بِسۡمِ ٱللَّهِ ٱلرَّحۡمَٰنِ ٱلرَّحِيمِ

MATH1321

Calculus 2

Chapter 8.4

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8.4 Integration by Partial Fractions

Ex. $\int \frac{8+x}{x^2-x-2}\,dx = \int \frac{8+x}{(x-2)(x+1)}\,dx$

$\frac{8+x}{x^2-x-2} = \frac{A}{x-2} + \frac{B}{x+1}$ (توحيد مقامات)

$8+x = A(x+1) + B(x-2)$

1) $x=2 \longrightarrow 8+2 = A(2+1) + B(2-2) \longrightarrow A = \frac{10}{3}$.

2) $x=-1 \longrightarrow 8-1 = A(-1+1) + B(-1-2) \longrightarrow B = \frac{-7}{3}$.

$\longrightarrow \int \frac{8+x}{x^2-x-2}\,dx = \int \left(\frac{\frac{10}{3}}{x-2} + \frac{\frac{-7}{3}}{x+1} \right) dx$

$= \frac{10}{3}\ln|x-2| - \frac{7}{3}\ln|x+1| + C$ #.

Important Trigonometric Identities:

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

* Cover Method *

لازم يكون المقام عبارة عن حاصل ضرب عوامل

مختلفة وجميعها خطية.

$\frac{x+7}{x(x-1)(x+3)(x+5)}$ (we can use cover method)

$\frac{x+7}{(x^2-1)(x-1)(x-1)}$ X (في تكرار)

Ex. $\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}\,dx = \int \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+3}\,dx$

cover method $\longrightarrow$ $A = \frac{1+4+1}{2 \cdot 4} = \frac{6}{8} = \frac{3}{4}$.

$B = \frac{1-4+1}{-2 \cdot 2} = \frac{-2}{-4} = \frac{1}{2}$.

$C = \frac{9-12+1}{-4 \cdot -2} = \frac{-2}{8} = \frac{-1}{4}$.

$= \int \frac{\frac{3}{4}}{x-1} + \frac{\frac{1}{2}}{x+1} - \frac{\frac{1}{4}}{x+3}\,dx$

$= \frac{3}{4}\ln|x-1| + \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln|x+3| + C$ #

Q. $\int \frac{f(x)}{g(x)}\,dx$, f and g are poly.

If $\deg f \geq \deg g$ we use long devision. $g(x)\overline{)f(x)}$

$\int \frac{7}{(x-1)(x+1)^2}\,dx = \int \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}\,dx.$

$\int \frac{x}{x^3(x^2+1)^2}\,dx = \int \underbrace{\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3}} + \frac{Dx+E}{x^2+1} + \frac{Fx+G}{(x^2+1)^2}\,dx$

or $\frac{Ax^2+Bx+C}{x^3}$

$\int \frac{x}{(x^2+1)(x^3+2)}\,dx = \int \frac{Ax+B}{x^2+1} + \frac{Cx^2+Dx+E}{x^3+2}\,dx$

$\int \frac{x^2}{(x-1)^3(x^2+1)^2}\,dx = \int \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{Dx+E}{x^2+1} + \frac{Fx+G}{(x^2+1)^2}\,dx$

* Note *

لما أوصل عند المعادلة هاي

$8+x = A(x+1) + B(x-2)$

بقدر أوجد الثوابت بعدة طرق

1) cover method. 3) الاشتقاق.

2) مقارنة المعاملات. 4) تعويض أي رقم.

Ex. $\displaystyle\int \frac{3x+2}{(x+1)^2}\,dx = \int \frac{A}{x+1} + \frac{B}{(x+1)^2}\,dx$

$3x+2 = A(x+1) + B \longrightarrow A = 3$

$\longrightarrow A + B = 2 \longrightarrow B = -1$

$= \displaystyle\int \frac{3}{x+1} + \frac{-1}{(x+1)^2}\,dx$ $\qquad u = x+1$

$= 3\ln|x+1| + \dfrac{1}{x+1} + C$ # $\qquad du = dx$

Ex. $\displaystyle\int_0^1 \frac{x^3}{x^2+2x+1}\,dx$ $\qquad x^3$ deg $>$ x^2+2x+1 deg

