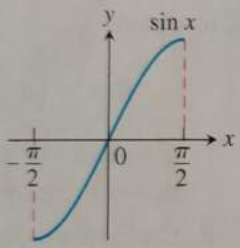
Inverse trigonometric functions arise when we want to calculate angles from side measurements in triangles. They also provide useful antiderivatives and appear frequently in the solutions of differential equations. This section shows how these functions are defined, graphed, and evaluated, how their derivatives are computed, and why they appear as important antiderivatives.

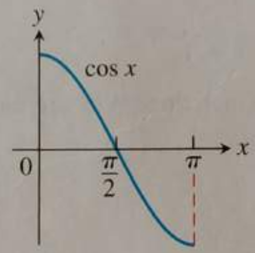
Defining the Inverses

The six basic trigonometric functions are not one-to-one (their values repeat periodically). However, we can restrict their domains to intervals on which they are one-to-one. The sine function increases from -1 at $x = -\pi/2$ to $+1$ at $x = \pi/2$. By restricting its domain to the interval $[-\pi/2, \pi/2]$, we make it one-to-one, so that it has an inverse $\sin^{-1}$ (Figure 7.14). Similar domain restrictions can be applied to all six trigonometric functions.

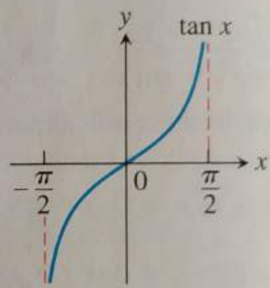
Domain restrictions that make the trigonometric functions one-to-one



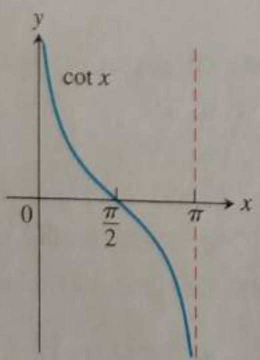
$y = \sin x$
Domain: $[-\pi/2, \pi/2]$
Range: $[-1, 1]$



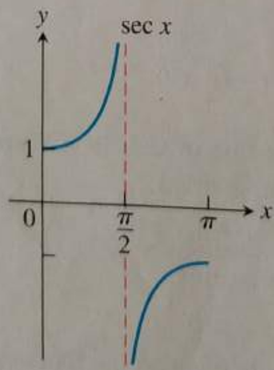
$y = \cos x$
Domain: $[0, \pi]$
Range: $[-1, 1]$



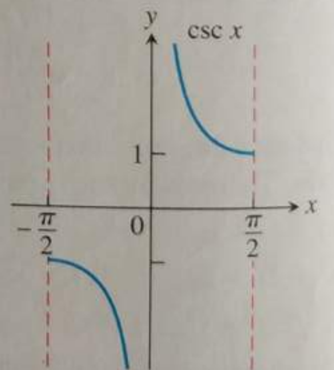
$y = \tan x$
Domain: $(-\pi/2, \pi/2)$
Range: $(-\infty, \infty)$



$y = \cot x$
Domain: $(0, \pi)$
Range: $(-\infty, \infty)$



$y = \sec x$
Domain: $[0, \pi/2) \cup (\pi/2, \pi]$
Range: $(-\infty, -1] \cup [1, \infty)$



$y = \csc x$
Domain: $[-\pi/2, 0) \cup (0, \pi/2]$
Range: $(-\infty, -1] \cup [1, \infty)$

... y equals arcsin x" and so on.

Caution The -1 in the expressions for the inverse means "inverse." It does *not* mean reciprocal. For example, the *reciprocal* of $\sin x$ is $(\sin x)^{-1} = 1/\sin x = \csc x$.

The graphs of the six inverse trigonometric functions are shown in Figure 7.15. We can obtain these graphs by reflecting the graphs of the restricted trigonometric functions through the line $y = x$, as in Section 7.1. We now take a closer look at these functions and their derivatives.

