int \csc x \cot x \, dx = -\csc x + c$

⑤ $\int \tan x \, dx = \ln|\sec x| + c$

⑥ $\int \cot x \, dx = + \ln|\sin x| + c$

⑦ $\int \sec x \, dx = + \ln|\sec x + \tan x| + c$

⑧ $\int \csc x \, dx = - \ln|\csc x + \cot x| + c$

Proof

$\boxed{5}$ $\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx$

$= - \int \frac{-\sin x}{\cos x} dx$

$= - \ln|\cos x| + c$

$= \ln|\sec x| + c$

$\boxed{6}$ نفس الطريقة

$\boxed{7}$ $\int \sec x \, dx = \int \sec x \left[\frac{\sec x + \tan x}{\sec x + \tan x}\right] dx$

$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$

$= \ln|\tan x + \sec x| + c$

$\boxed{9}$ نفس طريقة 7

Exp: Find ?

① $\int_{-3}^{-2} \frac{dx}{x} = \ln|x| \Big|_{-3}^{-2} = \ln 2 - \ln 3$

$= \ln \frac{2}{3}$

② $\int_{0}^{\pi/2} \tan\frac{x}{2}\, dx = 2\int_{0}^{\pi/4} \tan u \, du$

$u = \frac{x}{2}$
$du = \frac{1}{2}dx$
$x = 0 \rightarrow u = 0$
$x = \frac{\pi}{2} \rightarrow u = \frac{\pi}{4}$

$= 2 \ln|\sec u| \Big|_{0}^{\pi/4}$

$= 2\ln|\sec\frac{\pi}{4}| - 2\ln|\sec 0|$

$= 2\ln\sqrt{2} - 2\ln 1$

$= \ln 2$

③ $\int_{2}^{4} \frac{dx}{x \ln x}$

$u = \ln x$
$du = \frac{1}{x}dx$
$x = 2 \Rightarrow u = \ln 2$
$x = 4 \Rightarrow u = \ln 4$

$\int_{\ln 2}^{\ln 4} \frac{du}{u}$

$= \ln u \Big|_{\ln 2}^{\ln 4} = \ln(\ln 4) - \ln(\ln 2)$

$= \ln\left(\frac{\ln 4}{\ln 2}\right) = \ln\left(\frac{2\cancel{\ln 2}}{\cancel{\ln 2}}\right)$

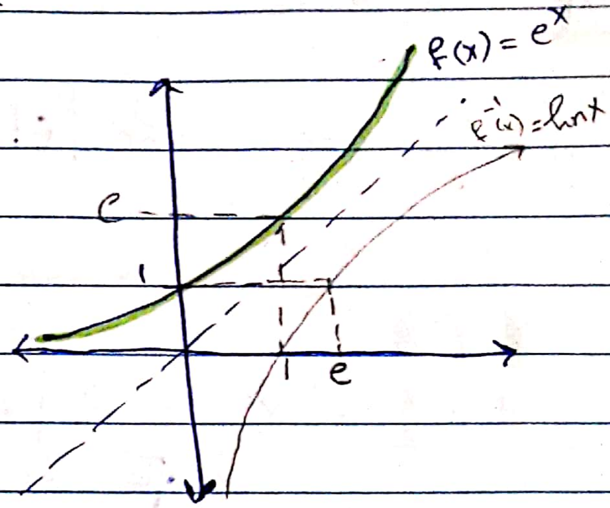
$= \ln 2$

7.3 Exponensial function :-

→ $f(x) = e^x$

$D = \mathbb{R}$

$R = (0, \infty)$



→ $y = e^x$

$\ln y = x \ln e$

$\ln y = x$

$x = \ln y$

$y = \ln x$

$f^{-1}(x) = \ln x$

Remarks :

① $e^{\ln x} = x \quad \forall x > 0$

② $\ln e^x = x$

solve :

① $e^{\ln 2x} = 8 \Rightarrow 2x = 8 \Rightarrow x = 4$

② $e^{(\ln 2)x} = 8$

$\ln e^{(\ln 2)x} = \ln 8$

$(\ln 2)x = \ln 8$

$x = \dfrac{\ln 8}{\ln 2} = \dfrac{\ln 2^3}{\ln 2} = 3$

Remark suppos: $a>0$ and $u(x)$ is diff

If $y = a^{u(x)}$ then $y' = a^{u(x)} u'(x) \ln a$

Hence, $\boxed{\int a^{u(x)} u'(x)\, dx = \frac{1}{\ln a} a^{u(x)} + C}$

Exp. Find y' if

1) $y = 3^x$

S1) $\ln y = x \ln 3 \Rightarrow \frac{y'}{y} = \ln 3 \Rightarrow y' = 3^x \ln 3$

S2) $y' = 3^x (1) \ln 3$

2) $y = e^{5-7x}$

$y' = e^{5-7x} * (-7) \ln e$

$y' = -7 e^{5-7x}$

3) $f(x) = \pi^{ex}$

$f'(x) = \pi^{ex} (e) \ln \pi$

$= \pi^{ex} \ln \pi^e$

4) $f(x) = x^{\pi+e}$

$f'(x) = (\pi+e) x^{(\pi+e-1)}$

5) $y = (\ln 2)\, 2^{\sin 3t}$

$y' = (\ln 2)\, 2^{\sin 3t} \cos 3t\, (3) \ln 2$

$= (\ln 2)^2 (3)\, 2^{\sin 3t} \cos 3t$

Exp. Find ① $\int 5^{\sec\theta} \ln 5 \sec\theta \tan\theta \, d\theta = 5^{\sec\theta} + C$

② $\frac{\ln 5}{\ln 5} \int 5^{\sec\theta + 4} \sec\theta \tan\theta \, d\theta$

$\frac{1}{\ln 5} 5^{\sec\theta+4} + C$

③ $\int 7^x \, dx$

$= \frac{\ln 7}{\ln 7} \int 7^x \, dx$

$\frac{1}{\ln 7} 7^x + C$

- properties ~~of~~ Expo

$\forall \; x_1, x_2, x_3 \Rightarrow$

1. $e^{x_1} e^{x_2} = e^{x_1 + x_2}$

2. $\frac{e^{x_1}}{e^{x_2}} = e^{x_1 - x_2}$

3. $[e^{x_1}]^{x_2} = e^{x_1 x_2}$

4. $e^{\ln y} = y \quad y > 0$

Exp :- $y = 2^x = e^{\ln 2^x}$

$= e^{x \ln 2}$

$y' = e^{x\ln 2} * \ln 2$

Exp :- $\int \frac{dx}{1+e^x} = ?$

$\int \frac{dx}{1+e^x} * \left(\frac{e^{-x}}{e^{-x}}\right) = \int \frac{e^{-x}}{1+e^{-x}} dx$

$u = e^{-x} + 1$

$du = -e^{-x} dx$

$\int \frac{\cancel{e^{-x}}\, du}{\cancel{(-e^{-x})}\, u} = -\int \frac{du}{u}$

$= -\ln u$

$= -\ln(e^{-x}+1)$

$= \ln \frac{1}{e^{-x}+1} + C$

If $y = \ln u(x) \rightarrow y' = \frac{1}{u(x)} u'(x)$

Exp: Find

① $\int_{\ln 2}^{\ln 3} e^x \, dx = e^x \Big|_{\ln 2}^{\ln 3} = e^{\ln 3} - e^{\ln 2}$

$= 3 - 2 = 1$

② $\int \frac{e^{\sqrt{r}}}{\sqrt{r}} \, dr$ $\quad u = \sqrt{r}$

$du = \frac{1}{2\sqrt{r}} \, dr$

$2\int \frac{e^u}{\cancel{2\sqrt{r}}} \, du \, \cancel{2\sqrt{r}}$

$= 2e^u + c = 2e^{\sqrt{r}} + c$

③ $\int t e^{-t^2} \, dt$ $\quad u = -t^2$

$du = -2t \, dt$

$-\frac{1}{2}\int e^u \, du$

$= -\frac{1}{2} e^{-t^2} + c$

Exp Find y' if :

