chapter(1): Signals "continuous".

def: signals:

physical phenomenon that depends on time.

"function of time". , $x(t) = 5$ continuous one ~

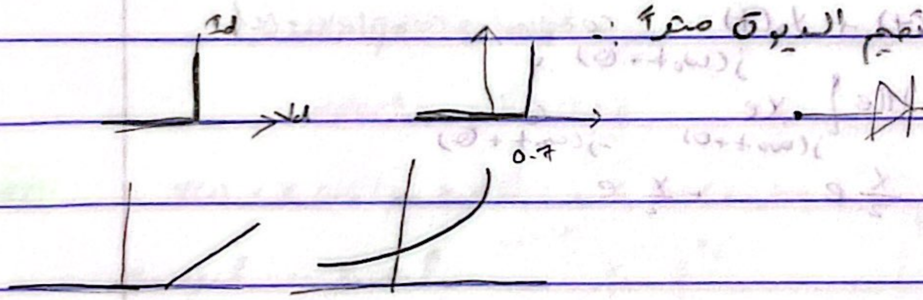
that drive a system or comes out of a system.

system:

aggregation of physical simple component according to a tecded topology to chive adeoinal objective.

excitation → T → response
"signal input".

. model not unique.



. اختلاف طريقة تطور الظاهرة

Elementary signals.

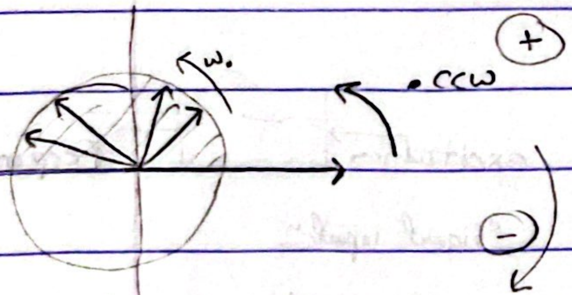
1) Sinosoidal signal. فقط freq واحدة .

* $x(t) = X \cos(\omega_0 t + \theta)$

Amplitude. $\in \mathbb{R}^+$

frequency $\to \in \mathbb{R}$

rotational ω_0 linear f_0



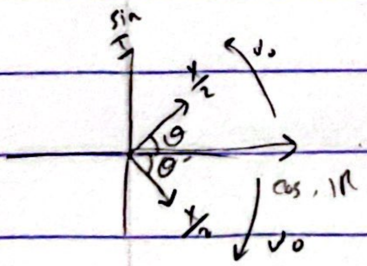
$\omega_0 \in \mathbb{R}^+$

$\omega_0 \in \mathbb{R}^-$

$x(t) = \mathrm{Re}[x_1(t)]$. complex

$x(t) = x_1(t) + x_2(t)$ complex + complex = real.

- $x(t) = \mathrm{Re}\{X e^{j(\omega_0 t + \theta)}\}$.
- $x(t) = \frac{X}{2} e^{j(\omega_0 t + \theta)} + \frac{X}{2} e^{-j(\omega_0 t + \theta)}$.

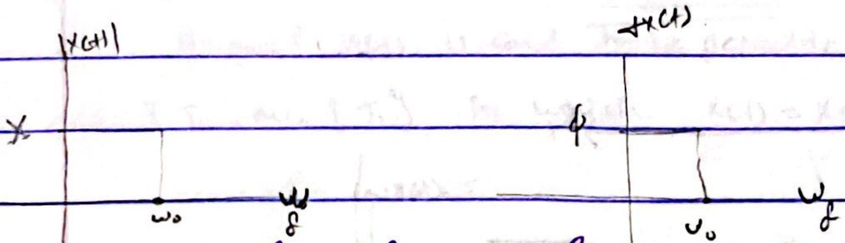


المحصلة

ونفس المقدار ولكن باتجاهين

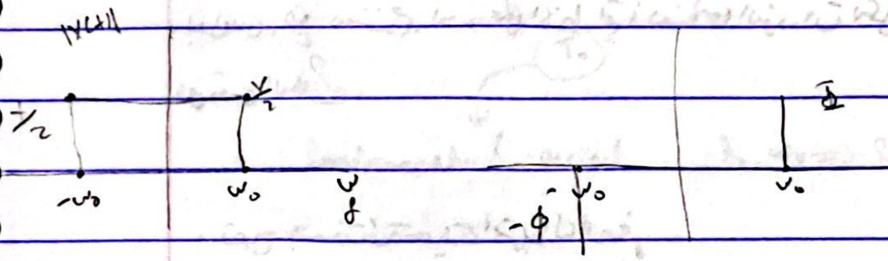
فيبقى صافي حقيقي .

• $\mathbb{R}\{X e^{j(\omega_0 t+\phi)}\}$



single-sided spectral representation.

• $\frac{X}{2}e^{j(\omega_0 t+\phi)} + \frac{X}{2}e^{-j(\omega_0 t+\phi)}$

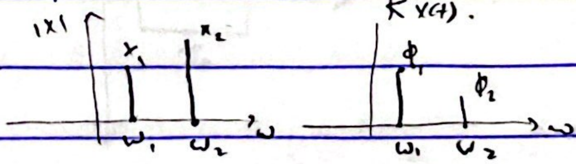


• double sided.

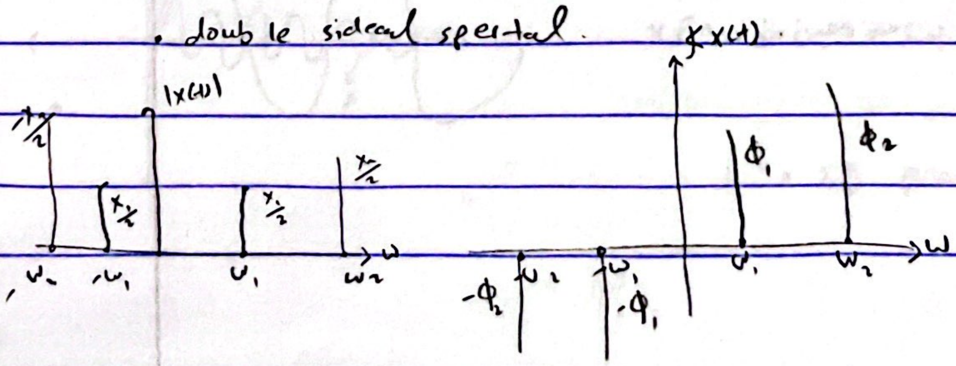
spectral rep.

ex:- $x(t) = X_1\cos(\omega_1 t+\phi_1) + X_2\cos(\omega_2 t+\phi_2)$.

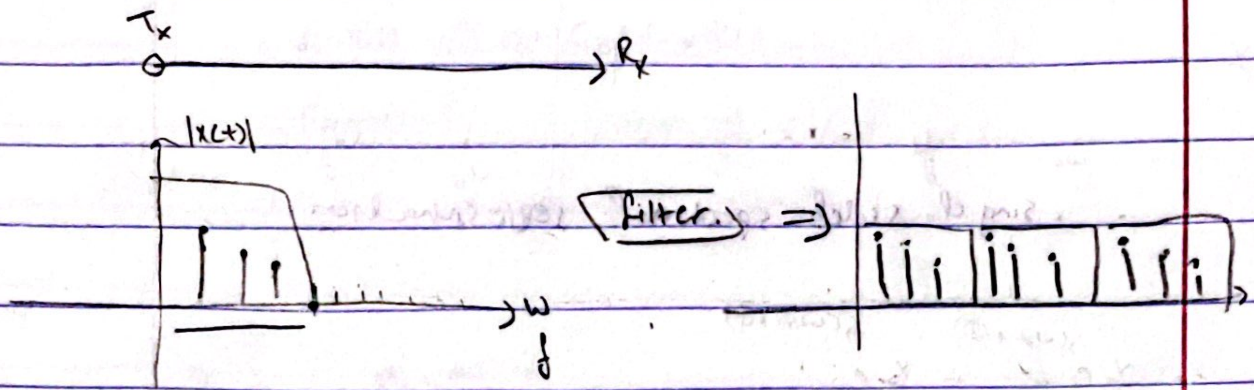
- singel-sideal



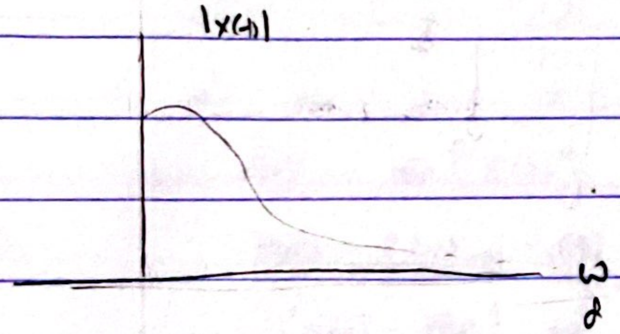
• double sideal spectral.



• عدد دورات الأول بيخرق نفس العدد في دورات الآخر.



• بدل من وضع مستخدم واحد على الخط تم أخذ أعلى الترددات لعمل موجة "دوجها" ووضع أكثر من واحد على نفس الخط.

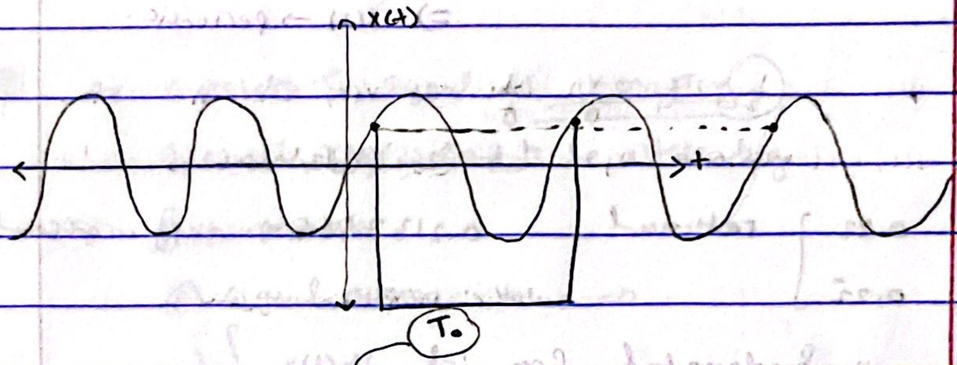


• تصبح بعد نقطة معينة أقرب إلى الصفر.

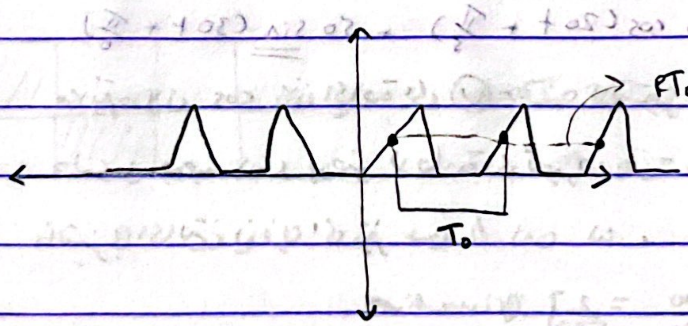
$\boxed{\text{II}}$ the sinusoidal signal is periodic signal

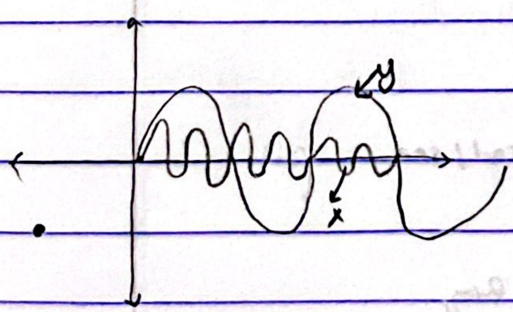
A signal $x(t)$ is said to be periodic

$\Leftrightarrow \exists\, T_0 = \min\{T_i\}$ for which $x(t) = x(t + nT_0)$ $\forall t, \forall n$



fundamental period , $f_0, w_0 \Rightarrow$ fundamental freq.





• كل دورة من y يوجد بها عدد معين من دورات x وهذا العدد ثابت.

• x,y periodic.

Elementary signals.

$$x(t) = X_1 \cos(\omega_1 t + \phi_1) + X_2 \cos(\omega_2 t + \phi_2)$$

عددين اي $\frac{int}{int}$

$$\frac{\omega_1}{\omega_2} = \frac{f_1}{f_2} = \frac{T_2}{T_1} \Longleftrightarrow \text{rational number.}$$

$$\Rightarrow x(t) \rightarrow \text{periodic.}$$

$\frac{1}{2}$ $\frac{2}{4}$ $\frac{3}{6}$ $\frac{4}{8}$ ...

min one. عندما يكون بأبسط صورة .

0.22] rational   0.2137568...] irrational

$0.2\bar{2}$

fundamental freq : f_0.

$$\frac{\omega_1}{\omega_2} = \frac{f_1}{f_2} = \frac{T_2}{T_1} = \frac{n}{n_2} = \boxed{\frac{\bar{n}_1}{\bar{n}_2}} \text{ primitive.}$$

ابسط صورة .

ex:- 1) $x(t) = 30\cos(20t + \frac{\pi}{3}) + 50\sin(30t + \frac{\pi}{6})$.

ما نحولها لـ cos لأن الاجابة بالـ ω هذه لا تتغير وتبقى ثابتة في الحالتين .

ولو كان يوجد (−) برضو لا حاجة اني تغيره .

فالتغير في الحالتين يؤدي الى تغير على φ ، ليس ω ،

$$\Rightarrow \frac{\omega_1}{\omega_2} = \frac{20}{30} = \frac{2}{3} \text{] primitive.}$$

"periodic, rational".

$30 = 3 \cdot X \quad \therefore X = 10 \; (\omega_0) \text{ rad/sec.}$

$\Rightarrow 2 \cdot \omega_0 = 20 \quad \therefore \boxed{\omega_0} = 10$ too.

fundamental freq.

2) $x(t) = 10\cos(\pi t + \frac{\pi}{4}) + 20\cos(40t + \frac{\pi}{6})$

$\frac{\omega_1}{\omega_2} = \frac{\pi}{40} \Rightarrow$ irrational , not periodic .

لانه قسمة عدد غير صحيح على عدد صحيح .

$\boxed{2}$ the sinusoidal signal is alternating

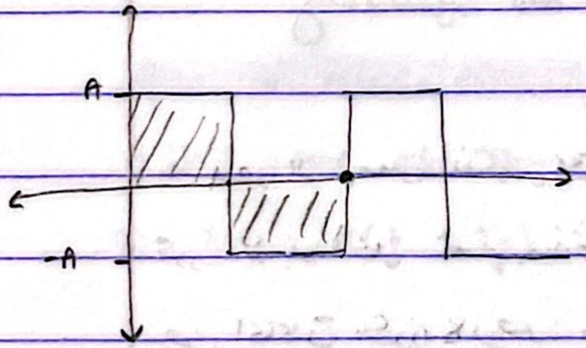
A signal $x(t)$ is said to be alternating

$\Longleftrightarrow$ 1) $x(t)$ periodic .

2) signal average value $= 0$.

$$\frac{1}{T_0}\int_{T_0} x(t)\,dt = 0$$

3) $\frac{1}{NT_0}\int_{NT_0} x(t)\,dt = 0$.



تكامل المساحة في الفترة $T_0 = 0$.

جزء منها موجب وجزء سالب

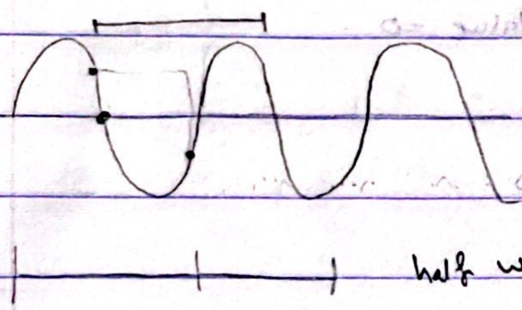
[3] The sinosoidel signal in the direct form of sin/cos has <u>absolute</u> symmetry properties.

$$x(t) = X \cos(\omega_1 t)$$

$$\tilde{x}(t) = \tilde{X} \sin(\omega_1 t).$$

- $x(t)$ has even symmetry $\Leftrightarrow$ $x(t) = x(-t)$ $\forall t$.
- $x(t)$ has odd symmetry $\Leftrightarrow$ $x(t) = -x(-t)$. $\forall t$

$x(-t) = -x(t)$. $\forall t$



Relative symmetry.

half wave odd symmetry

absolute:- عند فحص الـ symm نتفحص في المحاور الأصلية

محور y إذا كان even ونقطة الأصل إذا كان odd ولكن في

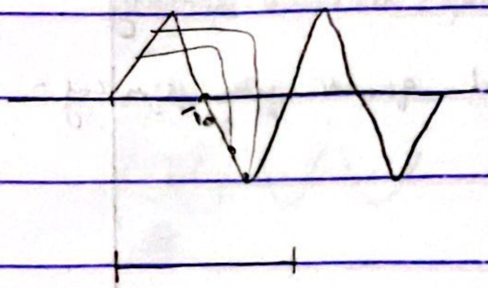
بعض الحالات يمكن لا يوجد symm حولهم فإذا وجدنا نقطة أخرى

بعد إجراء الحسابات نطرح الـ <u>delay</u> "مقدار الإزاحة" منه.

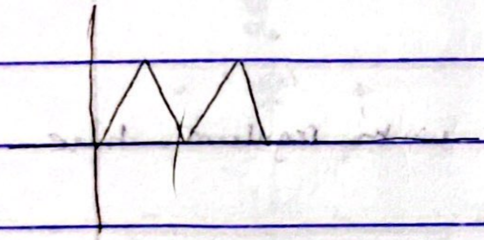
Half wave odd symmetry.

relative symmetry → signal intrinsic symmetry.

$x(t) = -x(t + T/2) \quad \forall t, \in t_0$.



Half odd symmetry.

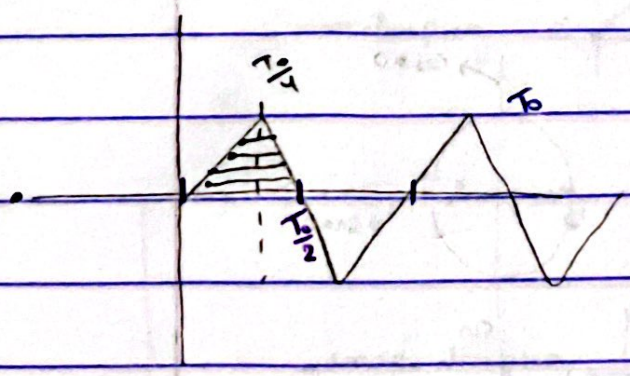


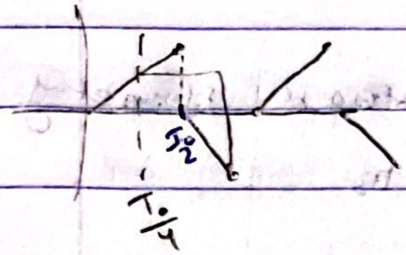
Half even symmetry.

Quarter-wave even symmetry.

1) Half-wave odd symmetry.

2) $x(t) = x(t \pm \frac{T_0}{4})$.

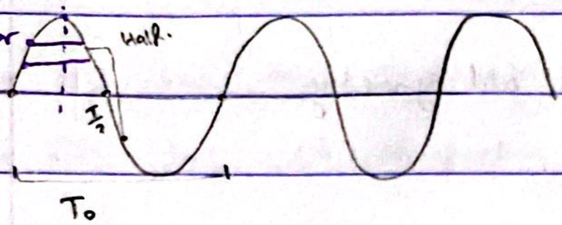




"not quarter even symmetry."

but it's Half odd symmetry.

• Sinisodal.

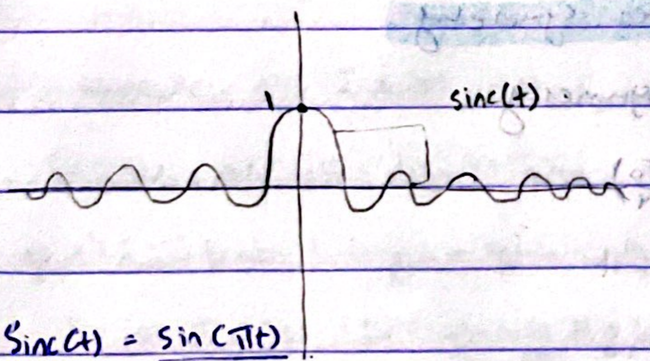


∴ so it's Half odd semmetry.

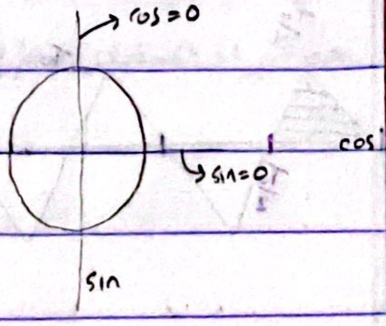
and quarter even symmetry.

oscillitary signals.

has positive and negative value with regular time repetition of signal shape.



$$\text{Sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$



- $\lim_{t \to 0} \text{sinc}(t) = 1$
- $\lim_{t \to \infty} \text{sinc}(t) = 0$
- even symmetry.
- oscillatory
- zeros of sinc(t)

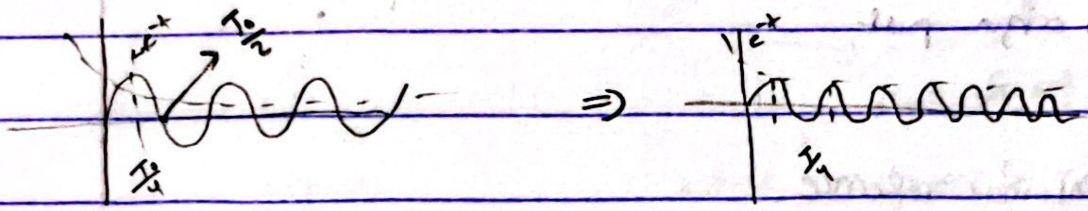
$$\sin(\pi t) = 0 \Leftrightarrow \pi t = n\pi$$
$$\therefore t = n.$$

- عند رفع الـ order تزيد سرعة الوصول.

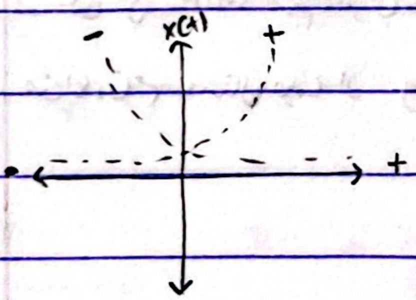
- freq point: لا يوجد نقاط انعكاس في الإشارة.

pos always or negative.

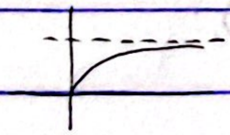
$\Rightarrow y(t) = e^{t} \cos(yt + \frac{T_2}{2})$

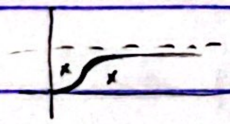


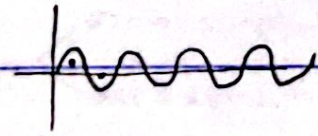
$\Rightarrow \sin(\pi t)$, $T_0 = \frac{2\pi}{\omega_0} = \frac{2\pi}{\pi} = \boxed{2}$ time periode.



$x(t) = e^{\alpha t}$. +, − , α إشارة

- — over damping  • not oscillitary

— critical damping  • 50%

— under damping  • oscillitary

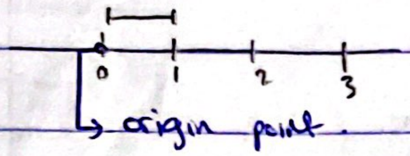
→ يوجد نقطة واحدة فقط عندما الغاوس، تتشكل بين التشتر لاسي وأسني
وليس دائماً مثل الـ under damping

so it's have just one freq point.

Refernce operation.

- linear refernce.

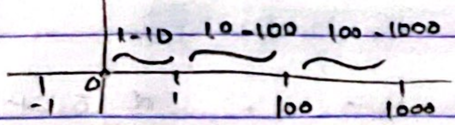
unit, scale.



> direction.

- nonlinear refernce.

→ $\log_{10} \frac{x_{n+1}}{x_n} = 1$



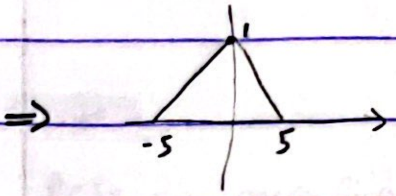
⇐ نفس المسافة بين الارقام عى الخط ، ولكن كل unit منه تحتوي عدد مختلف من الارقام ، مثال عليها ال log.

operation:

→ origin → shift.

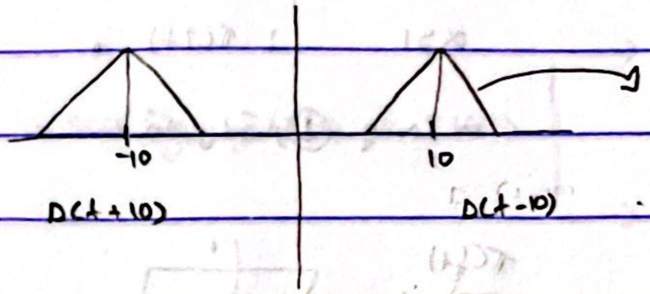
scale → scaling.

Direction → folding.



- $D(t) \to D(t-10)$, $t-10=0$ $\therefore t=10$

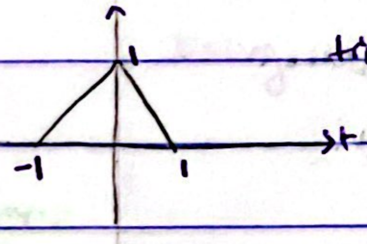
  $D(t+10)$, $t+10=0$ $\therefore t=-10$



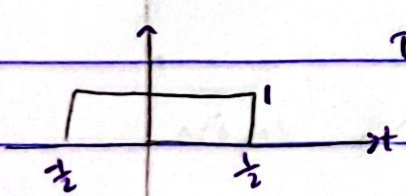
$D(t+10)$ $D(t-10)$

• الازاحة بالسالبة لل ref تكون لليسار

اذا بالسالبة لل signals تكون لليمين .



triangle of (t).



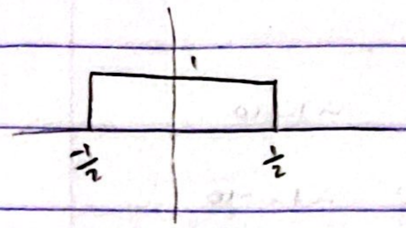
$\Pi(t) \to$ finite pulse

ex:-

- $x_1(t) = \cos(\underbrace{2\pi}_{w_0} t) \to T_1 = \frac{2\pi}{2\pi} = 1$
- $x_2(t) = \cos(\underbrace{4\pi}_{w_0} t) \to T_2 = \frac{2\pi}{4\pi} = \frac{1}{2}$
- $x_3(t) = \cos(\underbrace{\tfrac{1}{2}\pi}_{w_0} t) \to T_3 = \frac{2\pi}{\frac{1}{2}\pi} = 4$

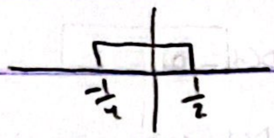
$x(t) \xrightarrow{\text{scaling}} x(\alpha t)$ <
- $\alpha > 1 \to$ compressions : تنضغط
- $\alpha < 1 \to$ expansion . تتمدد

$\Rightarrow \Pi(t)$.



$\Pi(2t)$

$\Rightarrow$

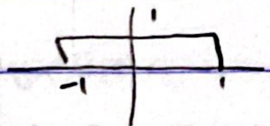


• $\Pi(2t)$: $\alpha > 1$

تنكمش بمقدار 2 ∴ comp .

$\Pi(\frac{1}{2}t)$

$\Rightarrow$



• $\Pi(\frac{1}{2}t)$: $\alpha < 1$

تتمدد بمقدار 2 ∴ exp

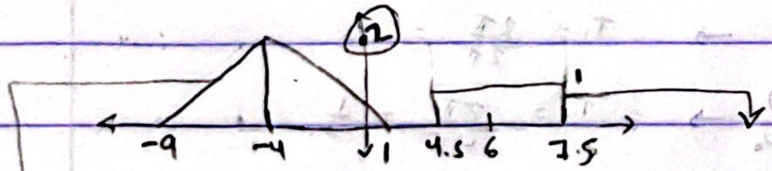
ex:-

$$x(t) = \Pi(\tfrac{t}{3} - 2) + 2D\left(\frac{t+4}{5}\right)$$

$$= \Pi\left(\frac{t-6}{3}\right) + 2D\left(\frac{t+4}{5}\right)$$

$$= \Pi\left(\tfrac{1}{3}(t-6)\right) + 2D\left(\tfrac{1}{5}(t+4)\right)$$

$\Downarrow$ pulse — $\Downarrow$ triangle.



• بداية نعمل الازاحة لليمين بمقدار 6 من نقطة الأصل.

• ازاحة لليسار بمقدار 4

• $\alpha < 1$ ∴ exp بمقدار 5

$5 \cdot 1 = 5$

أي 5 بدل 1

بما أن $\alpha < 1$ يجب أن يكون تمدد بمقدار 3

والطول ال pulse يكون بمقدار $\frac{1}{2}$

∴ $3 \cdot \frac{1}{2} = \frac{3}{2}$

أي 1.5 بدل 0.5

note:-

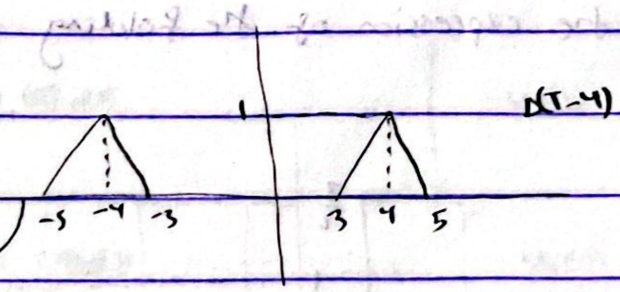
Reference operation:-

- origin → shift → $x(t) \to x(t-\tau)$
  - → $\tau>0$, signal shift right
  - → $\tau<0$, signal shift left

- unit → scaling → $x(t) \to x(\alpha t)$
  - → $\alpha>1$, compression.
  - → $\frac{1}{\alpha}$
  - → $\alpha<1$, expansion.

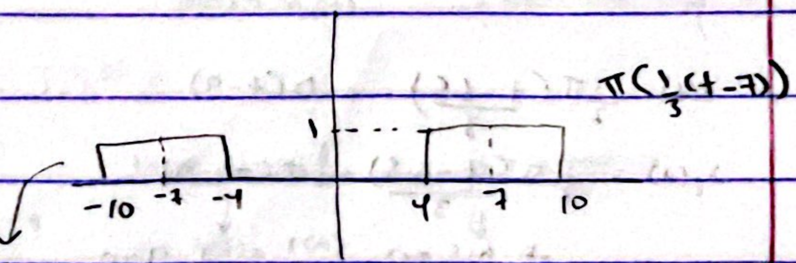
- folding → Direction → $x(t) \to x(-t)$ → direction reversal.

ex:-



→ $-t-4=0$

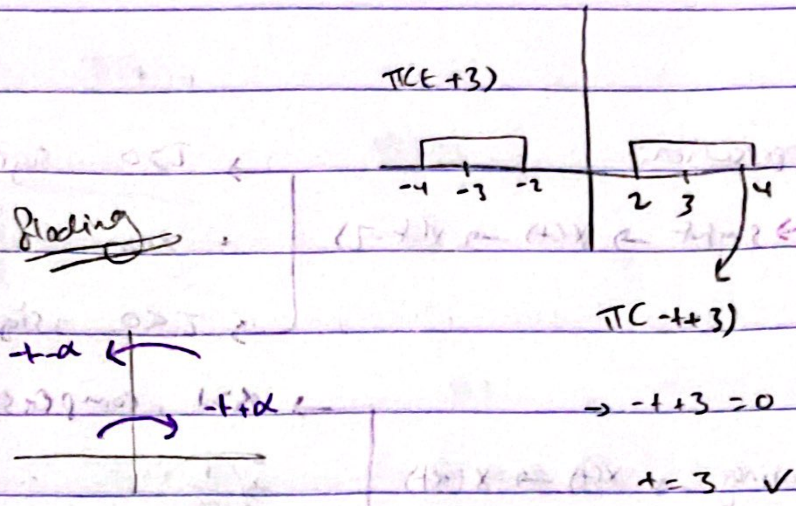
$\therefore -t=-4$ ✓



→ $-t-7=0$

$\therefore t=-7$ ✓

Folding :



Note:-

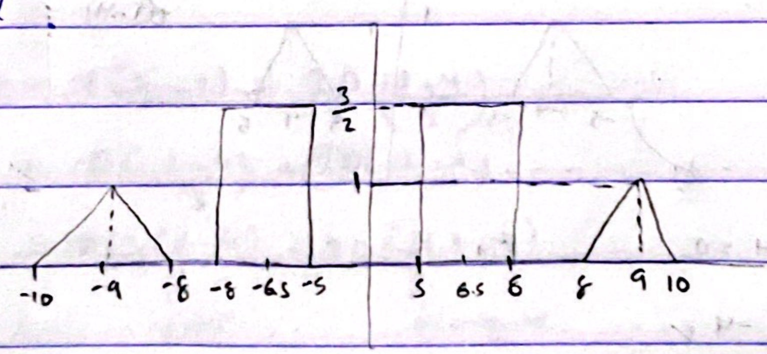
$\Rightarrow -t+3=0$

$t=3$ ✓

ex:-

1. write the signal expression.