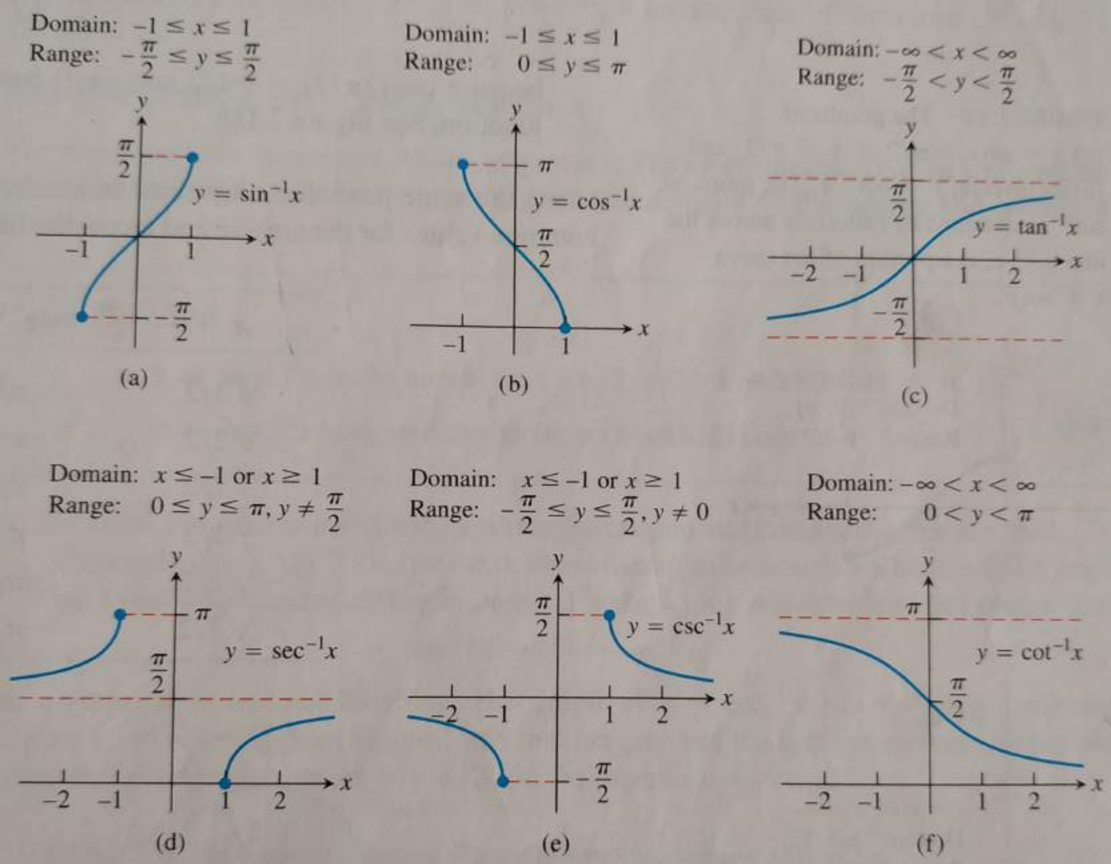


FIGURE 7.15 Graphs of the six basic inverse trigonometric functions.

The Arcsine and Arccosine Functions

We define the arcsine and arccosine as functions whose values are angles (measured in radians) that belong to restricted domains of the sine and cosine functions.

DEFINITION

$\mathbf{y = \sin^{-1} x}$ is the number in $[-\pi/2, \pi/2]$ for which $\sin y = x$.

$\mathbf{y = \cos^{-1} x}$ is the number in $[0, \pi]$ for which $\cos y = x$.

...inverse trigonometric functions are summarized in Table 7.3.

