4.5 : Extensions of the change of variable Technique.

exp: $X_1, X_2, \ldots, X_{k+1}$ Gamma $(\alpha_i, 1)$, $i = 1, \ldots, k+1$

$X_1, \ldots, X_{k+1}$ independent.

let $k = 2$

$X_1, X_2, X_3 \sim$ Gamma $(\alpha_i, 1)$, $i = 1, 2, 3$.

X_1, X_2, X_3 independent.

$$f(x_1, x_2, x_3) = \frac{1}{\Gamma(\alpha_1)} x^{\alpha_1 - 1} e^{-x_1} \cdot \frac{1}{\Gamma(\alpha_2)} x_2^{\alpha_2 - 1} e^{-x_2} \frac{1}{\Gamma(\alpha_3)} x_3^{\alpha_3 - 1}$$

$$A = \{ (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 > 0, x_2 > 0, x_3 > 0 \}$$

$$Y_1 = \frac{X_1}{X_1 + X_2 + X_3} \rightarrow X_1 = Y_1 Y_3$$

$$Y_2 = \frac{X_2}{X_1 + X_2 + X_3} \rightarrow X_2 = Y_2 Y_3$$

$$Y_3 = X_1 + X_2 + X_3 \rightarrow X_3 = Y_3 - Y_1 Y_3 - Y_2 Y_3.$$

$$\rightarrow J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \frac{\partial x_1}{\partial y_3} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \frac{\partial x_2}{\partial y_3} \\ \frac{\partial x_3}{\partial y_1} & \frac{\partial x_3}{\partial y_3} & \frac{\partial x_3}{\partial y_3} \end{vmatrix} = \begin{vmatrix} \overset{+}{y_3} & \overset{-}{0} & \overset{+}{y_1} \\ \overset{-}{0} & y_3 & y_2 \\ \overset{+}{-y_3} & -y_3 & 1-y_1-y_2 \end{vmatrix}$$

$$= y_3 \begin{vmatrix} y_3 & y_2 \\ -y_3 & 1-y_1-y_2 \end{vmatrix} - y_3 \begin{vmatrix} 0 & y_1 \\ y_3 & y_2 \end{vmatrix}$$

$$= y_3 (y_3 - y_1 y_3 - \cancel{y_2 y_3} + \cancel{y_2 y_3}) - y_3 (- y_1 y_3)$$

$$= y_3^2 - \cancel{y_1 y_3^2} + \cancel{y_1 y_3^2}$$

$$J = y_3^2$$

$$\rightarrow g(y_1, y_2, y_3) = \begin{cases} f[\,w_1(y_1, y_2, y_3),\ w_2(y_1, y_2, y_3),\ w_3(y_1, y_2, y_3)\,]\,|J|, & (y_1, y_2, y_3) \in B \\ 0 & , \text{else} \end{cases}$$

$$\rightarrow B = \{ (y_1, y_2, y_3) \in \mathbb{R}^3 : 0 < y_1 < 1,\ 0 < y_2 < 1,\ 0 < y_1 + y_2 < 1,\ y_3 > 0 \}.$$

$$\rightarrow g(y_1, y_2, y_3) = f(\,y_1 y_3,\ y_2 y_3,\ y_3 - y_1 y_3 - y_2 y_3\,) \cdot y_3^2$$

$$= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} (y_1 y_3)^{\alpha_1 - 1} (y_2 y_3)^{\alpha_2 - 1} (y_3 - y_1 y_3 - y_2 y_3)^{\alpha_3 - 1}\, e^{-(\cancel{y_1 y_3} + \cancel{y_2 y_3} + y_3 - \cancel{y_1 y_3} - \cancel{y_2 y_3})} \cdot y_3^2$$

$$= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)}\, y_1^{\alpha_1 - 1} y_2^{\alpha_2 - 1} (1 - y_1 - y_2)^{\alpha_3 - 1}\, y_3^{\alpha_1 - 1} y_3^{\alpha_2 - 1} y_3^{\alpha_3 - 1}\, e^{-y_3}\, y_3^2$$