2. write the expression of the folding version of the signal :



$$x(t) = \frac{3}{2}\Pi\left(\frac{t-6.5}{3}\right) + \Delta(t-9)$$

$$x_f(t) = \frac{3}{2}\Pi\left(\frac{-t-6.5}{3}\right) + \Delta(-t-9)$$

$-t-6.5=0 \qquad -t-9=0$

$\therefore t=-6.5$ ✓ $\qquad \therefore t=-9$ ✓

singularity signals:-

Recursion . $f: D \to CD$.

$n! \xrightarrow{\text{generation}} n! = n(n-1)!$ generator.

$0! = 1$. base case.

- Integration generator $\int$
- Derivative generator.

$$u_{-k}(t) = \int_{-\infty}^{t} u_{-k+1}(\tau)\,d\tau$$

Derivative generator.

$$u_k = \frac{d\,u_{k-1}(t)}{dt}$$

$$u_{-1}(t) = \int_{-\infty}^{t} u_{-1+1}(\tau)\,d\tau = \int_{-\infty}^{t} u_0(\tau)\,d\tau .$$

$$u_0(t) = \frac{d\,u_{-1}(t)}{dt}$$

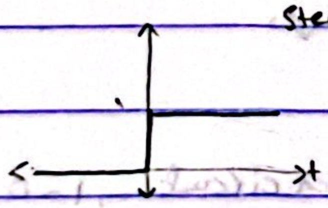
$$u_{-2}(t) = \int_{-\infty}^{t} u_{-1}(\tau)\,d\tau .$$

$$u_1(t) = \frac{d\,u_0(t)}{dt}$$

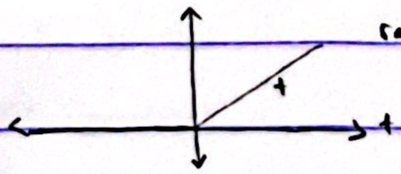
$$u_{-3}(t) = \int_{-\infty}^{t} u_{-2}(\tau)\,d\tau .$$

$$u_2(t) = \frac{d\,u_1(t)}{dt}$$

step signal.

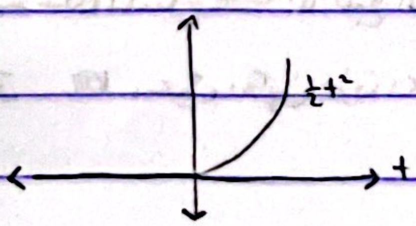
⇒ 

$u(t) = u_{-1}(t)$. , not continuous at $\underline{\underline{0}}$



ramp signal. , continuous.

$u_{-2}(t) = r(t)$.



$p(t)$: parabolic signal , continuous.

$\Rightarrow$ step:- $u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$

ramp:- $r(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases}$

parabolic:- $p(t) = \begin{cases} \frac{1}{2}t^2 & , t \geq 0 \\ 0 & , t < 0 \end{cases}$

$\Rightarrow$ $r(t) = t\, u(t)$.

$p(t) = \frac{1}{2}t^2 u(t)$

لتسهيل الكتابة ولكن
عند الحل نستخدم التعريفين .
أخرين .

$u(t) = \delta(t)$ Dirac impulse.

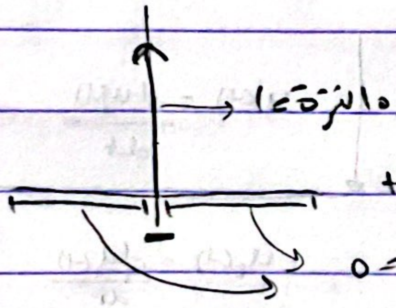
unit impulse

Generation calculus. J . δ بعد اكتشاف ال

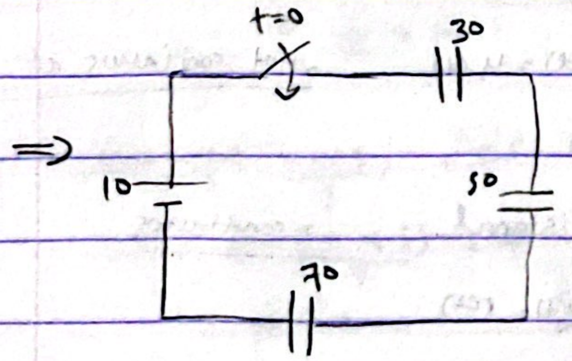
$$\begin{cases} \int_{-\infty}^{\infty} \delta(t)\,dt = 1. \\ \delta(t) = 0 \quad \forall t \neq 0 . \end{cases}$$

. it's functional not function.

هو عكس ال function الذي يعرف ولم تحت خصائصه ال δ وجد من خلال خصائصه
بحيث ان تكامله عند النقطة الواحدة = 1 هي عكس ال function الذي يساوي 0



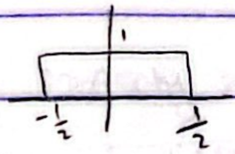
دي 0.



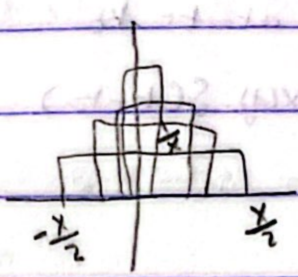
open circuit , i = 0 . ∀ V.

• عند غلق المفتاح يجب ان تنتج البطارية تيار كبير جدا حتى يستطيع موازنة بقية الاجزاء
دكتيق KVL ، حيث يكون فرق الجهد بالحلقة المغلقة = 0

• $\Pi(t) \Rightarrow$

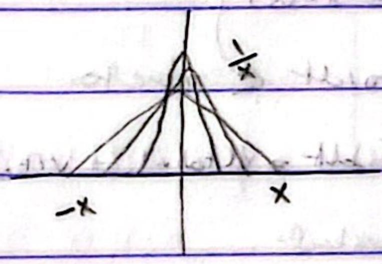


$\Downarrow$



$$\lim_{x \to 0} \frac{1}{x} \Pi\left(\frac{t}{x}\right) = \delta(t).$$

• $\Delta(t) \Rightarrow$



$$\lim_{x \to 0} \frac{1}{x} \Delta\left(\frac{t}{x}\right) = \delta(t).$$

$$x(t) = \begin{cases} 1 - |t| & , -1 \le t \le 1. \\ 0 & , \text{otewise}. \end{cases}$$

$$\Delta\left(\frac{t}{x}\right) = 1 - \frac{|t|}{x}.$$

→ properties of $\delta(t)$.

1) point property of $\delta(t)$.

if $x(t)$ is continous at $t = t_0$.

then $x(t)\,\delta(t-t_0) = x(t_0)\,\delta(t-t_0)$.

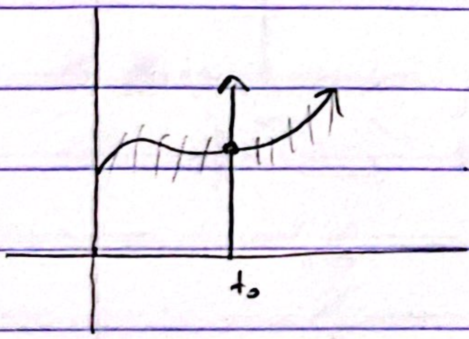
2) Sampling property of $\delta(t)$

$$\int_{-\infty}^{\infty} x(t)\,\delta(t-t_0)\,dt = x(t_0)$$

$$\int_{-\infty}^{\infty} x(t_0)\,\delta(t-t_0)\,dt \leftarrow \text{due to}$$

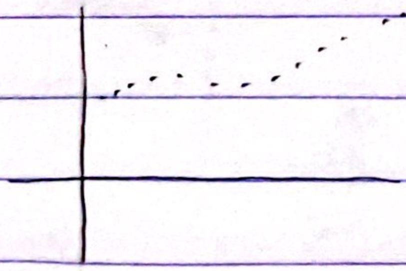
$$\underline{\underline{x(t_0)}} \int_{-\infty}^{\infty} \delta(t-t_0)\,dt = x(t_0) * 1 = x(t_0).$$

point, so it's constant.



• حنضرب ال $\delta(t)$ بال function بجال ال $\delta(t)$

بغير النقطة المحددة = 0 بالتالي بيصير كأنه (1).



• وضع أكثر من $\delta(t)$, بالتالي بيصير كأنه

ديسكريت.

3) Scaling property of $\delta(t)$.

$$\delta(at) = \frac{1}{|a|}\,\delta(t).$$

Proof:- $\int_{-\infty}^{\infty} \delta(at)\,dt$

$\boxed{t' = at}$

$dt' = a\,dt$ • $t \to -\infty \to \begin{cases} a>0, & t' \to -\infty \\ a<0, & t' \to \infty \end{cases}$

$dt = \frac{1}{a}\,dt'$.

• $t \to \infty \to \begin{cases} a>0, & t' \to \infty \\ a<0, & t' \to -\infty \end{cases}$

$\boxed{a>0}$

$$\int_{-\infty}^{\infty} \delta(t')\cdot\frac{1}{a}\,dt' = \frac{1}{a}\underbrace{\int_{-\infty}^{\infty} \delta(t')\,dt'}_{\delta(t)} \Rightarrow \frac{1}{|a|}\,\delta(t).$$

$\boxed{a<0} \to -|a|$

$$\int_{\infty}^{-\infty} \delta(t')\cdot\frac{-1}{|a|}\,dt' = \int_{-\infty}^{\infty} \frac{1}{|a|}\,\delta(t')\,dt'$$

$$= \frac{1}{|a|}\,\delta(t)\,dt.$$

(*) even symmetry of $\delta(t)$.

$$\delta(at) = \frac{1}{|a|}\,\delta(t)$$

$a = -1$

$\delta(-t) = \frac{1}{|-1|}\,\delta(t) = \delta(-t) = \delta(t)$ so it's has even sym.

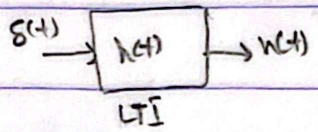
4) Convolution property of $\delta(t)$.

$x_1(t)$, $x_2(t)$.

$$x(t) = x_1(t) \circledast x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau .$$

$\Rightarrow$ for $x_2(t) = \delta(t)$.

$$x_1(t) \circledast \delta(t) = \int_{-\infty}^{\infty} x_1(\tau)\, \delta(t-\tau)\, d\tau = x_1(t) .$$

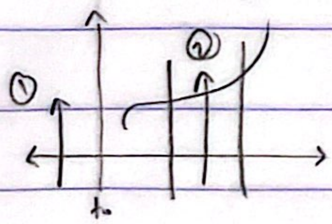


$$\int_{-\infty}^{\infty} x_1(\tau)\, \delta(\tau - t)\, dt = x_1(t) = \int_{-\infty}^{\infty} x_1(\tau)\, \delta(t-\tau)\, d\tau .$$

$$x_1(t) \circledast \delta(t)$$

5) Interval property of $\delta(t)$.

$$\int_{t_1}^{t_2} x(t_0)\, \delta(t - t_0)\, dt = \begin{cases} x(t_0) , & t_0 \in]t_1, t_2[\\ 0 , & \text{otherwise} . \end{cases}$$



① $\Rightarrow$ 0

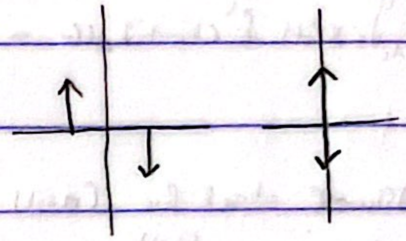
② $\Rightarrow x(t_0)$

Differentiation property of $\delta(t)$:

$$\delta(t) \xrightarrow{d/dt} \delta'(t) \xrightarrow{d/dt} \delta''(t) \;\dots\dots \xrightarrow{d/dt} \delta^{(n)}(t).$$

$\downarrow$

Kronecker impulse.



$x(t)$ continuous at $t = t_0$.

$t_0 \in]t_1, t_2[$.

$$\int_{t_1}^{t_2} x(t)\, \delta^{(n)}(t-t_0)\, dt = \frac{(-1)^n \, d^n x(t)}{dt^n}\Bigg|_{t=t_0}$$

$$\rightarrow \int_{t_1}^{t_2} x(t)\, \delta^{(n)}(t-t_0)\, dt = (-1)^n x^{n}(t)\Big|_{t_0} \qquad t_0 \in]t_1, t_2[.$$

Induction method:

1) prove true for $k=1$.

2) Assume true for $k = n-1$.

3) prove true for $k = n$.

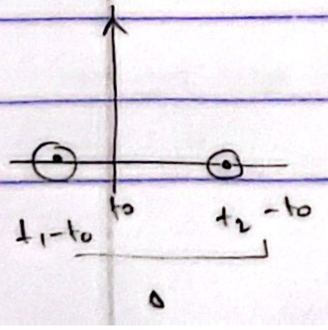
$n = 1$: $\Rightarrow$ true $\forall k$.

$$\int_{t_1}^{t_2} x(t)\, \delta'(t-t_0)\, dt' = (-1)' x'(t_0).$$

$$\int_{t_1}^{t_2} \frac{d}{dt}\left\{x(t)\, \delta(t-t_0)\right\} dt = \int_{t_1}^{t_2} x'(t)\, \delta(t-t_0)\, dt + \int_{t_1}^{t_2} x(t)\, \delta'(t-t_0)\, dt$$

$\delta(t_2-t_0) - \delta(t_1-t_0)$. $\quad x(t_0) \int_{t_1}^{t_2} \frac{d}{dt}\delta(t-t_0)\, dt = \delta(t_2-t_0) - \delta(t_1-t_0)$

$= 0$



$$\int_{t_1}^{t_2} x'(t)\,\delta(t-t_0)\,dt + \int_{t_1}^{t_2} x(t)\,\delta'(t-t_0)\,dt = 0.$$

$$\int_{t_1}^{t_2} x(t)\,\delta'(t-t_0)\,dt = -\int_{t_1}^{t_2} x'(t)\,\delta(t-t_0)\,dt.$$

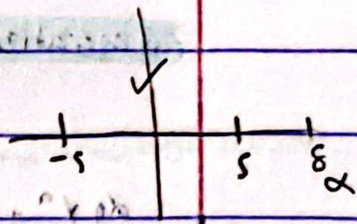
$$= -x'(t_0) \checkmark$$

(2) Assume that for $(n-1)$.

$$\int_{t_1}^{t_2} x(t)\,\delta^{(n-1)}(t-t_0)\,dt = (-1)^{n-1}\,x^{(n-1)}(t_0)$$

Example :- will the first dir.

- $\int_{-5}^{5} e^{-t} \cos(2t)\ \delta^{(1)}(t-8)\, dt = 0.$



- $\int_{-5}^{5} e^{-t} \cos(2t)\ \delta'(t-3)\, dt.$

$(-1)^1 \left(e^{-t} \cos(2t)\right)'$

$-\left[-e^{-t}\cos(2t) - 2\sin(2t)\right]$

$= e^{-t}\cos(2t) + 2e^{-t}\sin(2t) \Big|_3$

$= \underline{\underline{e^{-3}\cos(6) + 2e^{-3}\sin(6)}}$ rad

- $\int_{-10}^{10} e^{-2t^2} \sin\left(3t + \frac{\pi}{4}\right)\ \delta''(t)\, dt.$

$(-1)^2 \frac{d^2}{dt^2} x(t) \Big|_{t=0} \Rightarrow +1\, x''(t).$

- $x(t) = e^{-2t^2} \sin\left(3t + \frac{\pi}{4}\right).$

$x'(t) = -4t e^{-2t^2} \sin\left(3t + \frac{\pi}{4}\right) + 3e^{-2t^2} \cos\left(3t + \frac{\pi}{4}\right).$

$x''(t) = -4e^{-2t^2} \sin\left(3t + \frac{\pi}{4}\right) + \cancel{16te^{-2t^2} \sin\left(3t + \frac{\pi}{4}\right)} - \cancel{12te^{-2t^2} \cos\left(3t + \frac{\pi}{4}\right)} +$

$- \cancel{12te^{-2t^2} \cos\left(3t + \frac{\pi}{4}\right)} - 9e^{-2t^2} \sin\left(3t + \frac{\pi}{4}\right) \Big|_0$

$\boxed{\begin{array}{l} \cos(0) = 1 \\ \sin(0) = 0 \end{array}}$

$-4\sin\left(\frac{\pi}{4}\right) - 9\sin\left(\frac{\pi}{4}\right) = -13\sin\left(\frac{\pi}{4}\right)$

$= -9.19.$

Generalized identity of polynomials of singularity signals.

- $\alpha_n x^n + \alpha_{n-1} x^{n-1} + \dots + \alpha_1 x + \alpha_0 =$
  $\beta_n x^n + \beta_{n-1} x^{n-1} + \dots + \beta_1 x + \beta_0$
  $\alpha_k = \beta_k \quad \forall k.$

- $\alpha_{-k} u_{-k} + \alpha_{-k+1} u_{-k+1} + \dots + \alpha_{-3} u_{-3} + \alpha_{-2} u_{-2} + \alpha_{-1} u_{-1} + \alpha_0 u_0 =$
  $\beta_{-k} u_{-k} + \beta_{-k+1} u_{-k+1} + \dots + \beta_{-3} u_{-3} + \beta_{-2} u_{-2} + \beta_{-1} u_{-1} + \beta_0 u_0$
  $\alpha_r = \beta_r \quad \forall n.$

ex:- $5u(t) + A\delta(t) + 4r(t) + \frac{1}{2}p(t) = \alpha u(t) + 9\delta(t) + \beta r(t) + \gamma p(t)$

$\alpha = 5$

$A = 9$

$\beta = 4$

$\gamma = \frac{1}{2}$

• the signal can be power or energy at the time.

• ممكن ان لاتكون Energy or Power

• they are pos fun.

• الاعتماد على القيمة وليس على ال phase

Energy and power signal.

(*) x(t) is said to be an energy signal with energy E ⇔

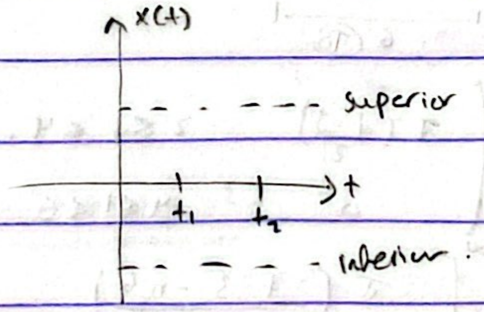
$$E = \lim_{T \to \infty} \int_{-T}^{T} |x(t)|^2 \, dt < \infty \Rightarrow P_{av} = 0.$$

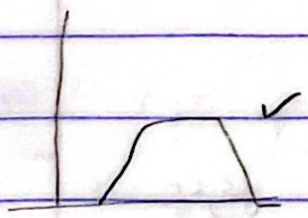
(*) x(t) is said to be a power signal with average power P_{av} ⇔

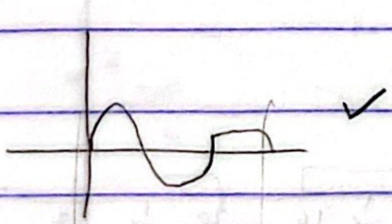
$$P_{av} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt < \infty \Rightarrow E \to \infty.$$

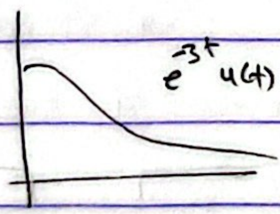
(*) x(t) may be neither power nor energy.

Theorem I) if x(t) is time limited and bounded then x(t) is an energy signal.




✓


✓


$e^{-3t}u(t)$

• not limited. so we can't judge.

= اذا افتقد الشرط لا يلغي النوع ولكن نحكم حسب التعريف لان النظرية فشلت

لأنها if وليست if & only if

theorem II) If $x(t)$ is periodic and bounded in the period then $x(t)$ is a power signal with average power:

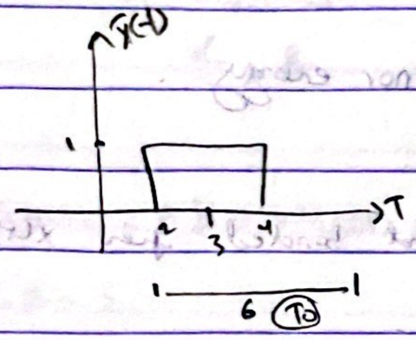
$$P_{av} = \frac{1}{T_0}\left[\int_{T_0} |x(t)|^2 dt\right] \quad T_0: \text{periode time}$$

$\Rightarrow$ energy.

$\Rightarrow \dfrac{\text{energy in one period}}{\text{period}}$

$\bar{x}(t)$ period of a periodic signal

$$x(t) = \sum_{n=0}^{\infty} \bar{x}(t - nT_0)$$

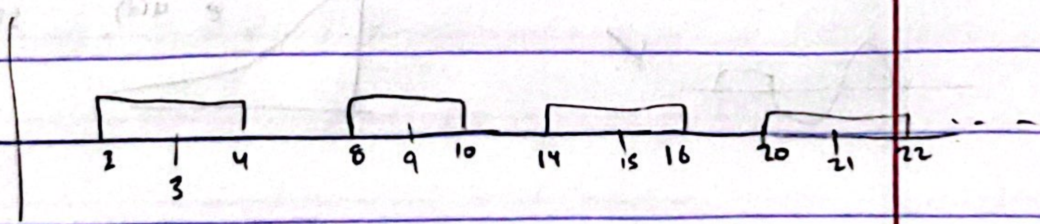


. time limited.

$$\bar{x}(t) \Rightarrow \begin{cases} \Pi\left(\frac{t-3}{2}\right) & 2 \le t < 4 \\ 0 & 4 \le t < 6 \end{cases}$$

$$x(t) = \sum_{n=-\infty}^{\infty} \Pi\left(\frac{t-3-n6}{2}\right)$$

المركز.



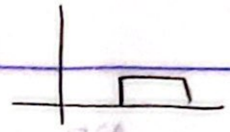
$n=0, \quad \frac{t-3}{2}$

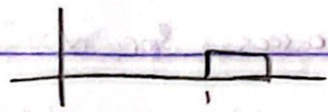
$n=1, \quad \frac{t-9}{2}$

$n=2, \quad t-15$

$n=3, \quad t-21$

دي عبارة عن :-





عبارة عن shift لنفس الـ energy sig.

ومجموع كل هذه الـ en.sig يكون عبارة عن pow.sig.

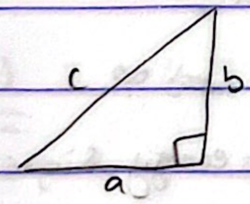
→ N - periodic → $N E_0$.

1-period	2-period	3-period.
$\frac{E_0}{T_0}$	$\frac{2E_0}{2T_0}$	$\frac{N E_0}{N T_0}$

$$\therefore P_{av} = \lim_{N\to\infty} \frac{N E_0}{N T_0} = \frac{E_0}{T_0}.$$

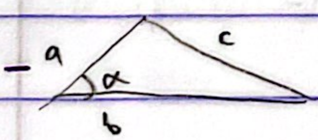
$$= \frac{1}{T_0} \int_{T_0} |x(t)|^2 dt.$$

note :-

—  $c^2 = a^2 + b^2$.

spichal case from

because $\alpha = 90°$.

—  $\Rightarrow c^2 = a^2 + b^2 - 2ab\cos(\alpha)$.

— $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{b}$

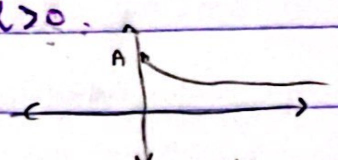
$= |a||b| \cos\theta$.

يكون الجتب orthogonal إذا كان حاصل الضرب ↑ = 0

Example:- $x(t) = Ae^{\alpha t}u(t)$, $A>0$.

we have three casses for α

1) $\alpha<0$, 2) $\alpha=0$, 3) $\alpha>0$.



1) $\boxed{\alpha<0}$ → $-|\alpha|$

$$\lim_{T\to\infty}\int_0^T |Ae^{\alpha t}|^2\,dt.$$

فقط الجزء الموجب لأنه معرّف بـ u(t)

$$\lim_{T\to\infty}\int_0^T A^2 e^{2\alpha t}\,dt.$$

$$A^2\lim_{T\to\infty}\int_0^T e^{2\alpha t}\,dt. \Rightarrow A^2\lim_{T\to\infty} \frac{e^{2\alpha t}}{2\alpha}\Big|_0^T$$

$$(A^2)\frac{-1}{2\alpha} \Rightarrow \frac{-A^2}{-2|\alpha|}$$

So it's energy signal. $\Leftarrow$ $= \frac{A^2}{2|\alpha|} < \infty$

2) $\boxed{\alpha=0}$ $A^2\lim_{T\to\infty}\int_0^T e^{2\alpha t}\,dt = A^2\lim_{T\to\infty}\int_0^T dt.$

$= A^2 \lim_{T\to\infty} T \Rightarrow \infty$

So it's not energy signal.

→ now we should check if it's power:-

$$\lim_{T\to\infty}\frac{TA^2}{2T} = \frac{A^2}{2} < \infty$$

So it's power one.

3) $\boxed{\alpha > 0}$.

عشان لما كانت $\alpha < 0$ كانت eng وهنا لما $\alpha = 0$ كانت pow

لأنه هنا لما اجت $\alpha > 0$ ما استخدم pow او لا شي .

$$P_{av} = A^2 \lim_{T \to \infty} \frac{1}{2T} \int_0^T e^{2\alpha t}\, dt .$$

$$= A^2 \lim_{T \to \infty} \frac{e^{2\alpha t}}{2\alpha} \cdot \frac{1}{2T} \Big|_0^T$$ $\Rightarrow \infty$ e^{αt} ازي لما T $= \infty$.

so it's neither energy nor power .

__theorem III)__ if $x(t) = \sum_{i=1}^{N} x_i(t)$ is an energy signal composed of orthogonal energy signal $x_i(t)$, $i = 1 \ldots N$

then $E_{x(t)} = \sum_{i=1}^{N} E_i$.

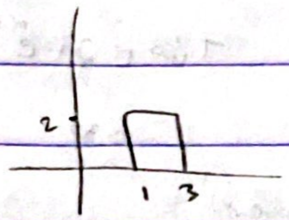
__theorem IV)__ // $\to$ power

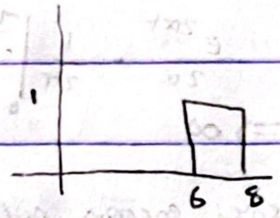
same as above but $\to$ $P_{av\, x(t)} = \sum_{i=1}^{N} P_{av\, i}$.

orthogonal signal

Two signals $x_1(t)$, $x_2(t)$ are said to be orthogonal $\Longleftrightarrow$ if and only if

$$\int_{-\infty}^{\infty} x_1(t) \cdot x_2(t)\, dt = 0 .$$



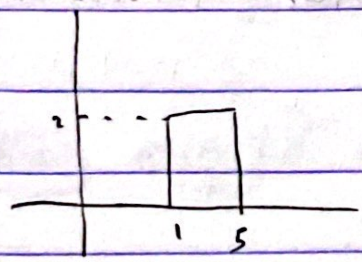


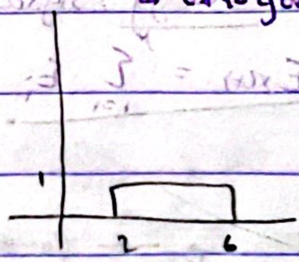
$\int x_1(t)\, (x_2(t))\, dt = 0$

$\therefore E_{T_0} = E_1 + E_2$

- $\cos(4\omega_0 t) + \cos(7\omega_0 t)$ ①, ①
  - $P_{tot} = P_{av1} + P_{av2}$
  - orthogonal

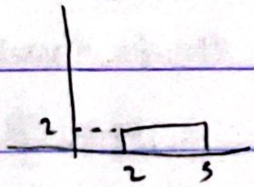
$\Rightarrow$





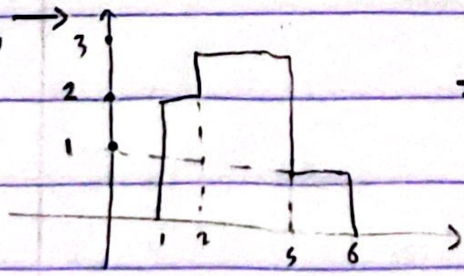
$\int x_1(t)\, x_2(t)\, dt \neq 0$ $\therefore$ not orth.

$x_1(t)\, x_2(t) =$

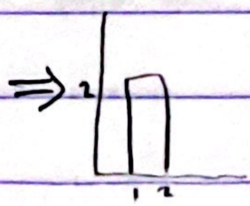


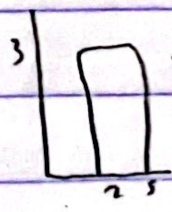
it's $|x_1(t)|^2 + |x_2(t)|^2 + 2|x_1(t)||x_2(t)|$

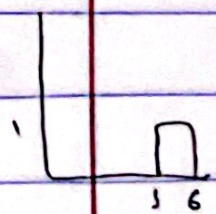
Add the Two signal to each other $\rightarrow$



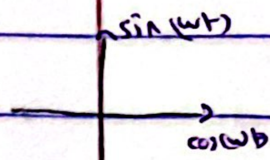
$\Rightarrow$







$E(t) = E_1 + E_2 + E_3$



Note:-

- $\cos\alpha \cos\beta = \frac{1}{2}\cos(\alpha+\beta) + \frac{1}{2}\cos(\alpha-\beta)$
- $\sin\alpha \sin\beta = \frac{1}{2}\cos(\alpha+\beta) - \frac{1}{2}\cos(\alpha-\beta)$.
- $\sin\alpha \cos\beta = \frac{1}{2}\sin(\alpha-\beta) + \frac{1}{2}\sin(\alpha+\beta)$.

Ex:-

1) $\cos(n\omega_0 t)$, $\cos(m\omega_0 t)$ $\rightarrow$ m,n positive integers.
2) $\sin(n\omega_0 t)$, $\sin(m\omega_0 t)$.
3) $\sin(n\omega_0 t)$, $\cos(m\omega_0 t)$ $\longrightarrow$ Homework.

1) $\longrightarrow \int_{T_0} \cos(n\omega_0 t) \cdot \cos(m\omega_0 t)\, dt$.

$$\underbrace{\int_{T_0} \frac{1}{2}\cos[(n+m)\omega_0 t]\, dt}_{=0} + \int_{T_0} \frac{1}{2}\cos[(n-m)\omega_0 t]\, dt .$$

alternating signal/period.

$$\begin{cases} \text{if } n \neq m, & 0 \\ \text{if } n = m & = \frac{1}{2}T_0 \end{cases}$$

2) $\longrightarrow$ Same as the first part but it's $\underline{\underline{-\frac{1}{2}T_0}}$

freq domain

Energy and power spectral density function

- If $x(t)$ is an energy signal with energy E then if
  $\exists\, G(f) > 0$ so that:

$$\int_{-\infty}^{\infty} G(f)\,df = E$$

then $G(f)$ is said to be the spectral energy desity function of $x(t)$

- If $x(t)$ is a power signal with average power P_{av} then if
  $\exists\, S(f) > 0$ so that:

$$\int_{-\infty}^{\infty} S(f)\,df = P_{av}$$

then $S(f)$ is said to be the power ~~density~~ spectral energy function of $x(t)$.

Ex:- $x(t) = A\cos(\omega_0 t + \phi)$.

$$P_{av} = \frac{1}{T_0}\int_{T_0} |x(t)|^2 \, dt.$$

$$= \frac{1}{T_0}\int_{T_0} A^2 \cos^2(\omega_0 t + \phi)\, dt$$

$$= \frac{A^2}{T_0}\int_{T_0} \frac{1 + \cos(2\omega_0 t)}{2}\, dt.$$

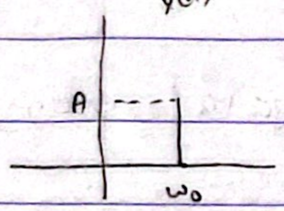
alternating.

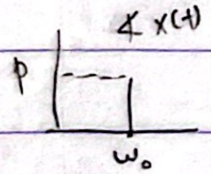
= integer number of period.

$$\frac{A^2}{T_0}\int_{T_0} \frac{1}{2}\, dt = \frac{A^2}{T_0}\,\frac{1}{2}\,T_0$$

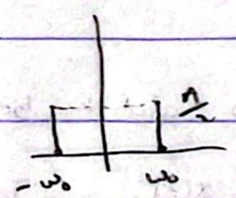
$$= \frac{A^2}{2}$$

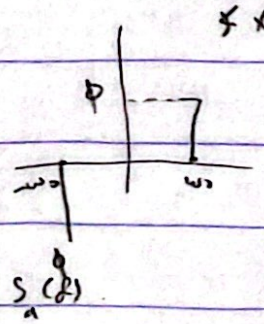
Single





double.





