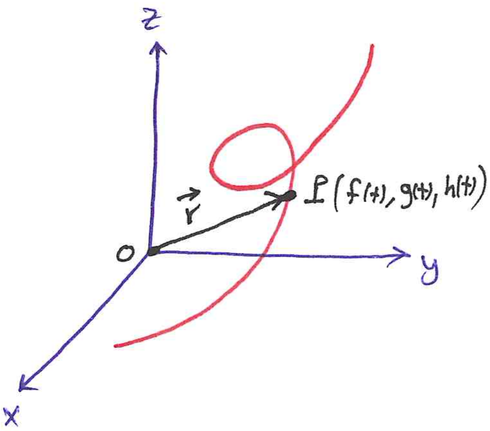
13.1 Curves in Space and Their Tangents



- If a particle moves in space, it makes a curve "particle path"
- The parameterization of this curve:

  $x = f(t), \quad y = g(t), \quad z = h(t), \quad t \in I$

- This curve in space can be represented in vector form (position vector)

  $$\vec{r}(t) = \overrightarrow{OP} = f(t)\,\vec{i} + g(t)\,\vec{j} + h(t)\,\vec{k}$$

  from origin to the particle's position $P(f(t), g(t), h(t))$ at time t.

  ⇒ f, g, h are the component functions.

  ⇒ $\vec{r}(t)$ is called also a vector-valued function or vector function

  ⇒ The domain of $\vec{r}(t)$ is the common domain of it's components.

  ⇒ f, g, h are real-valued functions "scalar functions"

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limits: Def Let $\vec{r}(t) = f(t)\,\vec{i} + g(t)\,\vec{j} + h(t)\,\vec{k}$ be a vector function with domain D. Let $\vec{L} = L_1\vec{i} + L_2\vec{j} + L_3\vec{k}$.

We say that $\vec{r}(t)$ has limit $\vec{L}$ as t approaches to and write $\lim\limits_{t \to t_0} \vec{r}(t) = \vec{L}$ if for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that

$$\text{if } \quad 0 < |t - t_0| < \delta \quad \text{then} \quad |\vec{r}(t) - \vec{L}| < \epsilon .$$

If $\lim\limits_{t \to t_0} f(t) = L_1$ and $\lim\limits_{t \to t_0} g(t) = L_2$ and $\lim\limits_{t \to t_0} h(t) = L_3$ then

$$\lim_{t \to t_0} \vec{r}(t) = \left(\lim_{t \to t_0} f(t)\right)\vec{i} + \left(\lim_{t \to t_0} g(t)\right)\vec{j} + \left(\lim_{t \to t_0} h(t)\right)\vec{k}$$

$$= L_1\,\vec{i} + L_2\,\vec{j} + L_3\,\vec{k}$$

$$= \vec{L}$$

Exp Let $\vec{r}(t) = (\sin t)\vec{i} + (\cos t)\vec{j} + 2t\,\vec{k}$.

Find $\lim\limits_{t\to\frac{\pi}{2}} \vec{r}(t) = \vec{i} + 0\,\vec{j} + \pi\,\vec{k} = \vec{i} + \pi\,\vec{k}$

Continuity : **Def** • A vector function $\vec{r}(t)$ is continuous at point $t = t_0 \in D$ if $\lim\limits_{t\to t_0} \vec{r}(t) = \vec{r}(t_0)$.

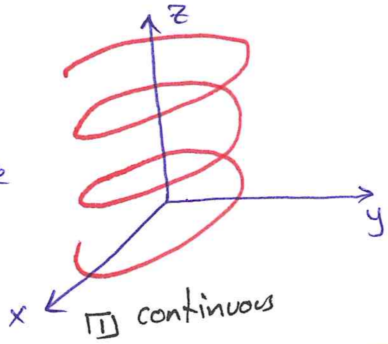
• The vector function $\vec{r}(t)$ is continuous if it is continuous at every point in its domain.

