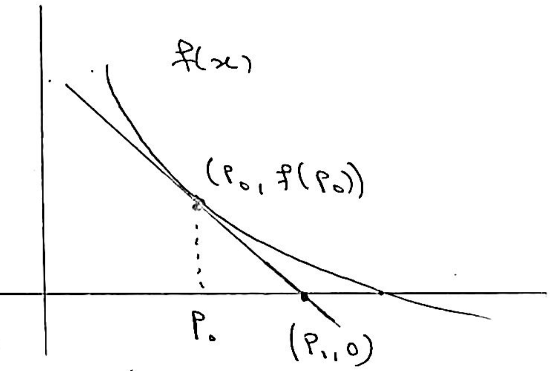
2.4 Newton - Raphson and secant method.

1) Newton - Raphson method:

(slope method for finding roots):

Graphical derivation of Newton Method:

$$f'(p_0) = \frac{0 - f(p_0)}{p_1 - p_0}$$, solving for p_1, then.



$$p_1 = p_0 - \frac{f(p_0)}{f'(p_0)}$$

we repeat this process to obtain a sequence $\{p_k\}$ that converges to p.

(i.e.) $$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}.$$

Example: $e^x - \cos x - 1 = 0$ on $[0, 1]$

If $\boxed{p_0 = 1}$, using Newton Method:

$$p_{n+1} = p_n - \frac{(e^{p_n} - \cos p_n - 1)}{e^{p_n} + \sin p_n}$$

$\Rightarrow$ $p_1 = 0.669083898$, $p_2 = 0.603760843$

$p_3 = \boxed{0.601349}991$, $p_4 = \boxed{0.601346}767$.

Note $|p_4 - p_3| < 10^{-5}$, which is good.

Thm : Newton - Raphson theorem. (Convergence).

Assume $f \in C^2[a,b]$ and $\exists\ p \in [a,b]$ s.t $f(p) = 0$

If $f'(p) \neq 0$, then $\exists\ \delta > 0$ s.t $\{p_n\}_{n=0}^{\infty}$ generated

by Newton - method $p_{n+1} = p_n - \dfrac{f(p_n)}{f'(p_n)}$, $n = 0, 1, 2, \ldots$

will converge to p for any initial point p_0

where $p_0 \in [p-\delta, p+\delta]$. (p_0 close to p)

proof: by hypothesis

$f(p) = 0 \Rightarrow$ Consider $g(x) =: x - \dfrac{f(x)}{f'(x)}$, $g(p) = p$

$$g'(x) = 1 - \frac{(f'(x))^2 - f(x) f''(x)}{(f'(x))^2} = \frac{f(x) f''(x)}{(f'(x))^2}$$

$$\Rightarrow \quad g'(p) = \frac{f(p) f''(p)}{(f'(p))^2} = 0 < 1.$$

beside : $g(p) = p - \dfrac{f(p)}{f'(p)} = p$

we conclude : 1) g is cont. on $[a,b]$ & $g' \in C[a,b]$

2) p is a fixed point and $g'(p) = 0 < 1$

therefore using fixed point thm , the sequence $\{p_k\}_{k=0}^{n}$, $k = 0, 1, \ldots$

will converge to the fixed point p , but p is

a root of $f(x)$, therefore $\{p_k\}$ Converges to p. ∎

The most Important Question is How fast the Iteration converge?

Def: If p is a root of f. The multiplicity of p is M iff $f(p) = f'(p) = \cdots = f^{(M-1)}(p) = 0$ & $f^{M}(p) \neq 0$.

Def: If p is a root of Multiplicity M, then $\exists$ continuous function $h(x)$ such that:

$$f(x) = (x-p)^M h(x) \quad , \text{where } h(p) \neq 0.$$

Def: If $M = 1$, we call p simple root.

Example: $f(x) = (x-1)\ln x$.

one root is $p = 1$, since $f(p) = 0$.

$$f'(p) = \frac{(p-1)}{p} - \ln p = \frac{(1-1)}{1} - \ln 1 = 0.$$

so we can't use Newton method here for convergence.

$f''(p) = 2 \Rightarrow p = 1$ of Multiplicity $\underline{2}$

Example: $f(x) = x^3 - 3x + 2$, $P = 1$ is a root.