$\Rightarrow S(f)$:-

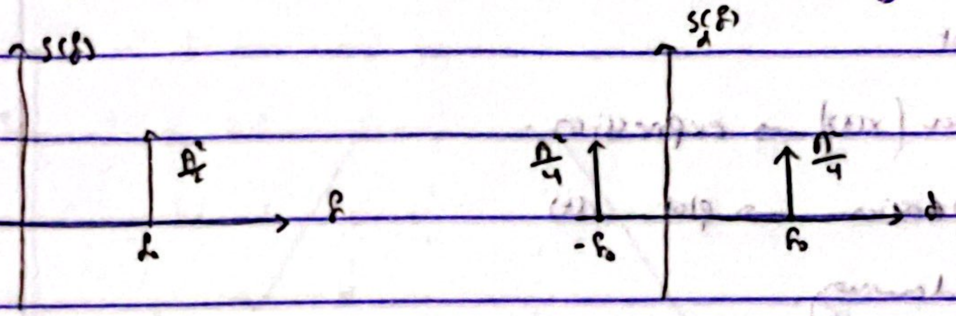
$$\int_{-\infty}^{\infty} S(f)\, df = \frac{A^2}{2}$$

$$S(f) = \frac{A^2}{2}\,\delta(f - f_0)$$

$\Rightarrow S_d(f)$

$$S_d(f) = \frac{A^2}{4}\,\delta(f - f_0) + \frac{A^2}{4}\,\delta(f - f_0)$$

plot the signal sided power spectral density function.



Example: $x(t) = 10\cos(10\pi t + \frac{\pi}{3}) + 15\sin(40\pi t + \frac{\pi}{3})$.

Signal expression using elementary signals.

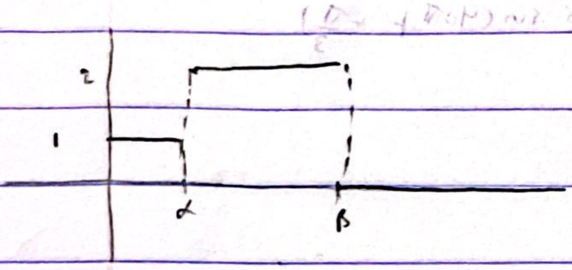
plot

given (x(t)) → expression.

expression → plot x(t).

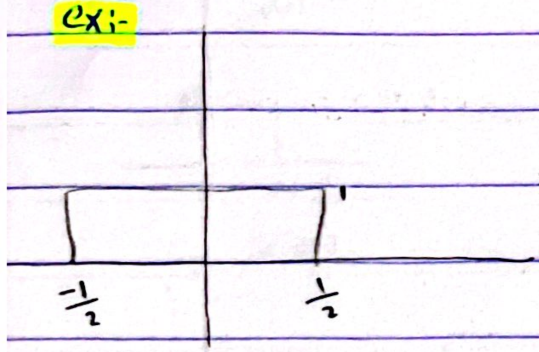
1) windowing

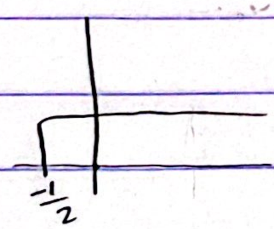
2) combination

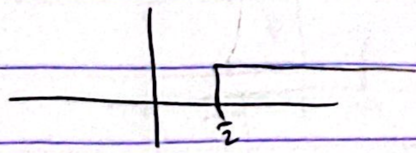


$u(t) + u(t-\alpha) - 2u(t-\beta)$.

ex:-



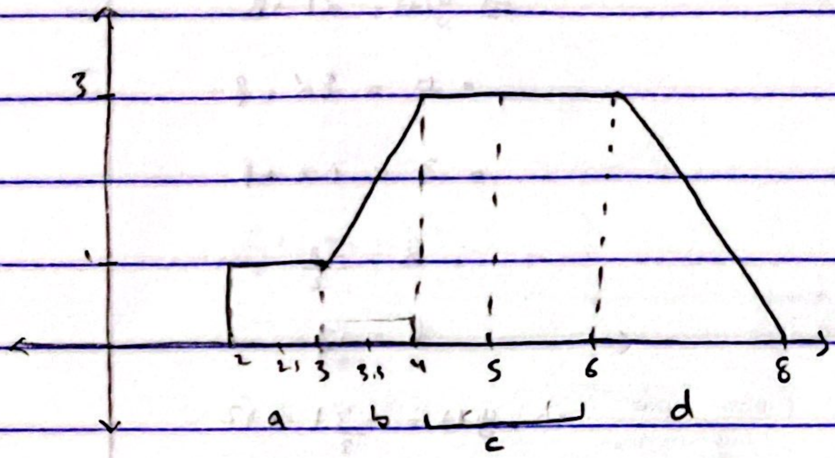
⇒  $u(t+\frac{1}{2})$.

 $u(t-\frac{1}{2})$

so it's ⇒ $u(t+\frac{1}{2}) - u(t-\frac{1}{2})$ ✓ and.

⇒ $u(t+\frac{1}{2}) + u(\bar{t}+\frac{1}{2})$.

Ex: (b)



Windowing

- $a(t) = \Pi(t - 2.5)$
- $b(t) = 2r(t - \frac{5}{2}) \cdot \Pi(t - \frac{7}{2})$

تم الضرب بها من أجل حذف الجزء

الاضافة حسب الرغبة

- $c(t) = 3\,\Pi(\frac{t-5}{2})$
- $d(t) = \frac{3}{2} r(-t+8) \cdot \Pi(\frac{t-7}{2})$

SO $x(t) = a(t) + b(t) + c(t) + d(t)$

(b)

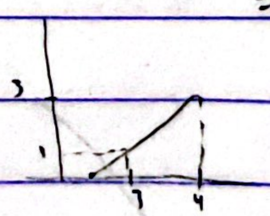
$\Rightarrow y(t) = \alpha t + \beta$

- $1 = 3\alpha + \beta$
- $3 = 4\alpha + \beta$

$\therefore \alpha = 2 , \beta = -5$

$\Downarrow \; y(t) = 2t - 5 = 0$

$\therefore t = \frac{5}{2}$



$\Rightarrow 2t - 5 , 2(t - \frac{5}{2})$.

$2r(t - \frac{5}{2})$. معادلة الخط كامل

(d)

$\Rightarrow y(t) = \alpha t + \beta$

$\cdot\ 0 = 8\alpha + \beta$

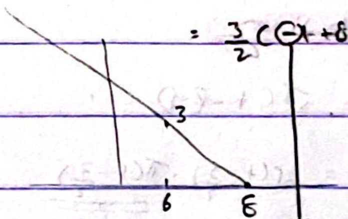
$\cdot\ 3 = 6\alpha + \beta$

$\therefore \alpha = \frac{-3}{2}$

$\beta = 12$

$\cdot\ y(t) = -\frac{3}{2}t + 12$

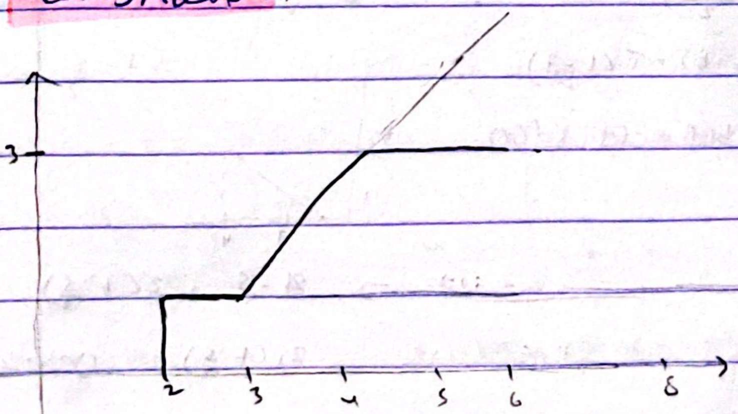
$= \frac{3}{2}(-t+8) = \frac{3}{2}(-(t-8))$

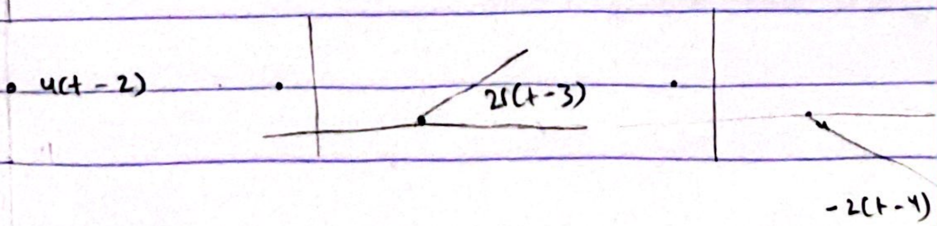


إذا كان السالب ما يكون ال Rn معكوس بـ $\underline{y}$

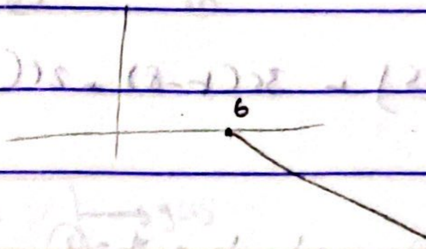
إما إذا كان خارج القوس بيعكس ال $\underline{x}$

Combination.

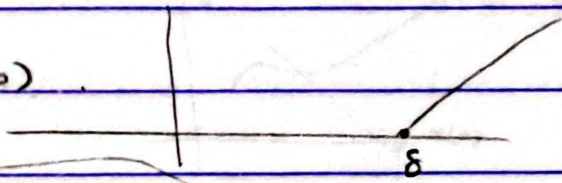




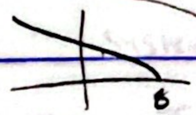
$-\frac{3}{2} r(t-6)$



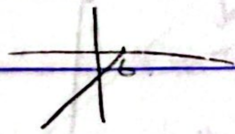
$\frac{3}{2} r(+t-8)$



$\frac{3}{2} r(-t-8)$



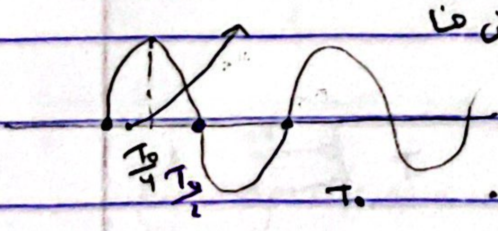
$-\frac{3}{2} r(-t-6)$



$\Rightarrow x(t) = u(t-2) + 2r(t-3) - 2r(t-4) - \frac{3}{2} r(t-6) + \frac{3}{2} r(t-8)$

$\Rightarrow$

نشكله انتصف هنا



عبارة عن $\frac{T_0}{8}$

$\frac{T_0}{4}$ $\frac{T_0}{2}$ T_0

$\frac{2.5}{2} = 1.25$

$\Pi\left(\frac{t-1.25}{2.5}\right)$ مقدار ربع الدورة.

$\Pi\left(\frac{t-1.25-10}{2.5}\right)$ نفس الوضع بالدورة الي بعدها

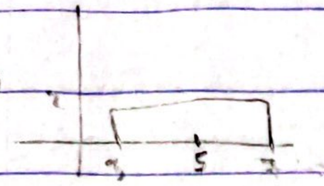
$\Pi\left(\frac{t-1.25-20}{2.5}\right)$ بعد دورتين

$\Rightarrow \sum_{n=0}^{\infty} \sin(4)\cdot \Pi\left(\frac{t-1.25-n10}{2.5}\right)$

Ex:- $2\Pi\left(\frac{t-5}{4}\right) + 3r(t-8) + 2r(-t+4)$.

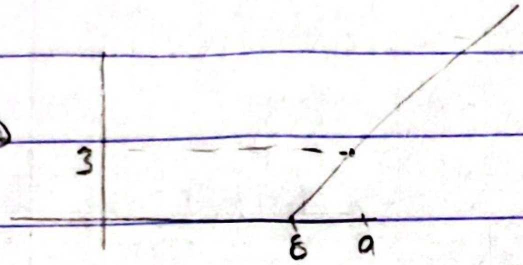
① ② ③

①

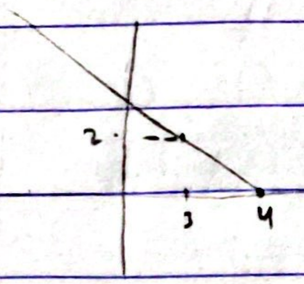


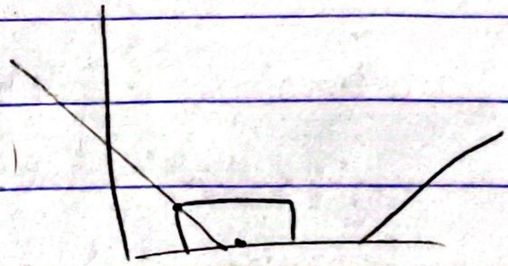
$\frac{1}{2} \cdot \frac{1}{\frac{1}{4}} = \frac{4}{2} = ②$

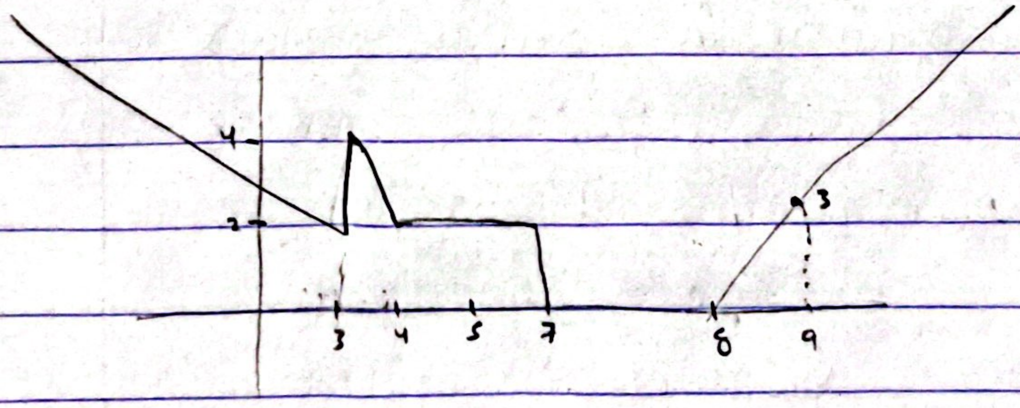
②



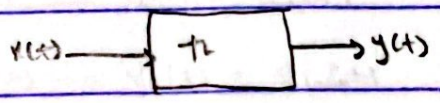
③

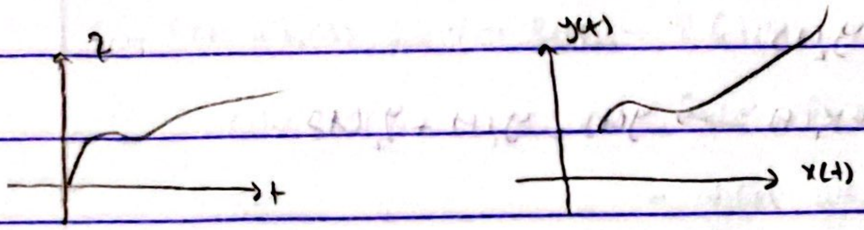






chapter(2):- Systems:



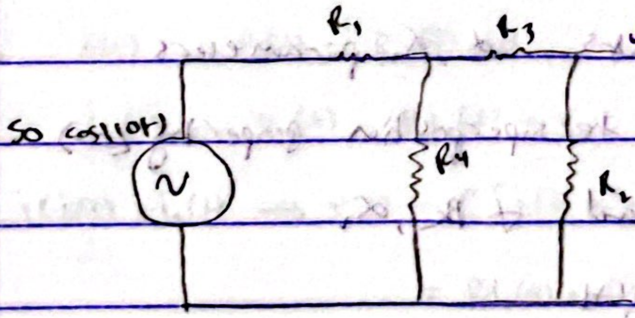


Signal. System.

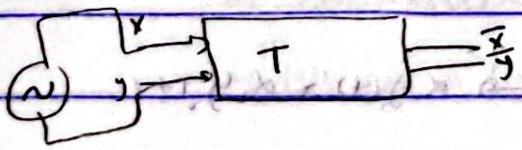
$y(t) = T[x(t)]$

Type of system models :-

1) linear / nonlinear
2) static / Dynamic.
3) time-invarient / time-varient.
4) casual / not casual



linear



Super position:

① Additivity.

$x_1(t) \to y_1(t)$.

$x_2(t) \to y_2(t)$

$x(t) = x_1(t) + x_2(t) \to y(t) = y_1(t) + y_2(t)$.

② proportionality.

$x(t) \to y(t)$

$k\,x(t) \to k\,y(t)$ $\forall k$.

① + ② $\Leftrightarrow$ superposition.

:يجب ان يكون linear من اجل أن يتحقق

def:- $\forall\, x_1(t), x_2(t)$: inputs, α_1, α_2 parameters

the system satisfies the superposition property $\Leftrightarrow$

$\forall\, x_1(t), x_2(t)$ and $\forall\, \alpha_1, \alpha_2$

$x_1(t) \to y_1(t)$

$x_2(t) \to y_2(t)$.

$x(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t) \to \alpha_1 y_1(t) + \alpha_2 y_2(t)$

Ex:- 1) $v(t) = Ri(t)$.

• $i_1(t) \rightarrow v_1(t) = Ri_1(t)$.

$i_2(t) \rightarrow v_2(t) = Ri_2(t)$

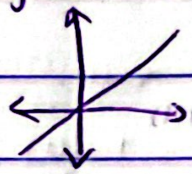
$i = i_1(t) + i_2(t) \rightarrow v = Ri(t) = R(i_1(t) + i_2(t))$

$= Ri_1(t) + Ri_2(t)$

$= v_1(t) + v_2(t)$.

• $i(t) \rightarrow v(t) = Ri(t)$. • $y(t) = \alpha x(t)$.

$\alpha i(t) \rightarrow v(t) = R[\alpha i(t)]$



Linear.

$= \alpha R i(t)$

$= \alpha v(t)$

So it's a linear model.

2) $v(t) = Ri(t) + \alpha$.

• $i_1(t) \rightarrow v_1(t) = Ri_1(t) + \alpha$.

$i_2(t) \rightarrow v_2(t) = Ri_2(t) + \alpha$.

$i(i_1(t) + i_2(t)) \rightarrow v(t) = R[i_1(t) + i_2(t)] + \alpha$

$= Ri_1(t) + Ri_2(t) \boxed{+\alpha}$. $\neq$

$\Rightarrow v_1(t) + v_2(t) = Ri_1(t) + Ri_2(t) + \boxed{2\alpha}$

so it's not a linear model

3) $y = x^2(t)$

- $x_1(t) \longrightarrow y_1(t) = x_1^2(t)$

$x_2(t) \longrightarrow y_2(t) = x_2^2(t)$

$x = x_1(t) + x_2(t) \longrightarrow y(t) = (x_1(t) + x_2(t))^2$

$= \underline{x_1^2(t) + x_2^2(t) + 2x_1(t)x_2(t)}$

$\Rightarrow y_1(t) + y_2(t) = \underline{x_1^2(t) + x_2^2(t)} \quad \neq$

so it's not a linear model.

4) $y = a^{\ln x(t)}$

- $x_1(t) \longrightarrow y_1(t) = a^{\ln x_1(t)}$

$x_2(t) \longrightarrow y_2(t) = a^{\ln x_2(t)}$

$x = x_1(t) + x_2(t) \longrightarrow y(t) = a^{\ln[x_1(t) + x_2(t)]}$

$= a^{(\ln x_1(t) + \ln(x_2(t))}$

$= \underline{a^{\ln x_1(t)} a^{\ln x_2(t)}}$

$y(t) = y_1(t) + y_2(t) \quad \neq$

$= \underline{a^{\ln x_1(t)} + a^{\ln x_2(t)}}$

so it's not a linear model.

Static / Dynamic.

↓

Instantaneous

↓

$y(t) = T[x(t)]$.

↓

Algebraic equation.

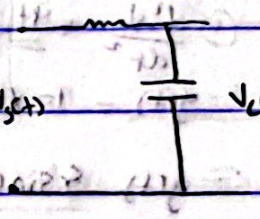
$$I(t) = I_0\left(e^{\frac{aV_D(t)}{KT}} - 1\right) \quad , \quad V(t) = R\,i(t)$$

↓ Algebric , so it's static .

Dynamic :-

↓

$$\frac{d}{dt} V_c(t) + \frac{1}{RC} V_c(t) = \frac{V_s(t)}{RC}$$



KVL

$$-V_s(t) + R\,i_c(t) + V_c(t) = 0$$

$$V_s(t) = R.\,C\frac{d}{dt}V_c(t) + V_c(t) = 0$$

$$\frac{V_s(t)}{RC} = \frac{d}{dt}V_c(t) + \frac{V_c(t)}{RC} \quad , \quad V_c(0) = V_0 .$$

- $\underline{\underline{\frac{dy^2}{dt}}} + 5y(t) = 7x(t)$. • first order differnical equation.
  • nonlinear.
- $\frac{dy}{dt} + 5y(t) + \underline{\underline{7}} = 8x(t)$.
  it's $7y^0$ so nonlinear . يجب انا يكونو كلهم نفس القوة
- $\frac{dy}{dt} + 5y(t) = 8x(t) + 7$, linear

Dynamic linear system:- يحتوي على مشتقات

linear diff. equation:

$$\frac{dy}{dt} + 4\sin(t)\,y(t) = 10\,x(t)$$

Dynamic linear time-variant system → linear with variable cofficient.

$$\frac{dy}{dt} + 5y(t) = 10x(t)$$

Dynamic linear time-invariant system → linear with constant cofficient.

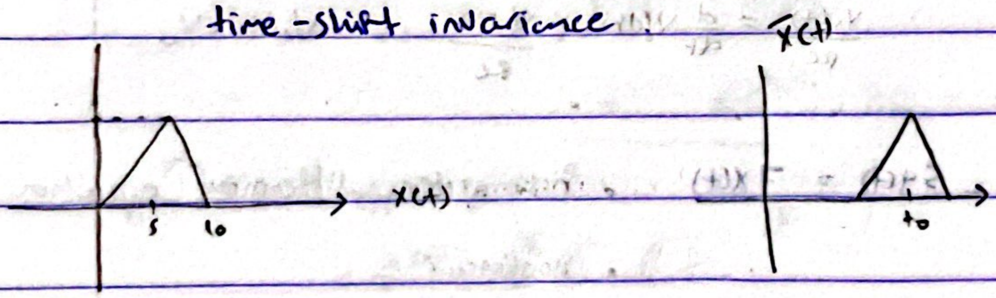
① $\frac{d^2y}{dt^2} + 4t\frac{dy}{dt}$ ① $y(t) = x(t)$ D. linear time-var system

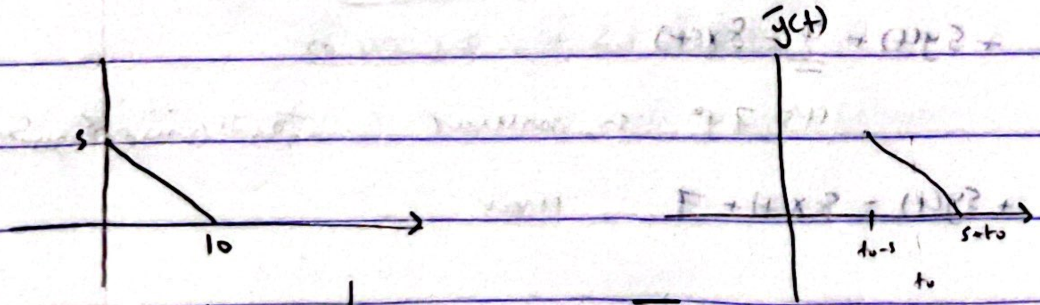
→ $y(t) = 10\,x(t)$ static time-invariant.

$y(t)$ $5\sin(t)$ $x(t)$ static time-variant.

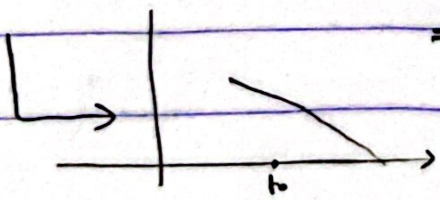
time-invariance

time-shift invariance:





shifted by t_0



= same as each other.

- بناخذ ال function متوامو ونوجد y(t) وبعدين بنعمل shift لل y(t) بمقدار معين
بعدين بناخذ نفس ال function ونقارن بين المقدارين
but already shifted by the same amount

- $y(t) = x(2t) \longrightarrow$ output shift, $y_{sh}(t) = \underline{x(2t - t_0)}$

$\tilde{x}(t) = x(t - t_0) \rightarrow \tilde{y}(t) = \tilde{x}(2t) = x(2(t - t_0))$ $\neq$

$= \underline{x(2t - 2t_0)}$

so it's time-variant.

- $y(t) = x^2(t) \longrightarrow$ output shift, $y_{sh}(t) = \underline{x^2(t - t_0)}$

$\tilde{x}(t) = x(t - t_0) \rightarrow \tilde{y}(t) = \tilde{x}(t) = \underline{x^2(t - t_0)}$ $=$

so it's time-invariant.

- $y(t) = x^2(\sqrt{t}) \longrightarrow y_{sh}(t) = \underline{x^2(\sqrt{t} - t_0)}$

$t = t - t_0 \rightarrow \tilde{y}(t) = \underline{x^2(\sqrt{t - t_0})}$ $\neq$

so it's time-invariant.

Causality

$$x(t) \longrightarrow \boxed{T} \longrightarrow y(t)$$

- T is said to be causal $\Leftrightarrow$ $\forall x_1(t), x_2(t)$ & $\forall t_0$.

  $x_1(t) = x_2(t) \quad \forall t \leq t_0 \Rightarrow$

  $y_1(t) = y_2(t) \quad \forall t \leq t_0$

- T is causal $\Leftrightarrow$ $\forall x(t)$ & $\forall t_0$

  $x(t) = 0 \quad \forall t \leq t_0 \Rightarrow$

  $y(t) = 0 \quad \forall t \leq t_0$

causaling test.

$$y(t_{resp}) = T(x(t_{exc}))$$

$$t_{exc} \leq t_{resp}.$$

ex:-

- $y(t) = x(\sqrt{t})$

  $\sqrt{t} \leq t$, $\forall t_0$ $\Downarrow$

  $y(\frac{1}{4}) = x(\frac{1}{2})$

  $\frac{1}{2} > \frac{1}{4}$ so it's not causal

- $y(t) = x(t^2)$

  $t^2 \leq t$ & that's wrong.

  so it's not causal

- $y(t) = e^{x(t-5)}$ $\quad t-5 < t$

  it's right $\forall t$

  so its causal.

- $\frac{dy}{dt} + 5y(t) = 10x(t+4)$.

  $t+4 < t$, no

  it's wrong

  so it's not causal.

- $y(t) = x(t-t_0)$, $t_0 \geq 0$.

  $t - t_0 \leq t$, it's right

  so it's causal.

- $y(t) = x(t+t_0)$, $t_0 \geq 0$.

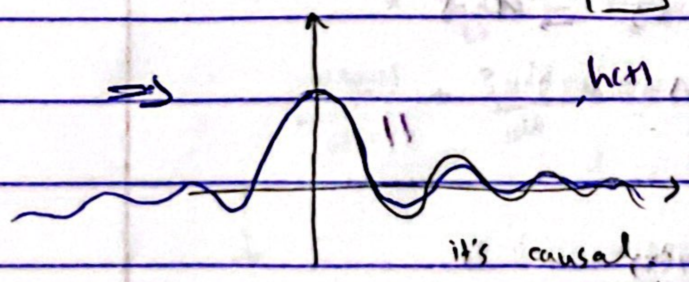
  $t + t_0 \leq t$, it's wrong $\forall t_0 - \{0\}$

  so it's not causal

Theorem: It is necessary and sufficient for a linear time invariant system to be causal that the impulse response $h(t) = 0$ $\forall t < 0$

$\delta(t) \rightarrow \boxed{T} \rightarrow h(t)$

$\Rightarrow$

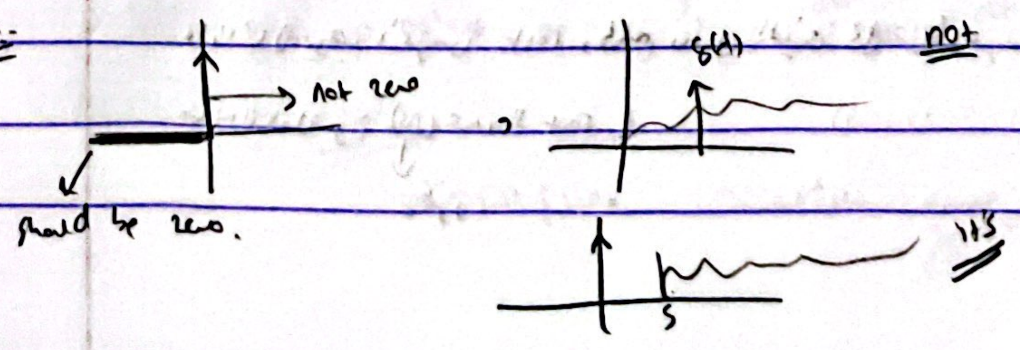


, $h(t)$ impulse response ↑

it's causal.

it's not causal.

note:



• LTI system -

$y(t)$ response to $x(t)$.

responce to $x(t-t_0) \to y(t-t_0)$.

$\to \frac{d\bar{y}(t)}{dt} + 5\bar{y}(t) = 10\,x(t-30)$.

$x(t) \to y(t)$ known $\to \bar{y}(t)$

$\underline{\underline{= y(t-30)}}$.

أي shift لل x بنعكس لل y أيضاً.

Linear time invariant system:

$h(t) \to$ impulse response.

Impulse response: the zero state system responce for

$x(t) = \delta(t)$

$\circ\ \lambda + 5\lambda^0 = 0 \Rightarrow \lambda + 5 \to \lambda = -5$.

$\uparrow\ \frac{dy}{dt} + 5y(t) = 5x(t)$

$\to \frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y(t) = x(t)$

$\lambda^2 + 3\lambda + 2 = 0 \quad (\lambda+2)(\lambda+1) \quad \therefore \lambda = -2, -1$

إذا كان يوجد الجذر root, فلم نفس الجذر كل مرة نضرب بـ t + زيادة

ما عدا أول مرة (على عدد الـ root)

يتكرر

1) characteristic algebric equation → roots ‖
↓
2) $y(t) = y_h(t) + y_p(t)$. eignvalues.

$y_h(t) = Ae^{-t} + Be^{-2t}$ → from $\frac{dy^2}{dt} + \frac{2dy}{dt} + 2y(t) = 0$.

$y(t) = Ae^{-t} + Be^{-2t} + y_p(t)$.

$y_p(t)$ → same type of excitations.

$y_{steady} \to \lim_{t\to\infty} y(t)$. | $\lim_{t\to\infty} y_h(t) = 0$.

then $y_{steady} = y_p(t)$.

$\Rightarrow y(t) = Ae^{t} + Be^{-2t} + y_p(t)$.

$\lim_{t\to\infty} y(t) = \infty$

الأهم من $y_h(t)$ مثل الوصول إلى $y_p(t)$ أسرع.

$\Rightarrow$ $y(t) = y_h(t) + y_p(t)$.

$= y_{zs}(t) + y_{zi}(t)$.

$\frac{d^2y(t)}{dt^2} + 3\frac{dy}{dt} + 2y(t) = x(t)$, $y(0) = y_0$, $y'(0) = y_0'$.

transued.

$\frac{d^2y_{zs}}{dt^2} + \frac{3dy_{zs}}{dt} + 2y_{zs}(t) = x(t)$ + $\frac{d^2y_{zi}}{dt^2} + \frac{3dy_{zi}}{dt} + 2y_{zi} = 0$.

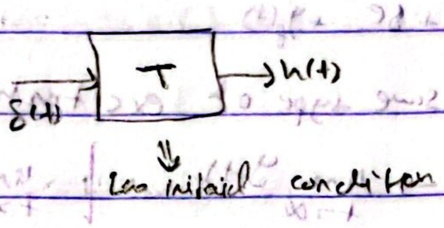
$y(0) = 0$, $y'(0) = 0$ $y(0) = y_0$, $y'(0) = y_0'$

steady

النوع الثاني يتغير حسب التحولات الخاصة في النظم

Impulse response : the zero state response of the system to excitation of the Type $x(t) = \delta(t)$.

$h(t)$?