① $\ln xy = e^{x+y}$

$\ln x + \ln y = e^{(x+y)}$

$\frac{1}{x} + \frac{y'}{y} = e^{(x+y)}(1 + y')$

بعد الترتيب

$\frac{1}{x} + \frac{y'}{y} = e^{(x+y)} + y' e^{(x+y)}$

$y' = \frac{y}{x}\left[\frac{xe^{x+y} - 1}{1 - ye^{x+y}}\right]$

(2) $y = [\ln x]^{\ln x}$

$$\ln y = \ln [\ln x]^{\ln x}$$

$$= \ln x \, \ln(\ln x)$$

$$\frac{y'}{y} = \ln x * \frac{1}{\ln x} * \frac{1}{x} + \ln(\ln x) \frac{1}{x}$$

$$\frac{y'}{y} = \frac{1}{x}\left(1 + \ln(\ln x)\right)$$

$$y' = \frac{(\ln x)^{\ln x}}{x}\left(1 + \ln(\ln x)\right)$$

Exp: Find $\displaystyle\int \frac{x\, 2^{x^2}}{1+2^{x^2}} \, dx$

$u = 1 + 2^{x^2}$

$du = 2^{x^2}(2x)\ln 2 \, dx$

$\frac{du}{2\ln 2} = x\, 2^{x^2} dx$

$$\int \frac{1}{u} \frac{du}{2\ln 2}$$

$$\frac{1}{2\ln 2}\int \frac{1}{u} du = \frac{1}{2\ln 2} \ln|u| + C$$

Exp Find $f'(e)$

if $f(x) = x^x$

① $\ln|f(x)| = x \ln x$

$$\frac{f'(x)}{f(x)} = x\frac{1}{x} + \ln x$$

$$\frac{f'(e)}{e^e} = 1 + 1$$

$$f'(e) = 2e^e$$

② $f(x) = e^{\ln x^x}$

$$= e^{x\ln x}$$

$$= e^{x\ln x}\left[x\frac{1}{x} + \ln x\right]$$

$$f'(e) = e^{e\ln e}[1+\ln e]$$

$$= 2e^e$$

Exp. Find $\lim_{x\to 0}(1+x)^{\frac{1}{x}} = ?$

$$= \lim_{x\to 0} e^{\ln(1+x)^{\frac{1}{x}}} = \lim \frac{\ln(1+x)}{e^x}$$

since exp (e^x) cont $\Rightarrow e^{\lim_{x\to 0}\frac{\ln(1+x)}{x}}$

$$e^{\lim_{x\to 0}\frac{\ln(1+x)}{x}} = e^{\lim_{x\to 0}\frac{\frac{1}{x+1}}{1}} = e^1 = e$$

$* \ \frac{0}{0} \to$ lopital

Logarithmic function

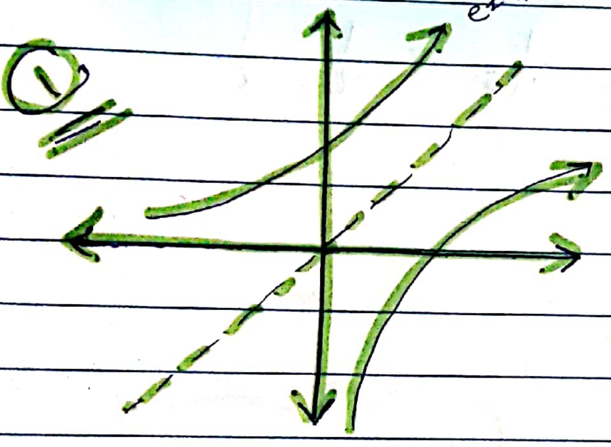
$$\log_a u(x) = \frac{\ln(u(x))}{\ln a} \qquad a>0,\ a\neq 1,\quad u(x)>0$$

Special case $a=e$

$$\log_e u(x) = \frac{\ln u(x)}{\ln e} = \ln u(x)$$

$$\boxed{\log_e x = \ln x}$$

1



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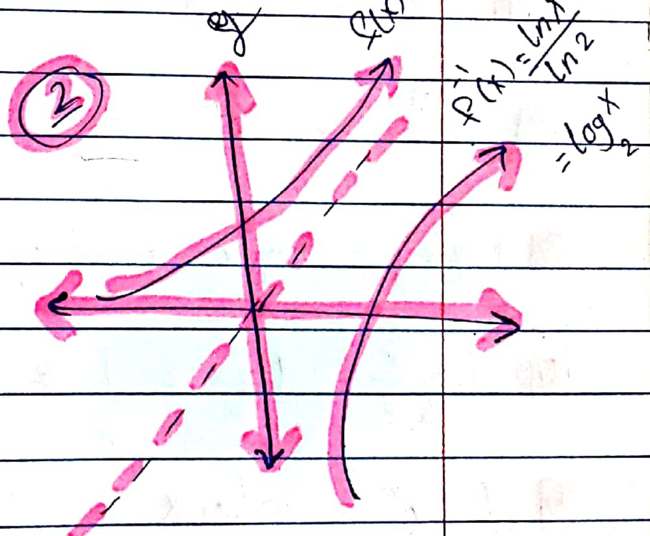

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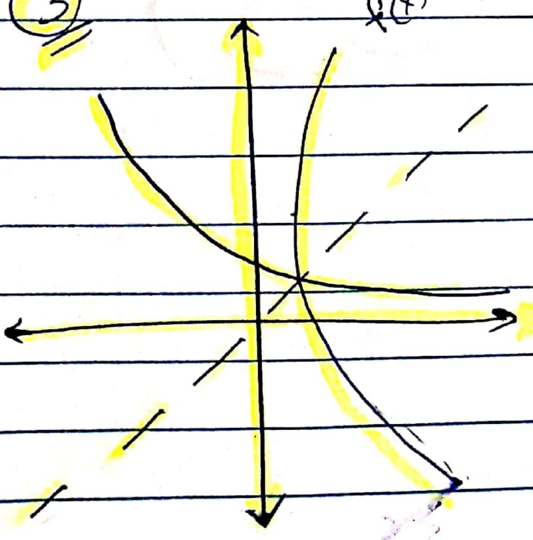


$$f^{-1} = -\log_2 x = -\frac{\ln x}{\ln 2} = \frac{\ln x}{-\ln 2} = \frac{\ln x}{\ln \frac{1}{2}} = \log_{\frac{1}{2}} x$$

Recall that if :

$\boxed{1}$ $y = \ln u(x)$ then $y' = \frac{dy}{dx} = \frac{1}{u(x)} u'(x)$

where $u(x) > 0$ and diff

$\boxed{2}$ $y = \log_a u(x) = \frac{\ln u(x)}{\ln a}$ then

$y' = \frac{1}{\ln a} \frac{1}{u(x)} u'(x)$ where $a > 0$ $a \neq 1$

$u(x) > 0$

$u(x)$ is diff

Exp :- $\frac{d}{dx}\left(\log_2 \sqrt{x}\right) = \frac{1}{\ln 2} \frac{1}{\sqrt{x}} \frac{1}{2\sqrt{x}} = \frac{1}{2\ln 2} \frac{1}{x}$

→ properites :-

$\boxed{1}$ $\log_a a^x = x$ $\forall x$

$\boxed{2}$ $a^{\log_a x} = x$ $\forall x > 0$

Exp: $2^{\log_2 x} = 8$

$x = ?$

$x = 8$

$\boxed{3}$ $\log_a xy = \log_a x + \log_a y$

$\boxed{4}$ $\log_a \frac{x}{y} = \log_a x - \log_a y$

$\boxed{5}$ $\log_a x^r = r \log_a x$

$\boxed{6}$ $\log_a \frac{1}{x} = -\log_a x$

Exp: $\int \frac{\log_2 x}{x}\,dx = \int \frac{\ln x}{x(\ln 2)}\,dx$

$$= \frac{1}{\ln 2}\int \frac{\ln x}{x}\,dx$$

$u = \ln x$

$du = \frac{dx}{x}$

$$= \frac{1}{\ln 2}\int u\,du$$

$$= \frac{1}{\ln 2}\,\frac{u^2}{2} + c$$

$$= \frac{1}{\ln 4}(\ln x)^2 + c$$

Exp: Find y' for

(1) $y = \log_2 (8x^{\ln 2})$

$$y' = \frac{1}{\ln 2}\,\frac{1}{8x^{\ln 2}}\left(8\ln 2\, x^{\ln 2 - 1}\right) = \frac{1}{x}$$

