

10.1 Sequences.

6. $a_n = \frac{2^n - 1}{2^n}$. Find the values of a_1, a_2, a_3 & a_4 .

$$\underline{a_1} = \frac{2^1 - 1}{2^1} = \frac{1}{2}, \quad \underline{a_2} = \frac{3}{4}, \quad \underline{a_3} = \frac{7}{8}, \quad \underline{a_4} = \frac{15}{16}, \dots$$

10. $a_1 = -2$, $a_{n+1} = \frac{na_n}{n+1}$, write the first ten terms of the sequence.

$$a_1 = -2, \quad \boxed{n=1} \quad a_{1+1} = \frac{1(a_1)}{1+1}$$

$$a_2 = \frac{-2}{2} = -1$$

$$\boxed{n=2} \quad a_{2+1} = \frac{2a_2}{2+1} = \frac{2(-1)}{3}$$

$$a_3 = -\frac{2}{3}$$

$$\boxed{n=3} \quad \frac{3a_3}{3+1} = a_{3+1}$$

$$a_4 = \frac{3(-\frac{2}{3})}{4} = -\frac{1}{2}$$

$$\boxed{n=4} \quad a_{4+1} = \frac{4a_4}{4+1} = \frac{4(-\frac{1}{2})}{5} = -\frac{2}{5}$$

22. 2, 6, 10, 14, 18, ... Find a formula for n th term of the sequence.

$$a_1 = 2$$

$$\begin{cases} 6-2=4 \\ 10-6=4 \\ 14-10=4 \end{cases} \Rightarrow d=4$$

$$\text{So, } a_n = 2 + (n-1)4$$

$$a_n = 2 + 4n - 4$$

$$a_n = -2 + 4n$$

$$n = 1, 2, \dots$$

Note arithmetic sequence

$$a_n = a_1 + (n-1)d$$

d
Common
difference

26 $0, 1, 1, 2, 2, 3, 3, \dots$, find a formula for n th term of the sequence

$$\left[\frac{0}{2}\right] = 0$$

$$\left[\frac{1}{2}\right] = \left[\frac{2}{2}\right] = 1$$

$$\left[\frac{3}{2}\right] = 1$$

$$\left[\frac{4}{2}\right] = \left[\frac{5}{2}\right] = 2$$

$$\left[\frac{6}{2}\right] = 2$$

$$\text{So } a_n = \left[\frac{n}{2}\right], \quad n = 1, 2, \dots$$

31 $a_n = \frac{1 - 5n^4}{n^4 + 8n^3}$, Is the sequence converges or diverges, find its limit, if.

$$a_n = \frac{\frac{1}{n^4} - 5}{1 + \frac{8}{n}}$$

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{\frac{1}{n^4} - 5}{1 + \frac{8}{n}}$$

$$= \frac{-5}{1} = -5 \quad \text{Converges.}$$

35 $a_n = 1 + (-1)^n$

$$a_n = 0, 2, 0, 2, \dots$$

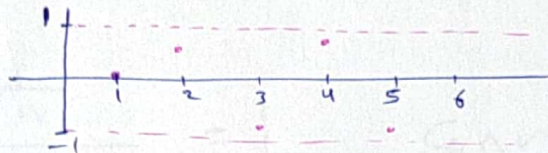
$$\lim_{n \rightarrow \infty} a_n \text{ does not exist}$$

diverges.

36 $a_n = (-1)^n \left(1 - \frac{1}{n}\right)$

$a_n = 0, \frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \dots$

$\lim_{n \rightarrow \infty} a_n$ does not exist. $\Rightarrow$ diverges



41 $a_n = \sqrt{\frac{2n}{n-1}}$

$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \sqrt{\frac{2n}{n-1}} = \sqrt{\lim_{n \rightarrow \infty} \frac{2n}{n-1}} = \sqrt{2}$ Converges.

44 $a_n = n\pi \cos(n\pi)$

$a_n = -\pi, 2\pi, -3\pi, 4\pi, \dots$, $\lim_{n \rightarrow \infty} n\pi \cos(n\pi) = \lim_{n \rightarrow \infty} n\pi(-1)^n$

$\lim_{n \rightarrow \infty} a_n$ does not exist $\Rightarrow$ diverges

48 $a_n = \frac{3^n}{n^3}$

$\lim_{n \rightarrow \infty} \frac{3^n}{n^3} \quad \frac{\infty}{\infty}$

$= \lim_{n \rightarrow \infty} \frac{\ln 3 \cdot 3^n}{3n^2}$

$= \lim_{n \rightarrow \infty} \frac{(\ln 3)^2 3^n}{6n}$

$= \lim_{n \rightarrow \infty} \frac{(\ln 3)^3 3^n}{6} = \infty$

diverges.