LTI
- System response to singularity signal. Impulse response.

$$\frac{dy}{dt} + 5y(t) = 10x(t).$$

$$h(t) \Rightarrow \frac{dh(t)}{dt} + 5h(t) = 10\delta(t).$$

① charact algebric equation

$$\lambda + 5 = 0 \quad , \lambda = -5$$

② $g(t) = Ae^{\lambda t}$

- $h(t) = g(t)\,u(t)$.
- $\frac{dh(t)}{dt} = g'(t)u(t) + \underset{g(0)}{\underline{\underline{g(t)}}}\,\delta(t).$

$$\Rightarrow g'(t)u(t) + \underline{\underline{g(0)}}\,\delta(t) + 5g(t)u(t) = \underline{\underline{10}}\,\delta(t).$$

$$\therefore g(0^+) = 10.$$

$$g(0^+) = Ae^{-5(0)} = A \quad \therefore A = 10.$$

$\Rightarrow$ $h(t) = g(t)\,u(t)$

$$= 10e^{-5t}u(t).$$

$$\Rightarrow \frac{dy(t)}{dt} + 5y(t) = 10\,\delta^{u}(t).$$

$$\lambda = -5$$

$$g(t) = Ae^{-5t}$$

$$y(t) = g(t)u(t) + B\,\delta(t).$$

$$\frac{dy(t)}{dt} = g'(t)u(t) + g(0)\delta(t) + B\,\overset{\circ}{\delta}(t).$$

$$\Rightarrow g'(t)u(t) + \underline{g(0)\delta(t)} + \underline{\underline{B\,\delta^{\circ}(t)}} + 5g(t)u(t) + 5B\delta(t)$$

$$= 10\,\underline{\underline{\delta^{\circ}(t)}}.$$

$$\therefore B = 10.$$

$$5B + g(0) = 0$$

$$\therefore g(0) = -5B = -50.$$

$$\Rightarrow g(0^{+}) = -50 = Ae^{0}$$

$$\therefore A = -50$$

$$\Rightarrow \delta(t) \xrightarrow{d/dt} \delta^{\circ}(t), \qquad y(t) = -50e^{-5t}u(t) + 10\delta(t)$$

$$h(t) \xrightarrow{d/dt} \frac{d\,h(t)}{dt}$$

$$\Rightarrow \frac{d}{dt}\left(10e^{-5t}u(t)\right)$$

$$= -50e^{-5t}u(t) + \underline{\underline{10e^{-5t}}}\,\delta(t).$$

$$= -50e^{-5t}u(t) + 10\,\delta(t).$$

Theorem:- Given a LTI system with y(t), the zero state response of the system to x(t), then the zero state responce to $\frac{d\,x(t)}{dt}$ is $\frac{dy(t)}{dt}$ and the zero state response to

$$\int_0^t x(\tau)\,d\tau \text{ is } \int_0^t y(\tau)\,d\tau .$$

Ex:- 1) $\frac{d^2 y(t)}{dt^2} + 3\frac{dy}{dt} + 2y(t) = 7\,\ddot{\delta}(t)$

$\therefore r^2 + 3r + 2 = 0$

$(r+2)(r+1) = 0$

$\therefore r = -2/-1$

$g(t) = Ae^{-t} + Be^{-2t}$.

$2^{\alpha} \Rightarrow y(t) = g(t)u(t) + \beta\,\delta(t)$

$3^{\alpha} \Rightarrow y'(t) = g'(t)u(t) + g(0)\,\delta(t) + \beta\,\dot{\delta}(t)$

$\Rightarrow y''(t) = g''(t)u(t) + g'(0)\,\delta(t) + g(0)\,\dot{\delta}(t) + \beta\,\ddot{\delta}(t)$.

$= 7\,\ddot{\delta} +$

from $\ddot{\delta}(t) \Rightarrow \therefore \beta = 7$.

$\dot{\delta}(t) \Rightarrow \therefore 3\beta + g(0) = 0 \quad \therefore g(0) = -21$.

$\delta(t) \Rightarrow \therefore 2\beta + 3g(0) + g'(0) = 0$

$\therefore g'(0) = -(3)(-21) - 2(7) = 49$

$g(0^+) = -21 = A + C \quad (1)$

$g'(t) = -Ae^{-t} - 2Ce^{-2t}$

$g'(0^+) = -A - 2C \quad (2)$

$\circ \; x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\Rightarrow \quad A + C = -21$

$+ \quad -A - 2C = 49$

$-C = 28 \quad \therefore C = -28 \quad A = 7$

$\therefore y(t) = (7e^{-t} - 28e^{-2t})u(t) + 7\delta(t)$

2) $\frac{d^2y(t)}{dt} + \frac{4\,dy}{dt} + 5y(t) = 7\,\delta^{oo}(t)$

$\lambda^2 + 4\lambda + 5 = 0 \qquad \lambda_{1,2} = -2 \pm j$

$g(t) = e^{\alpha t}\left[A\cos(\omega_d t) + B\sin(\omega_d t)\right] \qquad \alpha \pm \omega j$

$5 \wedge \Rightarrow y(t) = g(t)u(t) + \underline{B}\,\delta(t)$

$4 \wedge \Rightarrow y'(t) = g'(t)u(t) + \underline{\underline{g(0)}}\,\delta(t) + \underline{\underline{B}}\,\delta^{o}(t)$

$\Rightarrow y''(t) = g''(t)u(t) + g'(0)\,\delta(t) + g(0)\,\underline{\underline{\delta^{o}}}(t) + B\,\delta^{oo}(t)$

$= 7\,\delta^{oo}(t)$

$\therefore B = 7 \qquad \underline{\underline{\delta^{oo}(t)}}$

$4B + g(0) = 0 \qquad \underline{\underline{\delta^{o}(t)}}$

$\therefore g(0) = -28$

$g'(0) + 5B + 4g(0) = 0 \qquad \underline{\underline{\delta(t)}} \qquad$ Find A, B :-

$-28 + 35 + 4g(0) = 0$

$\therefore g(0) = -\frac{7}{4}$

3) $\frac{dy^2}{dt^2} + \frac{2dy}{dt} + y(t) = 10\,\delta^{\circ\circ}(t)$.

$\lambda^2 + 2\lambda + 1 = 0$

$(\lambda+1)(\lambda+1) = 0 \quad \therefore \lambda_1 = \lambda_2 = -1$.

$g(t) = Ae^{-t} + Bte^{-t}$

$\Rightarrow y(t) = g(t)u(t) + \beta\,\delta(t)$.

2 $\Rightarrow y'(t) = g'(t)u(t) + g(0)\,\delta(t) + \beta\,\delta^{\circ}(t)$

$\Rightarrow y''(t) = g''(t)u(t) + g'(0)\,\delta(t) + g(0)\,\delta^{\circ}(t) + \beta\,\delta^{\circ\circ}(t)$.

$= 10\,\delta^{\circ\circ}(t)$.

- $\beta = 10$.

$g(0) + 2\beta = 0$

- $\therefore g(0) = -20$

$2g(0) + g'(0) + \beta = 0$.

$-40 + g'(0) + 10$

$\therefore g'(0) = 30$ Find A, B:-

$\Rightarrow g(0) = A = -20$.

$g'(0) = -A + B - 0 = 30$

$20 + B = 30$

$\therefore B = 10$.

Theorem I:- Given a LTI system with impulse response $h(t)$, the zero state response $y(t)$ to any input $x(t)$ can be computed using

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

Theorem II:- Given a LTI system with step response $a(t)$, the zero state response $y(t)$ to any input $x(t)$ can be computed using $y(t) = \int_{-\infty}^{\infty} x'(\tau)\, a(t-\tau)\, d\tau$.

Note:-

$$u(t) \leftarrow \int_{-\infty}^{t} \delta(\tau)\, d\tau$$

$$a(t) \leftarrow \int_{-\infty}^{t} h(\tau)\, d\tau$$

Convolution theorems:- LTI

h(t) "impulse response" for any x(t)

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

zero step response.

Step response ← a(t)

for any x(t) $y(t) = \int_{-\infty}^{\infty} x'(\tau)\, a(t-\tau)\, d\tau$

property of convolution operation:-

1) $x_1(t) \circledast x_2(t) = x_2(t) \circledast x_1(t)$.

2) Distribution over ±

$x_1(t) \circledast [x_2(t) \pm x_3(t)]$

$= x_1(t) \circledast x_2 t \pm x_1(t) \circledast x_3(t)$.

3) if $x_1(t)$ is time limited to $(a, b]$

and $x_2(t)$ " " " $(c, d]$

then $x_1(t) \circledast x_2(t)$ is time limited to $(a+c, b+d]$

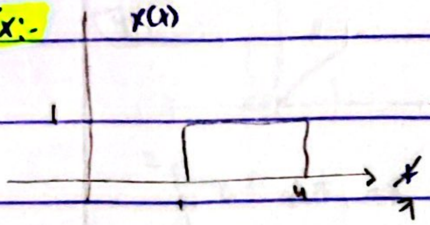
4) A_1 the area under $x_1(t)$

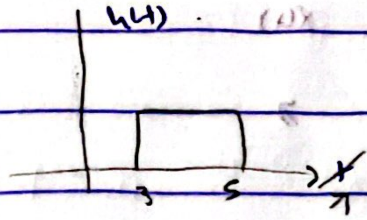
A_2 " " " $x_2(t)$.

then the area A under $x_1(t) \circledast x_2(t)$ is $A_1 A_2$.

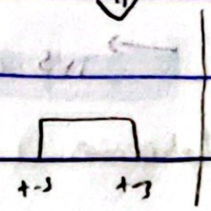
$$\int x(\tau)\,h(t-\tau)\,d\tau = \int h(\tau)\,x(t-\tau)\,d\tau$$

Ex:-



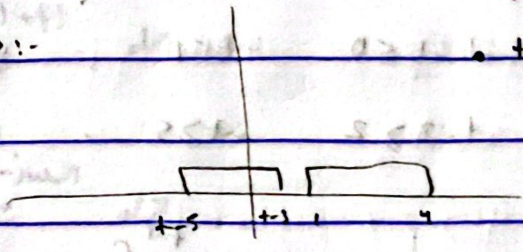


$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$



So:-

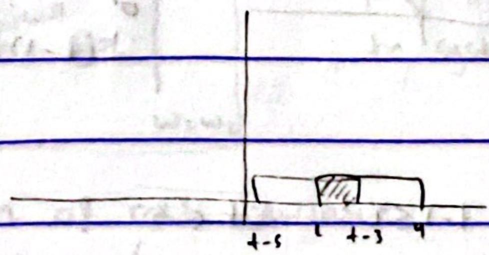
①



$t-3 \le 1 \Rightarrow t \le 4$

$y(t) = 0$

②



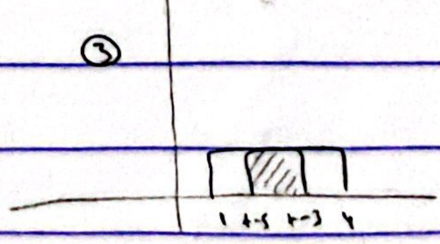
$t-3 \ge 1 \Rightarrow t \ge 4$

$t-5 \le 1 \Rightarrow t \le 6$

$\therefore 4 \le t \le 6$

$$\int_{1}^{t-3} 1\,d\tau = t-3-1 = t-4$$

③



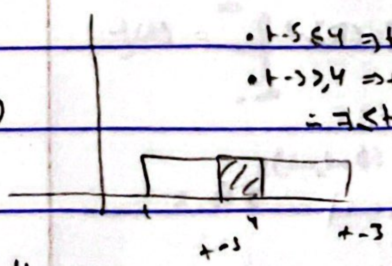
$t-5 \ge 1 \Rightarrow t \ge 6$

$t-3 \le 4 \Rightarrow t \le 7$

$\therefore 6 \le t \le 7$

$$\int_{t-5}^{t-3} 1\,d\tau = t-3-t+5 = 2$$

④



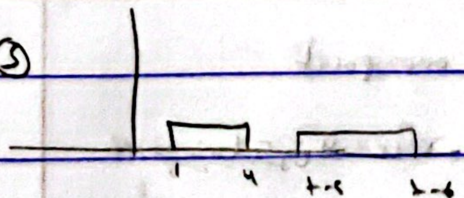
$t-5 \le 4 \Rightarrow t \le 9$

$t-3 \ge 4 \Rightarrow t \ge 7$

$\therefore 7 \le t \le 9$

$$\int_{t-5}^{4} d\tau = 4-t+5 = 9-t$$

⑤

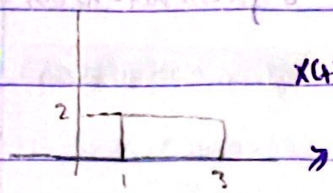


$t-5 > 4$

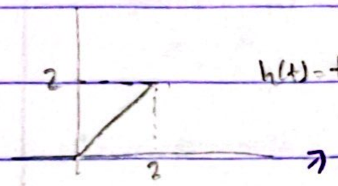
$\therefore t \ge 9 \quad y(t) = 0$

Ex:-

$x(t)$



$h(t) = t$



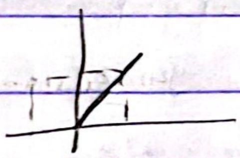
$\longrightarrow$ it's $r(t) - r(t-2) - 2u(t-2)$.

$\Rightarrow \int h(\tau)\, x(t-\tau)\, d\tau$.

[1] $t-1 \leq 0 \quad \therefore t \leq 1 \quad , \quad y(t) = 0$

$t-3 \geq 2 \quad \therefore t \geq 5 \quad , \quad y(t) = 0$.

[2]



$\int_0^{t-1} 2\tau \, d\tau$

$= (t-1)^2$.

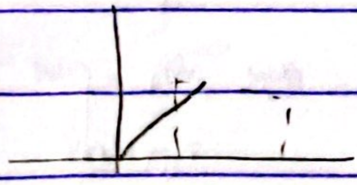
[3] $t-1 \leq 2 \quad \therefore t \leq 3$

$t-3 \geq 0 \quad \therefore t \geq 3$ it's just one point

عندما يكون ال $h(t)$, $x(t)$ متطابقان

$(t-1)^2 \Rightarrow (3-1)^2 = \underline{\underline{4}}$.

[4]



$$\int_{t-3}^{2} 2\tau \, d\tau \Rightarrow 4-(t-3)^2 \;//$$

Sinosidal Steady state response.

$h(t)$, $x(t) = Ae^{j(w_0 t+\phi)}$ sinosoidal rotating φ respon

$y(t) = ?$

$$y(t) = \int_{-\infty}^{\infty} h(\tau) A e^{j(w_0(t-\tau)+\phi)} \, d\tau$$

$$= Ae^{j(w_0 t+\phi)} \int_{-\infty}^{\infty} h(\tau) e^{-jw\tau} \, d\tau$$

$$= Ae^{j(w_0 t+\phi)} \boxed{\left. \int_{-\infty}^{\infty} h(\tau) e^{-jw\tau} \, d\tau \right|_{w=w_0}}$$

H(w) : frequency response of the system

→ H(w) = complex function of real variable.

Fourier transform of h(t). $w \in \mathbb{R}$, $H(w) \in \mathbb{C}$

H(S) = Transfer functions "Laplace transformation of h(t)"

↓ complex function of complex variable.

$s \in \mathbb{C}$, $H(w) \in \mathbb{C}$?!!

Sinosoidal State state.

$x(t) = Ae^{j(\omega_0 t + \phi)}$, $h(t)$

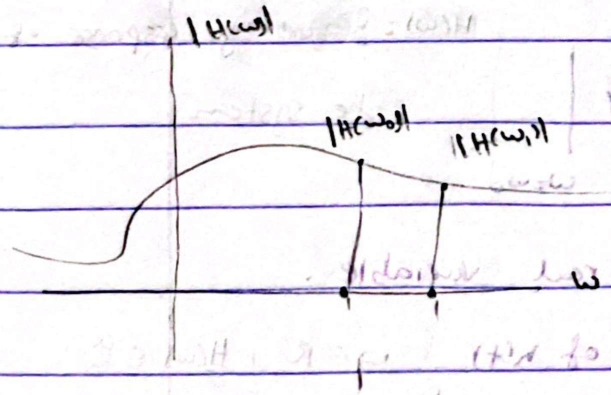
$y(t) = Ae^{j(\omega_0 t + \phi)}$. $H(\omega)\Big|_{\omega_0}$ freq. response spectral of $h(t)$.

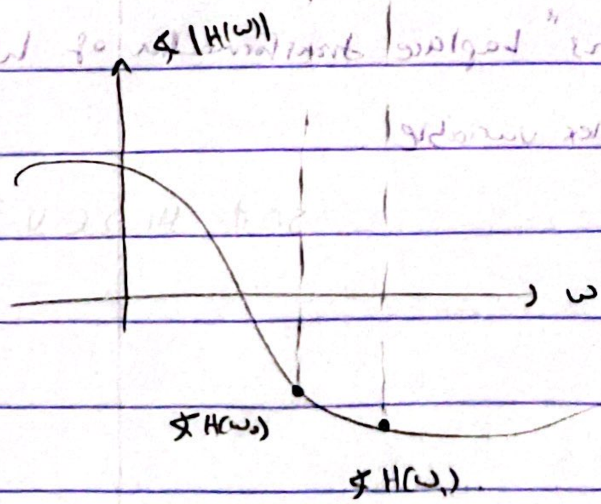
theorems.

Given a linear time invariant system with sinosoidal $x(t) = A\cos(\omega_0 t + \phi)$ and frequency responce $H(\omega)$, the response of the system is sinosoidal with same frequency.

$y(t) = Y\cos(\omega_0 t + \theta)$, $Y = A \cdot |H(\omega)|$

$\theta = \phi + \measuredangle H(\omega)\Big|_{\omega_0}$





Ex:- 1) $h(t) = Ae^{\alpha t} u(t)$ $\underline{\underline{\alpha < 0}}$

$H(\omega) = ?$ $\Rightarrow$ Asyptotic stabilioity.

$$H(\omega) = \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau$$

$$= \int_{0}^{\infty} Ae^{\alpha\tau} e^{-j\omega\tau} d\tau = \int_{0}^{\infty} Ae^{(\alpha - j\omega)\tau} d\tau$$

$\alpha < 0$, $\alpha = -|\alpha|$.

$$= \int_{0}^{\infty} Ae^{-(|\alpha| + j\omega)\tau} d\tau$$

$$\left. \frac{-Ae^{-(|\alpha| + j\omega)\tau}}{-(|\alpha| + j\omega)} \right|_{0}^{\infty}$$

$\lim e^{-(\,)} = 0$

then $\Rightarrow H(\omega) = 0 + \frac{A}{|\alpha| + j\omega}$

. freq. response : $e^{-j\omega(\omega_0)}$

$\therefore H(\omega) = \frac{A}{|\alpha| + j\omega}$

$\underline{\underline{\text{example}}}$: $h(t) = 10e^{-2t} u(t) \Rightarrow H(\omega) = \frac{10}{2 + j\omega}$.

$H(\omega) = \frac{10}{2+j\omega}$

2) Determine the response of the system to

$x(t) = 20\cos(10t + \frac{\pi}{4})$. مقدار زاوية هاي.

$y(t) = Y\cos(10t + \theta)$

$\therefore Y = 20 \cdot \frac{10}{\sqrt{4+\omega^2}} \Big|_{\omega=10} = \frac{200}{\sqrt{104}}$

$\therefore \theta = \frac{\pi}{4} - \tan^{-1}(\frac{\omega}{2}) = \frac{\pi}{4} - \tan^{-1}(\frac{10}{2})$

$= \frac{\pi}{4} - \tan^{-1}(5)$

$\Rightarrow y(t) = \frac{200}{\sqrt{104}} \cos[10t + \frac{\pi}{4} - \tan^{-1}(5)]$

3) $H(\omega) = \frac{10}{\omega^2 - 4 - 3j\omega}$ $\quad H(\omega) = |H(\omega)| \cdot e^{j\measuredangle H(\omega)}$

Determine the system sinosoidal steady state response to :-

$x(t) = 10\cos(10t + \frac{\pi}{4}) + 30\cos(20t + \frac{\pi}{6})$.

$|H(\omega)| = \frac{10}{\sqrt{(\omega^2-4)^2 + (-3\omega)^2}}$

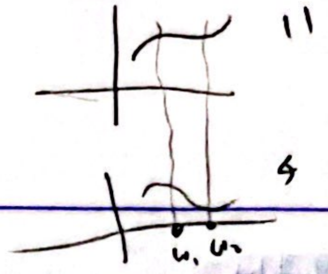
$\measuredangle H(\omega) = 0 - \tan^{-1}(\frac{-3\omega}{\omega^2-4})$. "phase - phase" بسط مقام

"البسط" phase of the const. مقام

$= \tan^{-1}(\frac{3\omega}{\omega^2-4})$.

$|H(10)|$. $\measuredangle H(10)$

$|H(20)|$, $\measuredangle H(20)$.



$\Rightarrow |H(10)| = \dfrac{10}{\sqrt{(10^2-4)^2 + 9(10)^2}}$, $\measuredangle H(10) = \tan^{-1}\left(\dfrac{30}{96}\right)$

$\Rightarrow |H(20)| = \dfrac{10}{\sqrt{(20^2-4)^2 + 9(20)^2}}$, $\measuredangle H(20) = \tan^{-1}\left(\dfrac{60}{396}\right)$

response:-

$\Rightarrow |Y_1| = 10 \cdot |H_{10}|$, $\measuredangle y_1(t) = \frac{\pi}{4} + \measuredangle H_{10}$

$y_1(t) = |Y_1| \cos(10t + \measuredangle y_1(t)) \rightarrow y(t) = y_1(t) + y_2(t)$

$y_2(t) = 30 \cdot |H(20)| \cos\left(20t + \frac{\pi}{6} + \measuredangle(H_{20})\right)$

$y = 10|H(10)| \cos(10t + \measuredangle H(10)) + 30|H(20)| \cos(20t + \measuredangle H(20))$.

Stability

Stability → steady state

transient → 0
↓
h(t)

$x(t) \to \boxed{T} \to y(t)$

$\lim_{t\to\infty} y(t) = y_p(t) = y(t)$ steady

. AS. stability.

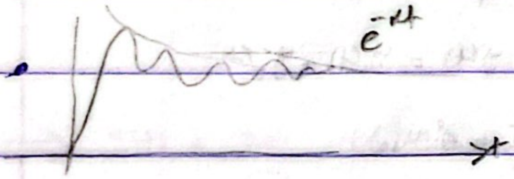
"Strong stability" Asymptotic stability : $\lim_{t\to\infty} h(t) = 0$

- $h(t) = 10e^{-t}u(t)$ ✓
- $h_2(t) = 10e^{t}u(t)$ X

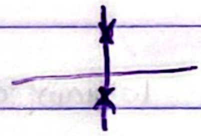
note!-

$\sin(w_0 t) \longleftrightarrow \frac{w_0}{s^2+w_0^2}$

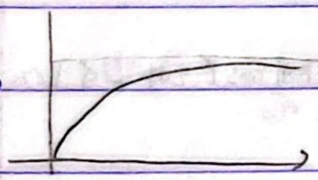
$\cos(w_0 t) \longleftrightarrow \frac{s}{s^2+w_0^2}$

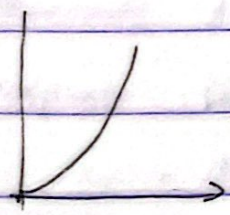
- 

✓

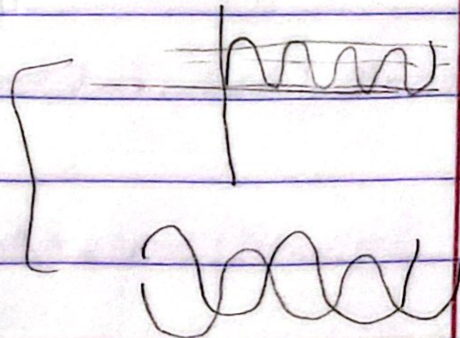


$\frac{\frac{1}{LC}}{s^2+\frac{1}{LC}}$

- 
overdamped ✓

- 
X

bounded



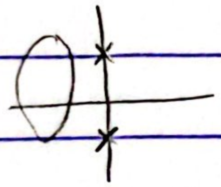
- $\frac{Y(s)}{s} \to \int y(t)\,dt$
- $\frac{1}{s^2}$, $\frac{1}{s}$

$sY(s) = \frac{dy(t)}{dt}$ $u(t) * u(t)$ $u(t)$

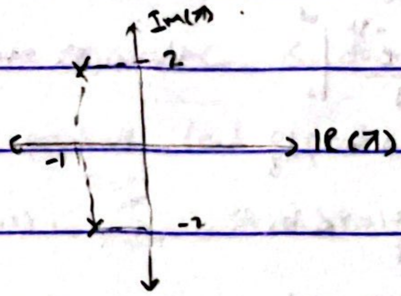
$\delta(t) \leftarrow 1$ $r(t)$

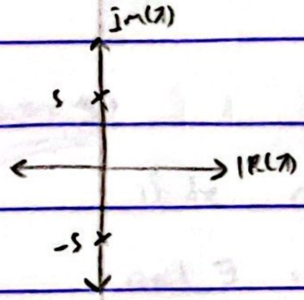
$L(h(t))$

$h(t) \leftrightarrow H(s)$



$Im(z)$ $z_{1,2} = 1 \pm j2$





$z = \mp j5$

↓

BIBO stability

A system is said to be BIBO stable: Bounded input / Bounded output

$\Leftrightarrow \forall x(t) \ \exists N, M$ so that

$$|x(t)| \leq N \Leftrightarrow |y(t)| < M.$$

Theorem I

Given a LTI system with impulse $h(t)$, the system is

BIBO stable $\Longleftrightarrow \int_{-\infty}^{\infty} |h(t)| \, dt < \infty$.

Ex:- $h(t) = 10e^{-3t}u(t)$.

$$\int_0^\infty |10e^{-3t}|\,dt = \int_0^\infty 10e^{-3t}\,dt$$

$$= \left.\frac{10e^{-3t}}{-3}\right|_0^\infty$$

$$= 0 - -\frac{10}{3}$$

$$= \frac{10}{3} < \infty$$

$\Rightarrow$ BIBO sta

and it's Asymp. Sta

$$\lim_{t\to\infty} 10e^{-3t} = 0 \checkmark$$

- Asymptotic $\rightarrow$ BIBO.

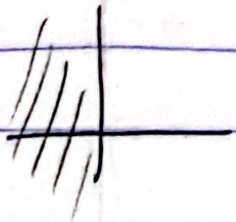
Note:-

characteristic equation →
- ch. Algebric equation of the diff. equation.
- ||
- denomevalue of the transfer eq.

<u>theorem II</u>

given a LTI system with characteristic equation $f(\gamma) = 0$
the system is asymptotically stable $\Leftrightarrow$
all the roots of the characteristic equation have
negative real part.

"roots at the semi-left plane"



theorem III)

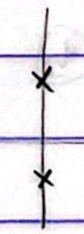
Given a LTI with char. eq. If ∃ λ a root of the char. eq, time eigenvalue, S-domain system pole with a positive real part then the system is unstable.

- $(\lambda+1)(\lambda-1)$ < $\lambda=1$, $\lambda=-1$ so it's not stabel

theorem IV)

if the char. Algebric eq has no-poles at the right-side and ∃ roots poles with $\mathbb{R}(\lambda)=0$

then if the roots poles are not repeted roots poles then the system is BIBO stable.



other wise if ∃ a λ_i with $\mathbb{R}(\lambda_i)=0$ and repeted then the system is unstabel

- $H_1 = (S+4)(S+5+jW)^2(S+5-jW)^2$ stabel
- $H_2 = (S+4)(S+5jW)^2(S-5jW)^2$. not stabel

Modeling and Simulation

Seperate and Integrate "observer representation".

Ex: 1) $\frac{d^4y}{dt^4} + 5\frac{d^3y}{dt^3} + 4\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 7y(t) = 4\frac{d^3x}{dt^3} + 3\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 6x(t)$

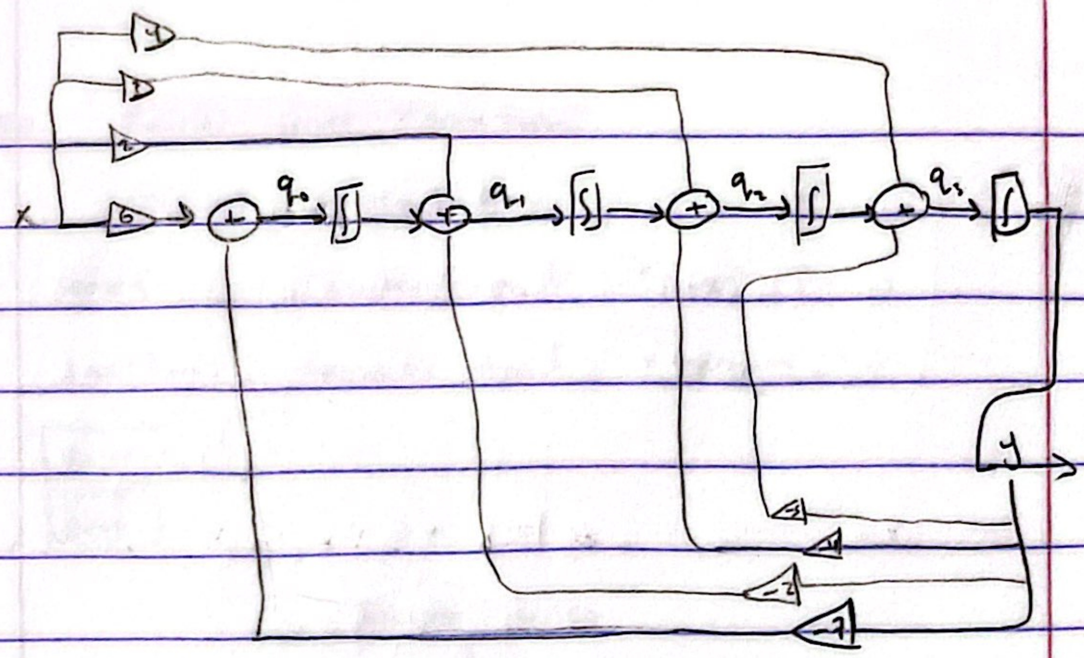
$\rightarrow \frac{d^4y}{dt^4} + 5\frac{d^3y}{dt^3} + 4\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 4\frac{d^3x}{dt^3} - 3\frac{d^2x}{dt^2} - 2\frac{dx}{dt} = \underbrace{6x(t) - 7y(t)}_{q_0}$

I.t.s $\frac{d^3y}{dt^3} + 5\frac{d^2y}{dt^2} + 4\frac{dy}{dt} - 4\frac{d^2x}{dt^2} - 3\frac{dx}{dt} = \underbrace{2x(t) - 2y(t) + \int q_0\,dt}_{q_1}$

I.t.s $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} - 4\frac{dx}{dt} = \underbrace{3x(t) - 4y(t) + \int q_1\,dt}_{q_2}$

I.t.s $\frac{dy}{dt} = \underbrace{4x(t) - 5y(t) + \int q_2}_{q_3}$

$\Rightarrow y(t) = \int q_3$



2) $\frac{d^4y}{dt^4} + \frac{3d^3y}{dt^2} - \frac{5\,dy}{dt} + 2y(t) = 7\frac{d^2x}{dt^2} + 3x(t).$

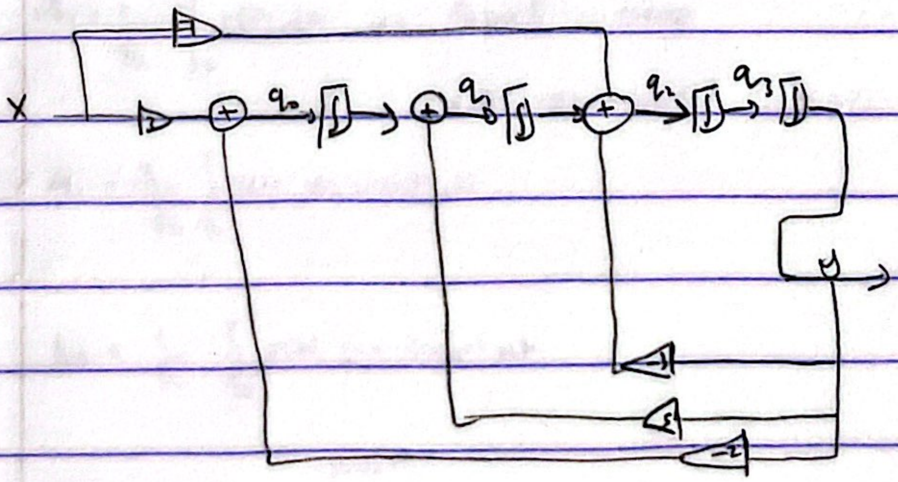
$\frac{d^4y}{dt^4} + \frac{3d^2y}{dt^2} - \frac{5\,dy}{dt} - 7\frac{d^2x}{dt^2} = \underline{\underline{3x(t) - 2y(t)}}$ q_0

Its $\frac{d^3y}{dt^3} + \frac{3dy}{dt} - \frac{7dx}{dt} = \underline{\underline{5y(t) + \int q_0}}$ q_1

Its $\frac{d^2y}{dt^2} = \underline{\underline{\int q_1 + 7x(t) - 3y(t)}}$ q_2

$\frac{dy}{dt} = \underline{\underline{\int q_2\,dt}}$ q_3

$\Rightarrow y(t) = \int q_3$



chapter(3):- Fourier series / transform.

Spectral representation.

series → periodic signal "power"

transform → Aperiodic signal "energy".

Fourier integral series

Fourier series.

$x(t)$: periodic with period T_o.

Trigonometric form.

$$x(t) = a_o + \sum_{n=1}^{\infty} a_n \cos(n\omega_o t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_o t)$$

$a_o, a_n, b_n \in \mathbb{R}$.

Singel = convergien.

complex exponential form.

$$x(t) = \sum_{n=0}^{\infty} x_n e^{jn\omega_o t}$$

$x_n \in \mathbb{C}$

doubel.

$$a_o = \frac{1}{T_o}\int_{T_o} x(t)\,dt$$ → Signal average.