$$= \left[\frac{\Gamma(\alpha_1 + \alpha_2 + \alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)}\, y_1^{\alpha_1 - 1} y_2^{\alpha_2 - 1} (1 - y_1 - y_2)^{\alpha_3 - 1} \right] \frac{1}{\Gamma(\alpha_1 + \alpha_2 + \alpha_3)}\, y_3^{\alpha_1 + \alpha_2 + \alpha_3 - 1}\, e^{-y_3}$$

$$g(y_1, y_2, y_3) = \begin{cases} g_{12}(y_1, y_2) \cdot g_3(y_3) & , (y_1, y_2, y_3) \in B \\ 0 & , \text{elsewhere} \end{cases}$$

where $g_{12}(y_1, y_2) = \begin{cases} \dfrac{\Gamma(\alpha_1 + \alpha_2 + \alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)}\, y_1^{\alpha_1 - 1} y_2^{\alpha_2 - 1} (1 - y_1 - y_2)^{\alpha_3 - 1} & , B_{12} \\ 0 & , \text{else} \end{cases}$

$$B_{12} = \{ (y_1, y_2) \in \mathbb{R}^2 : 0 < y_1 < 1,\ 0 < y_2 < 1,\ 0 < y_1 + y_2 < 1 \}.$$

$g_{12}(y_1, y_2)$ is called Dirichlet distribution with parameters $\alpha_1, \alpha_2, \alpha_3$

$\rightarrow$

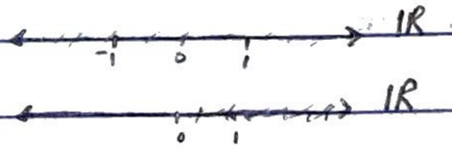
and $g_3(y_3) = \begin{cases} \dfrac{1}{\Gamma(\alpha_1+\alpha_2+\alpha_3)} \, y_3^{\alpha_1+\alpha_2+\alpha_3-1} \, e^{-y_3} & , \; y_3 \in B \\ 0 & , \; \text{else} \end{cases}$

$B_3 = \{ y_3 \in \mathbb{R} \, , \; y_3 > 0 \}$.

exp: $X \sim$ cauchy

$f(x) = \dfrac{1}{\pi(1+x^2)} \quad , \; -\infty < x < \infty$

$Y = X^2$, Find distribution Y.



$A = A_1 \cup A_2$

$A_1 = \{ x : \; x > 0 \}$.

$A_2 = \{ x : \; x < 0 \}$

$B = \{ y : \; y > 0 \}$.

$u_1(x) = x^2 \rightsquigarrow w_1(y) = \sqrt{y}$

$u_2(x) = x^2 \rightsquigarrow w_2(y) = -\sqrt{y}$

$\rightsquigarrow g(y) = \begin{cases} f(w_1(y)) \, |\bar{w}_1(y)| + f(w_2(y)) \, |\bar{w}_2(y)| & , \; y \in B \\ 0 & , \; \text{else} \end{cases}$

$= \begin{cases} f(\sqrt{y}) \, \left|\frac{1}{2\sqrt{y}}\right| + f(-\sqrt{y}) \, \left|-\frac{1}{2\sqrt{y}}\right| & , \; y \in B \\ 0 & , \; \text{else} \end{cases}$

$$\Rightarrow \quad g(y) = \begin{cases} \frac{1}{\pi(1+y)} \cdot \frac{1}{2\sqrt{y}} + \frac{1}{\pi(1+y)} \cdot \frac{1}{2\sqrt{y}} & , y \in B \\ 0 & , \text{else} \end{cases}$$

$$g(y) = \begin{cases} \frac{1}{\pi\sqrt{y}\,(1+y)} & , y \in B \\ 0 & , \text{else} \end{cases}$$

Exercises .