

Chapter 4: Interpolation and polynomial approximation

4.1+4.2: Given $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$

we are looking for the relation (function) between these points.

To know this function that passes through these points we approximate it.

Def: Interpolation: is an estimation of the unknown function by polynomial of degree at most n which passes through all the given points.

$$(i.e) \quad P_n(x_i) \approx f(x_i)$$

where f is the unknown function and P_n is the approximation polynomial. . . (we have $(n+1)$ points)

Exempl: $(1, 2), (2, 5), (3, 10)$ a polynomial passes through 3 point. is quadratic.

$$\Rightarrow f(x) = y = Ax^2 + Bx + C$$

$$2 = A + B + C$$

$$5 = 4A + 2B + C$$

$$10 = 9A + 3B + C$$

$$\left. \begin{array}{l} 2 = A + B + C \\ 5 = 4A + 2B + C \\ 10 = 9A + 3B + C \end{array} \right\} \Rightarrow \begin{array}{l} A = 1 \\ B = 0 \\ C = 1 \end{array}$$

$$\Rightarrow y = x^2 + 1$$

There fore we can estimate any value of any point in $[1, 3]$

Note: Sometimes we use Taylor polynomial to approximate function but sometimes it's hard to compute higher derivatives and sometimes not available.

4.3. Lagrange Interpolation:

Given $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$, we need to find the polynomial $P_n(x)$ which satisfies

$$P_n(x_i) = y_i, \quad i = 0, 1, \dots, n$$

• Given $(x_0, y_0), (x_1, y_1)$.

slope: $m = \frac{y_1 - y_0}{x_1 - x_0}$

then: $y - y_0 = m(x - x_0) = \left(\frac{y_1 - y_0}{x_1 - x_0} \right) (x - x_0)$

$$\Rightarrow y = y_0 + y_1 \frac{(x - x_0)}{x_1 - x_0} - y_0 \frac{(x - x_0)}{x_1 - x_0}$$

$$= \frac{(x_1 - x)}{x_1 - x_0} y_0 + \frac{(x - x_0)}{(x_1 - x_0)} y_1$$

$$\Rightarrow y = P_1(x) = \frac{(x - x_1)}{(x_0 - x_1)} y_0 + \frac{(x - x_0)}{(x_1 - x_0)} y_1$$

where $P_1(x)$ has degree ≤ 1 & has two linear factors.

we will denote:

$$L_{1,0}(x) = \frac{x - x_1}{x_0 - x_1}$$

$$(s.t.) \quad L_{1,0}(x_0) = 1, \quad L_{1,0}(x_1) = 0$$

$$\& \quad L_{1,1}(x) = \frac{x - x_0}{x_1 - x_0}$$

$$(s.t.) \quad L_{1,1}(x_1) = 1, \quad L_{1,1}(x_0) = 0$$

Note that $P_1(x)$ passes through $(x_0, y_0), (x_1, y_1)$.

$$P_1(x_0) = y_0$$

$$\& \quad P_1(x_1) = y_1$$

The terms $L_{1,0}(x)$ and $L_{1,1}(x)$ are called
Lagrange coefficient polynomials.

$$\& \text{ we can write: } P_1(x) = \sum_{k=0}^1 L_{1,k}(x) y_k$$

Note: If we use $P_1(x)$ to approximate $f(x)$ over
the Interval $[x_0, x_1]$, we call $P_1(x)$ Interpolation

while If we use $P_1(x)$ to approximate $f(x)$ over
 $x < x_0$ or $x > x_1$, the $P_1(x)$ is called Extrapolation.

Now Given $(x_0, y_0), (x_1, y_1), (x_2, y_2)$. (3 points)

$$P_2(x) := \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2$$

Similarly: If we have $(x_0, y_0), (x_1, y_1), (x_2, y_2), (x_3, y_3)$

$$P_3(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

In General: Given $(x_0, y_0), \dots, (x_n, y_n)$.