• DC → component, $f=0$

$$a_n = \frac{2}{T_o}\int_{T_o} x(t)\cos(n\omega_o t)\,dt.$$

$$b_n = \frac{1}{T_o}\int_{T_o} x(t)\sin(n\omega_o t)\,dt.$$

$$x_n = \frac{1}{T_o}\int_{T_o} x(t)\,e^{-jn\omega_o t}\,dt$$

$$\Rightarrow \begin{cases} a_0 = X_0 \\ a_n = 2\,\mathrm{Re}(X)_n \\ b_n = -2\,\mathrm{Im}(X_n) \end{cases} \qquad X_n \Rightarrow \begin{cases} a_0, & n=0 \\ \frac{a_n - j b_n}{2}, & n \geq 1 \\ \frac{a_n + j b_{-n}}{2}, & n \leq 1 \end{cases}$$

$$\Rightarrow |X_n| = \sqrt{\left(\frac{a_n}{2}\right)^2 + \left(\frac{b_n}{2}\right)^2} = \frac{1}{2}\sqrt{a_n^2 + b_n^2}.$$

Dirchlet theorem:

It's sufficient for $x(t)$ to have a fourier series representation that $x(t)$ be generally continuos and absolutely Integrable $\longrightarrow \int_{T_0} |x(t)|\, dt.$

$[4,7]$. $\{5, 7, 11, 9, ---\}$.

not countabel, countabel set

- $x(t) \leq |x(t)|$.
- $z = 5 + j3$
- $z^* = 5 - j3$.

Symmetry properites of fourier series cofficients.

1) if $x(t)$ is real then $X_n = X^*_{-n}$

→ $|X_n| = |X_{-n}|$ ⊖

Amplitude spectra even symmetry.

→ $\measuredangle X_n = -\measuredangle X_{-n}$ ⊖

Phase spectra odd symmetry.

2) if $x(t)$ is real and even then X_n is real and even

$\Rightarrow b_n = 0 \ \forall n$

3) if $x(t)$ is real and odd then $\underline{X_n}$ is imaginary and odd

$\Rightarrow a_n = 0 \ \forall n$ $\quad X_n = \dfrac{a_n - jb_n}{2}$

4) If $x(t)$ is real and alternating then $X_0 = a_0 = 0$

5) If $x(t)$ has half-wave odd symmetry then $X_n = 0$

$\forall n$ even

⇒ Parseval theorem:- periodic

let $x(t)$ be a signal that has a Fourier series representation that the Pav of the signal enabble computed by $P_{av} = \sum_{-\infty}^{\infty} |X_n|^2 \xrightarrow[\text{res}]{x(t)} P_{av} = |X_0|^2 + 2\sum_{n=1}^{\infty} |X_n|^2$.

• $(e^{jn\omega_0 t})^* = e^{-jn\omega_0 t}$ • $(z_1 \cdot z_2)^* = z_1^* z_2^*$ • $\xrightarrow[\text{real}]{x(t)} X_n^* = X_{-n}$

• $e^{jn\omega_0 t} = \cos(n\omega_0 t) + j\sin(n\omega_0 t)$ • $(z_1 + z_2)^* = z_1^* + z_2^*$ • z z^*

• $e^{-jn\omega_0 t} = \quad " \quad -j \quad "$ $(a+jb)$ $(a-jb)$

$= a^2 + b^2 = |z|^2$

$$\Rightarrow P_{av} = \frac{1}{T_0}\int_{T_0} |x(t)|^2 dt.$$

• $x(t) = \sum_{-\infty}^{\infty} X_n e^{jn\omega_0 t}$

↳ $x^*(t) = \sum_{-\infty}^{\infty} X_n^* e^{-jn\omega_0 t}$

$$= \frac{1}{T_0}\int_{T_0} x(t) \cdot x^*(t) dt$$

$$P_{av} = \frac{1}{T_0}\int \left(x(t) \sum_{-\infty}^{\infty} X_n^* e^{-jn\omega_0 t} \right) dt$$

out of the Σ

$$= \sum_{-\infty}^{\infty} X_n^* \frac{1}{T_0}\int x(t) e^{-jn\omega_0 t} dt.$$

$$\Rightarrow X_n = \frac{1}{T_0}\int_{T_0} x(t) e^{-jn\omega_0 t} dt \qquad = \sum_{-\infty}^{\infty} X_n^* X_n$$

$$= \sum_{-\infty}^{\infty} |X_n|^2$$

Theorem.

The response of a LTI system with impulse response $h(t)$ and frequency response $H(\omega)$ for an input $x(t)$ that has Fourier freq. resp can be computed by $y(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega t} H(\omega)\Big|_{n\omega_0}$

$$= \sum_{-\infty}^{\infty} |X_n| \, |H(\omega_0)| e^{j(n\omega t + \measuredangle X_n + \measuredangle H(n\omega_0))}$$

$\Rightarrow$ $X_n e^{jn\omega_0 t}$. $X_n = \frac{1}{T_0} \int_{T_0} x(t) e^{-jn\omega_0 t} \, dt$

$y_n(t) = |X_n| \, |H(\omega_0 n)| e^{j(n\omega_0 t + \measuredangle H(n\omega_0) + \measuredangle X_n)}$

$y(t) = \sum_{-\infty}^{\infty} y_n(t) = \sum_{-\infty}^{\infty}$. $Ae^{j(n\omega_0 t + \phi)}$

. $H(\omega) = \frac{2}{2 + j\omega}$

. $x(t) = 10 \cos(10t + \frac{\pi}{3}) + 30 \cos(20t + \frac{\pi}{6})$

$\Rightarrow y(t) = (10)\left(\frac{2}{\sqrt{4+100}}\right) \cos\left(10t + \frac{\pi}{3} - \tan^{-1} \frac{10}{2}\right) + (30)\left(\frac{2}{\sqrt{4+30^2}}\right) \cos\left(20t + \frac{\pi}{6} - \tan^{-1}\left(\frac{20}{2}\right)\right)$

$\Rightarrow \quad x(t) = \cos^2(10t)$

F.S

$$\cos^2(10t) = \frac{1 + \cos(2 \cdot 10t)}{2}$$

$$= \underbrace{\tfrac{1}{2}}_{a_0 \,\to\, \text{DC component}} + \underbrace{\tfrac{1}{2}\cos 20t}_{n=1,\ \text{fundemental}}$$

note:-

$n=0$, DC component. $\quad \bullet\ W_0 T_0 = 2\pi$

$n=1$, fundemental $\quad \frac{2\pi}{T_0} \cdot T_0 = 2\pi$

$n=2$, first harmonic

$n=3$, second harmonic.

[redacted]

$$x(t) = a_0 + \sum_{n}^{\infty} a_n \cos(n w_0 t) + \sum^{\infty} b_n \sin(n w_0 t).$$

$$x(t) = \sum_{n=1}^{\infty} \frac{-4A}{n\pi} \sin(n w_0 t).$$

$n = 2k+1$

$1 = 2k + 1$

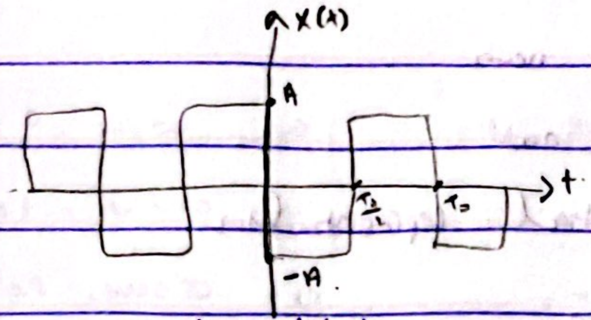
$\therefore k = 0$

$$\sum_{k=0}^{\infty} \frac{-4A}{(2k+1)\pi} \sin((2k+1) w_0 t)$$

$$\Rightarrow x(t) = \sum x_n e^{j n w_0 t}$$

$$= \sum \frac{2A}{n\pi} j\, e^{j n w_0 t}$$

EX:-



$X_n = X_{-n}^{*}$ → $|X_n| = |X_{-n}|$ → $\measuredangle X_n = -\measuredangle X_{-n}$

- odd symmetry → X_n is immaginary and has odd sym.

$a_n = 0$ $\forall n$ "alternating"

$b_n = 0$ $\forall n$: even "half wave odd sym."

<u>Sol:-</u>

$$b_n = \frac{2}{T_0} \int_{T_0} x(t) \sin(n\omega_0 t)\, dt$$

$$= \frac{2}{T_0} \left[\int_{-\frac{T}{2}}^{0} \sin(n\omega_0 t) dt - \int_{0}^{\frac{T}{2}} \sin(n\omega_0 t) dt \right]$$

لأنه x(t) في هذه المنطقة (A-) يعني (1-)

$$= \frac{2}{T_0} \left[\frac{-\cos(n\omega_0 t)}{n\omega_0} \Big|_{-\frac{T}{2}}^{0} + \frac{\cos(n\omega_0 t)}{n\omega_0} \Big|_{0}^{\frac{T}{2}} \right]$$

$$= \boxed{\frac{2}{n\omega_0 T}} \left[-\cos(n\omega_0 t) \Big|_{-\frac{T}{2}}^{0} + \cos(n\omega_0 t) \Big|_{0}^{\frac{T}{2}} \right]$$

$$\frac{1}{n\pi}$$

$$= -1 + \cos\left(\frac{n\omega_0 T_0}{2}\right) + \cos\left(\frac{n\omega_0 T_0}{2}\right) - 1$$

$$= -2 + 2\left(\frac{n\omega_0 T_0}{2}\right) \qquad " \frac{\omega_0 T_0}{2} = \frac{2\pi}{2} = \pi "$$

$$= \frac{-2}{n\pi}\left(1 - \cos(n\pi)\right)$$

- n odd $= \frac{-2}{n\pi}(1+1)$

$= \frac{-4}{n\pi} \boxed{A}$

$X_n = \begin{cases} X_0 = 0 = a_0 \\ X_n = \frac{a_n - j b_n}{2} = \frac{\frac{-4A}{n\pi} j}{-2} = \frac{2A}{n\pi} j \\ X_{-n} = X_n^{*} = -\frac{2A}{n\pi} j \end{cases}$

So $x(t) = \sum_{\substack{n=1 \\ n\ \text{odd}}}^{\infty} \frac{-4A}{n\pi} \sin(n\omega_0 t)$.

$$X_n = \begin{cases} 0 & , n\text{ even} \\ \frac{2A}{n\pi} j & , n\text{ odd} \end{cases}$$

Discrete line spectral representation.

$$|X_n| = \frac{2A}{|n|\pi}$$

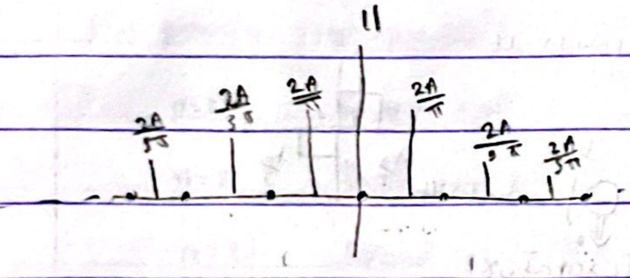
n, odd

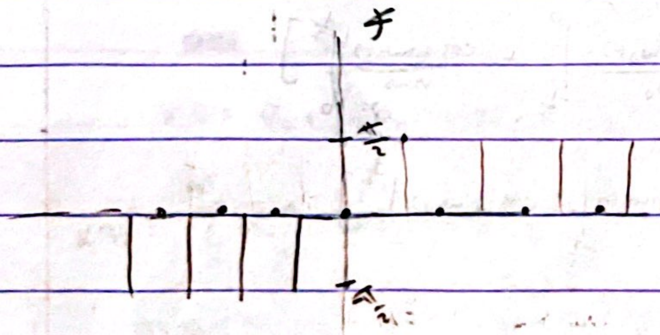
$$\angle X_n = \begin{cases} \frac{\pi}{2} & , n>0 \\ -\frac{\pi}{2} & , n<0 \end{cases}$$

$\frac{2A}{n\pi} j$

cons $= \angle = 0$

$\tan^{-1} \infty = \frac{\pi}{2}$





$$P_{av} = |X_0|^2 + 2\sum_{n=1}^{\infty} |X_n|^2$$

$$= 0 + \sum_{\substack{n=1 \\ odd}}^{\infty} \frac{4A^2}{n^2\pi^2}$$

$w_0 = 10$ $\qquad X_0 = 0$

determine Pav in the band range of freq.

$w \in [-45, 45]$ $\qquad w \in (0, 45]$

$n=0\ ,\ w_0=0 \quad =0$ $\Rightarrow \frac{e^{jr}+e^{-jr}}{2} = \cos$

$n=1\ ,\ w_0=10$

$n=2\ ,\ w_0=20 \quad =0$ $\Rightarrow \mathbb{C};\ \mathbb{R}\times\mathbb{R}$

$n=3\ ,\ w_0=30$ (a,b)

$n=4\ ,\ w_0=40 \quad =0$ $a\in\mathbb{R},\ b\in\mathbb{R}.$

Ex:

$$\therefore P_{av} = |X_0|^2 + 2\sum_{n=1}^{\infty}|X_n|^2$$

$$= 0 + 2\left[\frac{4A^2}{\pi^2} + 0 + \frac{4A^2}{9\pi^2} + 0\right] \text{ watt}$$

$\Rightarrow$ Steady state system response to $x(t)$ using F.S.

System LTI $\quad H(\omega) \rightarrow$ freq. resp.

$$y(t) = \sum_{-\infty}^{\infty} |X_n||H(n\omega_0)|\, e^{j(n\omega_0 t + \measuredangle X_n + \measuredangle H(n\omega_0))}$$

$$\Rightarrow \quad = \sum_{\substack{-\infty \\ \text{odd}}}^{\infty} \frac{2A}{|n|\pi}|H(n\omega_0)|\, e^{j\left(n\omega_0 t + \frac{\pi}{2} - \tan^{-1}\left(\frac{n\omega_0}{2}\right)\right)}$$

$$\rightarrow H(\omega) = \frac{2}{2+jn\omega_0}$$

$$|H(\omega)| = \frac{2}{\sqrt{4+n^2\omega_0^2}}$$

determine $\tilde{y}(t)$ approximated to the 1 harmonic

$$\tilde{y}(t) = \frac{2A}{\pi}\cdot |H(\omega)|_{\omega_0}\, e^{j\left(\omega_0 t + \frac{\pi}{2} - \tan^{-1}\frac{\omega_0}{2}\right)} + \frac{2A}{3\pi}|H(\omega)|_{3\omega_0}\, e^{j\left(3\omega_0 t + \frac{\pi}{2} - \tan^{-1}\left(\frac{3\omega_0}{2}\right)\right)}$$

pos

neg $\quad + \frac{2A}{\pi}|H(\omega)|_{\omega_0}\, e^{j\left(-\omega_0 t - \frac{\pi}{2} + \tan^{-1}\left(\frac{\omega_0}{2}\right)\right)} + \frac{2A}{3\pi}|H(\omega)|_{3\omega_0}\, e^{j\left(-3\omega_0 t - \frac{\pi}{2} + \tan^{-1}\left(\frac{3\omega_0}{2}\right)\right)}$

$$= \frac{2A}{\pi}|H(\omega)|_{\omega_0}\left[e^{j\left(\omega_0 t + \frac{\pi}{2} - \tan^{-1}\frac{\omega_0}{2}\right)} + e^{-j\left(\omega_0 t + \frac{\pi}{2} - \tan^{-1}\frac{\omega_0}{2}\right)}\right]$$

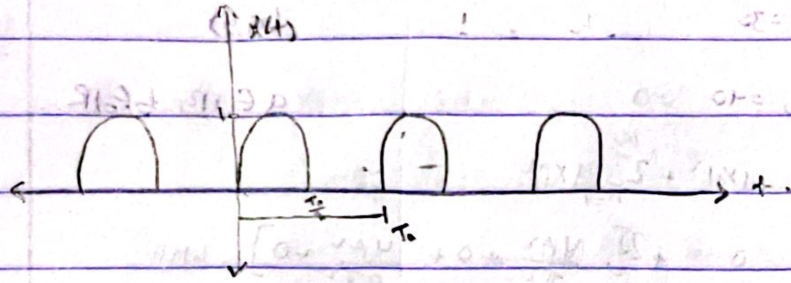
$$+ \frac{2A}{3\pi}|H(\omega)|_{3\omega_0}\left[e^{j\left(3\omega_0 t + \frac{\pi}{2} - \tan^{-1}\frac{3\omega_0}{2}\right)} + e^{-j\left(3\omega_0 t + \frac{\pi}{2} - \tan^{-1}\frac{3\omega_0}{2}\right)}\right]$$

$$= \frac{4A}{\pi}\left|H(\omega_0)\right|_{\omega_0} \cos\left[\omega_0 t + \frac{\pi}{2} - \tan^{-1}\left(\frac{\omega_0}{2}\right)\right] +$$

$$\frac{4A}{3\pi}\left|H(\omega_0)\right|_{3\omega_0} \cos\left[3\omega_0 t + \frac{\pi}{2} - \tan^{-1}\left(\frac{3\omega_0}{2}\right)\right]$$

↳ singel sided

Ex:-



$x(t) = \sin(\omega_0 t)$

$$x_n = \frac{1}{T_0}\int_0^{T/2} \sin(\omega_0 t)\, e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{2jT_0}\int_0^{T/2} \left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right) e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{2jT_0}\left[\int_0^{T/2} e^{j\omega_0(1-n)t}\, dt - \int_0^{T/2} e^{-j\omega_0 t(1+n)}\, dt\right]$$

$$= \frac{1}{2jT_0\, j\omega_0}\left[\frac{e^{j\omega_0 t(1-n)}}{(1-n)}\Bigg|_0^{T/2} + \frac{e^{-j\omega_0 t(1+n)}}{(1+n)}\Bigg|_0^{T/2}\right] \cdot \frac{T}{2} \cdot j\omega_0$$

$$= \frac{-1}{4\pi}\left[\frac{e^{j(1-n)\pi} - 1}{(1-n)} + \frac{e^{-j(1+n)\pi} - 1}{(1+n)}\right] \qquad \underline{\underline{n \neq \pm 1}}$$

- $e^{j(1-n)\pi} = \cos((1-n)\pi) + j\sin((1-n)\pi)$
- $e^{j(1+n)\pi} = \cos((1+n)\pi) + j\sin((1+n)\pi)$

when n odd cos = 1 → 0

" " even cos = −1 → $\underline{\underline{-2}}$

$X_n = 0$, n odd , $n \neq \pm 1$

n even $\Rightarrow \frac{-1}{4\pi} \cdot \left[\frac{-2}{(n-1)} + \frac{-2}{(n+1)} \right]$

$$\frac{1}{2\pi} \cdot \frac{2}{(1-n^2)} = \frac{1}{\pi} \cdot \frac{1}{(1-n^2)}$$

$$= \frac{1}{\pi(1-n^2)}$$

neg. con

$\frac{\pi}{}$ plus , cos

$\frac{\pi}{}$ cos $\mp$

now when $n = 1$.

$$= \frac{1}{2jT_0} \int_0^{\frac{T_0}{2}} (e^{j\omega_0 t} - e^{-j\omega_0 t}) e^{-j\omega_0 t} \, dt$$

$$= \frac{1}{2jT_0} \left[\int_0^{\frac{T_0}{2}} e^0 \, dt - \int_0^{\frac{T_0}{2}} e^{-2j\omega_0 t} \, dt \right]$$

$$= \frac{1}{2jT_0} \left[t \Big|_0^{\frac{T_0}{2}} + \frac{e^{-2j\omega_0 t}}{j\omega_0 t} \Big|_0^{\frac{T_0}{2}} \right] \Rightarrow \frac{e^{-j2\pi} - 1}{-j2\omega_0}$$

$$= \frac{\cancel{\cos(2\pi)}^{\,1} - j\sin(2\pi)^{\,0} \cancel{-1}}{-j\omega_0 2}$$

$$= 0$$

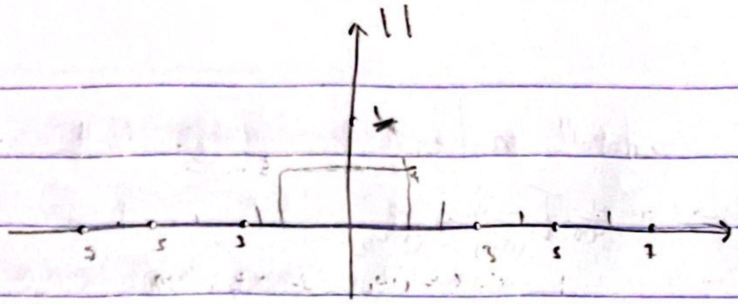
$$= \frac{1}{2jT_0} \cdot \left[\frac{T_0}{2} - 0 \right]$$

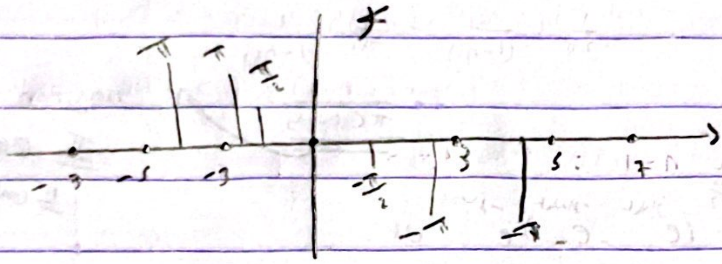
$$= \frac{1}{4j} = \frac{-j}{4}$$

$\therefore X_{n=1} = \frac{-j}{4}$

$X_{n=-1} = \frac{+j}{4}$

$$\therefore X_n = \begin{cases} 0 & , n \text{ odd } + n \neq \pm 1 \\ \frac{1}{\pi(1-n^2)} & , n \text{ even} \\ \mp \frac{j}{4} & , n = \pm 1 \end{cases}$$





Ex:-

now compute the signal power in the band rang [0, 80] Hz knowing that $f_0 = 15$.

$n=0 \rightarrow f=0 \rightarrow |X_0| = \frac{1}{\pi}$

$n=1 \rightarrow f=15 \rightarrow |X_1| = \frac{1}{4}$

$n=2 \rightarrow f=30 \rightarrow |X_2| = \frac{1}{3\pi}$

$n=3 \rightarrow f=45 \rightarrow |X_3| = 0$

$n=4 \rightarrow f=60 \rightarrow |X_4| = \frac{1}{15\pi}$

$n=5 \rightarrow f=75 \rightarrow |X_5| = 0$

can be ± . $n=6 \rightarrow f=90 \rightarrow$ out of range.

note

$$P_{av} = |X_0|^2 + (|X_1|^2 + |X_2|^2 + 0 + |X_4|^2 + 0)\,2$$

$$= \frac{1}{\pi^2} + \left(\frac{1}{16} + \left(\frac{1}{3\pi}\right)^2 + \left(\frac{1}{15\pi}\right)^2\right)2$$

$\tilde{y}(t)$ approximated → fundimintal.

$n=0$ | $n=\pm 1$

$$y(t) = x_0 H(0)\Big|_{w=0} + \frac{1}{4}|H(w_0)|\, e^{+j(w_0t-\frac{\pi}{2}+\measuredangle H(w_0))} + \frac{1}{4}|H(w_0)|\, e^{-j(w_0t-\frac{\pi}{2}+\measuredangle H(w_0))}$$

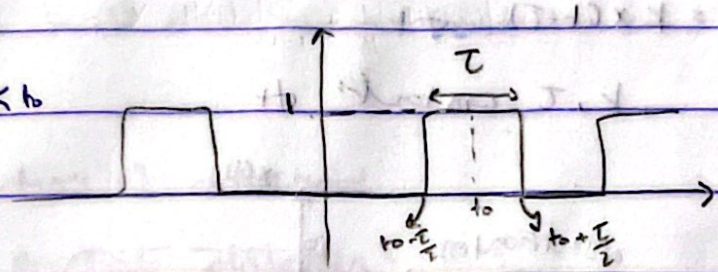
$$= \frac{1}{\pi}H(0) + \frac{2}{4}|H(w_0)|\cos\left(w_0t-\frac{\pi}{2}+\measuredangle H(w_0)\right)$$

- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
- $\measuredangle\left(\frac{z_1}{z_2}\right)$

$\measuredangle z_1 - \measuredangle z_2$

Ex:-

$\tau < t_0$



$$x_n = \frac{1}{T_0}\int_{t_0-\frac{\tau}{2}}^{t_0+\frac{\tau}{2}} e^{-jnw_0t}\,dt$$

$$= \frac{1}{T_0}\frac{e^{-jnw_0t}}{-jnw_0}\Bigg|_{t_0-\frac{\tau}{2}}^{t_0+\frac{\tau}{2}} = \frac{-1}{2\pi nj}\left[e^{-jnw_0(t_0+\frac{\tau}{2})} - e^{-j(nw_0(t_0-\frac{\tau}{2})}\right].$$

$$= +\frac{1}{n\pi}e^{-jnw_0t_0}\left[\frac{e^{-jnw_0\frac{\tau}{2}} - e^{jnw_0\frac{\tau}{2}}}{2j}\right]$$

- $\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$

$$\therefore x_n = \frac{-1}{n\pi}e^{-jnw_0t}\sin\left(nw_0\frac{\tau}{2}\right)$$

$$= e^{-jnw_0t}\frac{\sin(n\pi f_0\tau)}{n\pi f_0\tau}\cdot f_0\tau$$

$$x_n = f_0\tau\, e^{-jnw_0t}\,\text{sinc}(nf_0\tau).$$

note ⇒

Same solution but whithout e^{-jnw_0t} → shifting زمني

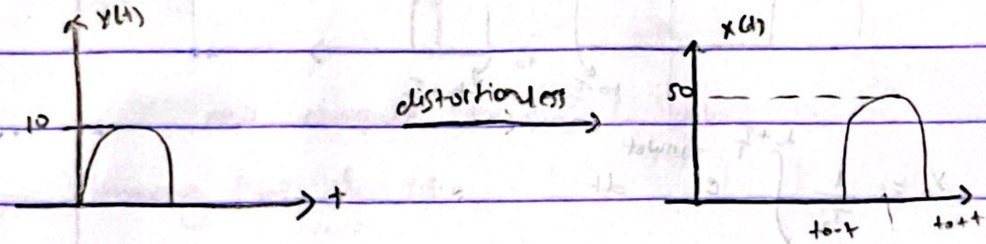
$\Rightarrow \quad H(\omega) = \mathcal{F}[h(t)] = L[h(t)]\Big|_{s=j\omega} = H(s)\Big|_{s=j\omega}$

System and signal distortion:-

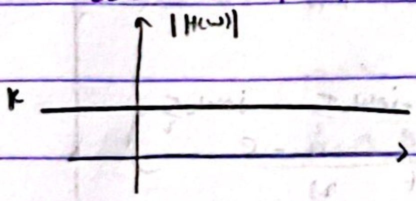
A system is said to be distortionless

$\Leftrightarrow \quad y(t) = k\, x(t-\tau) \quad \forall t$.

k, τ constant $\forall t$.

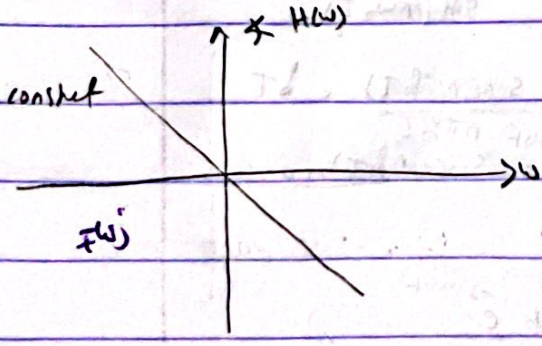


Ideal channel:-



$|H(\omega)| = \text{constant} \quad \forall \omega$.

$\measuredangle H(\omega) = -\omega j$



$\therefore H(\omega) = k e^{j\omega j}$.

ex:- $H(\omega) = \frac{2}{4+j\omega}$

$|H(\omega)| = \frac{2}{\sqrt{16+\omega^2}}$, function of (ω) "Amplitude distortion".

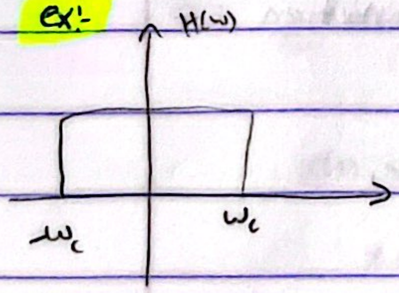
$\angle H(\omega) = -\tan^{-1}\left(\frac{\omega}{4}\right)$, "phase distortion".

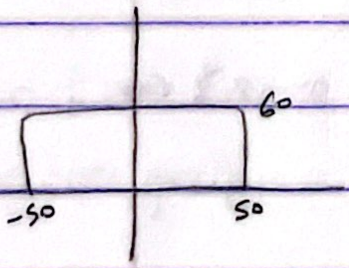
↳ So, its distortion not distortionless

3-types of distortion:-

1) Amplitude distortion
2) phase distortion. } LTI
3) frequency distortion. } NL

ex:-





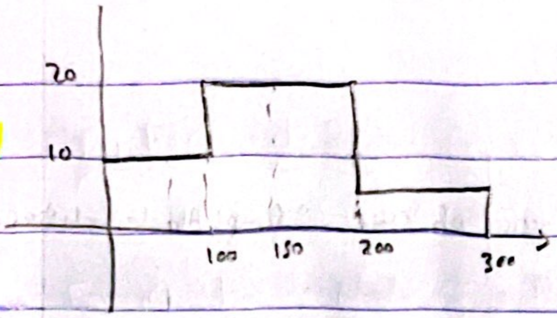
1 ⇒ $x(t) = 10\cos(70t+\frac{\pi}{6}) + 30\cos(30t+\frac{\pi}{4})$

2 ⇒ $\hat{x}(t) = 10\cos(30t+\frac{\pi}{6}) + 40\cos(70t+\frac{\pi}{4})$

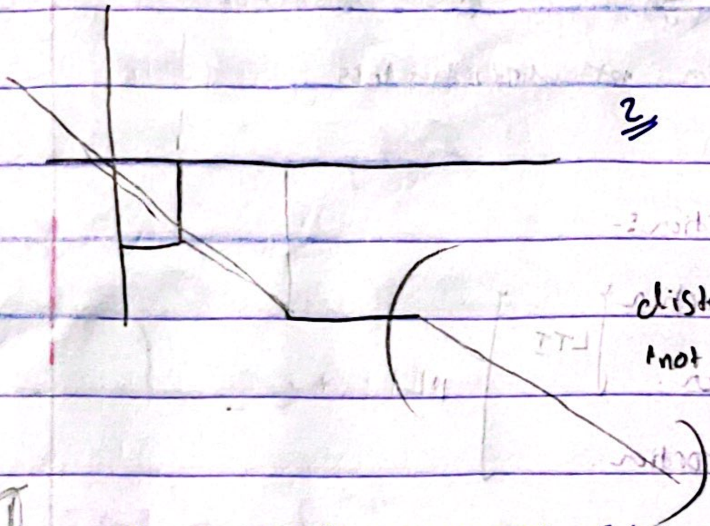
1 → distortion less

2 → distortion

ex:-



1



2

distortion
not pass the origin.

- $x(t) = 30\cos(20t) + 40\sin(30t + \frac{\pi}{5})$ 1 : distortionless

  2 : distortion

- $x(t) = 30\cos(60t + \frac{\pi}{4}) + 20\cos(80t + \frac{\pi}{6})$.

  1 : distortion less

  2 : " "

chapter(4):-

Fourier Transform:

Spectral representation $\xrightarrow[\text{Integral}]{\text{Fourier}}$ Energy Signals

↓

not F.S integral

↓

$\delta(t)$.

F. transform.

$x(t)$ a spectral function $X(f)$ is said to be fourier transform of $x(t)$ $\Leftrightarrow$ $\int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt = \underline{\underline{X(f)}}$ • convergent

$x(t)$ is said to be the inverse laplace transform of $X(f)$ $\Leftrightarrow$ $\int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df = \underline{\underline{x(t)}}$ • convergent.

$x(t), X(f)$ is a pair of fourier transform.

$$x(t) \underset{F^{-1}}{\overset{F}{\longleftrightarrow}} X(f)$$

$\Rightarrow$ F.S $\rightarrow$ $x_p(t)$ $\rightarrow$ discrete spectral rep.

periodic $\quad f \in \mathbb{N}$.

Dirchlet theorem:

It's sufficient for $x(t)$ to have a Fourier transform rep. that $x(t)$ be.

1- generally continuous.

2- absoletely integrable.

- $x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} \, df$.

- $[X(f)] = \text{unit} / Hz$

- $|X(f)| \rightarrow$ density of magnitude

- Symmetry

properties of Fourier transform.

1- if $x(t)$ is real then

$X(f) = X^{*}(-f)$
- $\rightarrow |X(f)| = |X(-f)|$
- $\rightarrow \measuredangle X(f) = -\measuredangle X(-f)$.

2- if $x(t)$ is real and even then $X(f)$ is real and even

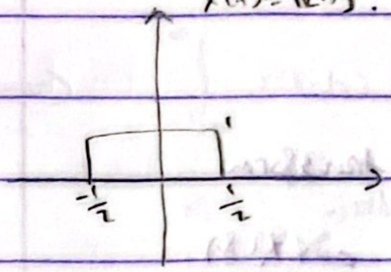
3- if $x(t)$ is real and odd then $X(f)$ immaginary and odd

- $\sin(z) = \frac{e^{jz} - e^{-jz}}{2j}$

- series $f = nf_0 \rightarrow N$.

Ex:-

$x(t)=\Pi(t)$.