50) $a_n = \frac{\ln n}{\ln 2n}$

$$\lim_{n \rightarrow \infty} \frac{\ln n}{\ln 2n} = \frac{\infty}{\infty}$$

$$= \lim_{n \rightarrow \infty} \frac{\frac{1}{n}}{\frac{2}{2n}} = 1 \quad \text{Converges.}$$

54) $a_n = \left(1 - \frac{1}{n}\right)^n$
 $a_n = \left(1 + \frac{(-1)}{n}\right)^n$

$$\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x$$

$$\lim_{n \rightarrow \infty} \left(1 + \frac{(-1)}{n}\right)^n = e^{-1} = \frac{1}{e} \quad (\text{converges}).$$

70) $\lim_{n \rightarrow \infty} \left(\frac{n}{n+1}\right)^n$

$$= \lim_{n \rightarrow \infty} e^{n \ln \left(\frac{n}{n+1}\right)} = \lim_{n \rightarrow \infty} e^{n (\ln n - \ln(n+1))}$$

$$= \lim_{n \rightarrow \infty} e^{\left[\frac{\ln n - \ln(n+1)}{\frac{1}{n}} \right]}$$

$$= \lim_{n \rightarrow \infty} e^{\left[\frac{\frac{1}{n} - \frac{1}{n+1}}{-\frac{1}{n^2}} \right]}$$

$$= \lim_{n \rightarrow \infty} e^{\left[-n^2 \left(\frac{n+1-n}{n(n+1)} \right) \right]}$$

$$= \lim_{n \rightarrow \infty} e^{\frac{-n^2}{n(n+1)}}$$

$$= e^{-1} = \frac{1}{e} \quad \text{Converges.}$$

$$58) a_n = (n+4)^{\frac{1}{n+4}}$$

$$\text{let } u = n+4.$$

$$n \rightarrow \infty \Rightarrow u \rightarrow \infty.$$

$$\lim_{u \rightarrow \infty} u^{\frac{1}{u}} = 1$$

$$\text{So } \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} (n+4)^{\frac{1}{n+4}} = 1$$

Converges,

$$60) a_n = \ln n - \ln(n+1).$$

$$a_n = \ln\left(\frac{n}{n+1}\right)$$

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \ln\left(\frac{n}{n+1}\right)$$

$$= \ln\left(\lim_{n \rightarrow \infty} \left(\frac{n}{n+1}\right)\right)$$

$$= \ln(1) = 0. \quad \text{Converges.}$$

$$63) a_n = \frac{n!}{n^n}$$

$$a_n = \frac{n}{n} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdots \frac{2}{n} \cdot \frac{1}{n}$$

$$0 \leq a_n \leq \frac{1}{n}$$

$$\lim_{n \rightarrow \infty} 0 \leq \lim_{n \rightarrow \infty} a_n \leq \lim_{n \rightarrow \infty} \frac{1}{n}$$

$$\text{by Sandwich Theorem } \lim_{n \rightarrow \infty} a_n = 0.$$

Converges.

72) $a_n = \left(1 - \frac{1}{n^2}\right)^n$

$$a_n = \left[\left(1 - \frac{1}{n}\right) \left(1 + \frac{1}{n}\right) \right]^n$$

$$= \left(1 - \frac{1}{n}\right)^n \left(1 + \frac{1}{n}\right)^n$$

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right)^n \cdot \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$$