$\Rightarrow f(x) = (x-1)(x^2+x-2)$

$= (x-1)(x-1)(x+2)$

$\Rightarrow$ 1 is double root

& 2 is a simple root.

$$
\begin{array}{r|l}
 & x^2 + x - 2 \\ \hline
x - 1 & x^3 - 3x + 2 \\
 & -x^3 \mp x^2 \\ \hline
 & x^2 - 3x + 2 \\
 & -x^2 \pm x \\ \hline
 & -2x + 2 \\
 & -2x + 2 \\ \hline
 & 0
\end{array}
$$

Note: Newton method for simple root $(M=1)$ is faster than for multiple root $(M>1)$.

Speed of Convergence : (order of Convergence):

Def: Assume that $\{P_n\}_{n=0}^{\infty}$ converges to P and set $E_n = P - P_n$ for $n \geq 0$.

If there exist two positive constants A, R such that:

$$\lim_{n\to\infty} \frac{|P - P_{n+1}|}{|P - P_n|^R} = \lim_{n\to\infty} \frac{|E_{n+1}|}{|E_n|^R} = A$$

Then the sequence is said to converge to P with order of convergence R., A is asymptotic error constant

- If $R = 1$, then the convergence of $\{P_n\}_{n=0}^{\infty}$ is linear.
- If $R = 2$, then the convergence of $\{P_n\}_{n=0}^{\infty}$ is Quadratic.

Example: Show that $P_n = \frac{1}{n^3}$ converges to 0 linearly?

$$\lim \frac{|e_{n+1}|}{|e_n|} = \lim_{n\to\infty} \frac{\left|0 - \frac{1}{(n+1)^3}\right|}{\left|0 - \frac{1}{n^3}\right|} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^3}}{\left(\frac{1}{n^3}\right)}$$

$$= \lim_{n\to\infty} \frac{n^3}{(n+1)^3} = \left(\lim_{n\to\infty} \frac{n}{n+1}\right)^3 = \left(\lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)\right)^3 = 1$$

$\Rightarrow \quad \frac{1}{n^3} \longrightarrow 0$ linearly.

Find order of Convergence for $P_n = 10^{-n}$

Example: $P_n = 10^{-n} \longrightarrow 0$ linearly??

$$\lim_{n\to\infty} \frac{|0 - 10^{-(n+1)}|}{|0 - 10^{-n}|^R} \overset{??}{=} A$$

$$\Rightarrow \lim_{n\to\infty} \frac{10^{-(n+1)}}{10^{-nR}} = \lim_{n\to\infty} \frac{10^{nR}}{10^{n+1}} = \lim_{n\to\infty} \frac{10^{n(R-1)}}{10}$$

therefore if $R > 1$, then $\lim_{n\to\infty} \frac{10^{n(R-1)}}{10} = \infty$

if $R = 1$, then $\lim_{n\to\infty} \frac{10^{n(R-1)}}{10} = \frac{1}{10}$

if $R < 1$, then $\lim_{n\to\infty} \frac{10^{n(R-1)}}{10} = 0$

Note: for every R, $\exists!\, A$, Here $R = 1$ & $A = \frac{1}{10}$.

Note: If R is large, then $\{P_n\}$ converges rapidly to P

Note: we can write it as:

$$|E_{n+1}| \approx A\, |E_n|^R$$

3)Find the order of convergence of the sequence
1.5,1.373333333,1.365262015,1.365230014,1.365230013

$p_0 \quad p_1 \quad p_2 \quad p_3 \quad p_4$

$|E_1| = |p_1 - p_0| = 0.126666667$

$|E_2| = |p_2 - p_1| = 0.008071318$

$|E_3| = |p_3 - p_2| = 0.000032001$

$|E_4| = |p_4 - p_3| = 0.000000001$

2 or $\boxed{4}$

converges fast

$R = 2$

$\frac{|E_2|}{|E_1|^2} = 0.50306$, $\frac{E_3}{|E_2|^2} = 0.49122$, $\longrightarrow$

$\boxed{R = 2}$, $\boxed{A \simeq 0.5}$.

4)When estimating the root of the function $f(x) = (x-1)^2 lnx$ and using Newton Method, find the order of converge R and the asymptotic error constant A.

$p = 1$ is the root.