$= \displaystyle\int_0^1 x - 2 + \frac{3x+2}{x^2+2x+1}\,dx$ نفس المثال السابق

$$\begin{array}{r|l} & x-2 \\ \hline x^2+2x+1 & x^3 \\ & -x^3-2x^2-x \\ \hline & -2x^2-x \\ & +2x^2+4x+2 \\ \hline & 3x+2 \end{array}$$

$= \dfrac{x^2}{2} - 2x + 3\ln|x+1| + \dfrac{1}{x+1}\Big|_0^1$

$= -2 + 3\ln 2.$

Ex. $\displaystyle\int \frac{4-2x}{(x^2+1)(x-1)^2}\,dx = \int \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2}\,dx$

$4-2x = (Ax+B)(x-1)^2 + C(x-1)(x^2+1) + D(x^2+1)$

1) $x = 1 \longrightarrow 2 = 0 + 0 + 2D \longrightarrow D = 1.$

2) نشتق $\longrightarrow -2 = Ax + B(2x-2) + A(x-1)^2 + C(x-1)(2x) + C(x^2+1) + 2Dx.$

$x = 1 \longrightarrow -2 = 0 + 0 + 0 + 2C + 2 \longrightarrow C = -2.$

3) بشتق كمان مرة أو بعوض رقم ثاني

$\longrightarrow 0 = 2(Ax+B) + A(2x-2) + A(2x-2) + 2C(x-1) + 2Cx + 2Cx + 2D$ $\qquad B = 1.$

$\longrightarrow 0 = 2A + 2A + 2A + 2C + 2C + 2C \longrightarrow A = -C = 2.$

$= \displaystyle\int \frac{2x+1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}\,dx = \int \frac{2x}{x^2+1} + \frac{1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}\,dx$

$= \ln|x^2+1| + \tan^{-1}x - 2\ln|x-1| - \dfrac{1}{x-1} + C$ #

8.4 Discussion: (12, 14, 18, 20, 23, 29, 30, 34, 37, 42, 47, 49, 54)

Q.12 $\int \frac{2x+1}{x^2-7x+12}\,dx = \int \frac{2x+1}{(x-4)(x-3)}\,dx \qquad \frac{A}{x-4} + \frac{B}{x-3}$

using cover method $(A = 9,\ B = -7)$

$= \int \frac{9}{x-4} - \frac{7}{x-3}\,dx = 9\ln|x-4| - 7\ln|x-3| + C$ #

Q.14 $\int_{\frac{1}{2}}^{1} \frac{y+4}{y^2+y}\,dy = \int \frac{y+4}{y(y+1)}\,dx \qquad \frac{A}{y} + \frac{B}{y+1}$

using cover method $(A = 4,\ B = -3)$

$= \int \frac{4}{y} - \frac{3}{y+1}\,dy = 4\ln|y| - 3\ln|y+1| \Big|_{\frac{1}{2}}^{1}$

$= -3\ln 2 - (4\ln\frac{1}{2} - 3\ln\frac{3}{2}) = \ln\frac{27}{4}.$

Q.18 $\int_{-1}^{0} \frac{x^3}{x^2-2x+1}\,dx \qquad x^3 \deg > x^2-2x+1 \deg$

$= \int x+2 + \frac{3x-2}{x^2-2x+1}\,dx$

$$\begin{array}{r|l} & x+2 \\ \hline x^2-2x+1 & x^3 \\ & -(x^3-2x^2+x) \\ \hline & 2x^2-x \\ & -(2x^2-4x+2) \\ \hline & 3x-2 \end{array}$$

$= \int x+2 + \frac{3x-2}{(x-1)^2}\,dx$

$\frac{A}{x-1} + \frac{B}{(x-1)^2}$

$3x-2 = Ax - A + B \qquad (A=3,\ -A+B=-2 \rightarrow B=1)$

$= \int_{-1}^{0} x+2 + \frac{3}{x-1} + \frac{1}{(x-1)^2}\,dx \qquad u = x-1$

$= \frac{x^2}{2} + 2x + 3\ln|x-1| - \frac{1}{x-1} \Big|_{-1}^{0} \qquad du = dx$

$= 2 - 3\ln 2.$

Q.20 $\int \frac{x^2}{(x-1)(x^2+2x+1)}\,dx = \int \frac{x^2}{(x-1)(x+1)^2}\,dx$

$\frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$

$x^2 = A(x+1)^2 + B(x-1)(x+1) + C(x-1) \qquad 2x = A(2x+2) + 2xB + C$

$(x=-1) \rightarrow 1 = 0+0+-2C \rightarrow C = \frac{-1}{2}. \qquad (x=-1) \rightarrow -2 = 0+-2B - \frac{1}{2} \rightarrow B = \frac{3}{4}.$