Exp [1] $\vec{r}(t) = (\cos t)\vec{i} + (\sin t)\vec{j} + t\,\vec{k}$

[2] $\vec{r}(t) = (\cos t)\vec{i} + (\sin t)\vec{j} + \lfloor t \rfloor \vec{k}$

is discontinuous at every integer, where $\lfloor t \rfloor$ is the greatest integer function "floor function"

$\lfloor 5.9 \rfloor = 5$
$\lfloor 7 \rfloor = 7$
$\lfloor -8.7 \rfloor = -9$



[1] continuous

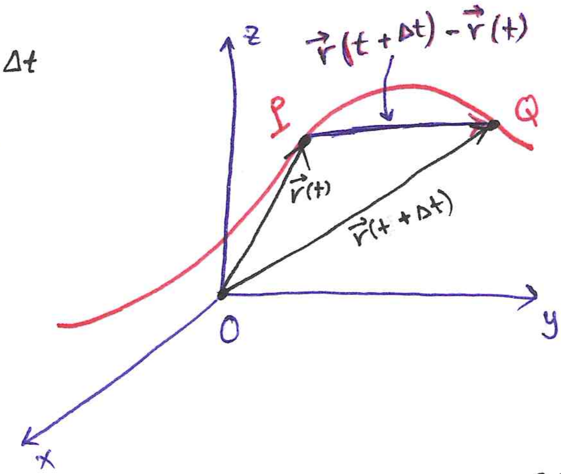
Derivatives and Motion

• Let $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ where f, g, h are differentiable

• If the particle moves from time t to $t + \Delta t$ then, the difference in positions is

$$\Delta\vec{r} = \vec{r}(t+\Delta t) - \vec{r}(t)$$

$$= f(t+\Delta t)\vec{i} + g(t+\Delta t)\vec{j} + h(t+\Delta t)\vec{k} - f(t)\vec{i} - g(t)\vec{j} - h(t)\vec{k}$$

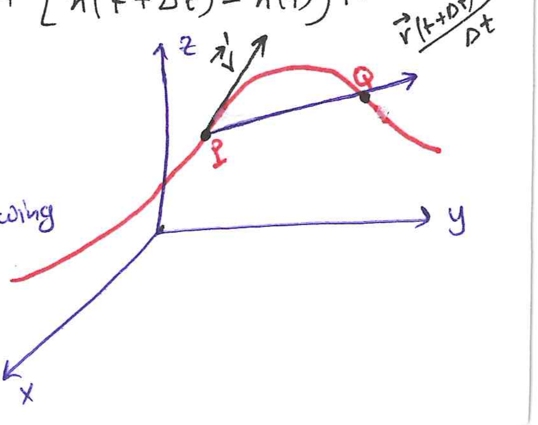


$$= \left[f(t+\Delta t) - f(t)\right]\vec{i} + \left[g(t+\Delta t) - g(t)\right]\vec{j} + \left[h(t+\Delta t) - h(t)\right]\vec{k}$$

• As $\Delta t \to 0$, [1] $Q \to P$

[2] The secant $PQ \to$ tangent $\vec{r}\,'$

[3] The quotient $\frac{\Delta\vec{r}}{\Delta t} \to$ the following Limit:



$$\lim_{\Delta t\to 0}\frac{\Delta\vec{r}}{\Delta t} = \left[\lim_{\Delta t\to 0}\frac{f(t+\Delta t)-f(t)}{\Delta t}\right]\vec{i} + \left[\lim_{\Delta t\to 0}\frac{g(t+\Delta t)-g(t)}{\Delta t}\right]\vec{j} + \left[\lim_{\Delta t\to 0}\frac{h(t+\Delta t)-h(t)}{\Delta t}\right]\vec{k}$$

$$= \left(\frac{df}{dt}\right)\vec{i} + \left(\frac{dg}{dt}\right)\vec{j} + \left(\frac{dh}{dt}\right)\vec{k}$$

Def The vector function $\vec{r}(t) = f(t)\,\vec{i} + g(t)\,\vec{j} + h(t)\,\vec{k}$ has derivative (is differentiable) at t if f, g, h have derivatives at t. The derivative is the vector function

$$\vec{r}\,'(t) = \frac{d\vec{r}}{dt} = \frac{df}{dt}\vec{i} + \frac{dg}{dt}\vec{j} + \frac{dh}{dt}\vec{k}$$

- A vector function $\vec{r}$ is differentiable if it is differentiable at every point of its domain.
- The curve $\vec{r}$ is smooth if $\frac{d\vec{r}}{dt}$ is continous and never $\vec{0}$.

That is, f, g, h have continuous first derivatives that are not simultaneously 0.

Def If $\vec{r}$ is the position vector of a particle moving along smooth curve in space, then

1. Velocity vector is $\vec{v} = \frac{d\vec{r}}{dt}$ "derivative of the position" "always tangent to the curve"
2. Speed is the magnitude of velocity : Speed $= |\vec{v}|$
3. Acceleration is the derivative of velocity: $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$
4. The unit vector $\frac{\vec{v}}{|\vec{v}|}$ is the direction of motion.
5. Velocity = (speed)(direction) $= |\vec{v}|\left(\frac{\vec{v}}{|\vec{v}|}\right)$.