TABLE 7.3 Derivatives of the inverse trigonometric functions

1. $\dfrac{d(\sin^{-1} u)}{dx} = \dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx}, \quad |u| < 1$

2. $\dfrac{d(\cos^{-1} u)}{dx} = -\dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx}, \quad |u| < 1$

3. $\dfrac{d(\tan^{-1} u)}{dx} = \dfrac{1}{1+u^2}\dfrac{du}{dx}$

4. $\dfrac{d(\cot^{-1} u)}{dx} = -\dfrac{1}{1+u^2}\dfrac{du}{dx}$

5. $\dfrac{d(\sec^{-1} u)}{dx} = \dfrac{1}{|u|\sqrt{u^2-1}}\dfrac{du}{dx}, \quad |u| > 1$

6. $\dfrac{d(\csc^{-1} u)}{dx} = -\dfrac{1}{|u|\sqrt{u^2-1}}\dfrac{du}{dx}, \quad |u| > 1$

Integration Formulas

The derivative formulas in Table 7.3 yield three useful integration formulas in Table 7.4. The formulas are readily verified by differentiating the functions on the right-hand sides.

TABLE 7.4 Integrals evaluated with inverse trigonometric functions

The following formulas hold for any constant $a \neq 0$.

1. $\displaystyle\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$ (Valid for $u^2 < a^2$)

2. $\displaystyle\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$ (Valid for all u)

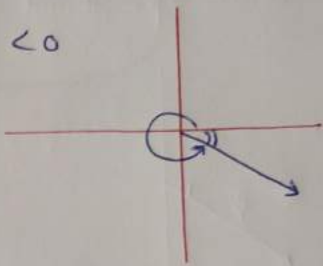
3. $\displaystyle\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$ (Valid for $|u| > a > 0$)

7.6 Inverse Trigonometric Functions.

$\boxed{2}$ c) Find the angle $\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

that is mean find x such that $\tan x = -\frac{1}{\sqrt{3}}$

- $-\frac{\pi}{2} < \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) < \frac{\pi}{2}$
- $\tan x$ is negative if $-\frac{\pi}{2} < x < 0$

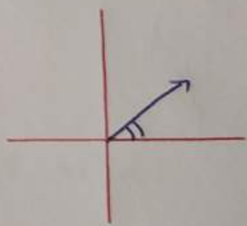


$\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$

$\boxed{5}$ a) Find the angle $\cos^{-1}\left(\frac{1}{2}\right)$

that is mean find x such that $\cos x = \frac{1}{2}$

- $0 \leq \cos^{-1}\left(\frac{1}{2}\right) \leq \pi$
- $\cos x$ is positive if $0 \leq x < \frac{\pi}{2}$



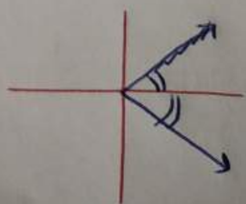
$\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

$\boxed{6}$ b) Find the angle $\csc^{-1}\left(-\frac{2}{\sqrt{3}}\right)$

that is mean find x such that $\sin x = -\frac{\sqrt{3}}{2}$

$-\frac{\pi}{2} \leq \csc^{-1}\left(-\frac{2}{\sqrt{3}}\right) \leq \frac{\pi}{2}$

$\csc x$ is negative ~~positive~~ if ~~$0 < x < \frac{\pi}{2}$~~ $-\frac{\pi}{2} < x < 0$

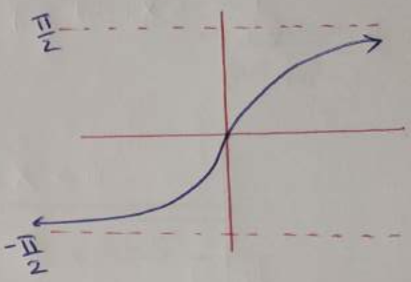


$\csc^{-1}\left(\frac{2}{\sqrt{3}}\right) = -\frac{\pi}{3}$

$\boxed{12}$ Find the value $\cot\left(\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$

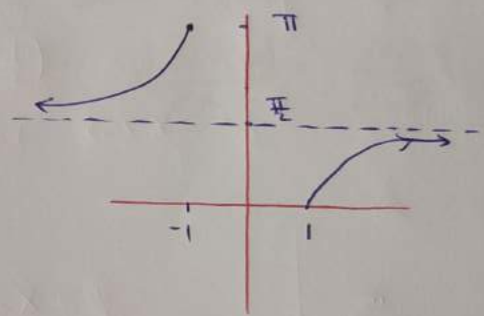
$= \cot\left(-\frac{\pi}{3}\right) = -\frac{1}{\sqrt{3}}$

$\boxed{16}$ Find $\lim\limits_{x \to -\infty} \tan^{-1} x$



$= -\frac{\pi}{2}$

$\boxed{18}$ Find $\lim\limits_{x \to -\infty} \sec^{-1} x$



$= \frac{\pi}{2}$

$\boxed{22}$ find the derivative of y with respect to x.

