Homework 7 (chapter 8) Reem Ali.

Abstract 1.

2. Show that $Z_2 \oplus Z_2 \oplus Z_2$ has 7 subgroups of order 2.

$\{0,1\} + \{0,1\} + \{0,1\} = \{(0,0,0),(1,0,0),(1,1,0),(1,0,1),(1,1,1),(0,1,0),(0,1,1),(0,1,0)\}$.

$a = (x,y,z)$ in $Z_2 \oplus Z_2 \oplus Z_2$

$a^2 = (x,y,z)(x,y,z) = (x+y,\ y+y,\ z+z) = (0,0,0)$ identity of $Z_2 \oplus Z_2 \oplus Z_2$.

So There are seven elements of $Z_2 \oplus Z_2 \oplus Z_2$ of order 2 (except $e=(0,0,0)$).
and for each such a there is a subgroup of order 2 : $\{e, a\}$
this gives 7 subgroups of order 2 has e and other element.

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4. Show that $G \oplus H$ is Abelian iff G and H are Abelian.

$\Rightarrow$ suppose that G and H are Abelian and $(g_1, h_1), (g_2, h_2) \in G \oplus H$, Then

$(g_1,h_1)(g_2,h_2) = (g_1 g_2,\ h_1 h_2)$

$= (g_2 g_1,\ h_2 h_1)$ since G and H are Abelian.

$= (g_2, h_2)(g_1, h_1)$.

Thus, $G \oplus H$ is Abelian.

$\Leftarrow$ suppose $G \oplus H$ is Abelian and let $g_1, g_2 \in G$, $h_1, h_2 \in H$, Then

$(g_1 g_2,\ h_1 h_2) = (g_1, h_1)(g_2, h_2)$

$= (g_2, h_2)(g_1, h_1)$ since $G \oplus H$ is Abelian.

$= (g_2 g_1,\ h_2 h_1)$.

Thus, $g_1 g_2 = g_2 g_1$ and G Abelian

and $h_1 h_2 = h_2 h_1$ So H is Abelian

6. Prove, By comparing orders of elements, that $Z_8 \oplus Z_2$ is not isomorphic to $Z_4 + Z_4$.

the element $(1,0) \in Z_8 \oplus Z_2$ with order 8

But $Z_4 \oplus Z_4$ doesn't have element of order 8.

So $Z_8 \oplus Z_2 \not\approx Z_4 \oplus Z_4$.

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8. Is $Z_3 \oplus Z_9$ isomorphic to Z_{27}?

No, since $Z_3 \oplus Z_9$ doesn't contains an element of order 27

But Z_{27} does have.

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10. How many elements of order 9 does $Z_3 \oplus Z_9$ have?

Z_9 contains 6 elements of order 9 : $\{1, 2, 4, 5, 7, 8\}$

and any of there with any element of Z_3 give an element of order 9

So we have $6 \times 3 = 18$ elements of order 9.

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14. suppose that $G_1 \approx G_2$ and $H_1 \approx H_2$. prove that $G_1 \oplus H_1 \approx G_2 \oplus H_2$.

* Assume $\alpha: G_1 \to G_2$ and $\beta: H_1 \to H_2$ are isomorphisms.

* Define a function $\phi: G_1 \oplus H_1 \longrightarrow G_2 \oplus H_2$ By $\phi(g,h) = (\alpha(g), \beta(h))$

* ϕ is 1-1 : assume $\phi(g,h) = \phi(x,y)$ then

$\Rightarrow (\alpha(g), \beta(h)) = (\alpha(x), \beta(y))$

$\Rightarrow \alpha(g) = \alpha(x)$ and $\beta(h) = \beta(y)$.

$\Rightarrow g = x$ and $h = y$ since α, β isomorphism.

$\Rightarrow (g,h) = (x,y) \rightsquigarrow$ 1-1 ✓

* ϕ is onto ✓

* ϕ is isomorphism: $\phi((g,h)(\bar{g},\bar{h})) = \phi(g\bar{g}, h\bar{h})$

$\Rightarrow (\alpha(g\bar{g}), \beta(h\bar{h}))$

$= (\alpha(g), \beta(h))(\alpha(\bar{g}), \beta(\bar{h}))$

$= \phi(g,h)\, \phi(\bar{g},\bar{h})$

So ϕ is an isomorphism that is $G_1 \oplus H_1 \approx G_2 \oplus H_2$

15. If $G \oplus H$ is cyclic prove that G and H are cyclic.

→ $G \cong G + \{e\}$ which is a subgroup of $G \oplus H$ and $G + \{e\}$ is cyclic.

(A subgroup of cyclic is cyclic).

Hence, G is cyclic.

→ For H the same above.

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16. In $Z_{40} \oplus Z_{30}$, Find two subgroups of order 12.

→ $10 \in Z_{40}$ and $|10| = 4$

$10 \in Z_{30}$ and $|10| = 3$

So $(10, 10) \in Z_{40} \oplus Z_{30}$ and $|(10,10)| = L.C.M(4,3) = 12$.

→ $10 \in Z_{40}$ and $|10| = 4$

$5 \in Z_{30}$ and $|5| = 6$

So $(10, 5) \in Z_{40} \oplus Z_{30}$ and $|(10,5)| = L.C.M(4,6) = 12$.

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18. Find a subgroup of $Z_{12} \oplus Z_{18}$ isomorphic to $Z_9 \oplus Z_4$.

$$Z_9 \oplus Z_4 \cong Z_4 \oplus Z_9 \cong \underbrace{\langle 3 \rangle \oplus \langle 2 \rangle}$$

↳ in $Z_{12} \oplus Z_{18}$.

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20. Determine the number of elements of order 15 and the number of cyclic subgroup of order 15 in $Z_{30} \oplus Z_{20}$.

→ Number of elements of order 15 is 48

→ Number of cyclic subgroups of order 15 is $\frac{48}{\underset{\phi(15)}{\underbrace{8}}} = 6$