(2) $y = \int_0^{\log_4 x} 2\ln 2\; 4^t\,dt$

$$y' = 2\ln 2\; 4^{\log_4 x}\left(\log_4 x\right)'$$

$$\ln 4\; x\, \frac{1}{\ln 4}\,\frac{1}{x}\,(1) = 1$$

Exp: Find y so that

$$\log_2 y = 3$$

$$2^{\log_2 y} = 2^3$$

$$y = 8$$

7.4 Separable Differential Equations:- (DE)

Exp: Solve $\dfrac{dy}{dx} = \dfrac{\cos x}{1+3y^2}$

If $\underset{x=0}{(y = 1)}$ (initial condition)

DE's: $\underset{\text{relations}}{\text{equations}}$ with $\underset{\text{Changes (rate)}}{\text{derivatives}}$.

$$\int (1+3y^2)\, dy = \int (\cos x)\, dx$$

$$y + y^3 = \sin x + c$$

$$1+1 = \sin(0) + c$$

$$c = 2$$

$\boxed{y + y^3 = \sin x + 2} \Rightarrow$ implicit solution.

Remark: $(\underset{\text{inital value proble}}{\text{IVP}})$ is DE with $(\underset{\text{intial condition}}{\text{IC}})$

~~Explicit sol.~~

Exp: solve the IVP

$$\frac{dy}{dt} = \sqrt{y} \rightarrow y(1) = 2$$

$$\frac{dy}{\sqrt{y}} = dt \Rightarrow \int \frac{dy}{\sqrt{y}} = \int dt$$

$$2y^{1/2} = t + C$$

* to find "c" $y(1) = 2 \Rightarrow 2 * \sqrt{2} = 1 + C$

$$C = 2\sqrt{2} - 1$$

$$\frac{2\sqrt{y} = t + 2\sqrt{2} - 1}{2}$$

$$\left(\sqrt{y} = \frac{t}{2} + \sqrt{2} - \frac{1}{2}\right)^2$$

$$y = \left(\frac{t}{2} + \sqrt{2} - \frac{1}{2}\right)^2 \longrightarrow$$ * "Explicit sol"

Exp $p(t)$: population size at time t

$$\frac{dp}{dt} \propto p$$

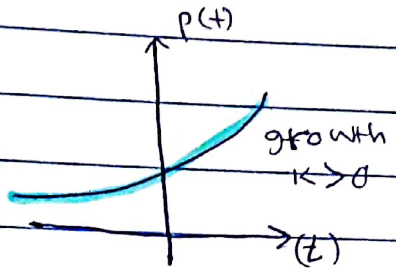
$$\frac{dp}{dt} = kp \quad (1) \qquad \text{[where } k \text{ is constant]}$$

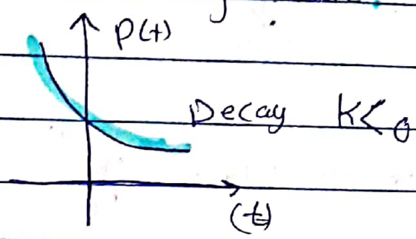
$p(0) = P_0$

and called:

(1) $k>0$ growth rate

(2) $k<0$ decay rate





Question: How to solve the (IVP) (1):

$$\frac{dp}{dt} = kp$$

$$\int \frac{dp}{p} = \int k\,dt \Rightarrow \ln|p| = kt + c$$

$$|p| = e^{kt+c}$$

$$|p| = \underbrace{\pm e^{c}}_{D} e^{kt}$$

$$p(t) = D e^{kt}$$

To Find D we use I.C: $p(0) = P_0$

$$p(0) = D e^{k(0)}$$

$$P_0 = D$$

Henec the solution of the IVP (1) is given by

$$\boxed{p(t) = P_0 e^{kt}} \quad (2)$$

Exp: Solve the IVP.

$$\frac{dy}{dt} = -2y \quad , \quad y(0) = 4$$

solution $y(t) = y_0 e^{kt}$

$y(t) = 4 e^{-2t}$

7.4 (complete) IVP: $\boxed{\frac{dy}{dt} = ky}$, $\boxed{y(0) = y_0}$

$\Downarrow$ DE $\qquad \Downarrow$ IC

has solution $y(t) = y_0 e^{kt}$

Exp: $(k>0)$: $y(t)$: populations size at time t

A poupulation increases Exponsially with time Such that the population after 3 years is 10,000 and the population after 5 years is 40,000. Find the initial size of population?

$$\frac{dP}{dt} = kP \qquad k>0$$

$\Downarrow$

$P(t) = P_0 e^{kt}$

$P(3) = P_0 e^{3k}$

$10{,}000 = P_0 e^{3k}$ — (1)

$40{,}000 = P_0 e^{5k}$ — (2)

$P(3) = 10{,}000$

$P(5) = 40{,}000$

Find P_0 ?

(2) ÷ (1)

$\hookrightarrow 4 = e^{2k}$

$\ln 4 = \ln e^{2k}$

$\cancel{2} \ln 2 = \cancel{2}k \Rightarrow \boxed{\ln 2 = k}$

k in (1) $\rightarrow 10{,}000 = P_0 e^{3\ln 2} = 8P_0$

$P_0 = \frac{10{,}000}{8} = 1250$

Radioactivity :-

$Q(t)$: amount of the quantity at time t present

$$\frac{dQ}{dt} = -kQ \quad , \quad Q(0) = Q_0 \qquad \boxed{K<0}$$

$$\boxed{Q(t) = Q_0 e^{-kt}}$$

Half-life time (t^*) is the time that make the initial quantity half.

$$Q(t^*) = \frac{1}{2} Q_0$$

$$\rightarrow Q(t^*) = \frac{1}{2} Q_0 \rightarrow Q_0 e^{-kt^*} = \frac{1}{2} Q_0$$

$$e^{-kt^*} = \frac{1}{2}$$

$$-kt^* = \ln\frac{1}{2} = -\ln 2 \rightarrow t^* = \frac{\ln 2}{k}$$

Exp: The half-life time of polonuim is 139 days If this sample is not useful after 95% of radioactivity then how many days will it be useful to use ?

$$t^* = 139 = \frac{\ln 2}{k} \Leftrightarrow k = \frac{\ln 2}{139} \quad (\text{find } \bar{t} \text{ s.t } Q(\bar{t}) = 0.05Q_0)$$

$$Q_0 e^{-k\bar{t}} = 0.05 Q_0$$

$$-k\bar{t} = \ln 0.05$$

$$\bar{t} = -\frac{\ln 0.05}{k}$$

$$\bar{t} \simeq 600 \text{ days}$$

Exp: A radio active material has half - life time $\ln 8$ years, How many years it will take to decay 80% ??

$$t^* = \ln 8 = \frac{\ln 2}{k} \rightarrow k = \frac{1}{3}$$

$$Q(t) = Q_0 \, e^{-kt}$$

$$\frac{20}{100} \cancel{Q_0} = \cancel{Q_0} \, e^{-kt}$$

$$\ln 0.02 = -kt$$

$$-3 \ln 0.02 = t \rightarrow t = 3 \ln \frac{10}{2} = 3 \ln 5.$$

7.5 Indeterminants' forms:

$\left(\left(\frac{0}{0}\right)\left(\frac{\infty}{\infty}\right)\right)$ $\left((\infty \cdot 0)(\infty - \infty)\right)$ $\left((0^0)(\infty^{1})(\infty^0)\right)$

$\downarrow$ L'Hopital Rule

$\downarrow$ Simple by
«توحيد مقامات»