$y = \cos^{-1}\left(\frac{1}{x}\right)$

$$\frac{d(\cos^{-1} u)}{dx} = \frac{-du/dx}{\sqrt{1-u^2}}$$

So $y' = \dfrac{-\left(\frac{-1}{x^2}\right)}{\sqrt{1-\left(\frac{1}{x}\right)^2}}$

$$= \frac{\frac{1}{x^2}}{\sqrt{\frac{x^2-1}{x^2}}} = \frac{1}{x^2} \cdot \sqrt{\frac{x^2}{x^2-1}} = \frac{1}{x^2} \cdot \frac{|x|}{\sqrt{x^2-1}}$$

$$= \frac{1}{|x|\sqrt{x^2-1}}$$

26 Find the derivative of y with respect to ~~x~~ s

$$y = \sec^{-1}(5s)$$

$$\frac{d(\sec^{-1}u)}{dx} = \frac{du/dx}{|u|\sqrt{u^2-1}}$$

$$\frac{dy}{d\cancel{x}s} = \frac{5}{|5s|\sqrt{(5s)^2-1}}$$

$$\frac{dy}{ds} = \frac{5}{|5s|\sqrt{25s^2-1}} = \frac{1}{|s|\sqrt{25s^2-1}}$$

30 Find the derivative of y with respect to ~~x~~ t

$$y = \cos^{-1}(e^{-t})$$

$$\frac{d\cos^{-1}(u)}{dx} = -\frac{du/dx}{\sqrt{1-u^2}}$$

$$\frac{dy}{dt} = -\frac{-e^{-t}}{\sqrt{1-(e^{-t})^2}} = \frac{+e^{-t}}{\sqrt{1-e^{-2t}}}$$

34 Find the derivative of y with respect to x

$$y = \tan^{-1}(\ln x)$$

$$\frac{d(\tan^{-1}u)}{dx} = \frac{du/dx}{1+u^2}$$

$$\frac{dy}{dx} = \frac{\frac{1}{x}}{1+(\ln x)^2} = \frac{1}{x\left(1+(\ln x)^2\right)}$$

$\boxed{35}$ Find the derivative of y with respect to t

$$y = \csc^{-1}(e^{t})$$

$$\frac{d(\csc^{-1}u)}{dx} = -\frac{du/dx}{|u|\sqrt{u^2-1}}$$

$$\frac{dy}{dt} = -\frac{e^{t}}{|e^{t}|\sqrt{(e^{t})^2-1}} = -\frac{e^{t}}{e^{t}\sqrt{e^{2t}-1}}$$

$$\frac{dy}{dt} = -\frac{1}{\sqrt{e^{2t}-1}}$$

$\boxed{40}$ Find the derivative of y with respect to x

$$y = \cot^{-1}\left(\frac{1}{x}\right) - \tan^{-1}x$$

$$\frac{d(\cot^{-1}u)}{dx} = -\frac{du/dx}{1+u^2}$$

$$\frac{d(\tan^{-1}u)}{dx} = \frac{du/dx}{1+u^2}$$

$$\frac{dy}{dx} = \frac{-\left(-\frac{1}{x^2}\right)}{1+\left(\frac{1}{x}\right)^2} - \frac{1}{1+(x)^2}$$

$$\frac{dy}{dx} = \frac{\frac{1}{x^2}}{\frac{x^2+1}{x^2}} - \frac{1}{1+x^2}$$

$$\frac{dy}{dx} = \frac{1}{x^2}\cdot\frac{x^2}{x^2+1} - \frac{1}{1+x^2}$$

$$\frac{dy}{dx} = \frac{1}{x^2+1} - \frac{1}{1+x^2} = 0$$

$\boxed{46}$ Evaluate $\int \frac{dx}{9+3x^2}$

Note that $\int \frac{du}{a^2+u^2} = \frac{1}{a} \tan^{-1}\left(\frac{u}{a}\right) + c$