Differentiation Rules for Vector Functions

- Let $\vec{u}$ and $\vec{v}$ be differentiable vector functions of t.

  $\vec{C} = a_1\vec{i} + a_2\vec{j} + a_3\vec{K}$ be a constant vector, $a_1, a_2, a_3 \in \mathbb{R}$.

  c any scalar

  f any differentiable scalar function.

[1] Constant Function Rule: $\frac{d}{dt}\vec{C} = \vec{0}$

[2] Scalar Multiple Rules: $\frac{d}{dt}\left[c\,\vec{u}(t)\right] = c\,\vec{u}'(t)$

$$\frac{d}{dt}\left[f(t)\,\vec{u}(t)\right] = f'(t)\,\vec{u}(t) + f(t)\,\vec{u}'(t)$$

[3] Sum Rule: $\frac{d}{dt}\left[\vec{u}(t) + \vec{v}(t)\right] = \vec{u}'(t) + \vec{v}'(t)$

[4] Difference Rule: $\frac{d}{dt}\left[\vec{u}(t) - \vec{v}(t)\right] = \vec{u}'(t) - \vec{v}'(t)$

[5] Dot Product Rule: $\frac{d}{dt}\left[\vec{u}(t)\cdot\vec{v}(t)\right] = \vec{u}'(t)\cdot\vec{v}(t) + \vec{u}(t)\cdot\vec{v}'(t)$

[6] Cross Product Rule: $\frac{d}{dt}\left[\vec{u}(t)\times\vec{v}(t)\right] = \vec{u}'(t)\times\vec{v}(t) + \vec{u}(t)\times\vec{v}'(t)$

[7] Chain Rule: $\frac{d}{dt}\left[\vec{u}(f(t))\right] = f'(t)\,\vec{u}'(f(t))$

Proof [5]

$$\frac{d}{dt}\left[\vec{u}(t)\cdot\vec{v}(t)\right] = \frac{d}{dt}\left[\left(u_1(t)\vec{i} + u_2(t)\vec{j} + u_3(t)\vec{K}\right)\cdot\left(v_1(t)\vec{i} + v_2(t)\vec{j} + v_3(t)\vec{K}\right)\right]$$

$$= \frac{d}{dt}\left[u_1v_1 + u_2v_2 + u_3v_3\right]$$

$$= u_1'v_1 + u_2'v_2 + u_3'v_3 + u_1v_1' + u_2v_2' + u_3v_3'$$

$$= \vec{u}'\cdot\vec{v} + \vec{u}\cdot\vec{v}'$$

Proof $\boxed{7}$ suppose that $\vec{u} = g_1(f(t))\vec{i} + g_2(f(t))\vec{j} + g_3(f(t))\vec{k}$ is differentiable vector function, where $f(t)$ is differentiable scalar function of t. Then g_1, g_2, g_3 are differentiable functions of t:

$$\frac{d}{dt}\left[\vec{u}(f(t))\right] = \frac{dg_1}{df}\frac{df}{dt}\vec{i} + \frac{dg_2}{df}\frac{df}{dt}\vec{j} + \frac{dg_3}{df}\frac{df}{dt}\vec{k}$$

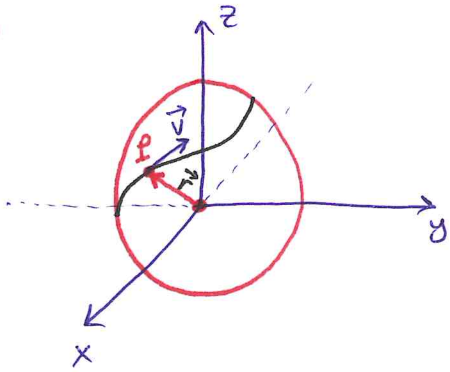
$$= \frac{df}{dt}\left(\frac{dg_1}{df}\vec{i} + \frac{dg_2}{df}\vec{j} + \frac{dg_3}{df}\vec{k}\right)$$

$$= \frac{df}{dt}\,\frac{d\vec{u}}{df}$$

$$= f'(t)\,\vec{u}'(f(t))$$

Vector Functions of Constant Length

Remark: If $\vec{r}(t)$ is a differentiable vector function of t of constant length, then $\vec{r}\cdot\vec{v} = 0$



Proof: Let $|\vec{r}(t)| = c$ "c is constant"

$$\vec{r}(t)\cdot\vec{r}(t) = c^2$$

$$\frac{d}{dt}\left[\vec{r}(t)\cdot\vec{r}(t)\right] = 0$$

$$\vec{r}'(t)\cdot\vec{r}(t) + \vec{r}(t)\cdot\vec{r}'(t) = 0$$

$$2\vec{r}'(t)\cdot r(t) = 0$$

$$r(t)\cdot\frac{d\vec{r}}{dt} = 0$$

The vectors $\vec{r}'(t)$ and $\vec{r}(t)$ are orthogonal because their dot product is 0.