$\downarrow$ we use this idea

$$\lim_{x \to c} f(x) = \lim_{x \to c} e^{\ln f(x)}$$

Th: (L'Hopital Rule):

Assume $g(x)$ and $f(x)$ are diff at c with $g(c) = f(c) = 0$ and $g'(c) \neq 0$

then $\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$ $\Rightarrow$ then put (c) instead (x)

Find :

(1) $\lim_{x\to -2} \frac{x+2}{x^2-4} \quad \left(\frac{0}{0}\right)$

$\lim_{x\to -2} \frac{1}{2x} = -\frac{1}{4}$

(2) $\lim_{x\to 0} \frac{x-\sin x}{x^3} \quad \left(\frac{0}{0}\right)$

$\lim_{x\to 0} \frac{1-\cos x}{3x^2} \quad \left(\frac{0}{0}\right)$

$\lim_{x\to 0} \frac{\sin x}{6x} \quad \left(\frac{0}{0}\right)$

$\lim_{x\to 0} \frac{\cos x}{6} = \frac{1}{6}$

(3) $\lim_{\theta\to 0} \frac{3^{\sin\theta}-1}{\theta} \quad \left(\frac{0}{0}\right)$

$\lim_{\theta\to 0} 3^{\sin\theta} \cos\theta \ln 3 = (1)(1) \ln 3$

$= \ln 3$

(4) $\lim_{x\to 0^+} [\ln x - \ln(\sin x)]$

$[-\infty - -\infty]$

$\lim_{x\to 0^+} \ln\left(\frac{x}{\sin x}\right) = \ln \lim_{x\to 0^+} \frac{x}{\sin x}$

$= \ln \lim_{x\to 0^+} \frac{1}{\cos x}$

$= \ln 1$

$= 0$

$\Rightarrow$ we can put the (limit) on the function if it is continuous.

7.5 Indeterminals forms:-

$\left(\frac{0}{0}, \frac{\infty}{\infty}\right) \to$ Derivative

Revision

$(\infty - \infty)$, $(0 \cdot \infty) \to$ simplify

0^0 , 1^∞ , $\infty^0 \longrightarrow f(x) = e^{\ln f(x)}$

Exp : Find ① $\lim_{x \to 1^+} \left(\frac{1}{x-1} - \frac{1}{\ln x} \right) = \infty - \infty$

$$\lim_{x \to 1^+} \left[\frac{\ln x - (x-1)}{(x-1)\ln x} \right] \quad \left(\frac{0}{0}\right)$$

$$\lim_{x \to 1^+} \left[\frac{\frac{1}{x} - 1}{\frac{(x-1)}{x} - \ln x} \right] \quad \left(\frac{0}{0}\right)$$

$$\lim_{x \to 1^+} \frac{\frac{-1}{x^2}}{\frac{1}{x^2} + \frac{1}{x}} = -\frac{1}{2}$$

Exp : $\lim_{x \to 0^+} (1+x)^{\frac{1}{x}}$ $\quad 1^\infty$

$$= \lim_{x \to 0} e^{\ln(1+x)^{\frac{1}{x}}} = \lim_{x \to 0} e^{\frac{1}{x}\ln(1+x)}$$

$$= e^{\lim_{x \to 0^+} \left(\frac{\ln(1+x)}{x} \right)} = e^{\lim_{x \to 0^+} \frac{\frac{1}{1+x}}{1}} = e$$

Exp: $\lim_{x\to 0^+} \left(1+\frac{1}{x}\right)^x$

$x\ln\left(1+\frac{1}{x}\right) \to 0 * \infty$

$$\lim_{x\to 0^+} e^{x\ln\left(1+\frac{1}{x}\right)} = \lim_{x\to 0^+} e^{\frac{\ln\left(1+\frac{1}{x}\right)}{\left(\frac{1}{x}\right)}} \quad \frac{\infty}{\infty}$$

$$e^{\lim_{x\to 0^+} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}} = e^{\lim_{x\to 0^+} \frac{\frac{\cancel{-1/x^2}}{1+\frac{1}{x}}}{\cancel{-1/x^2}}}$$

$$e^{\lim_{x\to 0^+} \frac{1}{1+\frac{1}{x}}} = e^0 = 1$$

Exp: $\lim_{x\to 0^+} \sqrt{x}\,\ln x$

① $\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{\sqrt{x}}} \quad \frac{-\infty}{\infty}$

$$= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{2}x^{-\frac{3}{2}}} = -2 \lim_{x\to 0^+} x^{-1} x^{\frac{3}{2}}$$

$$= -2 \lim_{x\to 0^+} \sqrt{x} = \text{zero}$$

Exp: $\lim_{x\to\infty} \frac{e^x}{x^2} \quad \left(\frac{\infty}{\infty}\right)$

$$\lim_{x\to\infty} \frac{e^x}{2x} \quad \left(\frac{\infty}{\infty}\right)$$

$$\lim_{x\to\infty} \frac{e^x}{2} = \infty$$

Exp: $\lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty} e^{\ln x^{\frac{1}{x}}}$

$= \lim_{x\to\infty} e^{\frac{\ln x}{x}}$

$= e^{\lim_{x\to\infty} \frac{\ln x}{x}} \left(\frac{\infty}{\infty}\right) = e^{\lim_{x\to\infty} \frac{\frac{1}{x}}{1}} = e^{0} = 1$

Exp: $\lim_{x\to-\infty} \frac{2^x + 4^x}{5^x - 2^x} * \frac{2^x}{2^x}$

* إذا x تقترب من ∞ فنقسم على أكبر عدد
* إذا x تقترب من $-\infty$ فنقسم على أصغر عدد

$\lim_{x\to-\infty} \frac{1 + 2^x}{\left(\frac{5}{2}\right)^x - 1} = \frac{1+0}{0-1} = -1$

Exp: $\lim_{x\to\infty} \frac{2^x + 4^x}{5^x - 2^x} * \frac{5^x}{5^x} = \lim_{x\to\infty} \frac{\left(\frac{2}{5}\right)^x + \left(\frac{4}{5}\right)^x}{1 - \left(\left(\frac{2}{5}\right)^x\right)}$

$\frac{0+0}{1-0} = 0$

Exp: Solve $2^{\log_4 5x} = x$

Sol: $2^{\log_2 5x}$ $= \frac{\ln 5x}{\ln 4} = \frac{\ln 5x}{2\ln 2} = \frac{1}{2} \frac{\ln 5x}{\ln 2}$

$= \frac{1}{2} \log_2 5x$

$= \log_2 \sqrt{5x}$

Hence $2^{\log_4 5x} = x$

$2^{\log_2 \sqrt{5x}} = x$

$\sqrt{5x} = x$

$5x = x^2 \rightarrow \cancel{x=0}$ or $x = 5$ ✓

S(2) $\ln 2^{\log_4 5x} = \ln x$

$\log_4 5x \ \ln 2 = \ln x$

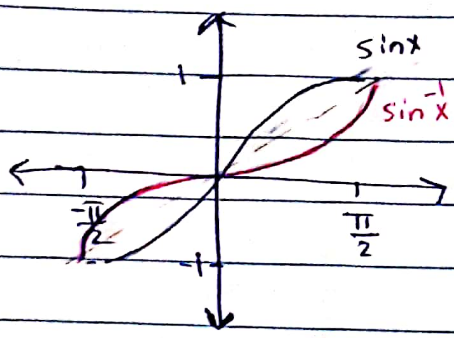
$\frac{\ln 5x}{2\cancel{\ln 2}} \cancel{\ln 2} = \ln x$

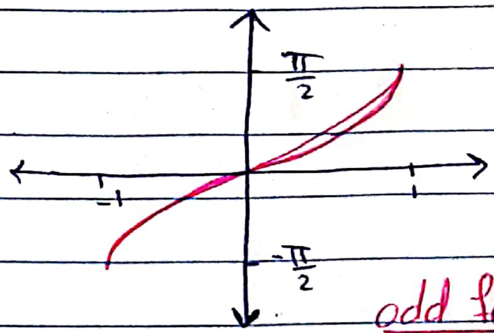
$\ln 5x = \ln x^2$

$5x = x^2 \rightarrow x = \cancel{0}, 5$

$\boxed{7.6}$ Inverse of Trigorometric Functions:

$\boxed{1}$ if $f(x) = \sin x$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ Then ?