so $\int \frac{dx}{9+3x^2} = \int \frac{dx}{(3)^2+(\sqrt{3}x)^2}$

$= \frac{1}{(\sqrt{3})} \int \frac{(\sqrt{3})\, dx}{(3)^2+(\sqrt{3}\,x)^2}$

$= \frac{1}{\sqrt{3}} \left[\frac{1}{3} \tan^{-1}\left(\frac{\sqrt{3}\,x}{3}\right) + c \right]$

$= \frac{1}{\sqrt{3}} \left[\frac{1}{3} \tan^{-1}\left(\frac{x}{\sqrt{3}}\right) \right] + c$

$\boxed{50}$ Evaluate $\int_0^{\frac{3\sqrt{2}}{4}} \frac{ds}{\sqrt{9-4s^2}}$

$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + c$

$\int \frac{ds}{\sqrt{9-4s^2}} = \int \frac{ds}{\sqrt{(3)^2-(2s)^2}}$

$= \frac{1}{2} \int \frac{2\,ds}{\sqrt{(3)^2-(2s)^2}} = \frac{1}{2}\left[\sin^{-1}\left(\frac{2s}{3}\right)\right]$

$\int_0^{\frac{3\sqrt{2}}{4}} \frac{ds}{\sqrt{9-4s^2}} = \frac{1}{2}\left[\sin^{-1}\left(\frac{2s}{3}\right)\right]\Big|_0^{\frac{3\sqrt{2}}{4}} = \frac{1}{2}\left[\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) - \sin^{-1}(0)\right]$

$= \frac{1}{2}\left[\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) - \sin^{-1}(0)\right] = \frac{1}{2}\left[\frac{\pi}{4} - 0\right] = \boxed{\frac{\pi}{8}}$

59 Evaluate $\int \frac{dx}{(2x-1)\sqrt{(2x-1)^2-4}}$

Note that $\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + c$

$$\int \frac{dx}{(2x-1)\sqrt{(2x-1)^2-4}} = \frac{1}{2}\int \frac{2\,dx}{(2x-1)\sqrt{(2x-1)^2-(2)^2}}$$

$$= \frac{1}{2}\left[\frac{1}{2}\sec^{-1}\left|\frac{2x-1}{2}\right|\right] + c$$

$$= \frac{1}{4}\sec^{-1}\left|\frac{2x-1}{2}\right| + c$$

64 Evaluate $\int_1^{e^{\frac{\pi}{4}}} \frac{4\,dt}{t(1+\ln^2 t)}$

$$= 4\int_1^{e^{\frac{\pi}{4}}} \frac{\frac{1}{t}\,dt}{(1+(\ln t)^2)}$$

Recall $\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)+c$

$$= 4\int_1^{e^{\frac{\pi}{4}}} \frac{\frac{1}{t}\,dt}{((1)^2+(\ln(t))^2)} = 4\left[\frac{1}{1}\tan^{-1}\left(\frac{\ln t}{1}\right)\right]\Big|_1^{e^{\frac{\pi}{4}}}$$

$$= 4\left[\tan^{-1}\left(\ln e^{\frac{\pi}{4}}\right) - \tan^{-1}(0)\right]$$

$$= 4\left[\tan^{-1}\left(\frac{\pi}{4}\right) - 0\right] = 4\tan^{-1}\left(\frac{\pi}{4}\right)$$

70 Evaluate $\displaystyle\int_{\frac{1}{2}}^{1} \frac{6\,dt}{\sqrt{3+4t-4t^2}}$

$$= \int_{\frac{1}{2}}^{1} \frac{6\,dt}{\sqrt{-(4t^2-4t-3)}} = \int_{\frac{1}{2}}^{1} \frac{6\,dt}{\sqrt{\left(\frac{-4}{2}\right)^2 - \left(4t^2-4t-3+\left(\frac{-4}{2}\right)^2\right)}}$$