Exp Let the position vector of particle moving along the curve $\vec{r}(t) = \left(4\cos\frac{t}{2}\right)\vec{i} + \left(4\sin\frac{t}{2}\right)\vec{j}$

1. Find the cartezian equation in the xy-plane describing the path.

$x = 4\cos\frac{t}{2} \quad , \quad y = 4\sin\frac{t}{2}$

$x^2 + y^2 = 16\cos^2\frac{t}{2} + 16\sin^2\frac{t}{2} = 16 \iff \boxed{x^2+y^2=16}$

2. Find the particle's velocity and acceleration vectors at $t = \pi$ and $t = \frac{3\pi}{2}$. sketch them as vectors on the curve.

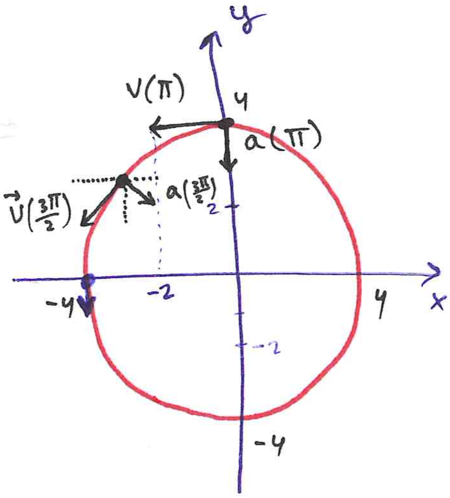
- $\vec{v} = \frac{d\vec{r}}{dt} = \left(-2\sin\frac{t}{2}\right)\vec{i} + \left(2\cos\frac{t}{2}\right)\vec{j}$

$\vec{v}(\pi) = -2\vec{i} + 0\vec{j} = -2\vec{i}$

$\vec{v}\left(\frac{3\pi}{2}\right) = \left(-2\sin\frac{3\pi}{4}\right)\vec{i} + \left(2\cos\frac{3\pi}{4}\right)\vec{j}$

$= \left(-2\frac{1}{\sqrt{2}}\right)\vec{i} - \left(2\frac{1}{\sqrt{2}}\right)\vec{j}$

$= -\sqrt{2}\,\vec{i} - \sqrt{2}\,\vec{j}$



- $\vec{a} = \frac{d\vec{v}}{dt} = \left(-\cos\frac{t}{2}\right)\vec{i} - \left(\sin\frac{t}{2}\right)\vec{j}$

$\vec{a}(\pi) = 0\vec{i} - \vec{j} = -\vec{j}$

$\vec{a}\left(\frac{3\pi}{2}\right) = \left(-\cos\frac{3\pi}{4}\right)\vec{i} - \left(\sin\frac{3\pi}{4}\right)\vec{j}$

$= \frac{1}{\sqrt{2}}\vec{i} - \frac{1}{\sqrt{2}}\vec{j}$

- at $t = \pi \Rightarrow$
$\vec{r}(\pi) = 0\vec{i} + 4\vec{j} = 4\vec{j}$

- at $t = \frac{3\pi}{2} \Rightarrow$
$\vec{r}\left(\frac{3\pi}{2}\right) = \left(-\frac{4}{\sqrt{2}}\right)\vec{i} + \left(\frac{4}{\sqrt{2}}\right)\vec{j}$
$= -2\sqrt{2}\,\vec{i} + 2\sqrt{2}\,\vec{j}$

3. Find the particle's speed

Speed $= |\vec{v}| = \sqrt{4\sin^2\frac{t}{2} + 4\cos^2\frac{t}{2}} = 2$

4. Find the direction of motion when $t = 2\pi$

Direction $= \frac{\vec{v}(2\pi)}{|\vec{v}(2\pi)|} = \frac{1}{\sqrt{4}}\left(0\vec{i} - 2\vec{j}\right) = -\vec{j}$

- at $t = 2\pi \Rightarrow$
$\vec{r}(2\pi) = -4\vec{i}$