$f(x) = (x-1)^2 \ln x = (x-1)^2\left((x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots\right)$

$= (x-1)^3\left(1 - \frac{(x-1)}{2} + \frac{(x-1)^2}{3} - \cdots\right)$

$= (x-1)^3 h(x)$, $h(1) = 1 \neq 0$ 2

$\Rightarrow \boxed{M = 3} \Rightarrow \boxed{R = 1}$. $\boxed{A = \frac{2}{3}}$ or 4

R: $f' = \frac{(x-1)^2}{x} + 2(x-1)\ln x \Rightarrow f'(1) = 0$

$f'' = \frac{x^2 - 1}{x^2} + 2\ln x + 2 - \frac{2}{x} \Rightarrow f''(1) = 0$

$f''' =$ [illegible]

$\left(\frac{2}{2}\right)^{41}$

Thm: Convergence Rate of Newton Method:

Assume
~~If~~ we use Newton-R- method such that $\{P_n\} \to P$

where $f(P) = 0$, then:

1) If P is simple root $(M=1)$, then $R=2$.

and $A = \dfrac{|f''(p)|}{|2f'(p)|}$ $\qquad P_{n+1} = P_n - \dfrac{f(P_n)}{f'(P_n)}$

2) If P is a multiple root $(M>1)$, then $R=1$

and $A = \dfrac{M-1}{M}$.

Example: Quadratic Convergence at a simple root.

$f(x) = x^3 - 3x + 2$, has -2 as a simple root.

Start with $P_0 = -2.4$, Using Newton method:

Numerically:

P_K	$P_{K+1} - P_K$	$E_K = P - P_K$	$\dfrac{\lvert E_{K+1}\rvert}{\lvert E_K\rvert^2}$
-2.40000000	0.323809524	0.40000000	0.476190475
-2.076190476	0.072594465	0.076190476	0.619469086
-2.003596011	0.003587422	0.003596011	0.664202613
-2.000008589	0.000008589	0.000008589	⋮
-2.00000000	0.0000000	0.0000000	

Theoritically: Simple root $\Rightarrow R = 2$ & $A = \dfrac{|f''(P)|}{2|f'(P)|}$

$A = \dfrac{|6(-2)|}{2\,|3(-2)^2-3|} = \dfrac{12}{18} \approx 0.666667$

Now for the same function we will check

. Linear convergence to a double Root. $\boxed{P=1}$

Numerically	P_k	$P_{k+1} - P_k$	$E_k = P - P_k$	$\frac{\lvert E_{k+1}\rvert}{\lvert E_k\rvert}$
0	1.20000000	-0.09696967	-0.20000000	0.515151515
1	1.103030303	-0.050673883	-0.103030303	0.508165253
2	1.052356420	-0.025955609	-0.052356420	0.496751115
3	1.02640081	-0.013143081	-0.026400811	0.509753688
4	1.013257730	-0.006614311	-0.013257730	0.501097775
5	1.006643419	-0.003318055	-0.00643419	0.500550093
⋮	⋮	⋮	⋮	⋮

Theoretically $R=1$ since we have double root $(M>1)$

and $A = \frac{M-1}{M} = \frac{2-1}{2} = \frac{1}{2}$.

Example: show that the Bisection method Converges Linearly to 0? on [a,b]

$$\lim_{n\to\infty} \frac{|P - P_{n+1}|}{|P - P_n|} \leq \frac{\frac{(b-a)}{2^{n+2}}}{\frac{(b-a)}{2^{n+1}}} = \frac{2^{n+1}}{2^{n+2}} = \frac{1}{2}.$$

$\frac{1}{2}$/43

Newton

(Q1) Consider the equation $x = \cos x$

(a) Use Newton's method with $P_0 = 0.2$ to estimate the solution of this equation with error less that 10^{-5}

(b) Find the order of convergence and the asymptotic error constant both numerically and theoretically.

Solution: (a) $f(x) = x - \cos x$, $f'(x) = 1 + \sin x$, $f''(x) = \cos x$

Newton's iteration: $P_{n+1} = P_n - \dfrac{P_n - \cos P_n}{1 + \sin P_n}$

n	P_n	$\lvert P_n - P_{n-1}\rvert$
0	0.2	——
1	0.850777122	0.650777122
2	0.741530193	0.109246929
3	0.739086449	0.002443744
4	0.739085133	0.000001316

So $P \approx P_4 = 0.739085133$

(b) Theoretically: $f'(P) = f'(0.739085133) = 1.673612029 \neq 0$,

So P is a simple root.