$(x=1) \rightarrow 1 = 4A+0+0 \rightarrow A = \frac{1}{4}. \qquad = \int \frac{1}{4}\left(\frac{1}{x-1}\right) + \frac{3}{4}\left(\frac{1}{x+1}\right) - \frac{1}{2}\left(\frac{1}{(x+1)^2}\right)dx$

$= \frac{1}{4}\ln|x-1| + \frac{3}{4}\ln|x+1| + \frac{1}{2x+2} + C$ #

Q.23 $\int \frac{y^2+2y+1}{(y^2+1)^2}\,dy \qquad \frac{Ay+B}{y^2+1} + \frac{Cy+D}{(y^2+1)^2}$

$y^2+2y+1 = (Ay+B)(y^2+1) + Cy+D$

$y^2+2y+1 = Ay^3 + Ay + By^2 + B + Cy + D$

$(A=0,\ B=1,\ C=2,\ D=0)$

$= \int \frac{1}{y^2+1} + \frac{2y}{(y^2+1)^2}\,dy = \tan^{-1}x - \frac{1}{y^2+1} + C$ #

$(u = y^2+1,\ du = 2y\,dy)$

Q.29 $\int \frac{x^2}{x^4-1}\,dx = \int \frac{x^2}{(x^2-1)(x^2+1)}$

$\frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$

$x^2 = A(x+1)(x^2+1) + B(x-1)(x^2+1) + (Cx+D)(x^2-1)$

$x=1 \rightarrow 1 = 4A+0+0 \rightarrow A = \frac{1}{4}. \qquad = \int \frac{1}{4}\left(\frac{1}{x-1}\right) - \frac{1}{4}\left(\frac{1}{x+1}\right) + \frac{1}{2}\left(\frac{1}{x^2+1}\right)dx$

$x=-1 \rightarrow 1 = 0+-4B+0 \rightarrow B = \frac{-1}{4}. \qquad = \frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+1| + \frac{1}{2}\tan^{-1}x + C$ #

$x=0 \rightarrow 0 = \frac{1}{4} + \frac{1}{4} - D \rightarrow D = \frac{1}{2}.$

$x=2 \rightarrow C = 0.$

Q.30 $\int \frac{x^2+x}{x^4-3x^2-4}\,dx = \int \frac{x^2+x}{(x^2-4)(x^2+1)}\,dx$

$\frac{A}{x-2} + \frac{B}{x+2} + \frac{Cx+D}{x^2+1}$

$x^2+x = A(x+2)(x^2+1) + B(x-2)(x^2+1) + (Cx+D)(x^2-4)$

$x=2 \longrightarrow 6 = 20A+0+0 \longrightarrow A = \frac{3}{10}.$

$x=-2 \longrightarrow 2 = 0+-20B+0 \longrightarrow B = \frac{-1}{10}.$

$x=0 \longrightarrow 0 = \frac{6}{10} + \frac{2}{10} - 4D \longrightarrow D = \frac{1}{5}.$

$x=1 \longrightarrow 2 = \frac{6}{10} - \frac{2}{10} - 3C - \frac{3}{5} \longrightarrow C = \frac{-1}{5}.$

$= \int \frac{3}{10}\left(\frac{1}{x-2}\right) - \frac{1}{10}\left(\frac{1}{x+2}\right) + \frac{1}{5}\left(\frac{1-x}{x^2+1}\right) dx$

$= \frac{3}{10}\ln|x-2| - \frac{1}{10}\ln|x+2| + \frac{1}{5}\tan^{-1}x - \frac{1}{10}\ln|x^2+1| + C$ #

Q.34 $\int \frac{x^4}{x^2-1}\,dx$ $\quad x^4$ deg > x^2-1 deg

$$\begin{array}{r|l} & x^2+1 \\ x^2-1 & x^4 \\ & -x^4+x^2 \\ \hline & x^2 \\ & -x^2+1 \\ \hline & 1 \end{array}$$

