Electromagnetic Theory I

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Chapter 1: Vector Analysis

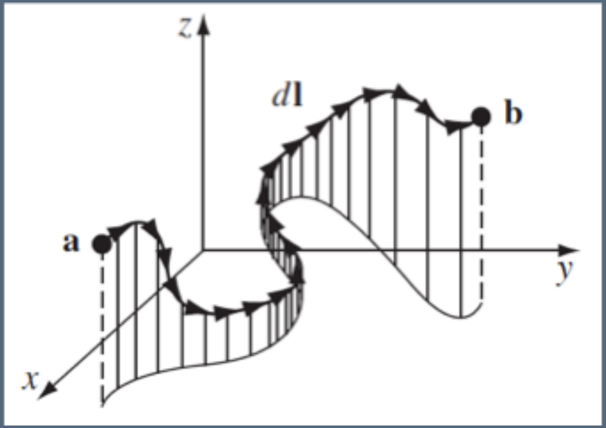
- ✓ Vector Algebra
- ✓ Differential Calculus
- ✓ Integral Calculus
- ✓ Curvilinear Coordinates
- ✓ The Dirac Delta Function
- ✓ The Theory of Vector Fields

Line, Surface and Volume Integrals

(i) Line Integrals

$$\int_{\vec{a}}^{\vec{b}} \vec{v} \cdot d\vec{l}$$

$$\oint_C \vec{v} \cdot d\vec{l}$$ If the path forms closed loop



Example: For $\vec{v} = y^2\hat{x} + 2x(y+1)\hat{y}$, find the line integral from (1, 1, 0) to (2, 2, 0)

Path 1:

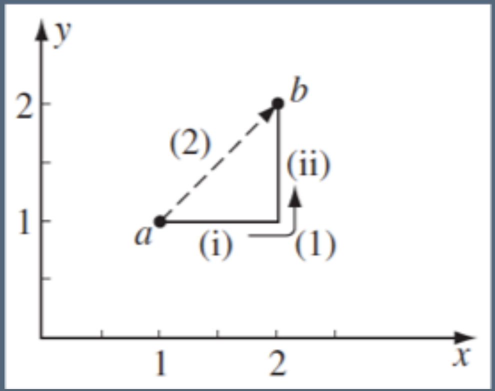
$(i)\ d\vec{l} = dx\,\hat{x}, y = 1 \rightarrow \int \vec{v} \cdot d\vec{l} = \int_1^2 dx = 1$

$(ii)\ d\vec{l} = dy\,\hat{y}, x = 2 \rightarrow \int \vec{v} \cdot d\vec{l} = 4\int_1^2 (y+1)dy = 10$

$$\int_{\vec{a}}^{\vec{b}} \vec{v} \cdot d\vec{l} = 1 + 10 = 11$$

Path 2:

$d\vec{l} = dx\hat{x} + dx\hat{y}, y = x \rightarrow \int \vec{v} \cdot d\vec{l} = \int_1^2 (3x^2 + 2x)dx = 10$



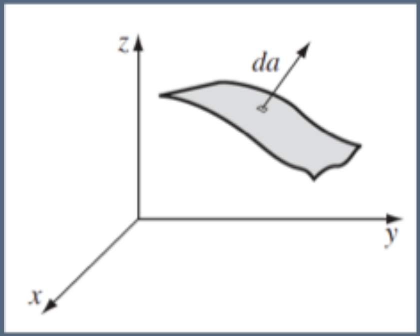
$$\oint_C \vec{v} \cdot d\vec{l} = 11 - 10 = 1$$

Line, Surface and Volume Integrals

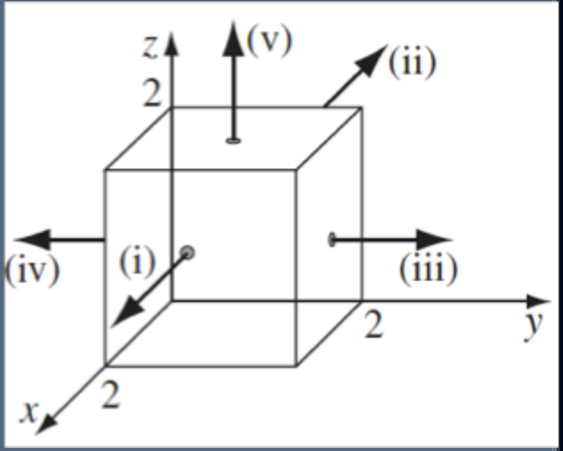
(ii) Surface Integrals

$$\int_S \vec{v} \cdot d\vec{a}$$

$$\oint_S \vec{v} \cdot d\vec{a}$$ If the surface forms closed surface



Example: For $\vec{v} = 2xz\hat{x} + (x+2)\hat{y} + y(z^2-3)\hat{z}$, find the surface integral over the five sides of the cube (excluding the bottom side)



$(i)\ d\vec{a} = dydz\,\hat{x}, x = 2 \rightarrow \int \vec{v} \cdot d\vec{a} = 4\int_0^2 dy \int_0^2 zdz = 16$

$(ii)\ d\vec{a} = -dydz\,\hat{x}, x = 0 \rightarrow \int \vec{v} \cdot d\vec{a} = -2(0)\int_0^2 dy \int_0^2 dz = 0$

$(iii)\ d\vec{a} = dxdz\,\hat{y}, y = 2 \rightarrow \int \vec{v} \cdot d\vec{a} = \int_0^2 (x+2)dx \int_0^2 dz = 12$

$(iv)\ d\vec{a} = -dxdz\,\hat{y}, y = 0 \rightarrow \int \vec{v} \cdot d\vec{a} = -\int_0^2 (x+2)dx \int_0^2 dz = -12$