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$$= \int_{\frac{1}{2}}^{1} \frac{6\,dt}{\sqrt{4-(4t^2-4t+1)}} = \int_{\frac{1}{2}}^{1} \frac{6\,dt}{\sqrt{(2)^2-(2t-1)^2}}$$

$$= 6\int_{\frac{1}{2}}^{1} \frac{dt}{\sqrt{(2)^2-(2t-1)^2}}$$

Note $\displaystyle\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right)+c$

$$= 6\left(\frac{1}{2}\right)\int_{\frac{1}{2}}^{1} \frac{2\,dt}{\sqrt{(2)^2-(2t-1)^2}} = 3\sin^{-1}\left(\frac{2t-1}{2}\right)\Bigg|_{\frac{1}{2}}^{1}$$

$$= 3\left[\sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}(0)\right]$$

$$= 3\left[\frac{\pi}{6} - 0\right] = \boxed{\frac{\pi}{2}}$$

$\boxed{74}$ Evaluate $\displaystyle\int_2^4 \frac{2\,dx}{x^2-6x+10}$

$$= \int_2^4 \frac{2\,dx}{x^2-6x+\left(\frac{-6}{2}\right)^2+10-\left(\frac{-6}{2}\right)^2}$$

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$$= \int_2^4 \frac{2\,dx}{(x^2-6x+9)+1} = \int_2^4 \frac{2\,dx}{(x-3)^2+(1)^2}$$

Note $\displaystyle\int \frac{du}{u^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)+c$

$$\int_2^4 \frac{2\,dx}{(x-3)^2+(1)^2} = 2\int_2^4 \frac{dx}{(x-3)^2+(1)^2}$$

$$= 2\left[\frac{1}{1}\tan^{-1}\left(\frac{x-3}{1}\right)\right]\Big|_2^4$$

$$= 2\left[\tan^{-1}(1) - \tan^{-1}(-1)\right]$$

$$2\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = 2\left(\frac{2\pi}{4}\right) = \boxed{\pi}$$

✓ $\boxed{79}$ Evaluate $\displaystyle\int \frac{dx}{(x+1)\sqrt{x^2+2x}}$

اكمال مربع $\left(\frac{\text{معامل } x}{2}\right)^2$

$$\int \frac{dx}{(x+1)\sqrt{x^2+2x}} = \int \frac{dx}{(x+1)\sqrt{x^2+2x+1-1}}$$

$$= \int \frac{dx}{(x+1)\sqrt{(x^2+2x+1)-1}} = \int \frac{dx}{(x+1)\sqrt{(x+1)^2-1}}$$

$$= \frac{1}{1}\sec^{-1}\left|\frac{x+1}{1}\right| + c = \sec^{-1}|x+1| + c$$

Since $\displaystyle\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + c$

✓ $\boxed{86}$ Evaluate $\displaystyle\int \frac{dy}{(\sin^{-1}y)\sqrt{1-y^2}}$

let $V = \sin^{-1}y \longrightarrow dV = \dfrac{dy}{\sqrt{1-y^2}}$

$$\int \frac{dy}{(\sin^{-1}y)\sqrt{1-y^2}} = \int \frac{dV}{V}$$

$$= \ln|V| + c = \ln|\sin^{-1}y| + c$$

$\boxed{90}$ Evaluate $\int \frac{e^x \sin^{-1} e^x}{\sqrt{1-e^{2x}}}\, dx$

Let $y = \sin^{-1} e^x \longrightarrow dy = \frac{e^x\, dx}{\sqrt{1-(e^x)^2}}$

$\rightarrow dy = \frac{e^x\, dx}{\sqrt{1-e^{2x}}}$

$$\int \frac{e^x \sin^{-1} e^x}{\sqrt{1-e^{2x}}}\, dx = \int y\, dy = \frac{y^2}{2} + c$$

$$= \frac{(\sin^{-1} e^x)^2}{2} + c$$