$$= e^{-1} \cdot e^1$$

$$= e^0$$

$$= \boxed{1}$$

remember $\rightarrow$

$$\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x$$

x is fixed

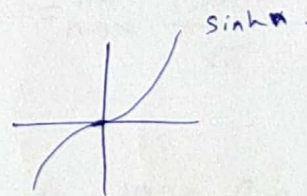
76) $a_n = \sinh(\ln n)$

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \sinh(\ln n)$$

$$= \lim_{n \rightarrow \infty} \frac{e^{\ln n} - e^{-\ln n}}{2}$$

$$= \lim_{n \rightarrow \infty} \frac{n - \frac{1}{n}}{2}$$

$$= \infty \text{ diverges.}$$



$$\sinh n = \frac{e^n - e^{-n}}{2}$$

82) $a_n = \frac{\tan^{-1} n}{\sqrt{n}}$

$$-\frac{\pi}{2} < \tan^{-1} n < \frac{\pi}{2}$$

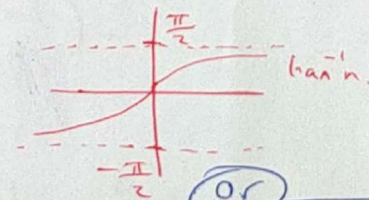
$$-\frac{\pi}{2\sqrt{n}} < \frac{\tan^{-1} n}{\sqrt{n}} < \frac{\pi}{2\sqrt{n}}$$

$$\lim_{n \rightarrow \infty} -\frac{\pi}{2\sqrt{n}} = 0$$

$$\lim_{n \rightarrow \infty} \frac{\pi}{2\sqrt{n}} = 0$$

So, by Sandwich Theorem

$$\lim_{n \rightarrow \infty} \frac{\tan^{-1} n}{\sqrt{n}} = 0.$$



or

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \cdot \tan^{-1} n$$

$$= 0 \cdot \frac{\pi}{2}$$

$$= 0$$

Converges.

86 $a_n = \frac{(\ln n)^5}{\sqrt{n}}$

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{(\ln n)^5}{\sqrt{n}} \quad \frac{\infty}{\infty}$$

$$= \lim_{n \rightarrow \infty} \frac{5(\ln n)^4}{\frac{1}{2\sqrt{n}}}$$

$$= \lim_{n \rightarrow \infty} \frac{10(\ln n)^4}{\sqrt{n}}$$

$$= \lim_{n \rightarrow \infty} \frac{40(\ln n)^3}{\frac{1}{2\sqrt{n}}}$$

$$= \lim_{n \rightarrow \infty} \frac{80(\ln n)^3}{\sqrt{n}}$$

$$= \lim_{n \rightarrow \infty} \frac{240(\ln n)^2}{\frac{1}{2\sqrt{n}}}$$

$$= \lim_{n \rightarrow \infty} \frac{480(\ln n)^2}{\sqrt{n}}$$

$$= \lim_{n \rightarrow \infty} \frac{960 \ln n}{\frac{1}{2\sqrt{n}}}$$

$$= \lim_{n \rightarrow \infty} \frac{1920 \ln n}{\sqrt{n}}$$

$$= \lim_{n \rightarrow \infty} \frac{1920}{\frac{1}{2\sqrt{n}}}$$

$$= \lim_{n \rightarrow \infty} \frac{3840}{\sqrt{n}}$$

$$= \boxed{0} \text{ Converges.}$$

92 $a_{n+1} = \frac{a_n + 6}{a_n + 2}$, $a_1 = -1$. (Assume the sequence is converges & find the limit).

Since a_n converges $\Rightarrow \lim_{n \rightarrow \infty} a_n = L$

$$\& \lim_{n \rightarrow \infty} a_{n+1} = \lim_{n \rightarrow \infty} \frac{a_n + 6}{a_n + 2}$$

$$L = \frac{L + 6}{L + 2}$$

$$L(L+2) = L+6$$

$$L^2 + 2L - L - 6 = 0$$

$$L^2 + 6 - 6 = 0.$$

$$(L+3)(L-2) = 0$$

$$L = -3 \text{ or } L = 2.$$

Since $a_n > 0$ for $n \geq 2$

$$\text{So, } \boxed{L = 2}$$

III) Determine if the following sequence is monotonic & if it is bounded.

$$a_n = \frac{3n+1}{n+1}$$

$$\left[2, \frac{7}{3}, \frac{10}{4}, \frac{13}{5}, \dots \right]$$

$$a_{n+1} > a_n$$

$$\frac{3(n+1)+1}{(n+1)+1} > \frac{3n+1}{n+1}$$

$$\frac{3n+4}{n+2} > \frac{3n+1}{n+1}$$

$$\underline{3n^2+7n+4} > \underline{3n^2+7n+2}.$$

$$4 > 2 \quad \checkmark$$

$\therefore$ The sequence is nondecreasing (monotonic).

$$\text{Now, } \frac{3n+1}{n+1} < 3.$$

$$3n+1 < 3n+3$$

$$1 < 3$$

$\therefore$ The sequence is bounded above by 3 & bounded below by 2.

$\Rightarrow a_n$ is bounded.