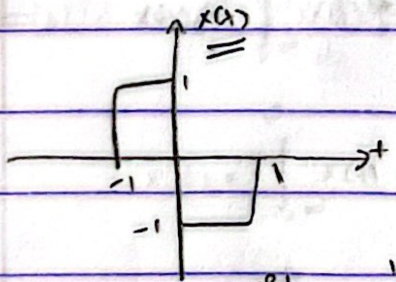


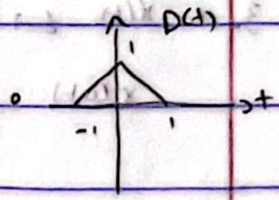
$$X(f) = \int_{-1/2}^{1/2} e^{-j2\pi f t}\,dt = \frac{e^{-2\pi f t j}}{-2\pi f j}\Bigg|_{-1/2}^{1/2}$$

$$= \frac{e^{-j\pi f} - e^{j\pi f}}{-j2\pi f} = \frac{\sin(\pi f)}{\pi f}$$

$$= \text{sinc}(f).$$

Ex:-





$x(t) = D'(t)$.

$$X(f) = \int_{-1}^{0} e^{-j2\pi f t}\,dt - \int_{0}^{1} e^{-j2\pi f t}\,dt.$$

$$= \frac{e^{-j2\pi f t}}{-j2\pi f}\Bigg|_{-1}^{0} + \frac{e^{-j2\pi f t}}{j2\pi f}\Bigg|_{0}^{1}$$

$$= \frac{-1+e^{j2\pi f}}{+j2\pi f} + \frac{e^{-j2\pi f}-1}{j2\pi f} = \frac{-2+e^{j2\pi f}+e^{-j2\pi f}}{j2\pi f}$$

$$= \frac{-2+2\cos(2\pi f)}{j2\pi f} = \frac{-2(1-\cos(2\pi f))}{j2\pi f}$$

$$= \frac{-2\sin^2(\pi f)}{j\pi f} \cdot \frac{j\pi f}{j\pi f}$$

$$= 2j\pi f\,\frac{\sin^2(\pi f)}{(\pi f)^2}$$

// $= j2\pi f\,\text{sinc}^2(f)$.

// f: odd
sinc: even
odd · even = odd

Fourier transform theorem:

1- Fourier transform is a linear transform

$x_1(t) \longleftrightarrow X_1(f)$, $x_2(t) \longleftrightarrow X_2(f)$.

$F[\alpha_1 x_1(t) + \alpha_2 x_2(t)] = \alpha_1 X_1(f) + \alpha_2 X_2(f)$.

2- $x(t) \longleftrightarrow X(f)$

$x(t-t_0) \longleftrightarrow X(f)e^{-j2\pi f t_0}$

3- $x(t) \longleftrightarrow X(f)$.

$x(at) \longleftrightarrow \frac{1}{|a|} X\left(\frac{f}{a}\right)$

4- $a=-1$, $X(f) = X(-f)$.

$x(-t) \longleftrightarrow X(-f)$.

5- duality.

$x(t) \longleftrightarrow X(f)$.

$X(t) \longleftrightarrow x(-f)$.

S_1 S_2

$y = f(x)$ $x = f(y)$

$\Pi(t) \longleftrightarrow \text{sinc}(f)$.

$\text{sinc}(t) \longleftrightarrow \Pi(-f) = \Pi(f)$. $t \longleftrightarrow -f$

$$\Longrightarrow \int_{-\infty}^{\infty} x(t-t_0)\, e^{-j2\pi f t}\, dt$$

2.

$$= \int_{-\infty}^{\infty} x(t')\, e^{-j2\pi f(t'+t_0)}\, dt'$$

- $t' = t - t_0$
- $dt' = dt$

$$= e^{-j2\pi f t_0} \underbrace{\int_{-\infty}^{\infty} x(t')\, e^{j2\pi f t'}\, dt'}_{X(f)}$$

$t \to \infty,\ t' \to \infty$

$t \to -\infty,\ t' \to -\infty$

$$= X(f)\, e^{-j2\pi f t_0}.$$

$$\Longrightarrow x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi f t}\, df.$$

3.

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{j2\pi f}\, dt.$$

$x \leftrightarrow f$

so it's $-f$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt = \mathcal{F}\{x(t)\}_{f=-f}$$

$$x(t) \longleftrightarrow X(-f)$$

- $x(t) \leftrightarrow X(-f)$
- $x(-t) \leftrightarrow X(-f)$

Frequency shift theorem:-

$$x(t-t_0) \longleftrightarrow X(f)\, e^{-j2\pi f t_0}.$$

$$x(t)\, e^{j2\pi f_0 t} \longleftrightarrow X(f-f_0).$$

$$X(f-f_0) \longleftrightarrow x(t)\, e^{-j2\pi f_0 t}.$$

$$\mathrm{Sinc}(t) \longleftrightarrow \Pi(f).$$

$$e^{j2\pi f_0 t}\ \mathrm{Sinc}(t) \longleftrightarrow \Pi(f-f_0)$$

Differentiation theorem:-

$$x(t) \longleftrightarrow X(f).$$

$$!! \quad \frac{dx}{dt} \longleftrightarrow j2\pi f\, X(f).$$

$$\frac{d^n x(t)}{dt^n} \longleftrightarrow (j2\pi f)^n X(f).$$

$\Longrightarrow$

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi f t}\, df.$$

$$\frac{dx(t)}{dt} = \int_{-\infty}^{\infty} X(f) \left[\frac{d}{dt} e^{j2\pi f t}\right] df$$

$$= \int_{-\infty}^{\infty} \underbrace{X(f) \cdot j2\pi f}_{\downarrow\ !!}\, e^{j2\pi f t}\, df.$$

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi f t}\, df$$

$t \leftrightarrow f$

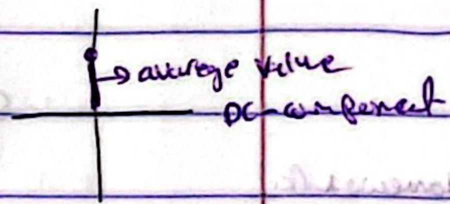
$$x(f) = \int_{-\infty}^{\infty} X(t)\, e^{j2\pi f t}\, dt$$

$$= \int_{-\infty}^{\infty} X(t)\, e^{-j2\pi f t}\, dt$$

$x(t) \leftrightarrow X(-f)$.

$F[\delta(t)] = ?$

$$F[\delta(t)] = \int_{-\infty}^{\infty} \delta(t)\, e^{-j2\pi f t}\, dt = 1$$



$\underline{\underline{\delta(t) \leftrightarrow 1}}$

$1 \leftrightarrow \delta(-f) = \delta(f)$.

$\lim_{t \to \infty} \cos wt$ "does not exist"

Integration theorem:

$x(t) \leftrightarrow X(f)$.

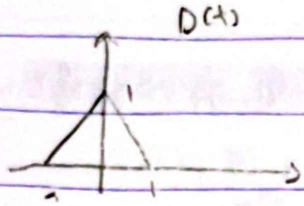
$$\int_{-\infty}^{\infty} x(\tau)\, d\tau \longleftrightarrow \frac{X(f)}{j2\pi f} + \boxed{\frac{1}{2} X(0)\, \delta(f)}.$$

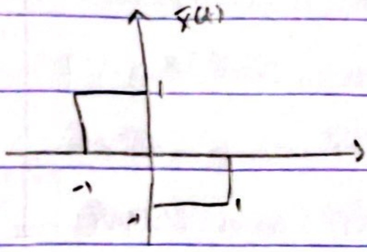
$$\Rightarrow u(t) = \int_0^t \delta(\tau)\, d\tau$$

$$F[u(t)] = \frac{F[\delta(t)]}{j2\pi f} + \frac{1}{2} F[\delta(t)]_0\, \delta(f).$$

$$= \frac{1}{j2\pi f} + \frac{1}{2}\delta(f)$$ "Super position"

Ex:-





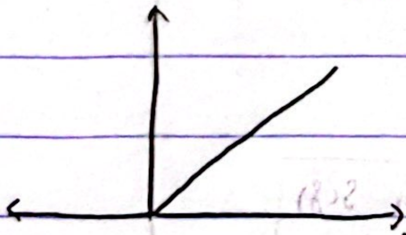
$\dot{x}(t) \longleftrightarrow j2\pi f \, \text{sinc}^2(f)$

$$F[\Delta(t)] = \frac{\cancel{j2\pi f} \, \text{sinc}^2(f)}{\cancel{j2\pi f}} + \frac{1}{2} \underbrace{j2\pi f \, \text{sinc}^2(f)\Big|_0}_{=0} \delta(f)$$

$= \text{Sinc}^2(f)$

Homework:-

$F[r(t)]$



$r(t) = t$

$$F(r(t)) = \int_0^{\infty} r(t) e^{-j2\pi f t} dt$$

$$= \int_0^{\infty} t e^{-j2\pi f t} dt$$

$$\Rightarrow + \; t \quad e^{-j2\pi f t}$$
$$- \; 1 \quad \frac{e^{-j2\pi f t}}{-j2\pi f}$$
$$+ \; 0 \quad \frac{e^{-j2\pi f t}}{(j2\pi f)^2}$$

$$= \left(\frac{-t e^{-j2\pi f t}}{(j2\pi f)} - \frac{e^{-j2\pi f t}}{(j2\pi f)^2} \right) \Bigg|_0^{\infty}$$

$$= \frac{1}{(j2\pi f)^2}$$

Convolution theorem:-

$x_1(t) \longleftrightarrow X_1(f)$. $\quad \cdot \ \mathcal{L}\left[x_1(t) \circledast y_1(t)\right] = X_1(s) \cdot Y_1(s)$

$x_2(t) \longleftrightarrow X_2(f)$.

$F\left[x_1(t) \circledast x_2(f)\right] = X_1(f) \cdot X_2(f)$

$- \ \Lambda(t) = \Pi(t) \circledast \Pi(t)$

$\Pi(t) \longleftrightarrow \text{sinc}(f)$.

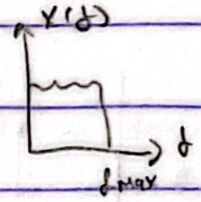
$F\left[\Pi(t) \circledast \Pi(t)\right] = \text{sinc}(f) \cdot \text{sinc}(f) = \text{sinc}^2(f)$.

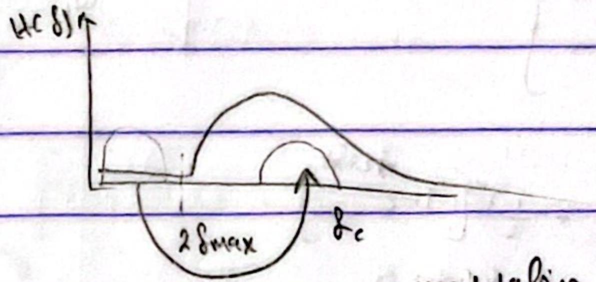
$x_1(t) \longleftrightarrow X_1(f)$

$x_2(t) \longleftrightarrow X_2(f)$.

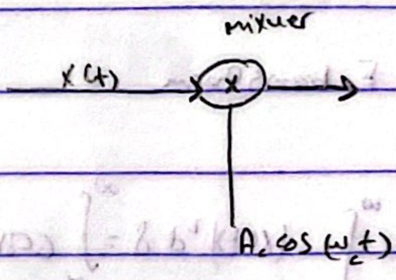
$x_1(t) \cdot x_2(t) \longleftrightarrow X_1(f) \circledast X_2(f)$

- $y(t) = x(t) \circledast h(t)$





modulation

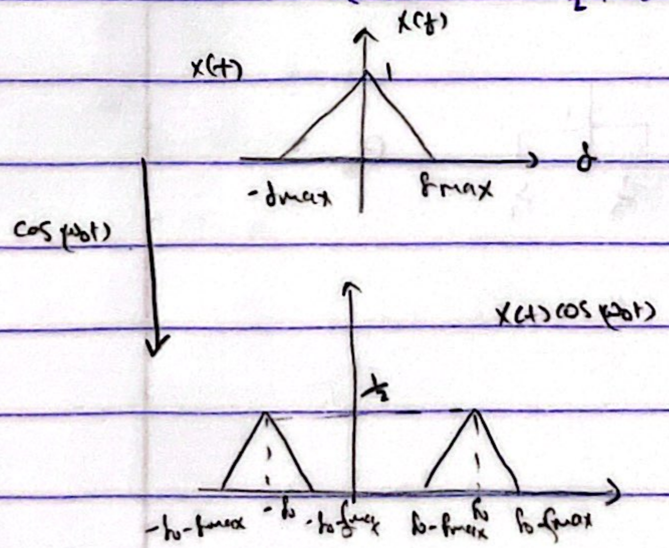


non-linear لأنه freq "يغيرها"

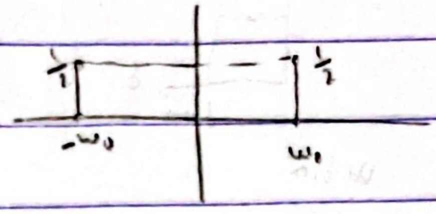
Modulation theorem:-

$x(t) \longleftrightarrow X(f)$

$x(t)\cos(\omega_0 t) \longleftrightarrow \frac{1}{2}X(f - f_0) + \frac{1}{2}X(f + f_0)$



$\Rightarrow$ $x(t) = \cos(\omega_0 t)$



$F[\cos(\omega_0 t)]$

$= F\left[\frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2}\right]$

$= \frac{1}{2} F[1 \cdot e^{j2\pi f_0 t}] + \frac{1}{2} F[1 \cdot e^{-j2\pi f_0 t}]$

$= \frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)$

$\Rightarrow$ Perseval's theorem for F. transform.

$x(t) \longleftrightarrow X(f)$.

$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |X(f)|^2 \, df = \int_{-\infty}^{\infty} G(f) \, df$

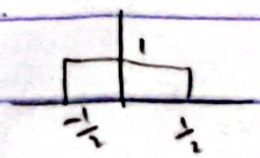
$\cdot G(f) = |X(f)|^2$

$x(t) = \text{Sinc}(t)$. Energy signal.

$E_x(t) = \int_{-\infty}^{\infty} \text{Sin}^2(t) \, dt$.

$\text{Sinc}(t) \longleftrightarrow \Pi(f)$.

$E = \int_{-\frac{1}{2}}^{\frac{1}{2}} (1)^2 \, df$.



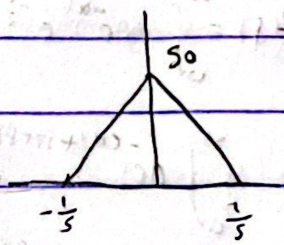
E_x

Homework:- E?

$x(t) = 10\ \text{sinc}^2(0.2t) \longleftrightarrow \frac{10}{0.2} \Delta(f/0.2)$

$50\ \Delta(f/0.2)$

$50\left[\int_{-\frac{1}{5}}^{0} (5t+1)^2\,dt + \int_{0}^{\frac{1}{5}} (-5t+1)^2\,dt\right]$



$\Rightarrow \frac{1}{5} \Rightarrow \frac{1}{\frac{1}{5}} = 5$

Ex:- $x(t) = Ae^{-\alpha t} u(t)$. $\alpha > 0$.

energy in the band $[0, B]$.

$$X(f) = \int_{0}^{\infty} Ae^{-\alpha t} e^{-j2\pi f t} \, dt$$

$$= \int_{0}^{\infty} Ae^{-(\alpha + j2\pi f)t} \, dt = \left. \frac{A \, e^{-(\alpha + j2\pi f)t}}{-(\alpha + j2\pi f)} \right|_{0}^{\infty}$$

$$= A\left(0 + \frac{1}{\alpha + j2\pi f}\right)$$

$$\therefore X(f) = \frac{A}{\alpha + j2\pi f}$$

$$\Rightarrow \int_{-B}^{B} \left[\frac{A}{\sqrt{\alpha^2 + (2\pi f)^2}} \right]^2 df$$

$$= A^2 \int_{-B}^{B} \frac{1}{\alpha^2 + (2\pi f)^2} \, df$$

let $v = \frac{2\pi f}{\alpha}$

$dv = \frac{2\pi}{\alpha} df$

$$= \frac{A^2}{\alpha^2} \int_{-B}^{B} \frac{1}{1 + \left(\frac{2\pi f}{\alpha}\right)^2} \, df$$

$f = -B \rightarrow v = \frac{-2\pi B}{\alpha}$

$f = B \rightarrow v = \frac{2\pi B}{\alpha}$.

$$= \frac{A^2}{\alpha^{\cancel{2}}} \int_{-\frac{2\pi B}{\alpha}}^{\frac{2\pi B}{\alpha}} \frac{1}{1 + v^2} \cdot \frac{\cancel{\alpha} \, dv}{2\pi}$$

$$= \frac{A^2}{2\pi\alpha} \int_{-\frac{2\pi B}{\alpha}}^{\frac{2\pi B}{\alpha}} \frac{1}{1 + v^2} \, dv = \frac{A^2}{2\pi\alpha} \left[\tan^{-1}(v) \right]_{-\frac{2\pi B}{\alpha}}^{\frac{2\pi B}{\alpha}}$$

$$= \frac{A^2}{\cancel{2}\pi\alpha} \cancel{2} \tan^{-1}\left(\frac{2\pi B}{\alpha}\right).$$

$$E_B = \frac{A^2}{\pi\alpha} \tan^{-1}\left(\frac{2\pi B}{\alpha}\right)$$

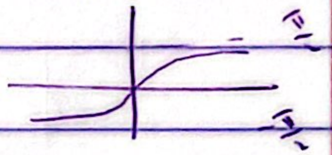
$\hookrightarrow \lim_{B \to \infty}$

$$= \frac{A^2}{\pi\alpha} \cdot \frac{\pi}{2} = \frac{A^2}{2\alpha}$$

$$\Rightarrow x(t) = \sum_{\infty} x_n e^{j2\pi f_0 t}$$

$$P_{av} = \sum_{n=-\infty} x_y$$

$\lim_{x \to \infty} \tan^{-1} x$



Hilbert transform.

$x(t)$ signal $x^H(t)$ is said to be the Hilbert transformed signal of $x(t)$ $\Longleftrightarrow$

$$x^H(t) = x(t) \circledast \frac{1}{\pi t}$$

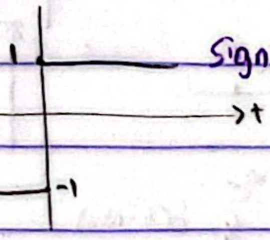
$$= \int_{-\infty}^{+} \frac{x(\tau)}{\pi(t-\tau)} d\tau \quad \text{"converge"}.$$

Hilber transformed signal is orthogonal to the signal

$$x^H(t) \perp x(t) \quad , \quad X^H(f) = X(f) \cdot F\left[\frac{1}{\pi t}\right]$$

$\text{Sign}(t) = 2u(t) - 1$

Signum(t) "Sign(t)" $= \begin{cases} 1 & t > 0 \\ -1 & t < 0 \end{cases}$



$$F[\text{Sign}(t)] = 2F[u(t)] - F[1]$$

$$= 2\left[\frac{1}{j2\pi f} + \frac{1}{2}\delta(f)\right] - \delta(f).$$

$$= \frac{1}{j\pi f}$$

By dualing:-

$F\left[\frac{1}{\pi t}\right] = j\,\text{sign}(-f) = -j\,\text{sign}(f)$

$\frac{1}{\pi t} \longleftrightarrow -j\,\text{sign}(f)$

$X^{H}(f) = X(f) \cdot -j\,\text{sign}(f)$

$\longrightarrow |X^{H}(f)| = |X(f)|$

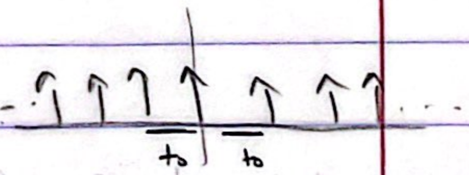
$\longrightarrow \measuredangle X^{H}(f) = \begin{cases} \measuredangle X(f) - \frac{\pi}{2}, & f>0 \\ \measuredangle X(f) + \frac{\pi}{2}, & f<0 \end{cases}$

note:- $x(t) \longrightarrow X(f)$

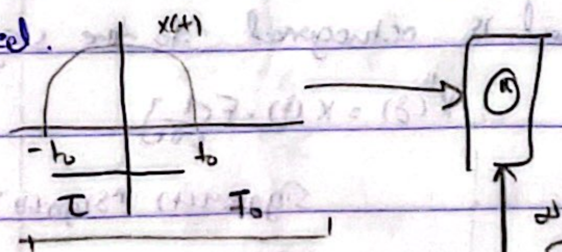
$\int x(\lambda)\,d\lambda \rightarrow \frac{X(f)}{j2\pi f}$. Orthog^n

train of $\delta(t)$

train of $\delta(t) = \sum_{n=-\infty}^{\infty} \delta(t-nt_0)$



time-limited.

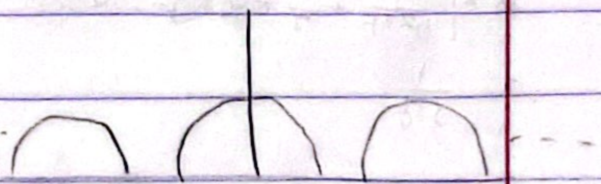


$\sum_{-\infty}^{\infty} \delta(t-nt_0)$.

$\Rightarrow x(t) \circledast \sum_{-\infty}^{\infty} \delta(t-nt_0)$

$= \sum_{-\infty}^{\infty}\left[x(t) \circledast \delta(t-nt_0)\right]$

$= \sum_{-\infty}^{\infty} x(t-nt_0)$ 'periodic'.



periodic signal.

$$x_p(t) = \sum_{-\infty}^{\infty} x(t - nt_0)$$ "Fourier transform"

$\downarrow$ F. series.

$$\sum_{-\infty}^{\infty} x_n e^{j2\pi n f_0 t}$$

$\downarrow$

$$F\left[\sum_{-\infty}^{\infty} x_n e^{j2\pi n f_0 t}\right]$$

$$= \sum_{-\infty}^{\infty} x_n F\left[e^{j2\pi n f_0 t}\right] = \sum_{-\infty} x_n \delta(f - nf_0).$$

Ex:-

$$x(t) = \sum_{-\infty}^{\infty} \delta(t - nt_0).$$

$F[x(t)]$

$$x(t) = \sum_{-\infty}^{\infty} x_n e^{j2\pi f_0 t}$$

$$x_n = \frac{1}{T_0} \int_{T_0} \delta(t) e^{-j2\pi n f_0 t} dt.$$

$$= \frac{1}{T_0} = f_0$$

$$\Rightarrow \sum_{-\infty}^{\infty} \delta(t - nt_0) \longleftrightarrow f_0 \sum_{-\infty}^{\infty} \delta(f - nf_0).$$

transform of a train of $\delta(t)$ is a scaled train (f_0) in frequency domain.

Summary :-

Hilbert transform → generation of an orthogonal signal.

$x^{(H)}(t) = x(t) \circledast \frac{1}{\pi t}$

$x^{H}(t) \to |X^{H}(f)| = |X(f)|$

$\to \measuredangle \begin{cases} \measuredangle X(f) - \frac{\pi}{2}, & f>0 \\ \measuredangle X(f) + \frac{\pi}{2}, & f<0 \end{cases}$

$\int_{D} x(t) \cdot x^{H}(t)\, dt = 0$

Ex:-

periodic signal / time limited signal.

Periodic

$x_p(t) = x(t) \circledast \sum_{-\infty}^{\infty} \delta(t - nT_0)$

- $F\left[\sum_{-\infty}^{\infty} x(t - nT_0)\right] \xrightarrow[\sum_{-\infty}^{\infty} x_n e^{j2\pi f_0 t}]{F.S} F[\text{series}]$

$= \sum_{-\infty}^{\infty} x_n \delta(f - nf_0)$.

- $F\left[\sum_{-\infty}^{\infty} \delta(t - nT_0)\right] \xrightarrow[x_n = \frac{1}{T_0} = f_0]{F.S} f_0 \sum_{-\infty}^{\infty} \delta(f - nf_0)$.

→ Forier theorem "convolution theorem".

$$F[x_p(t)] = F(x(t)) \cdot F\left[\sum_{-\infty}^{\infty} \delta(t - nT_0)\right]$$

$$= X(f) \cdot f_0 \sum_{-\infty}^{\infty} \delta(f - nf_0)$$

$$F[x_p(t)] = f_0 \sum_{-\infty}^{\infty} X(f)\, \delta(f - nf_0)$$

$$= f_0 \sum_{-\infty}^{\infty} X(n f_0)\, \delta(f - f_0)$$

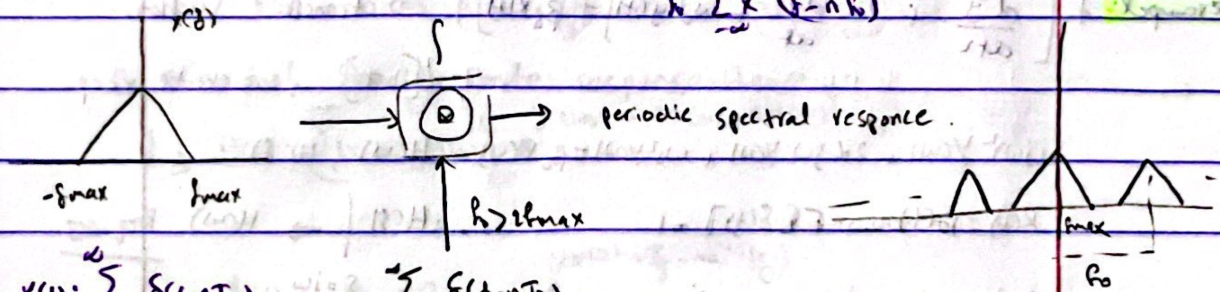
Sampling: Band-limited

$$x(t) \cdot \sum_{-\infty}^{\infty} \delta(t - nT_0) \longleftrightarrow X(f) \circledast F\left[\sum \delta(t - nT_0)\right]$$

$$\sum_{-\infty}^{\infty} X(f) \circledast f_0\, \delta(f - nf_0)$$

$$f_0 \sum_{-\infty}^{\infty} X(f - nf_0)$$

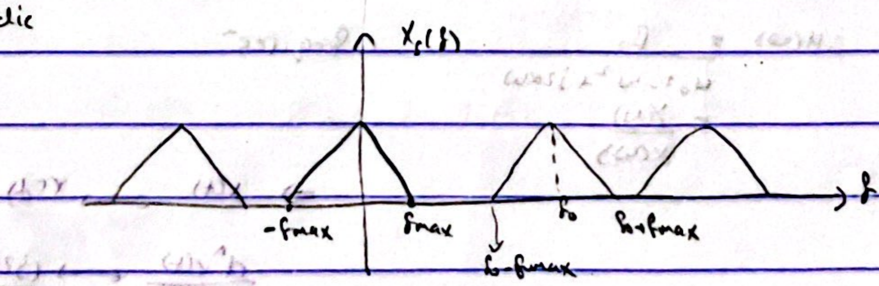
Ex:



$$x(t) \cdot \sum_{-\infty}^{\infty} \delta(t - nT_0) \longleftrightarrow f_0 \geq 2f_{max}$$

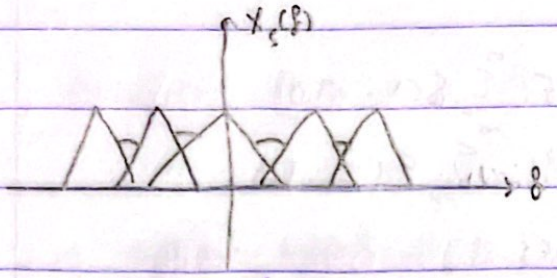
$$f_0 \sum_{-\infty}^{\infty} X(f - nf_0)$$

periodic



$$f_0 - f_{max} \geq f_{max}$$

$f_0 \geq 2f_{max} = f_N$: Nyquist sampling frequency.



Aliasing → No-signal reconstruction.

Freq. Dis

$$F[x_p(t)] = f_0 \sum_{-\infty}^{\infty} x(f)\, \delta(f - n f_0)$$
$$= f_0 \sum_{-\infty}^{\infty} x(n - f_0)\, \delta(f - n f_0).$$

Dynamic System:-

example:- $F\left[\frac{d^2y}{dt^2} + 2\alpha \frac{dy}{dt} + \omega_0^2 y(t)\right] = F[\beta_0 x(t)]$

$L\left[\frac{dx(t)}{dt}\right] = S\,X(S) - X(0)$

transfer function.

$L[h(t)] \rightarrow$ zero state resp.

$(j\omega)^2 Y(\omega) + 2\alpha j\omega Y(\omega) + \omega_0^2 Y(\omega) = \beta_0 X(\omega)$

$H(\omega)$, $H(f)$.

$x(t) = \delta(t) \rightarrow F[\delta(t)] = 1$

$H(S)\Big|_{S = j\omega} \rightarrow H(\omega)$ freq. res

↳

$(j\omega)^2 H(\omega) + 2\alpha j\omega H(\omega) + \omega_0^2 H(\omega) = \beta_0.$

$H(\omega)\left[\qquad\right] = \beta_0$

$\therefore H(\omega) = \frac{\beta_0}{\omega_0^2 - \omega^2 + j2\alpha\omega}$ ← freq. res

$= \frac{Y(\omega)}{X(\omega)}$

$\Rightarrow x(t) \longleftrightarrow X(f)$

$\frac{d^n x(t)}{dt^n} \longleftrightarrow \frac{(j2\pi f)^n}{j\omega} X(f)$

• Integ $\int \rightarrow \frac{1}{S}$

$\frac{1}{j2\pi f}$

$H(S) = \frac{Y(S)}{X(S)}$

$x(t) = \delta(t)$

$\delta(t) \longleftrightarrow 1$

Ex:-

- $\frac{dy}{dt} + 10\,y(t) = 12\,x(t)$

$(j\omega)Y(\omega) + 10\,Y(\omega) = 12\,X(\omega)$

$Y(\omega)\,[j\omega + 10] = 12\,X(\omega)$

- freq. resp

$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{12}{j\omega+10}$, $|H(\omega)| = \frac{12}{\sqrt{(j\omega)^2+100}}$

$\measuredangle H(\omega) = -\tan^{-1}\frac{\omega}{10}$

- Find $Y(t)$

$x(t) = 10\cos(40t + \frac{\pi}{3}) + 15\sin(30t + \frac{\pi}{6})$

- Sinisodal Steady state response theorem

$\hookrightarrow y(t) = \left\{ |X(t)|\,|H(\omega)|\, e^{j(n\omega_0 t + \measuredangle X(t) + \measuredangle H(\omega))} \right.$

$y_1(t) = 10 \cdot \frac{12}{\sqrt{(40j)^2+100}}\, e^{j(n40t + \frac{\pi}{3} - \tan^{-1}\frac{40}{10})}$

$y_2(t) = 15 \cdot \frac{12}{\sqrt{(30j)^2+100}}\, e^{j(n30t \quad - \tan^{-1}\frac{30}{10})}$.

$\Rightarrow \quad Y(s) = H(s) \cdot X(s)$.

$y(t) = \mathcal{L}^{-1}\left[H(s) \cdot X(s)\right]$

LTI $\quad H(s) = \dfrac{P(s) \rightarrow \text{polynomial}}{Q(s) \rightarrow \text{polynomial}} \qquad \mathcal{L}^{-1}$ "partial fraction".

proper system:-

dynamic of the system $y(t)$ > dynamic of $X(t)$.

[redacted]

ex:-

$$+ \frac{4d^2y}{dt^2} + \frac{6dy}{dt} + 5y(t) = \underbrace{\frac{5d^2x}{dt} + \quad + 7x(t)}_{\text{proper system}}.$$

$$Y(w)\left[-4w^2 + 6jw + 5\right] = (5jw + 7)X(w).$$

$$Y(w) = H(w) \cdot X(w) \quad \text{proper}$$

LTI

$$Y(w) = \frac{P(w) \rightarrow \text{excitation}}{Q(w) \rightarrow \text{system}} \qquad \text{order } P(w) < \text{order } Q(w)$$

$\rightarrow F^{-1}$ [partial fraction]

Steady state response to periodic signal.

by Fourier transform.

by F. series $\longrightarrow y(t) = \sum_{-\infty}^{\infty} |X_n| \, |H(n\omega_0)| \, e^{j(n\omega_0 t + \measuredangle X_n + \measuredangle H(n\omega_0))}$.

input signal $x(t)$ periodic

$$x(t) = \sum_{-\infty}^{\infty} X_n e^{j\omega_0 t n}$$

$$y(t) = x(t) \circledast h(t).$$

$$Y(f) = F[x(t)] \cdot F[h(t)] = H(f) \cdot \sum_{-\infty}^{\infty} X_n \, \delta(f - n f_0)$$

$$F\left[\sum_{-\infty}^{\infty} X_n e^{jn\omega_0 t}\right]$$

$$\Rightarrow Y(f) = \sum_{-\infty}^{\infty} X_n H(n f_0) \, \delta(f - n f_0)$$

$$y(t) = F^{-1}(Y(f))$$

$\Rightarrow$ F.T $\quad X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} \, dt.$

IFT $\quad x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} \, df$

$$y(t) = \sum_{-\infty}^{\infty} \left[X_n H(n f_0) \int_{-\infty}^{\infty} \delta(f - n f_0) e^{j2\pi f t} \, df\right]$$

$$= \sum_{-\infty}^{\infty} X_n H(n f_0) \, e^{j2\pi n f_0 t}$$

$$= \sum_{-\infty}^{\infty} |X_n| \, |H(n f_0)| \, e^{j(2\pi n f_0 + \measuredangle X_n + \measuredangle H(n f_0))}$$

Energy spectral density function.

$$x(t) \longleftrightarrow X(f)$$

parsevals theorem $\quad E = \int_{-\infty}^{\infty} |X(f)|^2 \, df.$

$$G_x(f) = |X(f)|^2. \quad = \int_{-\infty}^{\infty} G_x(f) \, df.$$

$$G_x(f) = |X(f)|^2 \xrightarrow{x(t)} \boxed{\begin{matrix} h(t) \\ H(f) \end{matrix}} \xrightarrow{y(t)} G_y(f).$$

$$Y(f) = H(f) \cdot X(f).$$

$$|Y(f)| = |X(f)| \cdot |H(f)|$$

$$G_y(f) = |Y(f)|^2$$

$$G_y(f), \; G_x(f)$$

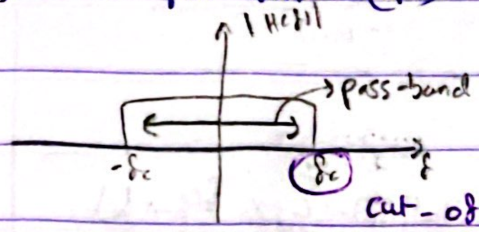
$$G_y(f) = |X(f)|^2 \cdot |H(f)|^2$$

$$G_y(f) = G_x(f) \, |H(f)|^2$$

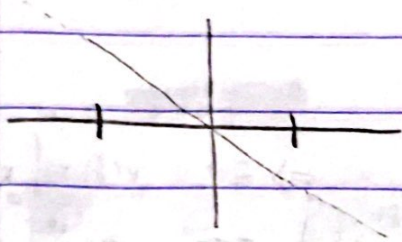
Filters:-

Ideal filters → Amplitude spectra.