$$P_n(x) = \frac{(x-x_1)\dots(x-x_n)}{(x_0-x_1)\dots(x_0-x_n)} y_0 + \dots + \frac{(x-x_0)\dots(x-x_{n-1})}{(x_n-x_0)\dots(x_n-x_{n-1})} y_n$$

$$P_n(x) = \sum_{k=0}^n L_{n,k}(x) y_k \quad \rightarrow \text{Lagrange polynomial.}$$

where: $L_{n,k}(x) = \frac{(x-x_0)\dots(x-x_{k-1})(x-x_{k+1})\dots(x-x_n)}{(x_k-x_0)\dots(x_k-x_{k-1})(x_k-x_{k+1})\dots(x_k-x_n)}$

OR
$$L_{n,k}(x) = \frac{\prod_{\substack{j=0 \\ j \neq k}}^n (x-x_j)}{\prod_{\substack{j=0 \\ j \neq k}}^n (x_k-x_j)}$$

Note that: $L_{n,k}(x_i) = 1 \quad \text{if } i = k$

$L_{n,k}(x_i) = 0 \quad \text{if } i \neq k$

Moreover:

$$P_n(x_0) = y_0$$

$$P_n(x_1) = 0y_0 + 1y_1 + 0y_2 + \dots + 0y_n = y_1$$

$\vdots$

$$P_n(x_j) = L_{n,0}(x_j)y_0 + \dots + L_{n,j}(x_j)y_j + \dots + y_n L_{n,n}(x_j)$$

$$= 0y_0 + \dots + 1y_j + \dots + 0y_n$$

$$= y_j$$

$$\Rightarrow P_n(x_k) = y_k, \quad \forall k = 0, 1, \dots, n.$$

Example: Given $f(x) = \cos x$ on $[0, 1.2]$

Find $P_1(x)$, $P_2(x)$, $P_3(x)$ and compare the

answers of $P_1(0.35)$, $P_2(0.35)$, $P_3(0.35)$ with the

exact answer.

To find $P_1(x)$, we need ~~two~~ points

$$\text{Let } (x_0, y_0) = (0, \cos 0), \quad (x_1, y_1) = (1.2, \cos 1.2)$$

$$(x_0, y_0) = (0, 1), \quad (x_1, y_1) = (1.2, 0.362358)$$

$$\text{Then } P_1(x) = \sum_{k=0}^1 L_{1,k}(x) y_k$$

$$= \frac{(x - x_1)}{(x_0 - x_1)} y_0 + \frac{(x - x_0)}{(x_1 - x_0)} y_1$$

$$= \frac{(x - 1.2)}{0 - 1.2} (1) + \frac{(x - 0)}{(1.2 - 0)} (0.362358)$$

$$\Rightarrow P_1(x) = -0.83333(x-1.2) + 0.301965x$$

$$\& P_1(0.35) = 0.8140208$$

$$\& f(0.35) = 0.9393727$$

error
 ≈ 0.1

$$P_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2$$

$$\text{Let } (x_0, y_0) = (0, \cos 0) = (0, 1)$$

$$(x_1, y_1) = (0.6, \cos 0.6) = (0.6, 0.8253361)$$

$$(x_2, y_2) = (1.2, \cos 1.2) = (1.2, 0.362258)$$

$$\Rightarrow P_2(x) = \frac{(x-0.6)(x-1.2)}{(0-0.6)(0-1.2)} (1) + \frac{(x-0)(x-1.2)}{(0.6-0)(0.6-1.2)} (0.825336)$$

$$+ \frac{(x-0)(x-0.6)}{(1.2-0)(1.2-0.6)} (0.362258)$$

$$\left[\frac{\text{length}}{\# \text{ points} - 1} \right]$$

or $\frac{1.2}{2} = 0.6$

$$\Rightarrow P_2(x) = 1.388889(x-0.6)(x-1.2) - 2.292599(x-0)(x-1.2) + 0.503275(x-0)(x-0.6)$$

$$\Rightarrow P_2(0.35) = 0.93315053$$

$$f(0.35) = 0.939372713$$

$$\Rightarrow \text{error} \approx 0.0062$$

To find $P_3(x)$, we need 4 points, we choose them regular nodes.

$$h = \frac{\text{length}}{\text{\textcircled{4}} - 1} = \frac{1.2 - 0}{3} = 0.4$$

of pts

$$\Rightarrow (x_0, y_0) = (0, 1), (0.4, \cos 0.4), (0.8, \cos 0.8), (1.2, \cos 1.2)$$

$$\Rightarrow (x_0, y_0) = (0, 1), (x_1, y_1) = (0.4, 0.921061)$$

$$(x_2, y_2) = (0.8, 0.696707), (x_3, y_3) = (1.2, 0.362358)$$

$$P_3(x) = \dots$$

$$\left. \begin{array}{l} P_3(0.35) = 0.939607167 \\ f_3(0.35) = 0.939372713 \end{array} \right\} \Rightarrow \text{error} \approx -2.3 \times 10^{-4}$$