Therefore, $R = 2$ and $A = \dfrac{|f''(P)|}{2|f'(P)|} = 0.220805395$

Numerically:

$\lvert E_n\rvert \approx \lvert P_n - P_{n-1}\rvert$	$\dfrac{\lvert E_{n+1}\rvert}{\lvert E_n\rvert^2}$
0.650777122	0.257955435
0.109246929	0.204756281
0.002443744	0.220365941
0.000001316	

Proof : part ① Recall $P_{n+1} = P_n - \frac{f(P_n)}{f'(P_n)}$

Let us define $g(x) := x - \frac{f(x)}{f'(x)}$

Note that $g(p) = p$.

Take Taylor second expansion of g about P.

$$g(x) = g(p) + g'(p)(x-p) + \frac{g''(c)(x-p)^2}{2}$$

substitute P_n in x:

$$g(P_n) = p + g'(p)(P_n - p) + \frac{g''(c_n)(P_n - p)^2}{2} \quad \cdots (*)$$

Now: $g'(x) = \frac{f(x) f''(x)}{(f'(x))^2} \Longrightarrow g'(p) = 0.$

$g''(x) = \; ----$

& $g''(p) = \frac{f''(p)}{f'(p)}$.

substitute in $(*)$, we have:

$$g(P_n) = p + g''(c_n) \frac{(P_n - p)^2}{2}$$

$$\Rightarrow \quad |P_{n+1} - p| = \frac{g''(c_n)}{2}(P_n - p)^2$$

$$\Rightarrow \lim_{n\to\infty} \frac{|E_{n+1}|}{|E_n|^2} = \lim_{n\to\infty} \left|\frac{g''(c_n)}{2}\right| = \left|\frac{f''(p)}{2f'(p)}\right|.$$

proof (2) We can prove it same way as (1), or

Since f has p as a multiple root $(M>1)$, then:

$$f(x) = (x-p)^M h(x) \quad \text{where} \quad h(p) \neq 0$$

Now, we know that:

$$P_{k+1} =: g(p_k) = p_k - \frac{f(p_k)}{f'(p_k)}$$

Can we use this Method for $M=1$ & $M>1$

1

(i-e) we define $g(x) = x - \dfrac{f(x)}{f'(x)} = x - \dfrac{(x-p)^M h(x)}{M(x-p)^{M-1} h(x) + (x-p)^M h'(x)}$

$$\Rightarrow g(x) = x - \frac{(x-p)h(x)}{Mh(x) + (x-p)h'(x)}, \quad \text{Note } \boxed{g(p) = p}$$

fixed point

$$g'(x) = 1 - \Big[(Mh(x) + (x-p)h'(x))((x-p)h'(x) + h(x)) -$$

3

$$- (x-p)h(x)\left[Mh'(x) + h'(x) + (x-p)h''(x)\right]\Big] \Big/ (\text{المقام})^2$$

$$\Rightarrow g'(p) = 1 - \frac{M h^2(p)}{M^2 h^2(p)} = 1 - \frac{1}{M} = \frac{M-1}{M}, \quad M \neq 1$$

need to show $= A$

Now, let's take the first order taylor poly. of $g(x)$ around p

$$\Rightarrow g(x) = g(p) + g'(c)(x-p) \qquad 2$$

$$\Rightarrow p_{n+1} = g(p_n) = p + g'(c)(p_n - p), \qquad \boxed{g'(c_k) \approx g'(p)}$$

$$\Rightarrow p - p_{n+1} \cong g'(p)(p - p_n) \qquad \lim_{k\to\infty} c_k = p$$

$$\Rightarrow |E_{n+1}| \approx \frac{M-1}{M} |E_n|^1$$

Thm: Acceleration of Newton Method: ($M>1 \Rightarrow R=1$) slow

If p is a root of multiplicity $\underbrace{M>1}_{R=1}$, then the iteration

$$P_{n+1} = P_n - M\frac{f(P_n)}{f'(P_n)}$$ will converge quadratically to P

proof: [Homework] We need to prove that the error term in taylor expansion of $g(x)$ starts from $(x-p)^2 g''(c)$.