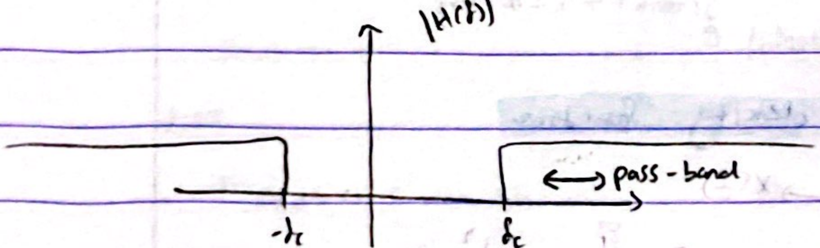
1) Low pass filter (LPF).



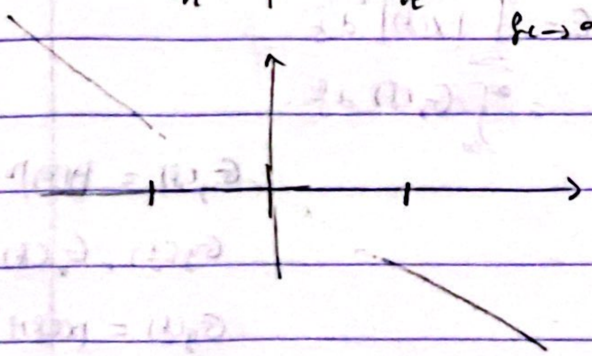
cut-off frequency.



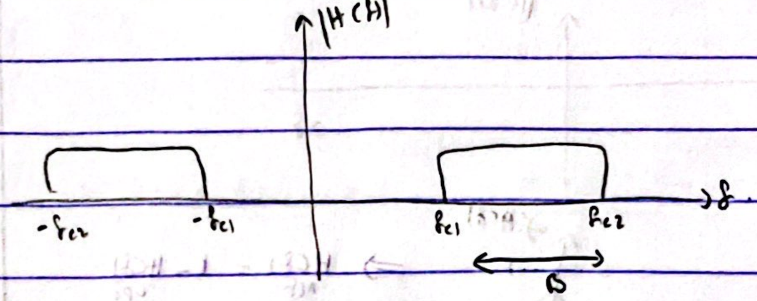
2) High pass filter (HPF).

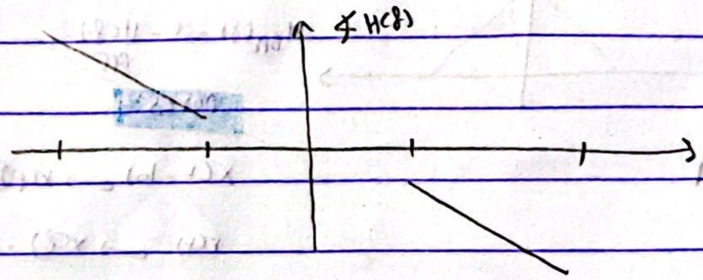


$f_c \to \infty$.

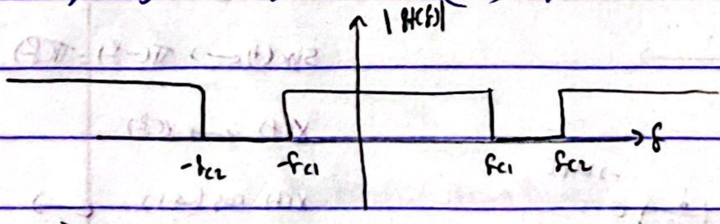


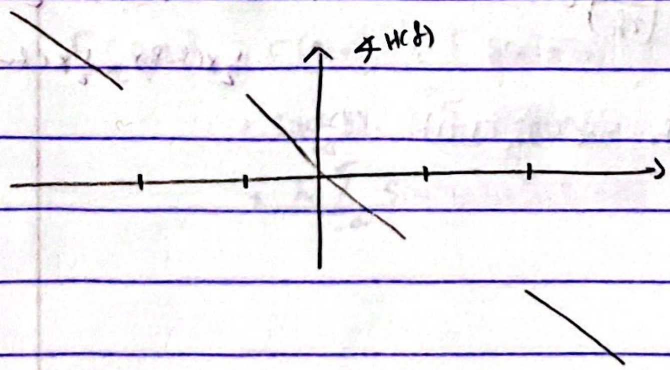
3) Band pass filter (BPF) :



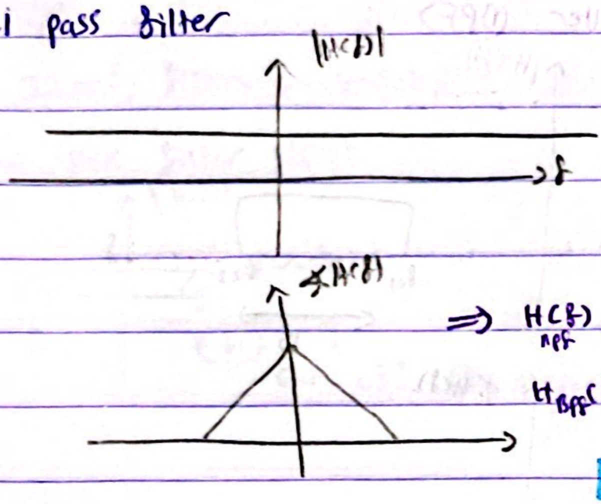


4) Band reject filter (BRF) Notch





5) All pass filter



$\Rightarrow H_{HPF}(f) = 1 - H_{LPF}(f)$

$H_{BPF}(f) = 1 - H_{BPF}(f)$:

Notes:-

$x(t - t_0) \longleftrightarrow X(f)\, e^{-j2\pi f t_0}$

$x(t) \longleftrightarrow X(f)$.

$x(\alpha t) \longleftrightarrow \frac{1}{|\alpha|} X(f/\alpha)$

$\Pi(t) \longleftrightarrow \text{sinc}(f)$

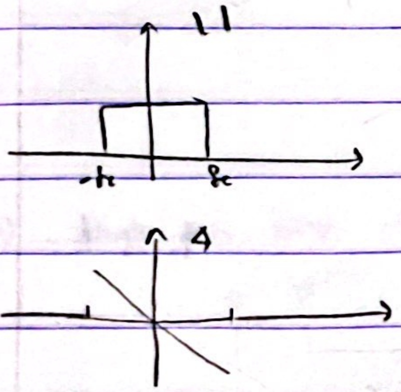
$\text{sinc}(t) \longleftrightarrow \Pi(-f) = \Pi(f)$.

$x(t) \longleftrightarrow X(f)$.

$x(t) \cos(\omega_0 t) \longleftrightarrow$

$\frac{1}{2} X(f - f_0) + \frac{1}{2} X(f + f_0)$.

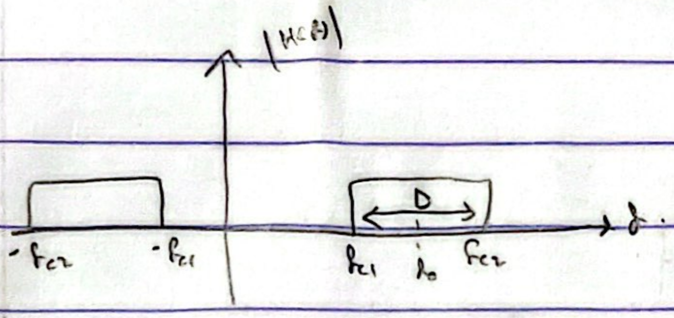
EX:-



$H_{LPF}(f) = \Pi\left(\frac{f}{2f_c}\right) e^{-j2\pi f T}$.

$h_{LPF}(t) = 2f_c \,\text{sinc}(2f_c(t - T))$

Ex:-



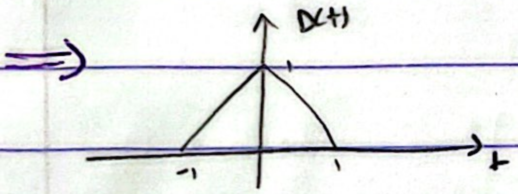
$\Rightarrow B = f_{c2} - f_{c1}$

$f_0 = \dfrac{f_{c1} + f_{c2}}{2}$

$$H_{BPF}(f) = \Pi\left(\frac{f - f_0}{B}\right) e^{-j2\pi f t} + \Pi\left(\frac{f + f_0}{B}\right) e^{-j2\pi f t}$$

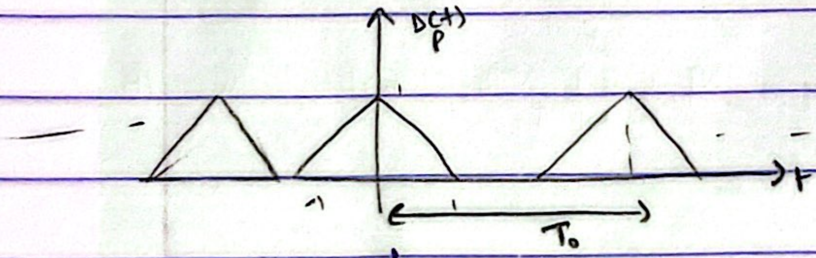
$$h_{BPF}(t) = h_{LPF}(t) + \cos\left[\underset{\omega}{2\pi f_0} (t - T)\right]$$

$\Rightarrow$



$$D_p(t) = D(t) \circledast \sum_{-\infty}^{\infty} \delta(t - nT_0)$$

$\Rightarrow D(t) \longleftrightarrow \text{sinc}^2(f)$



$\downarrow$ Fourier transform

$$F\{D_p(t)\} = F[D(t)] \cdot F[\delta(t - nT_0)]$$

$$= \text{sinc}^2(f) \cdot f_0 \sum_{-\infty}^{\infty} \delta(f - nf_0)$$

$$= f_0 \sum_{-\infty}^{\infty} \text{sinc}^2(nf_0)\, \delta(f - nf_0)$$

chapter(5):- Discrete Systems and Signals.

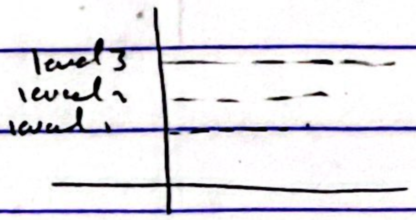
$x(t)$: $t \in \mathbb{R}$ → $\mathbb{R}$ → Anal[o]g.

$\mathbb{N}$ → Stair

$x(t)$: $t \in \mathbb{N}$ → $\mathbb{R}$ → Discrite

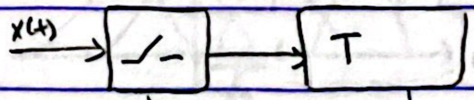
Discrite time signal $\mathbb{N}$ → discrite quantized signal

uniform sampling $t = nT_0$



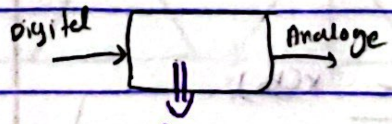
⇒ Analog to digital converges Digital signal

quatization.



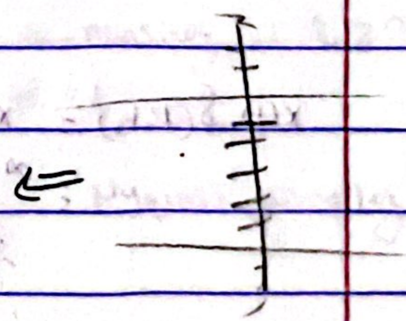
Sampel / Hold

→ threshold function quantization



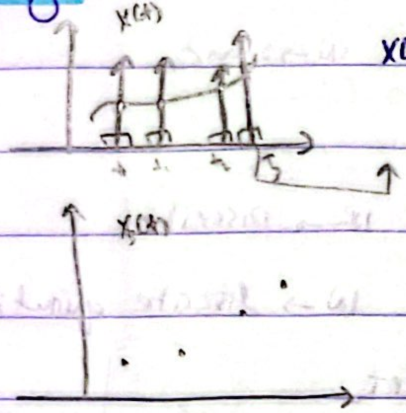
Digital to analong converter

bonded



[2,4] ⇒ not discrite.

Sampling:



$x(t) \longleftrightarrow x_s(t)$.

non ideal sampling.

reconstructed signal distortion.

من أجل الحصول على أ يعزز ب الΠ

$t \in \mathbb{N}$

integers.

$c(t) = \sum_{-\infty}^{\infty} \delta(t - nT_0)$ — ...↑ ↑ ↑.... uniform ideal sampling

$c(t) = \Pi\left(\frac{t-nT_0}{\tau}\right)$

$x(t)\,\delta(t-t_0) = x(t_0)\,\delta(t-t_0)$.

$= \int_{-\infty}^{\infty} x(t)\,\delta(t-t_0)\,dt = x(t_0)$

ضرب ال x(t) ب δ حتى يصبح Discrete

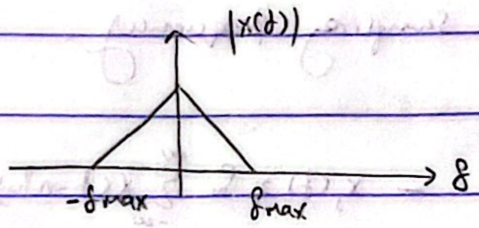
uniform ideal sampling :-

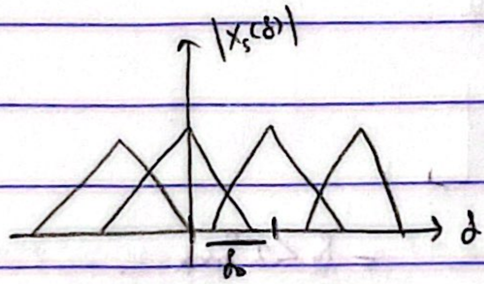
$$c(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0) \quad x(t) \longleftrightarrow X(f)$$

$$x_s(t) = x(t) \cdot \sum_{-\infty}^{\infty} \delta(t - nT_0)$$

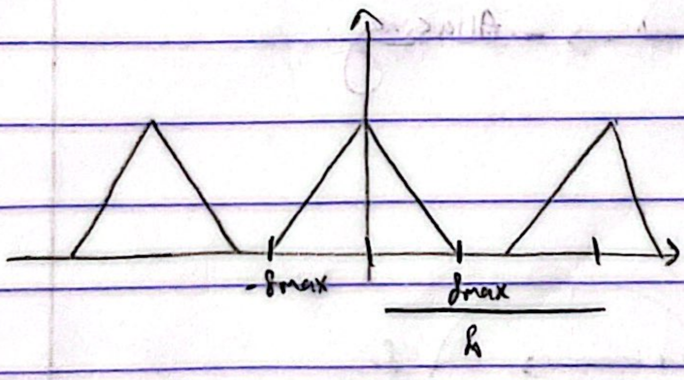
$$X_s(f) = X(f) \circledast f_0 \sum_{-\infty}^{\infty} \delta(f - nf_0)$$

$$= f_0 \, X(f - nf_0) .$$





Aliasing



No-Aliasing $\rightarrow f_0 \geq 2 f_{max} = f_N$

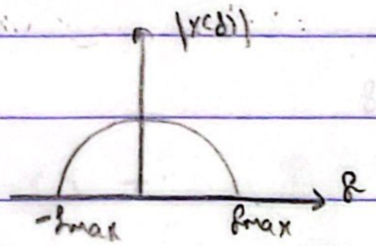
$f_N = 2 f_{max}$

" Nyquist sampling freq "

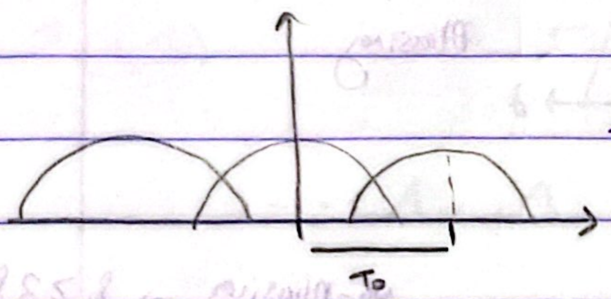
Sampling theorem:-

given a low-pass band limited signal with maximum frequency f_{max}, the minimum sampling frequency with which the signal can be reconstructed correctly is $f_N = 2 f_{max}$

Nyquist sampling frequency

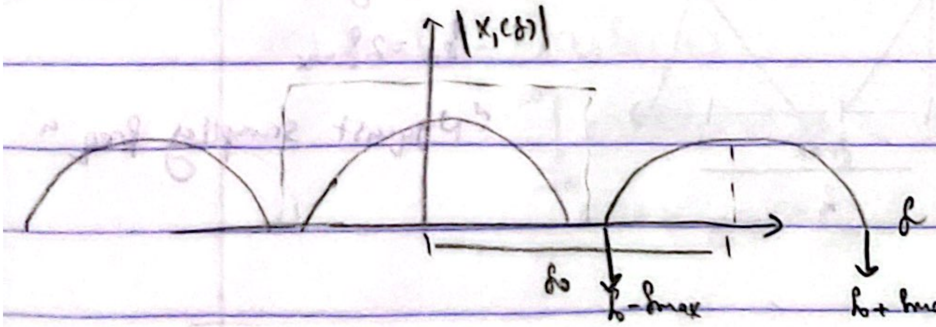


– $x_s(f) = f_0 \sum_{-\infty}^{\infty} x(f - n f_0)$



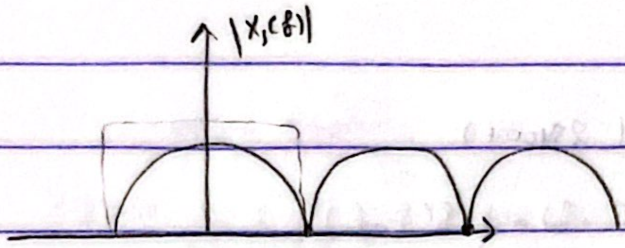
– $f_0 < 2 f_{max}$

– Aliasing



– $f_0 > 2 f_{max}$

– reconstruction.



$-X_s(f) = f_s \sum_{n}^{\infty} X(f - n f_s)$

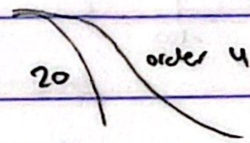
$- f_s = 2 f_{max}$

ideal low pass.

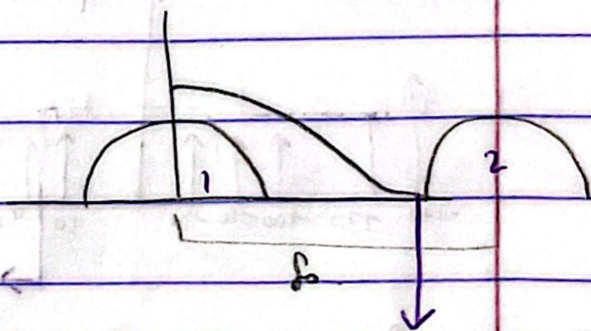
sampel /sec.

$\Rightarrow$ over sampling.

for filter.



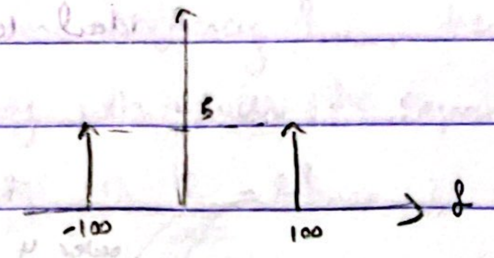
• كلما اصبح قريب من ال 90° بصير يحتاج order أعلى و جهد أكثر



• موقع ال 2 مناسب للفلتر، إذا وصلت أبعد من هيك يصبح بلا فائدة لأنه ال filter قريبة من ال 0 وكذلك يصبح عدد ال signal من نفس ال ... أقل.

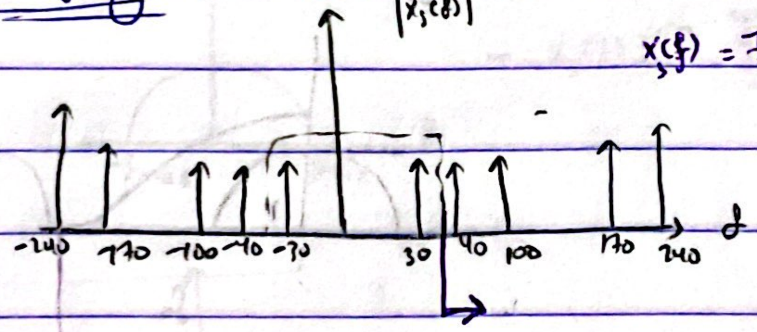
Ex:- $x(t) = 10\cos(2\pi 100 t)$

$$|X(f)| = \frac{1}{2}\delta(f-f_0) + \frac{1}{2}\delta(f+f_0)$$



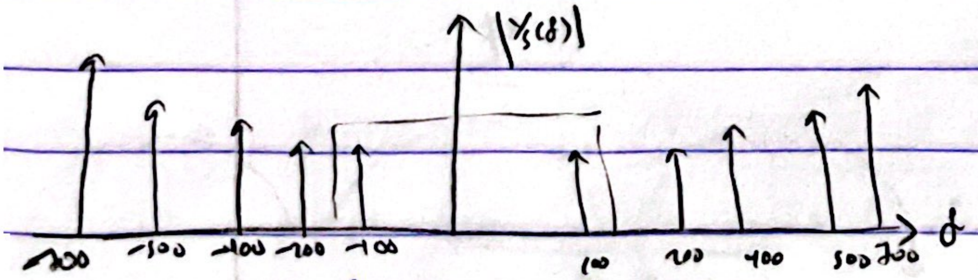
Sampling: $f_0 = 70$ $< 2f_{max}$

$$X_s(f) = 70 \sum_{-\infty}^{\infty} \delta(f-100-nf_0) + \delta(f+100-nf_0)$$



• يتم وضع الفلتر عند أقرب نقطة للأصل ومقارنة ال freq مع الأصلية.
إذا لم تكن نفسها يكون يوجد Aliasing.

$f_0 = 300$ $> 2f_{max}$



- ideal reconstruction filter.

$$\Pi\left(\frac{f}{f_0}\right) e^{-j2\pi f \tau}$$

• $\frac{1}{f_0} = T_0$

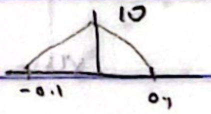
$$70 \rightarrow \frac{1}{f_0}\,\Pi\left(\frac{f}{70}\right) e^{-j2\pi f\tau}$$

$$300 \rightarrow \frac{1}{f_0}\,\Pi\left(\frac{f}{300}\right) e^{-j2\pi f\tau}$$

Ex:- $x(t) = \text{sinc}^2(t)$ (0.1)

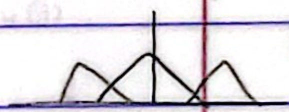
$X(f) = \frac{1}{0.1}\,\Delta\left(\frac{f}{0.1}\right)$ ~> scaling freq

$= 10\,\Delta(10f)$



- $\text{sinc}^2(t) \longleftrightarrow \Delta(-f) = \Delta(f)$.
- $\Delta(t) \longleftrightarrow \text{sinc}^2(f)$.
- $x(t) \longleftrightarrow X(-f)$
- $x(at) \longleftrightarrow \frac{1}{|a|} X(f/a)$

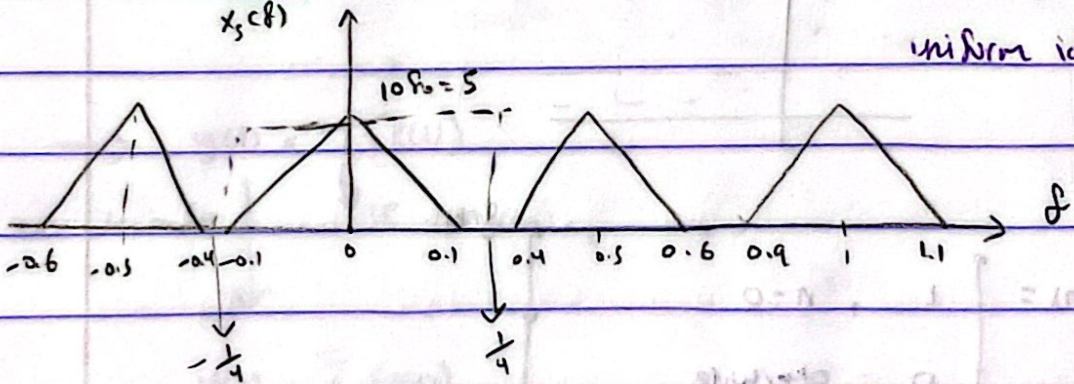
$f_N = 2 f_{max} = 2\times 0.1 = 0.2\ Hz$ "minimum".

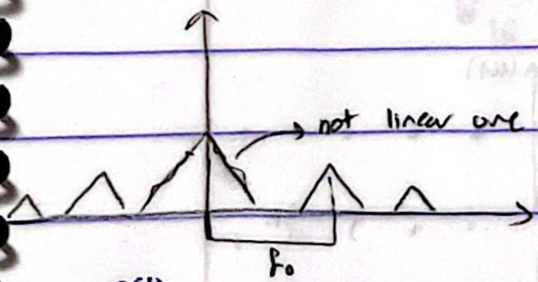


$f_N < 2 f_{max}$

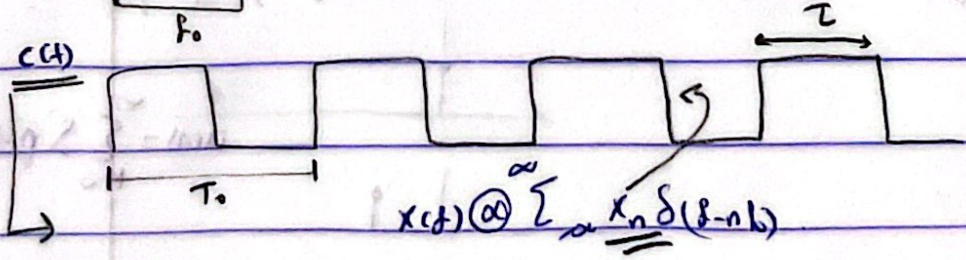
$f_0 = 0.5$ > 0.2

uniform ideal sampling.





$f_0 \geq f_N$.

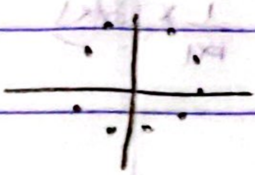


$x(t) \otimes \sum_{-\infty}^{\infty} x_n\,\delta(f - n f_0)$

$\sum \delta(t - nT_0)$.

Discrate Signal.

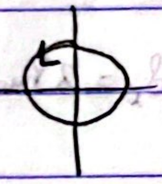
1) Sinosodal



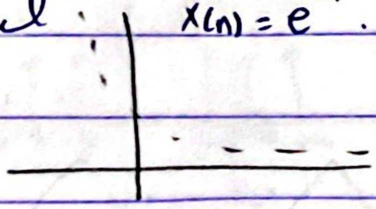
$x_n = X\cos(\omega n+\phi)$ discrate.

$x(t) = X\cos(\omega t+\phi)$

continuos.

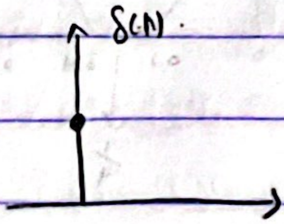


2) exponatial:

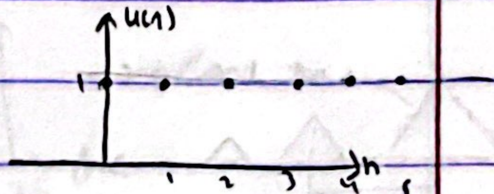


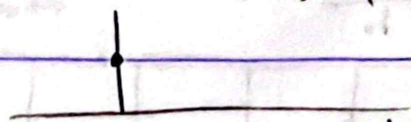
$x(n) = e^{\alpha n}$.

3) $\delta(n) = \begin{cases} 1 & , n=0 \\ 0 & , \text{otherwise} \end{cases}$

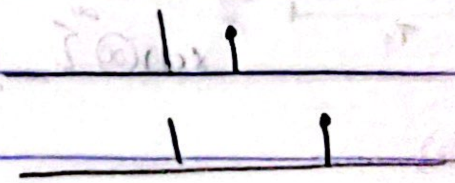


$u(n) = \begin{cases} 1 & , n\geq 0 \\ 0 & , \text{otherwise} \end{cases}$



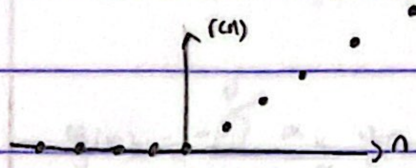


$u(n) = \sum_{k=0}^{\infty} \delta(n-k)$

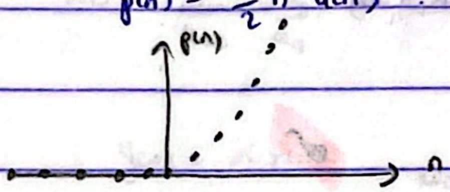


$r(n) = n\,u(n)$, $r(t) = t\,u(t)$

$x(n) = n$, $n \geq 0$



$p(n) = \frac{1}{2} n^2 u(n)$, $p(t) = \frac{1}{2} t^2 u(t)$.



$\Longrightarrow$ $y(t) = T\{x(t)\}$

$\downarrow$ $\mathbb{R}$ Domain

discrete $\downarrow$

$y(n) = \bar{T}\{x(n)\}$

$\downarrow$ $\mathbb{N}$ Domain operation

Classification of discrete system:-

1) linear / nonlinear.

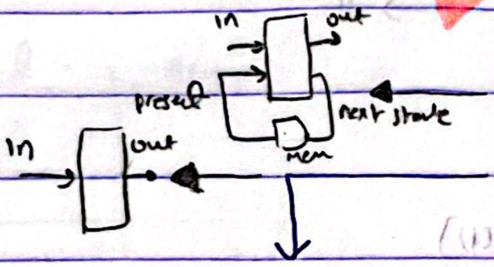
$x_1(n) \to y_1(n)$.

$x_2(n) \to y_2(n)$.

$x(n) = \underline{\alpha_1 x_1(n) + \alpha_2 x_2(n)} = y(n)$, α_1, α_2 parameter $\in \mathbb{R}$.

$y(n) = \alpha_1 y_1(n) + \alpha_2 y_2(n)$.

linear.



Seq $\Rightarrow$ Discrete.

comp $\Rightarrow$ static.

$y(n) = 3x^2(n)$, static

, finite impulse response " $\xrightarrow{FIR}$ $y(n) = 3x^2(n) + 4x^2(n-1)$, Dynamic

" Infinite impulse response " $\xrightarrow{IIR}$ $y(n) = 3x^2(n) + 4y(n-1)$, Dynamic difference equation

$$\sum_{k=0}^{N} a_k\, y(n-k) = \sum_{k=0}^{M} b_k\, x(n-k)$$ → scalar form of a linear time invariant shift system

$$y(n) = -\sum_{k=1}^{N} \frac{\alpha_k}{\alpha_0}\, y(n-k) + \sum_{k=0}^{M} b_k\, x(n-k) \quad \text{(IIR)}$$ direct form

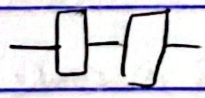
$$y(n) = \sum_{k=0}^{M} \frac{b_k}{a_0}\, x(n-k)$$ → FIR.

$= \quad y(n) = \alpha\, x(n)$

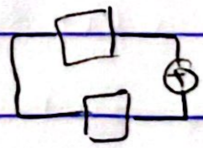
$$= \frac{1}{(s+1)(s+2)} = \frac{1}{(s+1)} \cdot \frac{1}{(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+2)}$$

second order implementation.