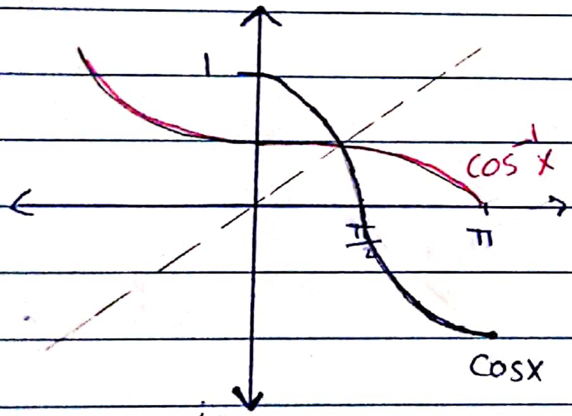
odd function

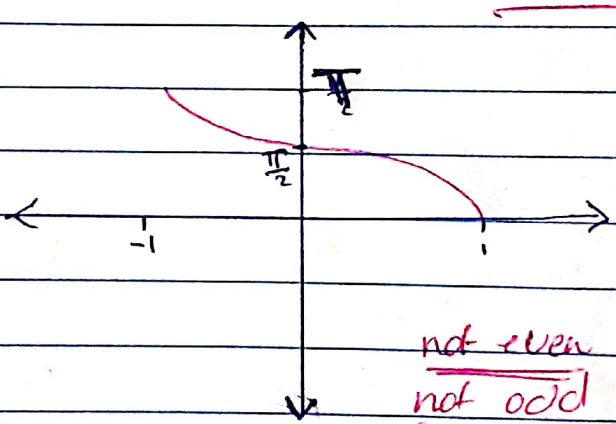
$\Rightarrow f^{-1}(x) = \sin^{-1} x = \text{arc } \sin$ on $[-1, 1]$

Exp: ① $\sin^{-1}(\frac{1}{2}) = \pi/6$ ② $\sin^{-1}(-\frac{1}{2}) = -\pi/6$

③ $\sin^{-1}(\frac{3}{2}) =$ Undefined ($\frac{3}{2}$ not in the domain).

$\boxed{2}$ if $f(x) = \cos x$ on $[0, \pi]$ Then ?





not even
not odd

$\Rightarrow f^{-1}(x) = \cos^{-1} x = \text{arc } \cos x$ on $[-1, 1]$

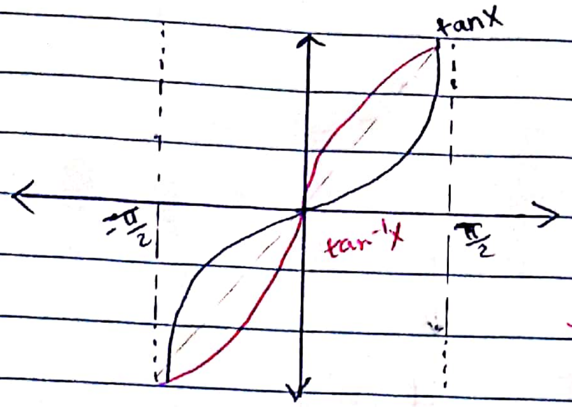
Exp: ① $\cos^{-1}\frac{1}{2} = \frac{\pi}{3}$ ② $\cos^{-1} -\frac{1}{2} = 2(\pi/3)$

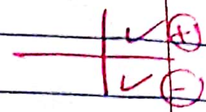
NOTE

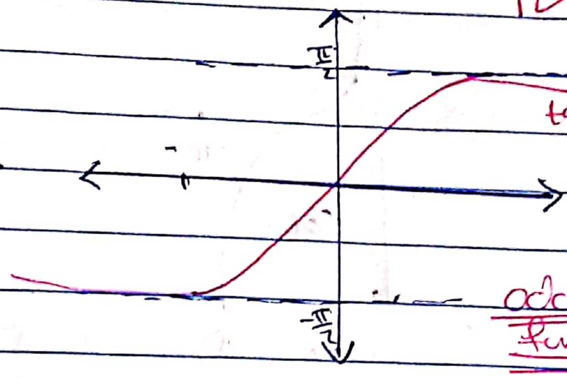
$$\cos^{-1} x + \cos^{-1}(-x) = \pi$$

3. if $f(x) = \tan(x)$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$ Then?

$\Rightarrow f^{-1}(x) = \tan x^{-1} = \arctan x$ on $\mathbb{R}$







odd function

Exp: (1) $\tan^{-1} 1 = \pi/4$ (2) $\tan^{-1} -1 = -\pi/4$ (3) $\tan^{-1}\sqrt{3} = \pi/3$

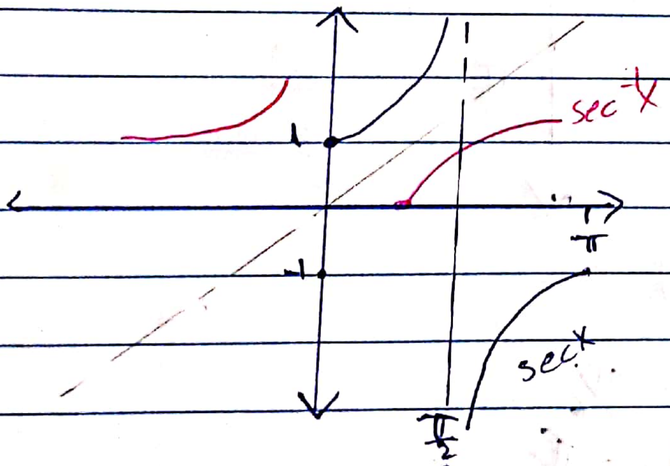
$\rightarrow \lim_{x \to \infty} \tan^{-1}x = \pi/2$

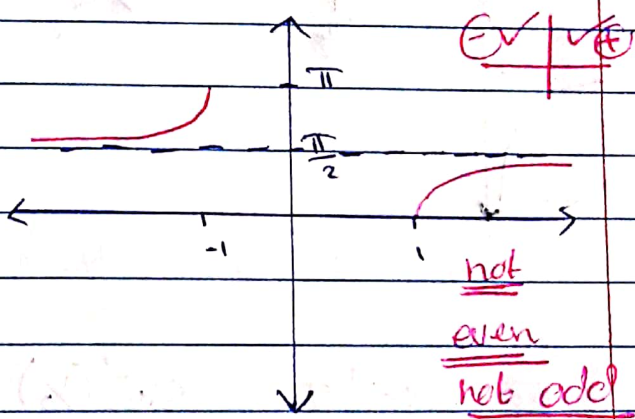
$\rightarrow \lim_{x \to -\infty} \tan^{-1}x = -\pi/2$

H. Asy

4. if $f(x) = \sec x$ on $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$ Then.

$\Rightarrow f^{-1}(x) = \sec^{-1}x = \text{arc sec}$ on $(-\infty, -1] \cup [1, \infty)$





not even
not odd

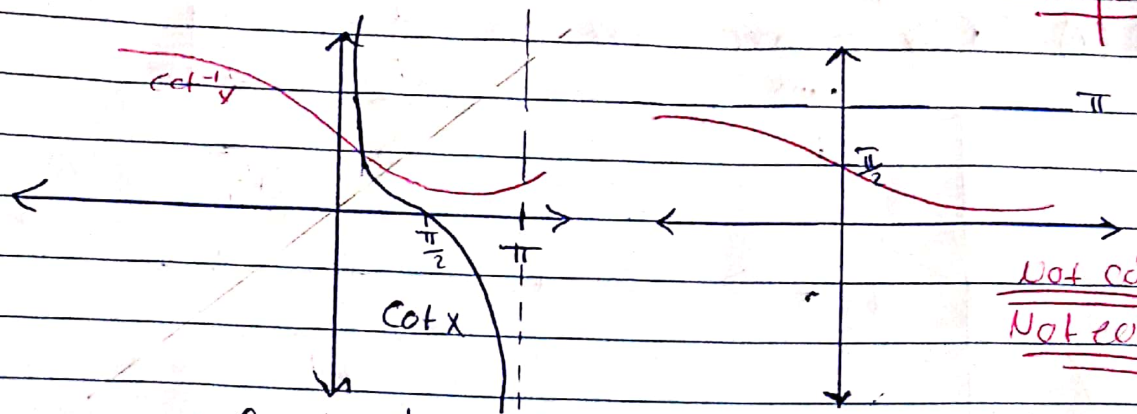
$\rightarrow \lim_{x \to \pm\infty} \sec^{-1}x = \pi/2$. H. Asy

Exp: (1) $\sec(1) = \text{zero}$ (2) $\sec^{-1}(\frac{1}{2}) = \text{undefined}$

(3) $\sec(-1) = \pi$

5 if $f(x) = \cot x$ on $(0, \pi)$ Then?