Thm: Lagrange polynomial approximation:

Assume that $f \in C^{n+1}[a, b]$, $x_0, \dots, x_n \in [a, b]$

are $n+1$ nodes. if $x \in [a, b]$, then:

$$f(x) = P_n(x) + E_n(x)$$

$$(i.e) \quad f(x) \approx P_n(x)$$

$$\text{where } E_n(x) = \frac{(x-x_0)(x-x_1) \dots (x-x_n)}{(n+1)!} f^{(n+1)}(c)$$

$$c \in [a, b].$$

$$(b) \quad (2.5-2)(2.5-3)(2.5-3.5) \cdot \frac{1}{27} \cdot \frac{1}{3!} \approx 0.02469$$

Question 5 (5+5 points). Let $f(x) = \frac{1}{1-2x}$.

(a) Use Lagrange interpolation polynomial with nodes $x_0 = 2, x_1 = 3, x_2 = 3.5$ to find an estimation for $f(2.5)$

(b) Find an upper bound of the error when Estimating $f(2.5)$

(a) Find an upper bound of the error in the estimation in (a).. for $x \in [2, 3.5]$

$$f(x) \approx P_2(x) = f_0 \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + f_1 \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + f_2 \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

$$f(2.5) \approx f(2) \frac{(2.5-3)(2.5-3.5)}{(2-3)(2-3.5)} + f(3) \frac{(2.5-2)(2.5-3.5)}{(3-2)(3-3.5)} + f(3.5) \frac{(2.5-2)(2.5-3)}{(3.5-2)(3.5-3)}$$

$$= \frac{-1}{3} \left[\frac{0.5}{+1.5} \right] + \frac{-1}{5} \left[\frac{-0.5}{-0.5} \right] + \frac{-1}{6} \left[\frac{-0.25}{0.75} \right]$$

$$= -\frac{1}{9} - \frac{1}{5} + \frac{1}{18} = \frac{-10-18+5}{90} = \frac{-23}{90}$$

$$(b) \quad E_2(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{3!} f^{(3)}(c)$$

$$f^{(3)}(x) = \frac{48}{(1-2x)^4}, \quad x \in [2, 3.5]$$

① find max $f^{(3)}(x)$ over $[2, 3.5]$

$\frac{48}{(1-2x)^4}$ is decreasing, so it has

$$\text{a max at } x=2, \quad \text{Max} \left[\frac{48}{(1-2x)^4} \right] = \frac{48}{(-3)^4} = \frac{16}{27}$$

② we find $\text{Max} |(x-2)(x-3)(x-3.5)|$ over $[2, 3.5]$

$$g(x) = (x-2)(x-3)(x-3.5)$$

$$g'(x) = (x-3)(x-3.5) + (x-2)(x-3+3.5) = (x-3)(x-3.5) + (x-2)(2x-6.5)$$

$$= x^2 - 6.5x + 10.5 + 2x^2 - 10.5x + 13 = 3x^2 - 17x + 23.5$$

$$g'(x) = 0 \Rightarrow x = \frac{17 \pm \sqrt{(17)^2 - 12 \times 23.5}}{6} = \frac{17 \pm 2.646}{6} \Rightarrow x_1 = 3.274, x_2 = 2.392$$

$$g(2) = 0, g(3.5) = 0, |g(3.274)| = 0.0789, |g(2.392)| = 0.264$$

$$\text{so } \text{Max} |g(x)| = 0.264$$

$$\text{STUDY (S) HUB.com} \quad \frac{0.264 \times \frac{16}{27}}{6} = 0.0260740741$$

Note: $E_n(x_i) = 0$ (i.e.) The error at the nodes is equal zero

proof: We will prove it for $n=1$, the general case in the exercise:
(i.e.) $E_1(x) = \frac{(x-x_0)(x-x_1)}{2!} P^{(2)}(c)$