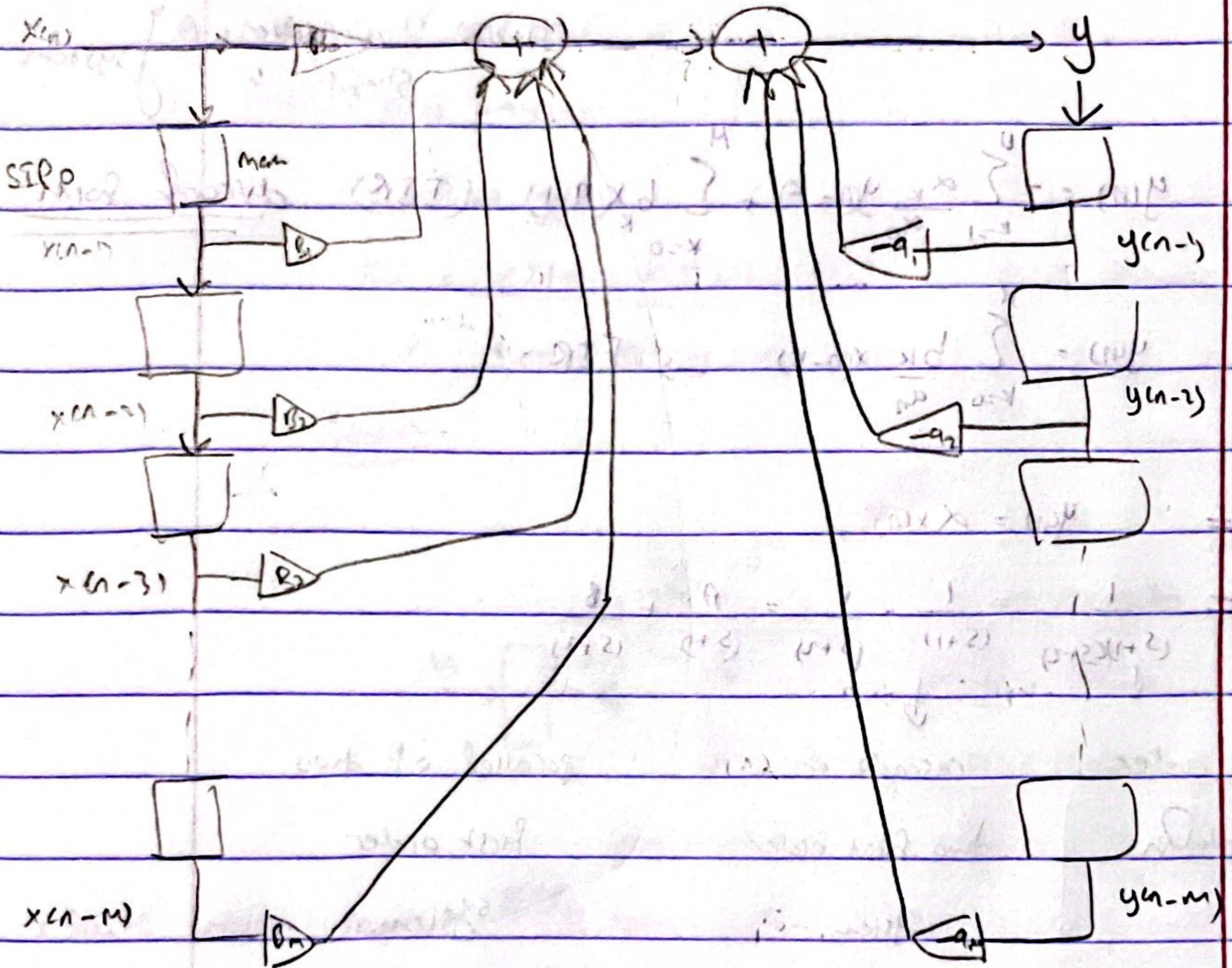
cascade of two first order system



parallel of two first order system



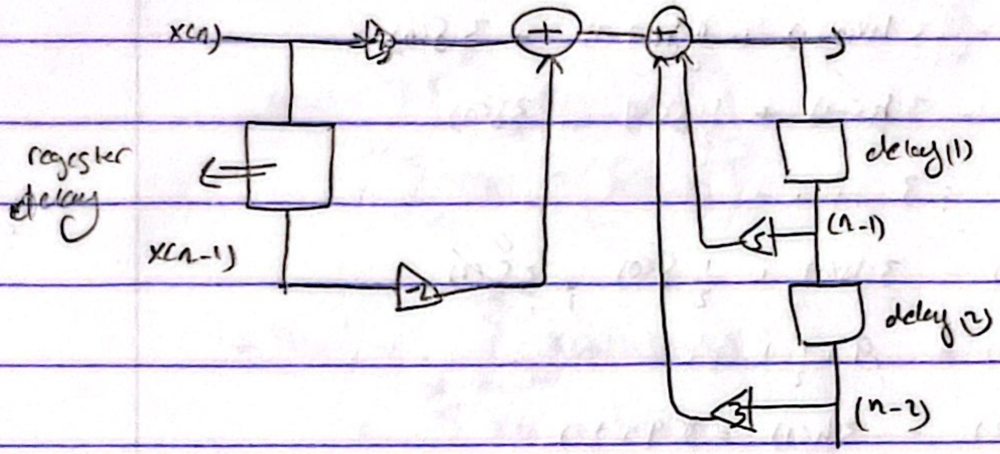
System Implementation "Direct form I".



$y(n) = 5\,y(n-1) + 3\,y(n-2) - 2\,x(n-1) - 3\,x(n)$. IIFR , order , linear .

Direct form I "

" implementation



Solution of difference equation

Impulse response .

$h(n)$ $\quad x(n)$

$y(n) = \frac{1}{2} y(n-1) + 2\,\delta(n)$

iterative process ; causal "

$h(0) = \frac{1}{2} h(-1) + \delta(0)$ 2

$h(0) = 2$

$h(1) = \frac{1}{2} h(0) + \delta(1)$

$h(1) = \frac{1}{2} h(0) = 2 \times \frac{1}{2} = 1$

$h(2) = \frac{1}{2} h(1) = \frac{1}{2}$

$h(3) = \frac{1}{2} h(2) = \frac{1}{4} = \left(\frac{1}{2}\right)^2$

$h(4) = \frac{1}{2} h(3) = \left(\frac{1}{2}\right)^3$.

$\vdots$

$h(n) = \left(\frac{1}{2}\right)^{n-1} u(n)$.

Ex:. $y(n) = 3y(n-1) + \frac{1}{2}x(n-1) + 3x(n)$: IIR

$h(n) = 3h(n-1) + \frac{1}{2}\delta(n-1) + 3\delta(n)$

$h(0) = 3\cancel{h(-1)} + \cancel{\frac{1}{2}\delta(-1)} + 3\delta(0)$

- $h(0) = 3$

$h(1) = 3h(0) + \frac{1}{2}\delta(0) + \cancel{3\delta(1)}$

- $h(1) = 9 + \frac{1}{2}$
- $h(2) = 3h(1) = 3(9 + \frac{1}{2})$
- $h(3) = 3h(2) = 3\cdot3(9+\frac{1}{2})$.

$\vdots$

$h(n) = 3^{n-1}(9+\frac{1}{2})u(n)$

Convolution theorem:

Given a LTI System with Impulse response $h(n)$ the response of the system to any input $x(t)$ can be computed by the convolution Sum:

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

$$\sum_{-\infty}^{\infty} a^k = \frac{1}{1-a}$$

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k)$$

$$\sum_{0}^{\infty} a^k = \frac{1-a^{k+1}}{1-a}$$

$$y(n) = T\left[\sum_{-\infty}^{\infty} x(k)\,\delta(n-k)\right]$$
$$= \sum_{-\infty}^{\infty} x(k)\,T\left[\delta(n-k)\right]$$

$$\therefore y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$
$$= \sum_{k=-\infty}^{\infty} x(n-k)h(k)$$

$x(n)$ unilateral. $x(n)=0,\ \forall n<0$ "causal system"

$$y(n) = \sum_{k=0}^{\infty} x(k)h(n-k)$$

$h(\text{negative})=0$.

$$y(n) = \sum_{k=0}^{\infty} x(k)h(n-k)\ ,\ k>0$$

"convergent series" $\sum_{0}^{\infty} a^k$.

$\Rightarrow x(n) = \left(\frac{1}{2}\right)^n u(n)$. First fn

$\Rightarrow h(n) = (2)^n u(n)$ sec s

sol

$$y(n) = \sum_{k=0}^{\infty} \left(\tfrac{1}{2}\right)^k \cdot 2^{n-k}$$
$$= 2^n \sum_{0}^{\infty} \tfrac{1}{2}^k\, 2^{-k}$$
$$= 2^k \sum_{0}^{\infty} \tfrac{1}{2}^k \cdot \tfrac{1}{2}^k = 2^k \sum_{0}^{\infty} \left(\tfrac{1}{4}\right)^k$$

$$= 2^n \left[\frac{1-\left(\frac{1}{4}\right)^{n+1}}{1-\frac{1}{4}}\right]$$

$$= \frac{4}{3}\cdot 2^n \left[1-\left(\tfrac{1}{4}\right)^{n+1}\right]$$

$x(n) = (2)^n\, u(n-2) \Rightarrow 2^{n-2}$

$h(n) = \left(\frac{1}{3}\right)^n u(n) \Rightarrow \left(\frac{1}{3}\right)^n$

$$y(n) = \sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^k \cdot 2^{n-2-k}$$

$$= 2^{n-2} \sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^k 2^{-k} = 2^{n-2} \sum_{k=0}^{\infty} \left(\frac{1}{6}\right)^k$$

$$= 2^{n-2} \frac{1-\left(\frac{1}{6}\right)^{k+1}}{1-\frac{1}{6}}$$

$$= \frac{6}{5} \cdot 2^{n-2} \left(1-\left(\frac{1}{6}\right)^{k+1}\right)$$

Z - transform.

$x(n)$ unilateral $x(n) = 0 \quad \forall n < 0$.

Single-sided Z-transform. $n \in \mathbb{N}$.

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n} . \quad z \in \mathbb{C} .$$

$$x(n) = \frac{1}{2\pi j} \oint X(z)\, z^{n-1}\, dz .$$

$$x_s(t) = \sum_{n=0} x(nT_s)\, \delta(t - nT_s) .$$

Laplace transform.

$x(t)$, $t \in \mathbb{R}$

$$\frac{A}{s+\alpha} = A e^{-\alpha t} u(t)$$

Singel sided Laplace transform

$$X(s) = \int_0^{\infty} x(t) e^{-st}\, dt , \quad x(t) = 0 \quad \forall t < 0 .$$

$X(s) : s \in \mathbb{C} \to \mathbb{C}$.

rational function in (s) $\to$ partial fraction.

" linear diff equation with constant cofficent " .

$$\frac{1}{2\pi j} \oint X(s) e^{st}\, ds .$$

$$\Rightarrow \quad x_s(t) = \sum_{n=0}^{\infty} x(nT_s)\,\delta(t-nT_s).$$

$$\mathcal{L}\left[x_s(t)\right] = \int_0^{\infty} \sum_0^{\infty} x(nT_s)\,\delta(t-nT_s)\,e^{-st}\,dt.$$

$$= \sum_0^{\infty} x(nT_s) \int_0^{\infty} \delta(t-nT_s)\,e^{-st}\,dt$$

$$= \sum_0^{\infty} x(nT_s)\,e^{-snT_s} \xrightarrow[\text{to } T_s]{\text{normalized}} \sum x(n)\,e^{-ns}$$

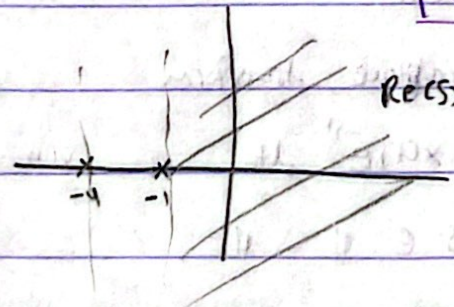
$$X(z) = \sum_0^{\infty} x(n)\,z^{-n} \qquad \Rightarrow \sum_0^{\infty} x(n)\,z^{-n} = X(z)$$

$X(z)$:

$\boxed{z = e^{s}}$ mapping

$$X(s) = \frac{1}{s+4} + \frac{1}{s+1}$$

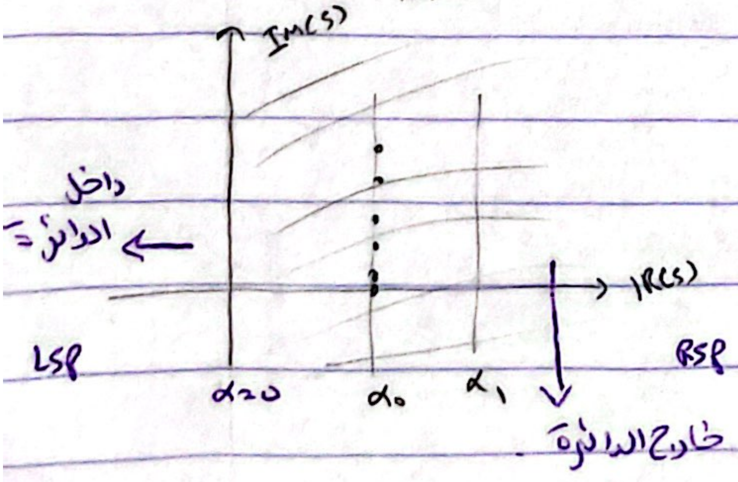
Singularity point



$Re(s) > -1$

$\Rightarrow$ x_0 is a singularity points of $f(x)$ $\Leftrightarrow$

$$\lim_{x \to \infty} |f(x)| = \infty$$



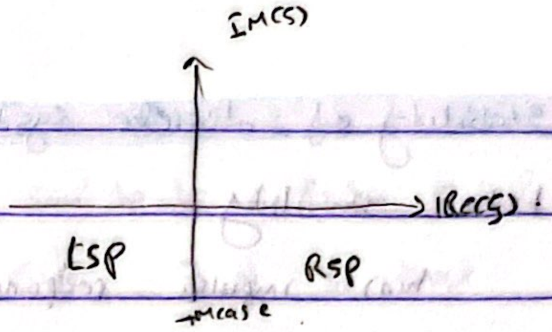
$$\Rightarrow s = \alpha + j\omega.$$

$$e^{-st} = e^{-(\alpha+j\omega)t}$$

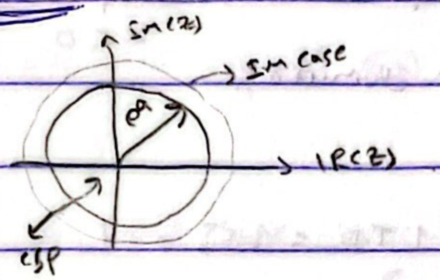
$$= e^{-\alpha t}\,e^{-j\omega t}.$$

$\Rightarrow z = e^{\alpha + j\omega}$

$= e^{\alpha} \cdot e^{j\omega}$



$\alpha < 0 \rightarrow e^{\alpha} < 1$



$\alpha = 0$

$e^{\alpha} e^{j\omega} = 1 \, e^{j\omega}$

$\alpha > 0$

$e^{\alpha} > 1$

$z = e^{s}$ → lines in s domain to circle in z domain.

LSP → Inside the circle of radius = 1

Im-axe → circle of radius = 1.

RSP → outside the circle of radious = 1.

Stability of discrete system:

=> Asymptotic stability.

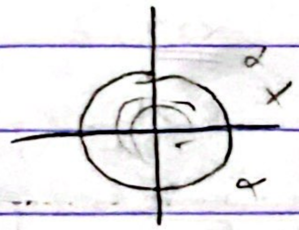
h(n) impulse response. LSP $\alpha < 0$

$\lim_{n \to \infty} h(n) = 0$ => time domain.

z - domain analysis. LTI = LSI.

1- the system is asymptotically stabel $\Leftrightarrow$ $|z_p| < 1$

2- is $\exists$ z_p so that $|z_p| > 1$ unstabel.

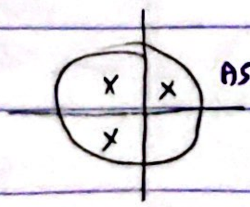


z_{p_i} → $|z_{p_i}| < 1$

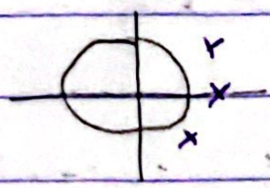
→ $|z_{p_i}| = 1$

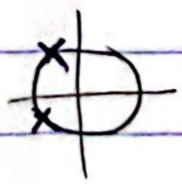
all the z_{p_i} with $|z_{p_i}| = 1$ and <u>Not</u> repeted

→ BIBO stable if $\exists$ a response pole with $|z_{p_i}| = 1$, then the system is unstabel.



AS. stability.





A discrete system is said to be BIBO stable $\Longleftrightarrow$

$\forall x(n)$, $|x(n)| \leq M$. $\exists N$ so that

$|y(n)| \leq N$.

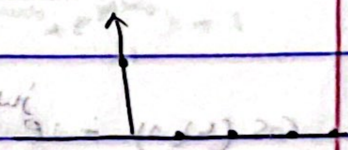
__theorem:__

A linear time invariant system with impulse response $h(n)$ is BIBO stable $\Longleftrightarrow \sum_{n=-\infty}^{\infty} |h(n)| < \infty$

- elementary signals computation of Z-transform.

  - $x(n) = \delta(n)$, $\sum_{n=0}^{\infty} \delta(n) z^{-n} = 1$

    $\delta(n) \longrightarrow 1$



  - $x(n) = u(n)$

    $X(z) = \sum_{n=0}^{\infty} z^{-n} = \sum_{n=0}^{\infty} 1 \cdot z^{-n} = \sum_{n=0}^{\infty} (z^{-1})^n$

$\sum_{n=0}^{\infty} z^{-n} = \sum_{n=0}^{\infty} (z^{-1})^n$ $\qquad \sum_{n=0}^{\infty} a^n \longrightarrow \frac{1}{1-a}$

converges $\Longleftrightarrow |z^{-1}| < 1$ $\qquad |a| < 1$

$\Rightarrow |z| > 1$

$\longrightarrow X(z) = \frac{1}{1-z^{-1}}$ خارج الدائرة وحدة z^{-1}

$X(z) = \frac{z}{z-1}$ داخل الدائرة وحدة z

$u(n) \longleftrightarrow \frac{z}{z-1}$

note:-

$x(n) = k^n u(n)$

$X(z) = \sum_{n=0}^{\infty} k^n z^{-n}$

"Single sided"

$= \sum_{n=0}^{\infty} (k z^{-1})^n$

converges

$|k z^{-1}| < 1$

$\Rightarrow |z| > |k|$

$X(z) = \frac{1}{1-kz^{-1}} = \frac{z}{z-k}$

$f(x) = x^1 \quad x \in (1,2]$

$f(x) = x^2 \quad x \in [3,4]$

$\Rightarrow$ differential relation

$\Rightarrow \cos(w_0 n) \longleftrightarrow$

$\frac{z^2 - z\cos w_0}{z^2 - 2z\cos w_0 + 1}$

$\Rightarrow$

$\cos(w_0 n) = \frac{e^{jw_0 n} + e^{-jw_0 n}}{2}$

$Z[\cos(w_0 n)] = \frac{1}{2} Z[e^{jw_0 n}] + \frac{1}{2} Z[e^{-jw_0 n}]$

$= \frac{1}{2} \frac{z}{z-e^{jw_0}} + \frac{1}{2} \frac{z}{z-e^{-jw_0}}$

$= \frac{z}{2} \frac{2z - (e^{jw_0} + e^{-jw_0})}{z^2 - z(e^{jw_0} + e^{-jw_0}) + 1}$

$\Rightarrow \frac{1}{2} \frac{z(2z - 2\cos w_0)}{z^2 - 2z\cos w_0 + 1}$

$\Rightarrow$ $\sin(w_0 n) = \dfrac{e^{jw_0 n} - e^{-jw_0 n}}{2j}$

$Z[\sin(w_0 n)] = \frac{1}{2j} Z[e^{jw_0 n}] - \frac{1}{2j} Z[e^{-jnw_0}]$

$= \frac{1}{2j} \dfrac{z}{z - e^{jnw_0}} - \frac{1}{2j} \dfrac{z}{z - e^{-jnw_0}}$

$= \frac{1}{2j} \dfrac{\cancel{z^2} - ze^{-jnw_0} \cancel{- z^2} + ze^{jnw_0}}{z^2 - ze^{-jnw_0} - ze^{jnw_0} + 1}$

$= \frac{1}{2j} \dfrac{z(e^{jnw_0} - e^{-jnw_0})}{z^2 - z(e^{-jnw_0} + e^{jnw_0}) + 1}$

$= \cancel{\frac{1}{2j}} \dfrac{z \cancel{2j} \sin(w_0 n)}{z^2 - 2z\cos(w_0 n) + 1}$

$= \dfrac{z \sin(w_0 n)}{z^2 - 2z\cos(w_0 n) + 1}$

Z transform theorem:

1) Linearity

$x_1(n) \longleftrightarrow X_1(z)$.

$x_2(n) \longleftrightarrow X_2(z)$.

$\Rightarrow x(n) = \alpha_1 x_1(n) + \alpha_2 x_2(n) \longleftrightarrow \alpha_1 X_1(z) + \alpha_2 X_2(z)$

2) time shift .

$x(n) \longleftrightarrow X(z)$

$\bar{x}(n) = x(n-m) \longleftrightarrow z^{-m} X(z)$

$\bar{X}(z) = \sum_{n=0}^{\infty} x(n-m) z^{-n}$. "shift right"

- $k = n-m$

$n=0 \to k=-m$.

$n=\infty \to k=\infty$

$\bar{X}(z) = \sum_{k=-m}^{\infty} x(k) z^{-(k+m)}$

$= z^{-m} \sum_{k=0}^{\infty} x(k) z^{-k} \rightsquigarrow x(n)=0 \quad \forall n<0$.

$= z^{-m} X(z)$.

$\bar{x}(n) = x(n+m)$ "shift left"

$k = n+m$.

$n=0 \rightarrow k=m$.

$n=\infty \rightarrow k=\infty$.

$$\bar{X}(z) = \sum_{n=0}^{\infty} x(n+m) z^{-n}.$$

$$= z^{m} \sum_{k=m}^{\infty} x(k) z^{-k} = -z^{m} \sum_{k=0}^{m-1} x(k) z^{-k} + z^{m} \sum_{n=0}^{\infty} x(k) z^{-k}.$$

$$= -z^{m} \sum_{k=0}^{m-1} x(k) z^{-k} + z^{m} X(z)$$

Continues

initial value theorem.

if $\exists \lim_{t \to 0} x(t) = x_0$.

then $x_0 = \lim_{s \to \infty} s X(s)$

final value theorem

$1 + \cos(n\omega_0)$

if $\exists \lim_{t \to \infty} x(t) = x_{\infty}$ $\quad \lim_{n \to \infty}$ (1) + dose not exist.

then $x_{\infty} = \lim_{s \to 0} s X(s) = x_{\substack{\text{steady} \\ \text{particular}}}$.

⟹ in this part of the course :-

initial value theorem.

$z = e^{s}$

if $\exists \lim_{n \to \infty} x(n) = x_0$

then $x_0 = \lim_{z \to \infty} X(z)$

$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n} = x_0$

Final value theorem

If $\exists \lim_{n \to \infty} x(n) = x_{\infty}$

then $x_{\infty} = \lim_{z \to 1} (1 - z^{-1}) X(z)$ → steady response.

x_{steady} = final value.

→ System Asymptitic stability.

3) Scaling

$x(n) \rightarrow \tilde{x}(n) = a^{n} x(n)$.

$x(n) \leftrightarrow X(z)$

$\tilde{X}(n) = \sum_{n=0}^{\infty} a^{n} x(n) z^{-n}$

$= \sum_{n=0}^{\infty} x(n) \left(\frac{a}{z}\right)^{n} = \sum_{n=0}^{\infty} x(n) \left(\frac{z}{a}\right)^{-n}$

$= X\left(\frac{z}{a}\right)$

transfer function of a discrete system. $H(z) = Z\{h(n)\}$.

- $x(n) \longleftrightarrow X(z)$
- $y(n) \longleftrightarrow Y(z)$

$$\sum_{k=0}^{\infty} a_k\, y(n-k) = \sum_{k=0}^{M} b_k\, x(n-k)$$

- $y(n) = -\sum_{k=1}^{\infty} \frac{a_k}{a_0} y(n-k) + \sum_{k=0}^{M} b_k\, x(n-k)$.

$$Z[\quad] = Z[\quad]$$

- $\delta(n) \longleftrightarrow 1$

System transfer function. $H(z)$.

- $\mathbb{C} = \mathbb{R} \times \mathbb{R}$

continuous

$x(n) \longleftrightarrow X(z)$.

$x(n-m) = z^{-m} X(z)$.

$Z\{y(n-k)\} = z^{-k} Y(z)$

continuo

$$\sum_{k=0}^{N} \left[a_k z^{-k} Y(z)\right] = \sum_{k=0}^{M} \left[b_k z^{-k} X(z)\right]$$

$$Y(z) \sum_{k=0}^{N} a_k z^{-k} = X(z) \sum_{k=0}^{M} b_k z^{-k}.$$

$$\therefore H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{\infty} b_k z^{-k}}{\sum_{k=0}^{\infty} a_k z^{-k}} \longrightarrow \text{IIR}.$$

$\Rightarrow$ System zeros

z_0 so that $|H(z_0)| = 0$

$\Rightarrow$ System poles, $z_p \longrightarrow \lim_{z \to z_p} |H(z)| \to \infty$.

• External response

$$H(z) = \frac{\cancel{(z-1)}(z-2)}{\cancel{(z-1)}(z-3)(z-4)}$$ "input-output relation"

• $z^2 - 2z + \frac{1}{2} = 0$

complex domain

$\exists$ sol $\forall z$.

$$Z\{y(n)\} = Z\left[\sum_{k=0}^{M} b_k x(n-k)\right] \rightarrow \text{FIR}$$

$$Y(z) = \sum_{k=0}^{M}\left[b_k z^{-k} X(z)\right]$$

$$H(z) = \frac{Y(z)}{X(z)} = \sum_{k=0}^{M} b_k z^{-k}$$. "origin singularity"

Ex.- Find $H(z)$:- 2'order //

$$Z\left[y(n) - 2y(n-1) + \frac{1}{2}y(n-2)\right] = Z\left[3x(n) - 4x(n-1)\right]$$

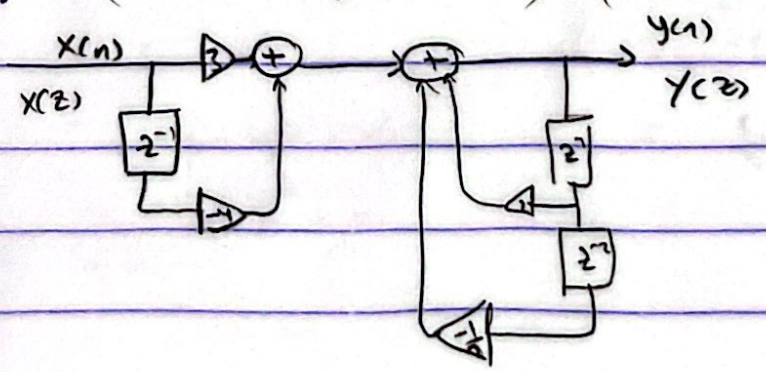
$$Y(z) - 2z^{-1}Y(z) + \frac{1}{2}Y(z)z^{-2} = 3X(z) - 4z^{-1}X(z)$$

$$\therefore H(z) = \frac{3-4z^{-1}}{1-2z^{-1}+\frac{1}{2}z^{-2}}$$

transfer function. $\frac{3-4z^{-1}}{z^{-2}\left[z^2-2z+\frac{1}{2}\right]} = \frac{3z^2-4z}{z^2-2z+\frac{1}{2}} = H(z)$

direct form I:-

$$y(z) = \left(2z^{-1}Y(z) - \frac{1}{2}z^{-2}Y(z)\right) + \left(3X(z) - 4z^{-1}X(z)\right)$$



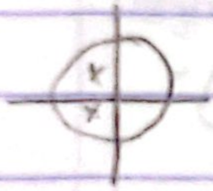
$y(n) = 3x(n) - 4x(n-1)$

$Y(z) = 3X(z) - 4z^{-1}X(z)$

$\frac{Y(z)}{X(z)} = 3 - 4z^{-1} = 3 - \frac{4}{z}$

$= \frac{3z - 4}{z}$

FIR



IIR

Inverse Z-transform

LTI $X(Z) \leftrightarrow x(n)$

$H(Z) = \frac{P(Z)}{Q(Z)}$ rational function of polinomyel

↓ ord $P(Z)$ < ord $Q(Z)$

partial fraction

↓

Inverse of basic function

$\Rightarrow u(t) \leftrightarrow \frac{1}{s}$

$u(n) \leftrightarrow \frac{z}{z-1}$

$H(Z) = \frac{5}{(Z+1)(Z+2)} = \frac{A}{Z+1} + \frac{B}{Z+2}$

$H(Z) \to \frac{H(Z)}{Z} = \frac{\frac{5}{2}}{Z} + \frac{-5}{Z+1} + \frac{\frac{5}{2}}{Z+2}$

$\to H(Z) = \frac{5}{2} - \frac{5Z}{Z+1} + \frac{5}{2}\frac{Z}{Z+2}$

$\Rightarrow H(Z) = \frac{5}{(Z+1)(Z+2)} \xrightarrow{?} u(n)$

$\frac{H(Z)}{Z} = \frac{5}{Z(Z+1)(Z+2)} = \frac{A}{Z} + \frac{B}{Z+1} + \frac{C}{Z+2}$

$\Rightarrow 1 \leftrightarrow \delta(n)$

$\frac{Z}{Z-1} \leftrightarrow u(n)$

$\frac{Z}{Z-(-2)} \leftrightarrow (-2)^n u(n)$

$\frac{Z}{Z-K} \leftrightarrow K^n u(n)$

$\Rightarrow A = \frac{5}{2}$

$B = -5$

$C = \frac{5}{2}$

Ex:-

$$h(n) = \frac{5}{2}\delta(n) - 5u(n) + \frac{5}{2}(-2)^n u(n)$$

$$= \frac{5}{2} - 5\frac{z}{z-1} + \frac{5}{2}\frac{z}{z-(2)}$$

Ex:-

$$Y(z) = \frac{2z^2 + 3z + 1}{z^3 + 4z^2 + 2z + 1}$$

defined number of samples of $y(n)$ by division.

$$
\begin{array}{r|l}
 & 2z^{-1} - 5z^{-2} + 17z^{-3} \\
\hline
z^3+4z^2+2z+1 & 2z^2 + 3z + 1 \\
 & -\ \ 2z^2 + 8z + 4 + 2z^{-1} \\
\hline
 & -5z - 3 - 2z^{-1} \\
 & -\ \ -5z - 20 - 10z^{-1} - 5z^{-2} \\
\hline
 & 17 + 8z^{-1} + 5z^{-2} \\
 & 17 + 68z^{-1} + 84z^{-2} + 17z^{-3} \\
\hline
 & 60z^{-1} - 24z^{-2} - 17z^{-3}
\end{array}
$$

by initial value theorem.

$$y(z) = \sum_{n=0}^{\infty} y(n) z^{-n}$$

$y(n): 0, 2, -5, 17, \ldots$

$Y(z) \xrightarrow{z\ \text{trans}} y(n)$

Steady state response "discrete domain"

Given a linear shift invariant system with frequency response $H(\omega)$, the response of the system for a sinosoidal discrete signal is sinosoidal with same input frequency and $Y = |H(\omega)|_{\text{input freq}}$ input Amplitude

phase $\phi = \measuredangle$ input signal $+ \measuredangle H(\omega)|_{\text{input freq}}$

$H(\omega)$

$$x(n) = X\cos(\omega_0 n + \phi)$$

$$y(n) = X \cdot |H(\omega_0)| \cos(\omega_0 n + \phi_x + \measuredangle H(\omega_0))$$

$\Rightarrow$ $h(n)$, $x(n) = Ae^{j(\omega_0 n + \phi)}$

$$y(n) = x(n) \circledast h(n) = \sum_{k=-\infty}^{\infty} h(k)\, A e^{j(\omega_0(n-k)+\phi)}$$

$$= Ae^{j(\omega_0 n+\phi)} \boxed{\underbrace{\sum_{k=-\infty}^{\infty} h(k) e^{-j\omega_0 k}}_{\text{freq. response}}}$$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

discrete fourier transfer:

$$X(f) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi f n}$$

Discrite Fourier transfer $h(n)$

$$= Ae^{j(\omega_0 n+\phi)} \cdot H(\omega)\Big|_{\omega_0}$$

Discrete Fourier transform from Z-transform.

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$X(f) = \sum_{n=0}^{\infty} x(n) e^{-j2\pi f n}$$

$$H(s)\Big|_{s=j\omega} = H(\omega)$$

"freq. resp"

$z = e^{s}$, $s = j\omega$

$= j2\pi f$.

$$H(z)\Big|_{z=e^{j\omega}} = \text{freq. resp discrete signal.}$$