Let $$g(x) = x - \frac{M f(x)}{f'(x)} = x - \frac{M(x-p)^M h(x)}{M(x-p)^{M-1}h(x) + (x-p)^M h'(x)}$$

$$\Rightarrow g(x) = x - \frac{M(x-p)h(x)}{Mh(x) + (x-p)h'(x)}$$

$$g'(x) = 1 - \frac{M\left[Mh(x) + (x-p)h'(x)\right]\left[(x-p)h'(x) + h(x)\right]}{\left(Mh(x) + h'(x)(x-p)\right)^2}$$

$$+ \frac{M(x-p)h(x)\left(Mh'(x) + (x-p)h''(x) + h'(x)\right)}{\left(Mh(x) + h'(x)(x-p)\right)^2}$$

$$g'(p) = 1 - \frac{M^2 h^2(p)}{M^2 h^2(p)} = 1 - 1 = 0.$$

$\Rightarrow$ Taylor expansion of g about p:

$$g(x) = g(p) + \cancel{g'(p)(x-p)} + \frac{g''(c)}{2!}(x-p)^2$$

$$\Rightarrow P_{n+1} = g(P_n) = g(p) + \frac{g''(c)}{2!}(P_n - p)^2$$

$$\Rightarrow |P - P_{n+1}| \approx \frac{g''(p)}{2!}|P - P_n|^2$$ Converges quadratically

Example: Let $f(x) = x^3 - 3x + 2 = (x-1)^2(x-2)$

1 has multiplicity 2

Using Acceleration Method, the Iteration is

$$P_{n+1} = P_n - 2\frac{f(P_n)}{f'(P_n)}$$

P_n	$P_{n+1} - P_n$	$E_n = P - P_n$	$\frac{\lvert E_{n+1}\rvert}{\lvert E_n\rvert^2}$
1.20000	-0.193939394	-0.20000	0.1515151515150
1.0060606	-0.006054519	-0.006060606	0.1657185578
1.000006087	-0.000006087	-0.000006087	
1.00 --	0.0--	0.0 --	

So, the Iteration Converges Quadratically.

Note: Using Newton method we need 6 Iteration.
but Using Acceleration method, we need only 4 Iterations

Example: Show that if $g(p) = p$ & $g'(p) = g''(p) = 0$, then the sequence generated by $P_{n+1} = g(P_n)$ Converges at least cubically.

proof: $g(x) = g(p) + g'(p)(x-p) + \frac{g''(p)}{2!}(x-p)^2 + \frac{g'''(c)(x-p)^3}{3!}$

$$P_{n+1} = g(P_n) = p + 0 + 0 + \frac{g'''(c)(P_n - p)^3}{3!}$$

$$\Rightarrow \quad |P_{n+1} - p| = \frac{|g'''(c)|}{3!}|P_n - p|^3$$

$$\Rightarrow \quad \lim_{n\to\infty}\frac{|P_{n+1} - p|}{|P_n - p|^3} = \frac{|g'''(p)|}{6}$$

At least ↓

If $g'''(p) = 0 \Rightarrow R > 3$

If $g'''(p) \neq 0 \Rightarrow R = 3$

Birzeit University
Mathematics Department

Order of convergence | Math330 | 2018/2019

(Q) Let p be a fixed point of $g(x)$.

(a) show that if $g'(p) = g''(p) = \cdots = g^{(k-1)}(p) = 0$ and $g^{(k)}(p) \neq 0$, then the fixed point iteration of $g(x)$ will converge to p with $R = k$ and $A = |\frac{g^{(k)}(p)}{k!}|$

(b) Use part (a) to show that if p is a simple root of $f(x)$, then Newton's iteration will converge to p with $R = 2$ and $A = |\frac{f''(p)}{2f'(p)}|$