Define: $g(t) = f(t) - P_1(t) - \frac{E_1(x)(t-x_0)(t-x_1)}{(x-x_0)(x-x_1)}$ ←

Notice that x, x_0, x_1 are constants in $g(t)$.

$$\begin{aligned} \text{and } g(x) &= f(x) - P_1(x) - \frac{E_1(x)(x-x_0)(x-x_1)}{(x-x_0)(x-x_1)} \\ &= f(x) - P_1(x) - E_1(x) = 0. \end{aligned}$$

No error at the nodes

Similarly: $g(x_0) = f(x_0) - P_1(x_0) - E_1(x_0) \cdot 0 = 0$

$g(x_1) = f(x_1) - P_1(x_1) - E_1(x_1) \cdot 0 = 0$

Consider the Interval (x_0, x_1) & let $x \in (x_0, x_1)$
 f cont & f' exist in (x_0, x_1)

By Mean value thm: on (x_0, x) , $\exists c_1 \in (x_0, x)$

(s.t) $g'(c_1) = \frac{g(x) - g(x_0)}{x - x_0} = 0$

Similarly: $\exists c_2 \in (x, x_1)$ such that

$$g'(c_2) = \frac{g(x_1) - g(x)}{x_1 - x} = 0$$

Now Consider the Interval (c_1, c_2) , by MVT

$\Rightarrow g' : \exists c \in (c_1, c_2)$ such that.

$$g''(c) = \frac{g'(c_2) - g'(c_1)}{c_2 - c_1} = \frac{0 - 0}{c_2 - c_1} = 0$$

Now go back to $g(t)$ & compute the derivative.

$$g'(t) = f'(t) - p_1'(t) - E_1(x) \frac{(t-x_0) + (t-x_1)}{(x-x_0)(x-x_1)}$$

$$g''(t) = f''(t) - 0 - E_1(x) \frac{2}{(x-x_0)(x-x_1)}$$

but: $\boxed{g''(c) = 0}$

$$\Rightarrow g''(c) = f''(c) - E_1(x) \frac{2}{(x-x_0)(x-x_1)} = 0$$

$$\Rightarrow E_1(x) = \frac{(x-x_0)(x-x_1) f''(c)}{2}$$

Example: for previous example: $f(x) = \cos x$, $x \in [0, 1.2]$

$$E_2(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{(3)!} f^{(3)}(c)$$

$$= \frac{(x-0)(x-0.6)(x-1.2)}{6} f^{(3)}(c)$$

$$\Rightarrow E_2(0.35) = \frac{(0.35)(0.35-0.6)(0.35-1.2)}{6} f^{(3)}(c)$$

We can bound the error:

$$|E_2(x)| \leq \left| \frac{x(x-0.6)(x-1.2)}{6} \right| \cdot \max_{x_0 \leq x \leq x_n} |f^{(3)}(x)|$$

$$f^{(3)}(x) = \sin x \quad \text{which is increasing function on } [0, 1.2]$$

$$\Rightarrow \max |f^{(3)}(x)| \leq \sin(1.2) = 0.932.$$

which implies:

$$|E_2(0.35)| \leq 0.01155.$$

in finding $P_2(0.35)$ we found 0.0062 which is better due to $\max |f^{(3)}(c)|$

Note: The next Result address the special case when the nodes for lagrange polynomial are equally spaced.

(i.e) $x_k = x_0 + hk$, for $k=0, 1, \dots, n$, $\boxed{h = \frac{b-a}{n}}$
 $n = \# \text{ points} - 1$
 & $P_n(x)$ used inside $[x_0, x_n]$.

Thm: (Error Bounds for Lagrange Interpolation Equally spaced Nodes). (Uniform partition).