10.2 Infinite Series.

Ex Find a formula for the n th partial sum & use it to find the series' sum:-

$$\frac{5}{1} + \frac{5}{2(3)} + \frac{5}{3(4)} + \dots + \frac{5}{n(n+1)} + \dots$$

$$a_n = \frac{5}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$$

$$\star A = 5 \text{ \& } B = -5$$

$$\text{So, } \frac{5}{n(n+1)} = \frac{5}{n} - \frac{5}{n+1}$$

$$S_n = \sum_{n=1}^{\infty} \left(\frac{5}{n} - \frac{5}{n+1} \right) \quad \text{Telescoping..}$$

$$S_n = \left(\frac{5}{1} - \frac{5}{2} \right) + \left(\frac{5}{2} - \frac{5}{3} \right) + \left(\frac{5}{3} - \frac{5}{4} \right) + \dots + \frac{5}{n} - \frac{5}{n+1}$$

$$\boxed{S_n = 5 - \frac{5}{n+1}} \quad \text{The } n\text{th partial sum.}$$

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left(5 - \frac{5}{n+1} \right) = 5.$$

$$\text{So, } \sum_{n=1}^{\infty} \frac{5}{n(n+1)} = 5 \quad \text{"That is Converges to 5"}$$

$$\boxed{14} \sum_{n=0}^{\infty} \frac{2^{n+1}}{5^n}$$

$$\sum_{n=0}^{\infty} \frac{2^{n+1}}{5^n} = \frac{2}{1} + \frac{4}{5} + \frac{8}{25} + \frac{16}{125} + \dots$$

its 'geometric series' with $a=2$ & $r=\frac{2}{5} < 1$

$$\text{So, } \sum_{n=0}^{\infty} \frac{2^{n+1}}{5^n} \text{ converges to } \frac{a}{1-r} = \frac{2}{1-\frac{2}{5}} = \frac{10}{3}$$

$$\sum_{n=0}^{\infty} \frac{2^{n+1}}{5^n} = \frac{10}{3}$$

$$\boxed{18} \left(-\frac{2}{3}\right)^2 + \left(-\frac{2}{3}\right)^3 + \left(-\frac{2}{3}\right)^4 + \dots$$

$$= \sum_{n=2}^{\infty} \left(-\frac{2}{3}\right)^n$$

its a geometric series with $a = \left(-\frac{2}{3}\right)^2 = \frac{4}{9}$

$$\& \ r = -\frac{2}{3}$$

$|r| < 1$ ($|-2/3| < 1$) , so the series converges

$$\sum_{n=2}^{\infty} \left(-\frac{2}{3}\right)^n = \frac{a}{1-r} = \frac{\frac{4}{9}}{1-\left(-\frac{2}{3}\right)} = \frac{4}{9} = \frac{4}{9} \cdot \frac{3}{5} = \boxed{\frac{4}{15}}$$

24 Express the following as the ratio of two integers.

$$1.\overline{414} = 1 + 0.414 + 0.000414 + \dots$$

$$= 1 + \frac{414}{1000} + \frac{414}{1000000} + \dots$$

★ Geometric series, with $a = \frac{414}{1000}$ & $r = \frac{1}{1000}$

$$\text{Converges to } \frac{a}{1-r} = \frac{\frac{414}{1000}}{1 - \frac{1}{1000}} = \frac{414}{999}$$

$$\rightarrow \text{So, } 1.\overline{414} = 1 + \frac{414}{999} = \frac{1413}{999}$$

32 $\sum_{n=0}^{\infty} \frac{e^n}{e^n + n}$

★ $\rightarrow$ by nth term test:

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{e^n}{e^n + n} = \lim_{n \rightarrow \infty} \frac{e^n}{e^n + 1}$$

$$= \lim_{n \rightarrow \infty} \frac{e^n}{e^n} = \lim_{n \rightarrow \infty} 1 = 1 \neq 0.$$

$\therefore \sum_{n=0}^{\infty} \frac{e^n}{e^n + n}$ diverges by nth term test.

38 $\sum_{n=1}^{\infty} (\tan n - \tan(n-1))$ A telescoping series.

$$S_n = (\tan 1 - \tan 0) + (\cancel{\tan 2} - \cancel{\tan 1}) + \dots + (\cancel{\tan n} - \cancel{\tan(n-1)})$$

$$= -\tan 0 + \tan n.$$

$$= \tan n.$$

by n^{th} Partial Sum:-

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \tan n = \text{DNE.}$$

$\therefore \sum_{n=1}^{\infty} (\tan n - \tan(n-1))$ diverges by n^{th} partial sum.