Proof: (a) Apply Taylor's expansion of $g(x)$ about $x = p$

$$g(x) = g(p) + g'(p)(x-p) + \frac{g''(p)}{2!}(x-p)^2 + \cdots + \frac{g^{(k-1)}(p)}{(k-1)!}(x-p)^{k-1} + \frac{g^{(k)}(c)}{k!}(x-p)^k$$

but $g(p) = p$ and $g'(p) = g''(p) = \cdots = g^{(k-1)}(p) = 0$

$\rightarrow \; g(x) = p + \frac{g^{(k)}(c)}{k!}(x-p)^k$

Substitute $x = p_n \rightarrow \; g(p_n) = p + \frac{g^{(k)}(c)}{k!}(p_n - p)^k$, c between p_n and p

but $g(p_n) = p_{n+1} \;\rightarrow p_{n+1} - p = \frac{g^{(k)}(c)}{k!}(p_n - p)^k \;\rightarrow\; \frac{p_{n+1}-p}{(p_n-p)^k} = \frac{g^{(k)}(c)}{k!} \;\rightarrow\; \frac{|E_{n+1}|}{|E_n|^k} = |\frac{g^{(k)}(c)}{k!}|$

now take limit an $n \rightarrow \infty$ and considering that $c \approx p$ when $n \rightarrow \infty$

$\rightarrow \; \lim\limits_{n\rightarrow\infty} \frac{|E_{n+1}|}{|E_n|^k} = |\frac{g^{(k)}(p)}{k!}| \;\rightarrow R = k$ and $A = |\frac{g^{(k)}(p)}{k!}|$

(b) We know that Newton's iteration is a special case of FPI with $g(x) = x - \frac{f(x)}{f'(x)}$

Therefore, based on part (a), we only need to prove that $g'(p) = 0$ but $g''(p) \neq 0$

Recall that p is a simple root of $f(x)$ mean $f(p) = 0$ and $f'(p) \neq 0$

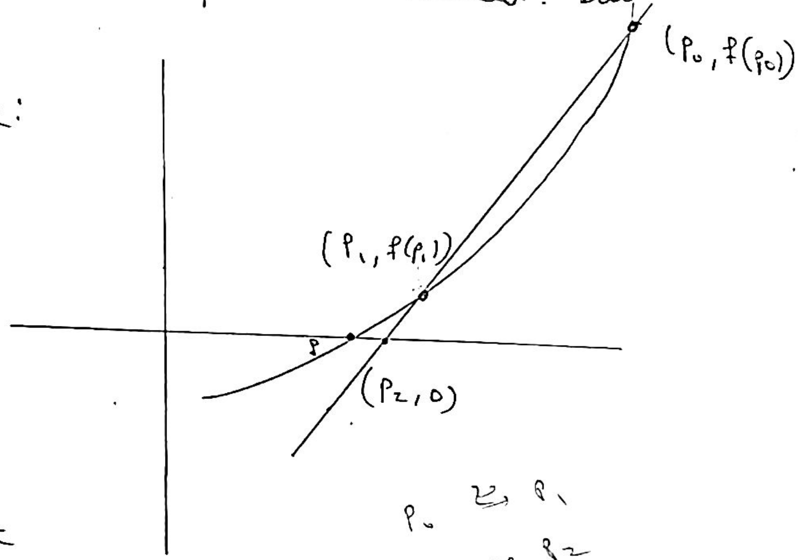
now, $g'(x) = 1 - \frac{(f'(x))^2 - f(x)f''(x)}{(f'(x))^2} = \frac{f(x)f''(x)}{(f'(x))^2} \;\rightarrow g'(p) = 0$

now, $g''(x) = \frac{(f'(x))^2\,[f(x)f'''(x) + f''(x)f'(x)]}{(f'(x))^4} = \frac{f(x)f'''(x) + f''(x)f'(x)}{(f'(x))^2} \;\rightarrow g''(p) = \frac{f''(p)}{f'(p)} \neq 0$

$\rightarrow \; R = 2$ and $A = |\frac{g''(p)}{2!}| = |\frac{f''(p)}{2f'(p)}|$

2) Secant method:

this method is similar to false position method! but faster. finding the slope:



$p_0 \rightsquigarrow p_1$

& $p_1 \rightsquigarrow p_2$

$$\frac{f(p_1)-0}{p_1-p_2} = \frac{f(p_1)-f(p_0)}{p_1-p_0}$$

solving for P_2, we get

$$P_2 = P_1 - \frac{f(P_1)(P_1-P_0)}{f(P_1)-f(P_0)}$$

$\vdots$

$$P_n = P_{n-1} - \frac{f(P_{n-1})(P_{n-1}-P_{n-2})}{f(P_{n-1})-f(P_{n-2})}$$

Note: In this method we need two Initial points P_0 & P_1

Moreover; this method is used ~~only~~ for simple roots with $\boxed{R=1.618}$ & for multiple root with $\boxed{R=1}$

Thm: If we use secant method so that $P_n \to P$ (simple root)

then: $$\lim_{n\to\infty} \frac{|E_{n+1}|}{|E_n|^{1.618}} = \left|\frac{f''(p)}{2f'(p)}\right|^{0.618}$$ for simple.