Assume that $f(x)$ is defined on $[a, b]$, which contains equally spaced nodes $x_k = x_0 + h k$, $h = \frac{b-a}{n}$

assume that $f(x)$ and the derivatives of $f(x)$ up to order $n+1$ ($f \in C^{(n+1)}[a, b]$) are continuous and

bounded on the special subintervals $[x_0, x_1], [x_0, x_2], \dots$

and $[x_0, x_3]$ resp. that is

$$|f^{(n+1)}(x)| \leq M_{n+1}, \quad \text{for } x_0 \leq x \leq x_n$$

Then: for $n=1, 2, 3$, the error terms have the following useful bounds:

$$(1) \quad |E_1(x)| \leq \frac{h^2 M_2}{8}, \quad x \in [x_0, x_1]$$

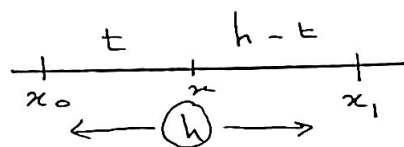
$$(2) \quad |E_2(x)| \leq \frac{h^3 M_3}{9\sqrt{3}}, \quad x \in [x_0, x_2]$$

$$(3) \quad |E_3(x)| \leq \frac{h^4 M_4}{24}, \quad x \in [x_0, x_3].$$

proof: (1) using change of variables:

$$x - x_0 = t$$

$$\& \quad x - x_1 = t - h$$



$$\hookrightarrow x - (x_0 + h) = \underline{x - x_0} - h = t - h.$$

Then: $E_1(x) = \frac{(x - x_0)(x - x_1) f^{(2)}(c)}{2!}$ by previous thm. $[x_0, x_1]$

then $E_1(t) = E_1(x_0 + t) = \frac{t(t-h) f^{(2)}(c)}{2!}$, $0 \leq t \leq h$ $[0, h]$

Assume $|f^{(2)}(x)| \leq M_2$, $\forall x_0 \leq x \leq x_1$

we need to determine a bound for $t(t-h) = t^2 - th$

Let $\phi(t) = t^2 - ht$

$$\phi'(t) = 2t - h$$



$\Rightarrow t = \frac{h}{2}$ is a critical point ($\phi'(t) = 0$)

The extreme value of $\phi(t)$ over $[0, h]$ occur either at the end points or at the critical point.

Note $\phi(0) = 0$, $\phi(h) = 0$ & $\phi(\frac{h}{2}) = -\frac{h^2}{4}$

$\Rightarrow |\phi(t)| = |t^2 - th| \leq \left| -\frac{h^2}{4} \right| = \frac{h^2}{4}, \quad 0 \leq t \leq h.$

$\Rightarrow |E_1(x)| = \left| \frac{(t^2 - th) f^{(2)}(c)}{2!} \right| \leq \frac{h^2}{8} \cdot M_2.$

$\leq \frac{h^2}{4} \cdot \frac{M_2}{2!}$

Note: if we keep the x , we find $E_1^*(\frac{x_0 + x_1}{2}) = \frac{h^2}{4!}$

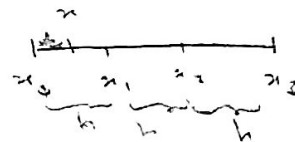
proof (3) : $|E_3(x)| \leq \frac{h^4 M_4}{24}, \quad x \in [x_0, x_3]$

$$E_3(x) = \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{4!} f^{(4)}(c)$$

Let $|f^{(4)}(x)| \leq M_4, \quad \forall x \in [x_0, x_3]$

then $|E_3(x)| \leq \frac{|(x-x_0)(x-x_1)(x-x_2)(x-x_3)|}{24} M_4$

we change the variables:



$$x = x_0 + t \Rightarrow t = x - x_0$$

$$\rightarrow x_1 = x_0 + h \Rightarrow x - x_1 = x - x_0 - h = t - h$$

$$x_2 = x_0 + 2h \Rightarrow x - x_2 = x - x_0 - 2h = t - 2h$$

$$x_3 = x_0 + 3h \Rightarrow x - x_3 = x - (x_0 + 3h) = t - 3h$$

$$\Rightarrow |E_3(t)| = |E_3(x_0 + t)| \leq \frac{t(t-h)(t-2h)(t-3h)}{24} M_4, \quad t \in [0, 3h]$$