$f^{-1}(x) = \cot^{-1} x = \text{arc} \cot x$ on $\mathbb{R}$



Not odd

Not even

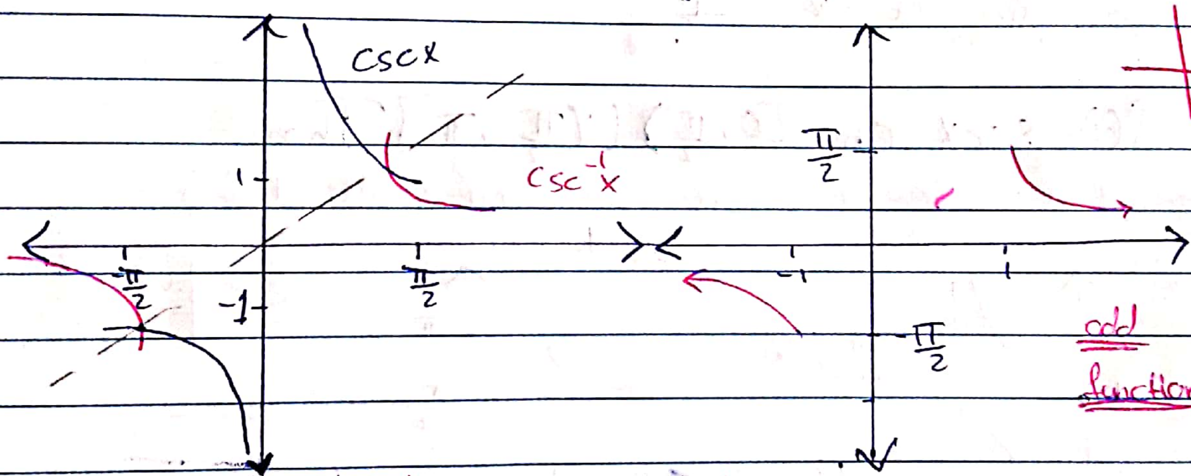
$\lim_{x \to -\infty} \cot^{-1} x = \pi$, $y = \pi$

$\lim_{x \to \infty} \cot^{-1} x = 0$, $y = \text{zero}$

H. Asy

6 if $f(x) = \csc x$ on $[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$

Then? $f^{-1}(x) = \csc^{-1} x = \text{arc} \csc x$ on $(-\infty, -1] \cup [1, \infty)$



odd function

$\lim_{x \to \pm\infty} \csc^{-1} x = \text{zero}$, H. Asy

Exp: $\csc^{-1} 2 = \sin^{-1}(\frac{1}{2}) = \frac{\pi}{6}$

$\csc^{-1} -2 = \sin^{-1}(-\frac{1}{2}) = -\frac{\pi}{6}$

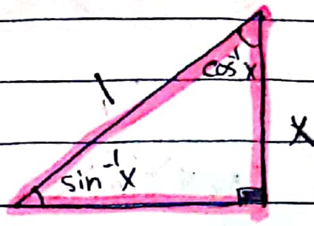
$\csc^{-1}(1) = \frac{\pi}{2}$

$\csc^{-1}(-\frac{2}{3}) =$ Undefind

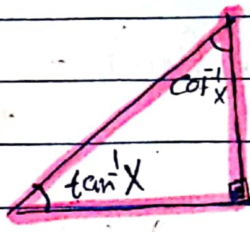
$\csc^{-1}(0) =$ Undefind

NOTE:

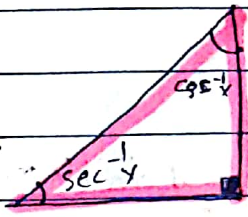
$\boxed{1}$ $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$



$\boxed{2}$ $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$



$\boxed{3}$ $\sec^{-1}x + \csc^{-1}x = \frac{\pi}{2}$



Exp: $\cot\left(\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right) = ?$

$$\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$$

$$\cot\frac{-\pi}{3} = \frac{\cos\frac{-\pi}{3}}{\sin\frac{-\pi}{3}} = \frac{-1}{\sqrt{3}}$$

Exp: $\sin^{-1}\left(\sin\frac{3\pi}{4}\right) = ?$

$$\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$$

Exp: Find the domain of $f(x) = \sin^{-1}(\ln x)$

$D = [e^{-1}, e]$

$\ln x = -1$	$\ln x = 1$
e^{-1}	e

Derivatives for the Inverse Of Trigonometric Functions :-

let $u(x)$ be a diff function :-

$\boxed{1}$ $\frac{d}{dx} \sin^{-1}(u(x)) = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx}$ $\quad -1 < u < 1$

$\boxed{2}$ $\frac{d}{dx} \cos^{-1}(u(x)) = \frac{-1}{\sqrt{1-u^2}} \frac{du}{dx}$

$\boxed{3}$ $\frac{d}{dx} \tan^{-1}(u(x)) = \frac{1}{1+u^2} \frac{du}{dx}$

$\boxed{4}$ $\frac{d}{dx} \cot^{-1}(u(x)) = \frac{-1}{1+u^2} \frac{du}{dx}$

$\boxed{5}$ $\frac{d}{dx} \sec^{-1}(u(x)) = \frac{1}{|u|\sqrt{u^2-1}} \frac{du}{dx}$

$\boxed{6}$ $\frac{d}{dx} \csc^{-1}(u(x)) = \frac{-1}{|u|\sqrt{u^2-1}} \frac{du}{dx}$

proof ① If $f(x) = \sin x \rightarrow f^{-1}(x) = \sin^{-1} x$

$f'(x) = \cos x$

$$\frac{d f^{-1}(x)}{dx} = \frac{1}{f'(f^{-1}(x))} = \frac{1}{\cos(\sin^{-1} x)} \sim ①$$

But $\rightarrow \boxed{\sin^2 x + \cos^2 x = 1}$

$\cos x = \sqrt{1 - \sin^2 x} \sim ②$

② in ①

$$\frac{1}{\sqrt{1 - \sin^2(\sin^{-1} x)}} = \frac{1}{\sqrt{1 - x^2}}$$

proof ③ If $f(x) = \tan x \longrightarrow f^{-1}(x) = \tan^{-1} x$

$f'(x) = \sec^2 x$

$$\frac{df^{-1}}{dx} = \frac{1}{f'(f^{-1}(x))} = \frac{1}{f'(\tan^{-1} x)} = \frac{1}{\sec^2(\tan^{-1} x)}$$

But $\rightarrow \boxed{1 + \tan^2 x = \sec^2 x}$ $= \dfrac{1}{1 + \tan^2(\tan^{-1} x)}$

$$= \frac{1}{1 + x^2}$$

proof ⑤ $y = \sec^{-1} x \rightarrow \sec y = x$

$\sec y \tan y \; y' = 1$

$\rightarrow y' = \dfrac{1}{\sec y \tan y}$

$\Rightarrow y' = \dfrac{1}{x \tan y}$

$1 + \tan^2 y = \sec^2 y$

$\tan y = \pm \sqrt{\sec^2 y - 1}$

$= \pm \sqrt{x^2 - 1}$

$$= \frac{1}{\pm x \sqrt{x^2 - 1}}$$

$$= \frac{1}{|u| \sqrt{x^2 - 1}}$$

Exp: Find $y'|_{x=e}$ if $y = \tan^{-1}(\ln x)$

$$y' = \frac{1}{1+(\ln x)^2} * \frac{1}{x}$$

$$y'(e) = \frac{\frac{1}{e}}{1+(\ln e)^2} = \frac{\frac{1}{e}}{2} = \frac{1}{2e} \quad m$$

