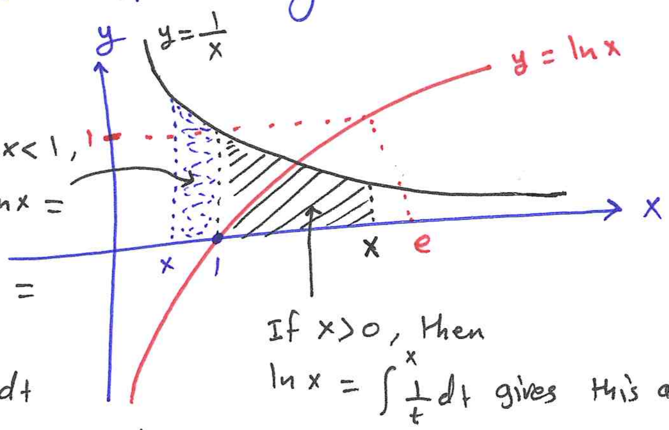
7.2 Natural logarithms

* The natural logarithm function $\ln x = \int_1^x \frac{1}{t}\,dt \;, x>0$



If $0<x<1$, then $\ln x = \int_1^x \frac{1}{t}\,dt = -\int_x^1 \frac{1}{t}\,dt$ gives the negative of this area

If $x>0$, then $\ln x = \int_1^x \frac{1}{t}\,dt$ gives this area

- $\ln 1 = 0 = \int_1^1 \frac{1}{t}\,dt$
- $D = (0, \infty)$
- $R = (-\infty, \infty)$
- $\ln e = 1$ where $e \approx 2.718$ is just a number.

---

* If $y = \ln x = \int_1^x \frac{1}{t}\,dt \;, x>0$, then $y' = \frac{dy}{dx} = \frac{1}{x}$

- If $y = \ln|u(x)| = \int_1^{|u(x)|} \frac{1}{t}\,dt$, then $y' = \frac{u'(x)}{u(x)}$, $u(x) \neq 0$ and different

- If $y = \ln|x|$, $x \neq 0$, then $y' = \frac{dy}{dx} = \frac{1}{x}$

---

* If u is differentiable that is never zero, then

$$\int \frac{1}{u}\,du = \ln|u| + C$$

Th2 (Properities of the Natural logarithm)

For any positive numbers a and b :

1. $\ln ab = \ln a + \ln b$ " Product Rule "
2. $\ln \frac{a}{b} = \ln a - \ln b$ " Quotient Rule "
3. $\ln \frac{1}{b} = -\ln b$ " Reciprocal Rule "
4. $\ln b^r = r \ln b$ " Power Rule " r is rational.

Example: Express $\ln\sqrt{13.5}$ in terms of $\ln 2$ and $\ln 3$

$$\ln\sqrt{13.5} = \ln\left(\frac{27}{2}\right)^{\frac{1}{2}} = \frac{1}{2}\ln\left(\frac{27}{2}\right) = \frac{1}{2}\left[\ln 27 - \ln 2\right]$$

$$= \frac{1}{2}\left[\ln 3^3 - \ln 2\right] = \frac{1}{2}\left[3\ln 3 - \ln 2\right]$$

* The integrals of $\tan x$, $\cot x$, $\sec x$, $\csc x$

1. $\int \tan x\, dx = \ln|\sec x| + C$
2. $\int \cot x\, dx = \ln|\sin x| + C$
3. $\int \sec x\, dx = \ln|\sec x + \tan x| + C$
4. $\int \csc x\, dx = -\ln|\csc x + \cot x| + C$

Remember:

$\int \sec x \tan x = \sec x + C$

$\int \csc x \cot x = -\csc x$

$\int \sec^2 x\, dx = \tan x + C$

$\int \csc^2 dx = -\cot x + C$

Proof $\boxed{1}$ $\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx = -\int \frac{du}{u}$

$u = \cos x > 0$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$du = -\sin x\, dx$

$$= -\ln|u| + C = -\ln|\cos x| + C = \ln\frac{1}{|\cos x|} + C = \ln|\sec x| + C$$

$\boxed{3}$ $\int \sec x\, dx = \int \sec x\left(\frac{\sec x + \tan x}{\sec x + \tan x}\right) dx = \int \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}\, dx$

$u = \sec x + \tan x$

$du = \sec x \tan x + \sec^2 x\ dx$

$$= \int \frac{du}{u} = \ln|u| + C$$

$$= \ln|\sec x + \tan x| + C$$

Example: Find $\boxed{1}$ $\int_{-3}^{-2} \frac{dx}{x} = \ln|x|\Big|_{-3}^{-2} = \ln 2 - \ln 3 = \ln\frac{2}{3}$

$\boxed{2}$ $\int_0^{\frac{\pi}{2}} \tan\frac{x}{2}\, dx = 2\ln\left|\sec\frac{x}{2}\right|\Big|_0^{\frac{\pi}{2}} = 2\ln\frac{1}{\cos\frac{x}{2}}\Big|_0^{\frac{\pi}{2}}$

$$= -2\ln\cos\frac{x}{2}\Big|_0^{\frac{\pi}{2}} = -2\left[\ln\cos\frac{\pi}{4} - \ln\cos 0\right]$$

$$= -2\left[\ln\frac{1}{\sqrt{2}} - \ln 1\right] = +2\ln\sqrt{2} = \ln 2$$

Example: Use logarthmic differentiation to find $\frac{dy}{dx}$ for

① $y = \sqrt{x(x+1)}$

$$\ln y = \ln \sqrt{x}\sqrt{x+1} = \ln \sqrt{x} + \ln \sqrt{x+1}$$

$$\ln y = \frac{1}{2}\ln x + \frac{1}{2}\ln(x+1)$$

$$\frac{y'}{y} = \frac{1}{2}\left[\frac{1}{x} + \frac{1}{x+1}\right] \Leftrightarrow y' = \frac{1}{2}y\left[\frac{1}{x} + \frac{1}{x+1}\right]$$

$$= \frac{1}{2}\sqrt{x(x+1)}\left[\frac{1}{x} + \frac{1}{x+1}\right]$$

---

② $y = t(t+1)(t+2)$

$$\ln y = \ln t + \ln(t+1) + \ln(t+2)$$

$$\frac{y'}{y} = \frac{1}{t} + \frac{1}{t+1} + \frac{1}{t+2}$$

$$y' = y\left[\frac{1}{t} + \frac{1}{t+1} + \frac{1}{t+2}\right]$$

$$y' = t(t+1)(t+2)\left[\frac{1}{t} + \frac{1}{t+1} + \frac{1}{t+2}\right]$$