(i.e) $R \cong 1.618 \approx \left(\frac{\sqrt{5}+1}{2}\right)$, for simple root.

For Multiple root only Numerically.

(48)

(Q2) Consider the equation $x = \cos x$.

(a) If $P_0 = 0.5$ and $P_1 = \frac{\pi}{4}$, use the secant method to approximate the root of the equation with accuracy of 10^{-4}.

(b) Find the order of convergence and the asymptotic error constant both theoretically and numericaly.

Solution: (a) $f(x) = x - \cos x$, $f'(x) = 1 + \sin x$, $f''(x) = \cos x$

Secant's iteration: $P_{n+2} = P_{n+1} - \dfrac{f(P_{n+1})(P_{n+1} - P_n)}{f(P_{n+1}) - f(P_n)}$

secant

n	P_n	$\lvert P_n - P_{n-1}\rvert$
0	0.5	——
1	0.785398163	0.285398163
2	0.736384138	0.049014025
3	0.739058138	0.002674
4	0.739085149	0.000027011338

So $P \approx P_4 = 0.739085149$

(b) Theoretically: $f'(P) = f'(0.739085149) = 1.673612041 \neq 0$,

So P is a simple root.

Therefore, $R = 1.618$ and $A = \left|\dfrac{f''(P)}{2f'(P)}\right|^{0.618} = 0.393185938$

Numerically:

$\lvert E_n\rvert \approx \lvert P_n - P_{n-1}\rvert$	$\dfrac{\lvert E_{n+1}\rvert}{\lvert E_n\rvert^{1.618}}$
0.285398163	0.372735166
0.049014025	0.351741034
0.002674	0.393005788
0.000027011338	

Table 2.8 Acceleration of Convergence at a Double Root

k	p_k	$p_{k+1} - p_k$	$E_k = p - p_k$	$\frac{\lvert E_{k+1}\rvert}{\lvert E_k\rvert^2}$
0	1.200000000	−0.193939394	−0.200000000	0.151515150
1	1.006060606	−0.006054519	−0.006060606	0.165718578
2	1.000006087	−0.000006087	−0.000006087	
3	1.000000000	0.000000000	0.000000000	

Table 2.9 Comparison of the Speed of Convergence

Method	Special considerations	Relation between successive error terms
Bisection		$E_{k+1} \approx \frac{1}{2}\lvert E_k\rvert$
Regula falsi		$E_{k+1} \approx A\lvert E_k\rvert$
Secant method	Multiple root	$E_{k+1} \approx A\lvert E_k\rvert$
Newton-Raphson	Multiple root	$E_{k+1} \approx A\lvert E_k\rvert$
Secant method	Simple root	$E_{k+1} \approx A\lvert E_k\rvert^{1.618}$
Newton-Raphson	Simple root	$E_{k+1} \approx A\lvert E_k\rvert^2$
Accelerated Newton-Raphson	Multiple root	$E_{k+1} \approx A\lvert E_k\rvert^2$

Example 2.17 (Acceleration of Convergence at a Double Root). Start with $p_0 = 1.2$ and use accelerated Newton-Raphson iteration to find the double root $p = 1$ of $f(x) = x^3 - 3x + 2$.

Since $M = 2$, the acceleration formula (31) becomes

$$p_k = p_{k-1} - 2\frac{f(p_{k-1})}{f'(p_{k-1})} = \frac{p_{k-1}^3 + 3p_{k-1} - 4}{3p_{k-1}^2 - 3},$$

and we obtain the values in Table 2.8. ■

Table 2.9 compares the speed of convergence of the various root-finding methods that we have studied so far. The value of the constant A is different for each method.