Let $\phi(t) = t(t-h)(t-2h)(t-3h) = t^4 - 6ht^3 + 11h^2t^2 - 6h^3t$
 $\phi'(t) = 4t^3 - 18ht^2 + 22h^2t - 6h^3$

$$\phi'(t) = 0 \Rightarrow t = 2.618033989 h$$

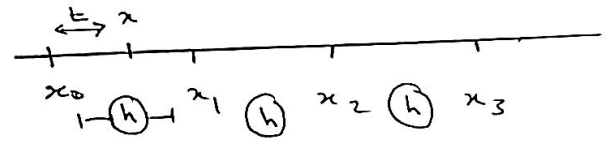
$$t = 0.381966 h$$

$$t = 1.5 h$$

$$\phi(2.618033989 h) = \phi(0.381966 h) = |-h^4| = h^4$$

$$\& \phi(0) = \phi(3h) = 0$$

then the $\max |\phi(t)| = h^4$



$$\Rightarrow |E_3(n)| \leq \frac{h^4 M_4}{24}$$

Example: back to $\cos x$, $[0, 1.2]$.

$$\bullet 1) |E_1(n)| \leq \frac{h^2}{8} M_2, \quad h = x_1 - x_0 = 1.2.$$

$$|f^{(2)}(c)| \leq M_2 : f^{(2)}(x) = -\cos x$$

$$\cos 0 = 1, \quad \cos(1.2) = 0.362357754$$

then $\cos x$ is decreasing $\Rightarrow \max_{0 \leq x \leq 1.2} |\cos x| = \cos 0 = 1$

$$\Rightarrow |E_1(n)| \leq \frac{(1.2)^2}{8} \cdot 1 = 0.180000$$

$$\bullet 2) |E_2(n)| \leq \frac{h^3 M_3}{9\sqrt{3}}, \quad h = \frac{1.2 - 0}{2} = 0.6$$

$$|f^{(3)}(x)| = |\sin x| \leq \sin(1.2) = 0.932039 = M_3$$

$$\Rightarrow |E_2(n)| \leq \frac{(0.6)^3 \cdot 0.932039}{9\sqrt{3}} = 0.012915.$$

4.4 Newton Polynomials:

In Lagrange Polynomial there is No Constructive relationship between $P_{n-1}(x)$ & $P_n(x)$

Each polynomial has to Construct Individually and the work requires to compute the higher-degree polynomials involves many computations.

Therefore: we need to find a new approach and Construct Newton Polynomials that have the recursive pattern.

Given $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$, $P_n(x_i) = y_i$

Let
$$P_1(x) = a_0 + a_1(x - x_0)$$

$$\begin{aligned} P_2(x) &= a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) \\ &= P_1(x) + a_2(x - x_0)(x - x_1) \end{aligned}$$

$$\begin{aligned} P_3(x) &= a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2) \\ &= P_2(x) + a_3(x - x_0)(x - x_1)(x - x_2) \end{aligned}$$

$\vdots$

$$P_n(x) = P_{n-1}(x) + a_n(x - x_0) \dots (x - x_{n-1})$$

$P_n(x)$ is called Newton polynomial with n centers

$x_0, x_1, \dots, x_{n-1}$ with degree $\leq n$

We need Now to determine a_i 's.

$$y_0 = P_1(x_0) = a_0 + a_1(x_0 - x_0) = a_0$$

$$\Rightarrow a_0 = y_0 = f(x_0).$$

Zero Divided difference

Beside: $y_1 = P_1(x_1) = f(x_0) + a_1(x_1 - x_0) = f(x_1)$

Solve for a_1 :

$$a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = f[x_0, x_1]$$

which is called first divided difference.

Moreover: $y_2 = f(x_2) = P_2(x_2) = f(x_0) + f[x_0, x_1](x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$

then $a_2 = \frac{f(x_2) - f(x_0) - f[x_0, x_1](x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)}$

Rearrange: $a_2 = \frac{\frac{f(x_2) - f(x_0)}{(x_2 - x_0)} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{(x_2 - x_1)}$

For computational purpose we will write it as:

$$a_2 = \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{(x_2 - x_0)} = f[x_0, x_1, x_2]$$

a_2 is the second divided difference.

$$\text{where } f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

In General: $a_k = f[x_0, x_1, \dots, x_k]$ is the k th divided difference.