$= \int x^2+1 + \frac{1}{x^2-1}\,dx$

$= \int x^2+1 + \frac{A}{x-1} + \frac{B}{x+1}\,dx$

using cover method

$(A = \frac{1}{2},\ B = -\frac{1}{2}) \longrightarrow \int x^2+1 + \frac{1}{2}\left(\frac{1}{x-1}\right) - \frac{1}{2}\left(\frac{1}{x+1}\right) dx$

$= \frac{x^3}{3} + x + \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C$ #

Q.37 $\int \frac{y^4+y^2-1}{y^3+y}\,dy$ $\quad y^4$ deg > y^3deg

$$\begin{array}{r|l} & y \\ y^3+y & y^4+y^2-1 \\ & -y^4-y^2 \\ \hline & -1 \end{array}$$

$= \int y - \frac{1}{y^3+y}\,dy$

$= \int y - \frac{1}{y(y^2+1)}\,dy$

$\frac{A}{y} + \frac{By+C}{y^2+1}$ $\quad 1 = A(y^2+1) + (By+C)y$

$y=0 \longrightarrow 1 = A+0 \longrightarrow A = 1.$

$= \int y - \frac{1}{y} + \frac{y}{y^2+1}\,dy$

$= \frac{y^2}{2} - \ln|y| + \frac{1}{2}\ln|y^2+1| + C$ #

$A+B=0 \longrightarrow B=-1.$

$y=1 \longrightarrow 1 = 2+B+C \longrightarrow C=0.$

Q.42 $\int \frac{\sin\theta}{\cos^2\theta+\cos\theta-2}\,d\theta$ $\quad u=\cos\theta$

$= -\int \frac{1}{u^2+u-2}\,du$ $\quad du = -\sin\theta\,d\theta$

$= -\int \frac{1}{(u+2)(u-1)}\,du$ $\quad \frac{A}{u+2} + \frac{B}{u-1}$

using cover method $(A = \frac{-1}{3},\ B = \frac{1}{3})$

$= -\int -\frac{1}{3}\left(\frac{1}{u+2}\right) + \frac{1}{3}\left(\frac{1}{u-1}\right) du$

$= \frac{1}{3}\ln|u+2| - \frac{1}{3}\ln|u-1| + C$

$= \frac{1}{3}\ln|\cos\theta+2| - \frac{1}{3}\ln|\cos\theta-1| + C$ #

Q.47 $\int \frac{\sqrt{x+1}}{x}\,dx$ $\quad u^2 = x+1$

$= \int \frac{2u^2}{u^2-1}\,du$ $\quad 2u\,du = dx$

$u^2 \geq u^2 \longrightarrow$

$$\begin{array}{r|l} & 2 \\ u^2-1 & 2u^2 \\ & -2u^2+2 \\ \hline & 2 \end{array}$$

$= \int 2 + \frac{2}{u^2-1}\,du$

$= 2u + \int \frac{2}{(u-1)(u+1)}$ $\quad \frac{A}{u-1} + \frac{B}{u+1}$ using cover method $(A=1,\ B=-1)$

$= 2u + \ln|u-1| - \ln|u+1| + C$

$= 2\sqrt{x+1} + \ln\left|\sqrt{x+1}-1\right| - \ln\left|\sqrt{x+1}+1\right| + C$ #

Q.49 $\int \frac{1}{x(x^4+1)}\,dx \cdot \frac{x^3}{x^3} = \int \frac{x^3}{x^4(x^4+1)}\,dx$ $\quad u = x^4$

$= \frac{1}{4}\int \frac{1}{u(u+1)}\,du$ $\quad du = 4x^3 dx$

$\frac{A}{u} + \frac{B}{u+1}$ using cover method $(A = 1,\ B = -1)$

$= \frac{1}{4}\int \frac{1}{u} - \frac{1}{u+1}\,du = \frac{1}{4}\ln|u| - \frac{1}{4}\ln|u+1| + C$

$= \frac{1}{4}\ln\left|\frac{x^4}{x^4+1}\right| + C$ #

Q.54 Find x?

$(t+1)\frac{dx}{dt} = x^2+1$ where $(t > -1)$, $x(0) = 0$.

$(t+1)\,dx = (x^2+1)\,dt \longrightarrow \int \frac{1}{t+1}\,dt = \int \frac{1}{x^2+1}\,dx$

$\tan^{-1}x = \ln|t+1| + C$ $\quad$ using $x(0) = 0 \rightarrow C = 0$

$\tan(\tan^{-1}x = \ln|t+1|)$

$x = \tan(\ln|t+1|)$.