44 $\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$

$$\frac{2n+1}{n^2(n+1)^2} = \frac{A}{n} + \frac{B}{n^2} + \frac{C}{n+1} + \frac{D}{(n+1)^2}$$

$$A=0, B=1, C=0, D=-1.$$

$$S_n = \sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2} = \sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{1}{(n+1)^2} \right)$$

$$S_n = \left(1 - \cancel{\frac{1}{2^2}} \right) + \left(\cancel{\frac{1}{2^2}} - \cancel{\frac{1}{3^2}} \right) + \dots + \left(\cancel{\frac{1}{n^2}} - \frac{1}{(n+1)^2} \right)$$

$$S_n = 1 - \frac{1}{(n+1)^2}$$

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{(n+1)^2} \right)$$

$$= 1 - 0 = 1$$

So, the series converges to 1.

54

$$\sum_{n=0}^{\infty} \frac{\cos(n\pi)}{5^n}$$

$$= \frac{\cos(0)}{1} + \frac{\cos(\pi)}{5} + \frac{\cos(2\pi)}{5^2} + \dots$$

$$= 1 - \frac{1}{5} + \frac{1}{5^2} + \dots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{5^n} = \sum_{n=0}^{\infty} \left(-\frac{1}{5} \right)^n$$

Geometric series with $a=1$ & $r=-\frac{1}{5}$.

$|r| = \left| -\frac{1}{5} \right| < 1$, so it converges to $\frac{a}{1-r}$

$$\sum_{n=0}^{\infty} \frac{\cos(n\pi)}{5^n} = \frac{1}{1 - -\frac{1}{5}} = \frac{1}{\frac{6}{5}} = \boxed{\frac{5}{6}}$$

62 $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

by nth term test

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n^n}{n!} = \lim_{n \rightarrow \infty} \frac{n \cdot n \cdot \dots \cdot n}{n \cdot (n-1) \cdot \dots \cdot 1}$$

$$= \lim_{n \rightarrow \infty} \frac{n}{n} \cdot \lim_{n \rightarrow \infty} \frac{n}{n-1} \cdot \dots \cdot \lim_{n \rightarrow \infty} \frac{n}{1}$$

$$= (1) (1) \dots \infty$$

$$= \infty \neq 0$$

$\therefore \sum_{n=1}^{\infty} \frac{n^n}{n!}$ diverges by nth term test.

63 $\sum_{n=1}^{\infty} \frac{2^n + 3^n}{4^n}$

$$= \sum_{n=1}^{\infty} \left(\frac{2}{4}\right)^n + \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n + \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$$

$$= \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots\right) + \left(\frac{3}{4} + \frac{9}{16} + \frac{27}{64} + \dots\right)$$

geometric

series

$$a = \frac{1}{2}$$

$$r = \frac{1}{2} < 1$$

$$a = \frac{3}{4}$$

$$r = \frac{3}{4} < 1$$

$$= \frac{\frac{1}{2}}{1 - \frac{1}{2}} - \frac{\frac{3}{4}}{1 - \frac{3}{4}}$$

$$= 1 + 3 = \boxed{4}$$

178 Find the values of x for which the series converges & find the sum.

$$\sum_{n=0}^{\infty} (\ln x)^n$$

$$= 1 + \ln x + (\ln x)^2 + \dots$$

geometric series with $a=1$ & $r=\ln x$

→ To be converges:

$$|r| < 1$$

$$\rightarrow -1 < r < 1$$

$$-1 < \ln x < 1$$

$$e^{-1} < x < e$$

So, if $\frac{1}{e} < x < e$, then the series

$$\sum_{n=0}^{\infty} (\ln x)^n \text{ converges to } \frac{a}{1-r} = \frac{1}{1-\ln x}$$

90 Find the values of b for which
 $1 + e^b + e^{2b} + e^{3b} + \dots = 9.$

$$\sum_{n=0}^{\infty} (e^b)^n = 9.$$

geometric series with $a=1$ & $r=e^b$.

$$\sum_{n=0}^{\infty} (e^b)^n = \frac{1}{1-e^b} = 9.$$

$$1 = 9 - 9e^b.$$

$$9e^b = 8.$$

$$e^b = \frac{8}{9}$$

$$b = \ln\left(\frac{8}{9}\right).$$