Def: The divided differences for a function $f(x)$ are defined:

$$f[x_k] = f(x_k) \quad \text{Zeroth divided difference.}$$

$$f[x_{k-1}, x_k] = \frac{f[x_k] - f[x_{k-1}]}{x_k - x_{k-1}}$$

$$f[x_{k-2}, x_{k-1}, x_k] = \frac{f[x_{k-1}, x_k] - f[x_{k-2}, x_{k-1}]}{x_k - x_{k-2}}$$

& In General: higher order D.D:

$$f[x_{k-j}, x_{k-j+1}, \dots, x_k] = \frac{f[x_{k-j+1}, \dots, x_k] - f[x_{k-j}, \dots, x_{k-1}]}{x_k - x_{k-j}}$$

The following table shows the constructive of the divided differences:

x_k	$f[x_k]$	$f[x_{k-1}, x_k]$	$f[x_{k-2}, x_{k-1}, x_k]$	$f[, ,]$	$f[, , ,]$
x_0	$f[x_0] = a_0$				
x_1	$f[x_1]$	$f[x_0, x_1] = a_1$			
x_2	$f[x_2]$	$f[x_1, x_2]$	$f[x_0, x_1, x_2] = a_2$		
x_3	$f[x_3]$	$f[x_2, x_3]$	$f[x_1, x_2, x_3]$	$f[x_0, x_1, x_2, x_3] = a_3$	
x_4	$f[x_4]$	$f[x_3, x_4]$	$f[x_2, x_3, x_4]$	$f[x_1, x_2, x_3, x_4]$	$f[x_0, x_1, x_2, x_3, x_4] = a_4$

Thm: (Newton polynomial): Suppose that $x_0, x_1, \dots, x_n$ are $(n+1)$ distinct numbers in $[a, b]$, There exists a unique polynomial $P_n(x)$ of degree at most n with the property: $f(x_j) = P_n(x_j)$, $j = 0, 1, \dots, n$

The Newton Polynomial is:

$$P_n(x) = a_0 + a_1(x-x_0) + \dots + a_n(x-x_0)\dots(x-x_{n-1})$$

where $a_k = f[x_0, \dots, x_k]$, $k = 0, 1, \dots, n$.

Corollary: $f(x) = P_n(x) + E_n(x)$

where $E_n(x)$ is the same error as in Lagrange Interpolation.

Example: Let $f(x) = x^3 - 4x$

Construct divided difference for $x_0 = 1, x_1 = 2, \dots, x_5 = 6$.

and find $P_3(x)$

The points are $(1, -3), (2, 0), (3, 15), (4, 48),$
 $(5, 105), (6, 192)$

x_k	$f[x_k]$	1st D.D	2nd D.D	3rd D.D.	4th D.D
1	-3				
2	0	3			
3	15	15	6		
4	48	33	9	1	
5	105	57	12	1	0
6	192	87	15	1	0

then: $P_3(x) = -3 + 3(x-1) + 6(x-1)(x-2) + 1(x-1)(x-2)(x-3).$

for example: $f[3, 4, 5] = \frac{f[4, 5] - f[3, 4]}{5 - 3} =$

$$= \frac{\frac{f[5] - f[4]}{5 - 4} - \frac{f[4] - f[3]}{4 - 3}}{5 - 3} = \frac{\frac{105 - 48}{1} - \frac{48 - 15}{1}}{2} = \frac{105 - 48 - 48 + 15}{2} = \frac{12}{1} = 12$$

Ex. Estimate $f(5.5)$ using Newton Interpolation polynomial P_1, P_2, P_3 for the following table.

x_k	$f[x_k]$	1st D.D	2nd D.D	3rd D.D	4th .D.D
1	3	—	—	—	—
3	4.5	0.75	—	—	—
4.25	6		0.138462	—	—
5.75	7.25			0.05722	—
6	8				0.10287

$$P_1(x) = a_0 + a_1(x-x_0) = 3 + 0.75(x-1)$$

$$P_1(5.5) = 6.375 \approx P_1(5.5)$$

$$P_2(x) = P_1(x) + a_2(x-1)(x-3)$$

$$P_2(5.5) \approx 7.93267$$

$$P_3(x) = P_2(x) + a_3(x-1)(x-3)(x-4.25)$$

$$P_3(5.5) \approx 8.7373$$

Note: ① If $f(x) = x^5 - 4x^3 + 3x^2 + 7$

then $a_6 = 0$ & $a_i = 0$, $i > 5$.

② $a_5 = 1$, but $a_3 \neq -4$ & $a_2 \neq 3$, $a_4 \neq 0$
and so on.

③ $P_5(x) = f(x)$ for any nodes.

This Note is valid only for polynomials