$(v)\ d\vec{a} = dxdy\,\hat{z}, z = 2 \rightarrow \int \vec{v} \cdot d\vec{a} = \int_0^2 dx \int_0^2 ydy = 4$

$$\oint_S \vec{v} \cdot d\vec{a} = 16 + 0 + 12 - 12 + 4 = 20$$

Line, Surface and Volume Integrals

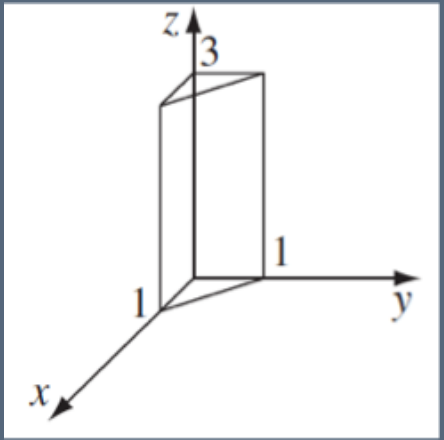
(iii) Volume Integrals

$$\int_V f\,d\tau$$

$$\int_V \vec{v}\,d\tau = \hat{x}\int_V v_x\,d\tau + \hat{y}\int_V v_y\,d\tau + \hat{z}\int_V v_z\,d\tau$$

Example: For $f(x,y,z) = xyz^2$, find the volume integral over the shown prism

$$\int_V f\,d\tau = \int_0^3\int_0^1\int_0^{1-y} xyz^2\,dx\,dy\,dz = \frac{3}{8}$$



The Fundamental Theorem of Divergence

$$\int_V (\vec{\nabla} \cdot \vec{v})\, d\tau = \oint_S \vec{v} \cdot d\vec{a}$$

Gauss's Theorem
Divergence Theorem

If the vector field represents the flow of an incompressible fluid, then the flux of the field is the total amount of fluid passing out through the surface, per unit time.

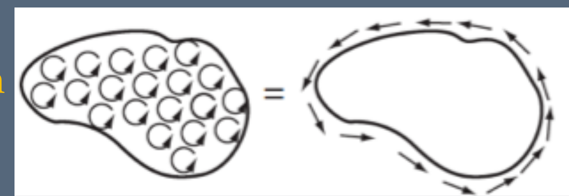
Example: For $\vec{v} = y^2\hat{x} + (2xy + z^2)\hat{y} + 2yz\,\hat{z}$, check the divergence theorem on unit cube

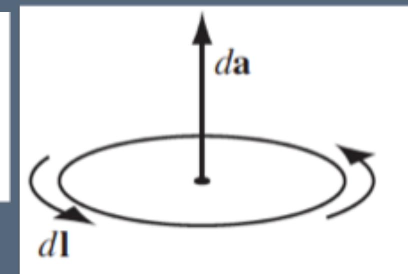
$$\int_V (\vec{\nabla} \cdot \vec{v})\, d\tau = \int_V (2x + 2y)\, d\tau = 2\int_0^1\int_0^1\int_0^1 (x + y) dx dy dz = 2$$

$$\oint_S \vec{v} \cdot d\vec{a} = 2$$

The Fundamental Theorem of Curls

$$\int_S (\vec{\nabla}\times\vec{v})\cdot d\vec{a} = \oint_C \vec{v}\cdot d\vec{l}$$ Stoke's Theorem



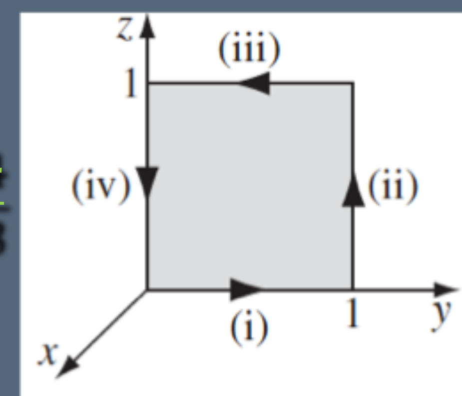


We keep the right rule-hand rule to determine the direction of area element and displacement vector

Example: For $\vec{v} = (2xz + 3y^2)\hat{y} + 4yz^2\,\hat{z}$, check Stoke's theorem on unit square

$$\int_S (\vec{\nabla}\times\vec{v})\cdot d\vec{a} = \int_S \left((4z^2 - 2x)\hat{x} + 2z\hat{z}\right)\cdot d\vec{a} = 2\int_0^1\int_0^1 4z^2\,dy\,dz = \frac{4}{3}$$

$$\oint_C \vec{v}\cdot d\vec{l} = \frac{4}{3}$$



The Fundamental Theorem of Curls

Corollary 1: $\int (\nabla \times \mathbf{v}) \cdot d\mathbf{a}$ depends only on the boundary line, not on the particular surface used.

Corollary 2: $\oint (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = 0$ for any closed surface, since the boundary line, like the mouth of a balloon, shrinks down to a point, and hence the right side of Eq. 1.57 vanishes.

Integration by Parts

$$\oint_S f\vec{A}\cdot d\vec{a} = \int_V \vec{\nabla}\cdot(f\vec{A})d\tau = \int_V \vec{\nabla}\cdot\vec{A}f\,d\tau + \int_V \vec{\nabla}f\cdot\vec{A}d\tau$$

$$\int_V \vec{\nabla}\cdot\vec{A}f\,d\tau = -\int_V \vec{\nabla}f\cdot\vec{A}d\tau + \oint_S f\vec{A}\cdot d\vec{a}$$