Find tangent of $y = \tan^{-1}(\ln x)$ at $x = e$

$$y(e) = \tan^{-1}(\ln e) = \tan^{-1} 1 = \frac{\pi}{4}$$

$$y - \frac{\pi}{4} = \frac{1}{2e}(x-e)$$

Inverse Integral:-

$\boxed{1}$ $$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a} \sec^{-1}\left|\frac{u}{a}\right| + c$$

$\boxed{2}$ $$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + c$$

$\boxed{3}$ $$\int \frac{du}{u^2+a^2} = \frac{1}{a} \tan^{-1}\left(\frac{u}{a}\right) + c$$

Exp: Find y' if

① $y = \cos^{-1} \sqrt{2}\, x$

$$y' = \frac{-1}{\sqrt{1-2x^2}} \sqrt{2}$$

Exp: $\int \frac{dx}{(2x-1)\sqrt{(2x-1)^2-4}}$

$u = 2x-1$
$du = 2dx$

$\frac{1}{2}\int \frac{du}{u\sqrt{u^2-2^2}} = \frac{1}{2} * \frac{1}{2} \sec^{-1}\left(\frac{u}{a}\right) + C$

$= \frac{1}{4} \sec^{-1}\left|\frac{2x-1}{2}\right| + C$

Exp: $y = \ln(\tan^{-1}\sqrt{x})$ Find $y'(1)$?

$y' = \frac{1}{\tan^{-1}\sqrt{x}} * \frac{1}{1+x^2} * \frac{1}{2\sqrt{x}}$

$y'(1) = \frac{1}{\tan^{-1} 1} * \frac{1}{2} * \frac{1}{2} = \frac{1}{\pi}$

Exp: $\int \frac{dx}{\sqrt{4x-x^2}}$

$4x - x^2 = -[x^2 - 4x]$
$= -[(x-2)^2 - 4]$
$= 4 - (x-2)^2$

$= \int \frac{dx}{\sqrt{2^2-(x-2)^2}}$

$u = x-2$
$du = dx$

$\int \frac{du}{\sqrt{2 - u^2}}$

$\sin^{-1}\left(\frac{u}{a}\right) + C = \sin^{-1}\left(\frac{x-2}{2}\right) + C$

Exp: $\int_{1/2}^{1} \frac{6\,dt}{\sqrt{3+4t-4t^2}}$

$3+4t-4t^2 =$
$-[4t^2-4t-3] =$
$-[(2t-1)^2-4]$
$[4-(2t-1)^2]$

$\int_{1/2}^{1} \frac{6\,dt}{\sqrt{2^2-(2t-1)^2}}$

$u = 2t-1$
$du = 2dt$

$t = \frac{1}{2} \to u = 0$
$t = 1 \to u = 1$

$\Rightarrow \frac{6}{2}\int_0^1 \frac{du}{\sqrt{2^2-(u)^2}} = 3\sin^{-1}\left(\frac{u}{2}\right)\Big|_0^1$

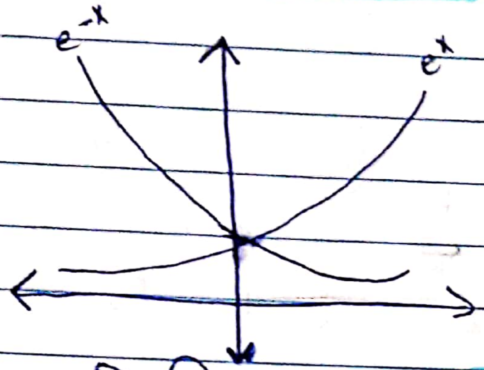
$3\left(\frac{\pi}{6} - 0\right) = \frac{\pi}{2}$

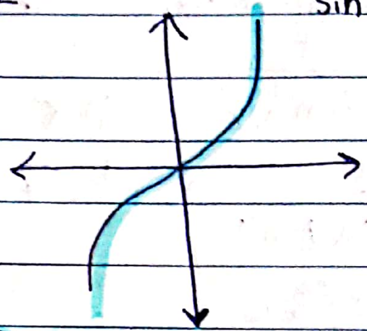
Exp: Find y' if $y = \csc^{-1}(x^2-3x)$

$$y' = \frac{-1}{|x^2-3x|\sqrt{(x^2-3x)^2-1}}\,(2x-3)$$

7.7 Hyperoplic functions:-

1 $f(x) = \sinh x = \dfrac{e^x - e^{-x}}{2}$

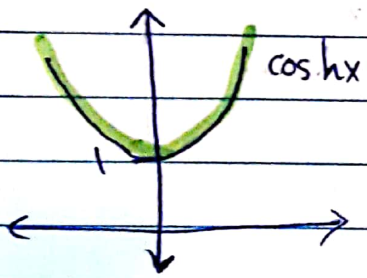




→ $D = \mathbb{R}$
→ $R = \mathbb{R}$

odd:
$\sinh -x = -\sinh x$

2 $f(x) = \cosh x = \dfrac{e^x + e^{-x}}{2}$



→ $D = \mathbb{R}$
→ $R = [1, \infty)$

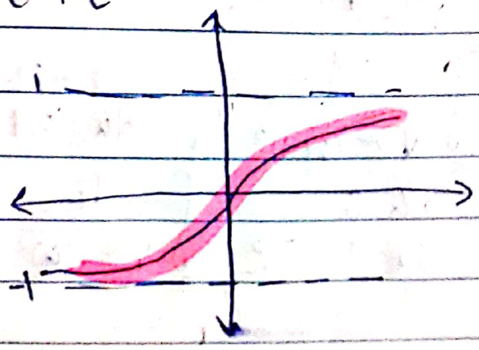
Even
$\cosh -x = \cosh x$

3 $f(x) = \tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

$\lim\limits_{x \to \infty} \tanh x = 1$

$\lim\limits_{x \to -\infty} \tanh x = -1$

$\tanh 0 = \frac{0}{1} = 0$



odd
$D = \mathbb{R}$
$R =]-1, 1[$

4. $\operatorname{sech} x = \dfrac{1}{\cosh x} = \dfrac{2}{e^{x} + e^{-x}}$

sech x even

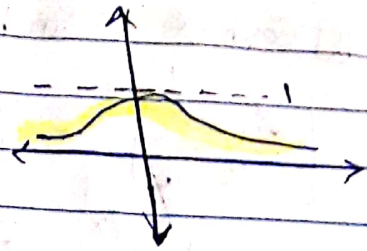
$\operatorname{sech}(0) = 1$

$\operatorname{sech} x > 0$

$y = 0$ is H. Asy

$D = \mathbb{R}$

$R = (0, 1]$

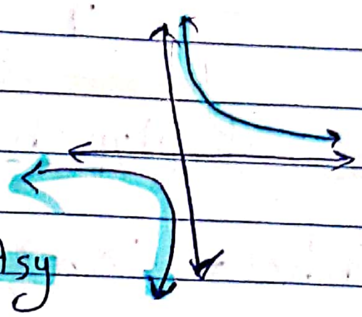


5. $\operatorname{csch} x = \dfrac{1}{\sinh x}$ odd

$D = \mathbb{R} \setminus \{0\}$

$R = \mathbb{R} \setminus \{0\}$

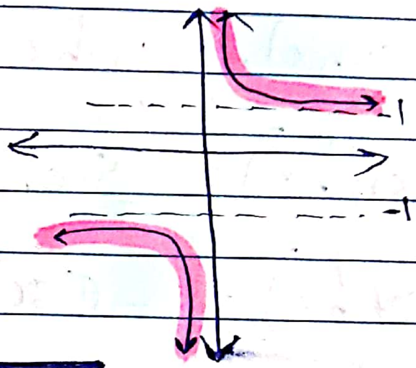
$x = 0$ is a V. Asy



6. $\coth x$ odd

$D = \mathbb{R} \setminus \{0\}$

$R = \mathbb{R} \setminus [-1, 1]$



Properities of hyperbolic functions:

1. $\cosh^2 x - \sinh^2 x = 1$

2. $\sinh 2x = 2 \sinh x \cosh x$

3. $\cosh 2x = \cosh^2 x + \sinh^2 x$
$= 2\cosh^2 x - 1$
$= 1 + 2\sinh^2 x$

4. $1 - \tanh^2 x = \operatorname{sech}^2 x$

5. $\coth^2 x - 1 = \operatorname{csch}^2 x$

proof

$\boxed{1}$ $\cosh^2 x - \sinh^2 x =$

$$\left[\frac{e^{x}+e^{-x}}{2}\right]^2 - \left[\frac{e^{x}-e^{-x}}{2}\right]^2$$

$$\frac{\cancel{e^{2x}}+2+\cancel{e^{-2x}}-\left(\cancel{e^{+2x}}-2+\cancel{e^{-2x}}\right)}{4} = \frac{4}{4} = 1$$

Derivatives of Hyperbolic Functions :.

1. $\frac{d}{dx}(\sinh x) = \cosh x$
2. $\frac{d}{dx}(\cosh x) = \sinh x$
3. $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$
4. $\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$
5. $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$
6. $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$

proof

$\boxed{2}$ $(\cosh x)'$

$$\frac{d}{dx}\left(\frac{e^{x}+e^{-x}}{2}\right) = \frac{e^{x}-e^{-x}}{2} = \sinh x$$

Integrals of Hyperbolic functions:

① $\int \cosh x \, dx = \sinh x + c$

② $\int \sinh x \, dx = \cosh x + c$

③ $\int \operatorname{sech}^2 x \, dx = \tanh x + c$

④ $\int \operatorname{csch}^2 x \, dx = -\coth x + c$

⑤ $\int \operatorname{sech} x \tanh x \, dx = -\operatorname{sech} x + c$

⑥ $\int \operatorname{csch} x \coth x \, dx = -\operatorname{csch} x + c$

Exp: Find y' if:

① $y = \ln \sinh x \rightarrow y' = \frac{1}{\sinh x} \cosh x = \coth x$

② $y = 4 \cosh \frac{x^3}{2} \rightarrow y' = 4 \sinh \frac{x^3}{2} * \frac{3x^2}{2}$
$= 2 \sinh \frac{x^3}{2} * 3x^2$

③ $y = \ln(\sinh x) - \frac{1}{2} \coth^2 x$
$y' = \coth x - \frac{1}{2}(2) \coth x (-\operatorname{csch}^2 x)$
$= \coth x [1 + \operatorname{csch}^2 x]$
$= \coth^3 x$

Exp: Find ① $\int_{2}^{4} 2\cosh(\ln x)\,dx$

$$= \cancel{2}\int_{2}^{4} \frac{e^{\ln x}+e^{-\ln x}}{\cancel{2}}\,dx$$

$$= \int_{2}^{4} x+\frac{1}{x}\,.\,dx$$

$$= \frac{x^2}{2}+\ln x\Big|_{2}^{4} = 6+\ln 2$$

② $\int_{0}^{\ln 2} 4e^{-\theta}\sinh\theta\,d\theta$

$$= 4\int_{0}^{\ln 2} e^{-\theta}\left(\frac{e^{\theta}-e^{-\theta}}{2}\right)d\theta$$

$$= 2\int_{0}^{\ln 2}\left(1-e^{-2\theta}\right)d\theta$$

$$= 2\left[\theta+\frac{1}{2}e^{-2\theta}\right]\Big|_{0}^{\ln 2}$$

$$= 2\ln 2+e^{-2\ln 2}\left(0+e^{0}\right)$$

$$= 2\ln 2+2^{-2}-1$$

$$= \ln 4+\frac{1}{4}-1$$

$$= \ln 4-\frac{3}{4}$$

7.8 Relative Rates of Growth:

* Assume (f) and (g) are positive for large X.

* If $\lim_{x\to\infty} \frac{f(x)}{g(x)} =$

1. $\to \infty$, then f(x) grows faster than g as $x \to \infty$
2. $\to 0$, then f(x) grows slower than g as $x \to \infty$
3. $\to 0 < L < \infty$, then both (f) and (g) grow in the same rate as $x \to \infty$

Exp: ① 4^x ; e^x

$$\lim_{x\to\infty} \frac{4^x}{e^x} = \lim_{x\to\infty} \left(\frac{4}{e}\right)^x = \infty$$

Hence, 4^x grows faster than e^x as $x \to \infty$

② $\left(\frac{3}{2}\right)^x$, e^x

$$\lim_{x\to\infty} \frac{e^x}{\left(\frac{3}{2}\right)^x} = \lim_{x\to\infty} \left(\frac{2e}{3}\right)^x = \infty$$

Hence, e^x grows faster than $\left(\frac{3}{2}\right)^x$ as $x \to \infty$

③ $\ln x$, 2^x

$$\lim_{x\to\infty} \frac{\ln x}{2^x} \left(\frac{\infty}{\infty}\right) \Rightarrow \lim_{x\to\infty} \frac{\frac{1}{x}}{2^x \ln 2} \Rightarrow \lim_{x\to\infty} \frac{1}{x\, 2^x \ln 2} = \left(\frac{1}{\infty}\right) = \text{zero}$$

Hence, $\ln x$ grows slower than 2^x as $x \to \infty$

④ $\log_3 x$, $\ln x$

$$\lim_{x\to\infty} \frac{\log_3 x}{\ln x} = \lim_{x\to\infty} \frac{\frac{\cancel{\ln x}}{\ln 3}}{\cancel{\ln x}} = \frac{1}{\ln 3}$$

Henec, $\log_3 x$, $\ln x$ grows in the same rate as $x\to\infty$

Exp: order the following functions from slowest to fastest as $x\to\infty$

$$* \lim_{x\to\infty} \frac{e^x}{e^{\frac{x}{2}}} = \lim_{x\to\infty} e^{\frac{x}{2}} = \infty$$

$e^x > e^{\frac{x}{2}}$

$(\ln x)^x > x^x$

$$* \lim_{x\to\infty} \frac{e^x}{(\ln x)^x} = \lim_{x\to\infty} \left(\frac{e}{\ln x}\right)^x \to (0^\infty)$$

$$\lim_{x\to\infty} e^{\ln\left(\frac{e}{\ln x}\right)^x} = e^{\lim_{x\to\infty} x \ln\left(\frac{e}{\ln x}\right)} = e^{\lim_{x\to\infty} x \cdot \lim_{x\to\infty} \ln\left(\frac{e}{\ln x}\right)}$$

$$e^{\lim_{x\to\infty} x \cdot \lim_{x\to\infty} [\ln e - \ln(\ln x)]} = e^{(-\infty)\infty} = e^{-\infty} = 0$$

$\ln x > e^x$

$\Rightarrow$ slowet to faster

$e^{\frac{x}{2}}, e^x, (\ln x)^x, x^x$

$$* \lim_{x\to\infty} \frac{x^x}{(\ln x)^x} = \lim_{x\to\infty} \left(\frac{x}{\ln x}\right)^x$$

$$= \lim_{x\to\infty} e^{\ln\left[\frac{x}{\ln x}\right]^x}$$

$$= e^{\lim_{x\to\infty} x \cdot \lim_{x\to\infty} \left[\ln \frac{x}{\ln x}\right]} = e^{\infty \cdot \lim_{x\to\infty} \ln \frac{1}{\frac{1}{x}}}$$

$$= e^{\infty(\infty)} = e^{\infty} = \infty$$

Exp: $5, x^3, x, x^5$

frome slowest to faster

$5, x, x^3, x^5$

Note: in polunomin functions the largest degree's function is the faster one