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BIBO Stability (Boundary Input Boundary output) Stability

Input Output Stability of LTI Systems

Theorem (BIBO Stability). A LTI system with proper rational transfer matrix $\mathbf{G}(s) = [\mathbf{G}_{ij}(s)]$ is BIBO stable if and only if every pole of every entry $\mathbf{G}_{ij}(s)$ of $\mathbf{G}(s)$ has negative real part.

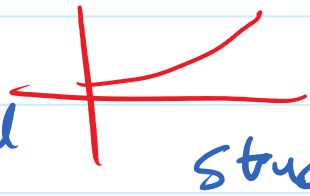
* This type of stability study the stability of the system under the effect of input $(u(t))$

BIBO stability < stable / unstable

$u(t) \longrightarrow \boxed{Sys} \longrightarrow x(t)$?

Ex:-

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$



study the BIBO stability

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

$$G(s) = C(sI - A)^{-1} B + D$$

$$(sI - A) = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} s & -1 \\ -1 & s-2 \end{bmatrix}$$

$$(sI - A)^{-1} = \begin{bmatrix} s-2 & 1 \\ 1 & s \end{bmatrix} \frac{1}{s^2 - 2s - 1}$$

$$C(sI - A)^{-1} = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{1\times 2} \underbrace{\begin{bmatrix} \frac{s-2}{\Delta} & \frac{1}{\Delta} \\ \frac{1}{\Delta} & \frac{s}{\Delta} \end{bmatrix}}_{2\times 2} = \begin{bmatrix} \frac{s-2}{\Delta} & \frac{1}{\Delta} \end{bmatrix}$$

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$$C(sI-A)^{-1} = \begin{bmatrix} \frac{s-2}{\Delta} & \frac{1}{\Delta} \end{bmatrix}$$

$$C(sI-A)^{-1}B = \underset{1\times 2}{\begin{bmatrix} \frac{s-2}{\Delta} & \frac{1}{\Delta} \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$

$$= \frac{1}{\Delta} = \frac{1}{s^2-2s-1}$$

$$s^2-2s-1=0 \quad \begin{cases} s_1 = -0.4142 \\ s_2 = 2.4142 \end{cases}$$

unstable in the sense of BIBO stability

$$G(s) = \begin{array}{c} \text{Transfer matrix} \end{array} \begin{bmatrix} \frac{1}{(s-2)(s+2)} & \frac{1}{(s+1)} \\ \frac{1}{(s+10)} & \frac{1}{(s+3)} \end{bmatrix}$$

$\to g_{11}$, $\to g_{12}$, $\to g_{21}$

$g_{11} \begin{cases} s_1 = 2 \\ s_2 = -2 \end{cases}$

$g_{12} = -1$

$g_{21} = -1$

$g_{22} = -3$

Based on that the sys is unstable in the sense of BIBO stability.

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BIBO stability of state equations. When the system is represented by state equations

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

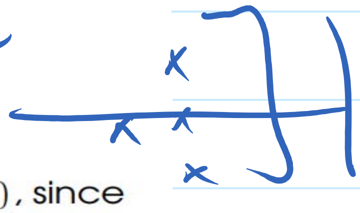
$\lambda_1 = -2$

$\lambda_2 = -3$

$\lambda_{3,4} = -2 \pm 3j$

the BIBO stability will depend on the eigenvalues of the matrix $\mathbf{A}$, since every pole of $\mathbf{G}(\mathbf{s})$ is an eigenvalue of $\mathbf{A}$.

stable



Input Output Stability of LTI Systems

Note that **not every eigenvalue of $\mathbf{A}$ is a pole of $\mathbf{G}(\mathbf{s})$**, since there may be pole-zero cancellations while computing $\mathbf{G}(\mathbf{s})$. Thus, a state equation may be BIBO stable even when some eigenvalues of $\mathbf{A}$ do not have negative real part.

Input Output Stability of LTI Systems

Note that **not every eigenvalue of $\mathbf{A}$ is a pole of $\mathbf{G}(\mathbf{s})$**, since there may be pole-zero cancellations while computing $\mathbf{G}(\mathbf{s})$. Thus, a state equation may be BIBO stable even when some eigenvalues of $\mathbf{A}$ do not have negative real part.

Example. Although the system

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -1 & 10 \\ 0 & 1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} -2 \\ 0 \end{bmatrix} \mathbf{u}(t)$$
$$\mathbf{y}(t) = \begin{bmatrix} -2 & 3 \end{bmatrix} \mathbf{x}(t) - 2\mathbf{u}(t)$$

has one eigenvalue with positive real part $\lambda = 1$, it is BIBO stable, since its transfer function

$$\mathbf{G}(\mathbf{s}) = \mathbf{C}(\mathbf{s}\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} = \frac{2(1-\mathbf{s})}{(\mathbf{s}+1)}$$

has a single pole at $\mathbf{s} = -1$. □

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Internal Stability "Lyapunov stability"

$$\left.\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}\right\} \text{LTI}$$

$\downarrow$ $x(0)$ IC

study the stability under $\boxed{\text{Sys}}$

the effect of IC $x(0)$. when $u(t) = 0$

$$\left.\begin{aligned} \dot{x} &= Ax \\ y &= Cx \end{aligned}\right\} u(t) = 0$$

$|\lambda I - A| = 0$

Internal stability:
- unstable
- m. stable
- Asy. stable

$$z = \begin{bmatrix} 0 & 1 \\ 0 & \end{bmatrix}$$

$z_1 = x$

$z_2 = \dot{x}$

Ex: Study the BIBO stability and the lyapunov stability for the sys. shown behind?

$$\dot{x}(t) = \begin{bmatrix} -1 & 10 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} -2 \\ 0 \end{bmatrix} u(t)$$

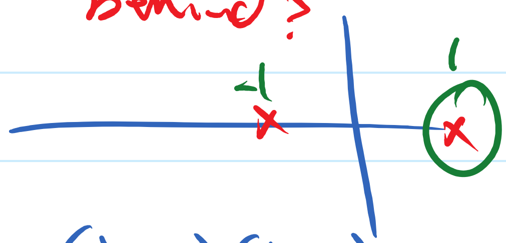
$$y(t) = \begin{bmatrix} -2 & 3 \end{bmatrix} x(t) - 2u(t)$$

* internal stability

$$|\lambda I - A| = \left| \begin{bmatrix} \lambda + 1 & -10 \\ 0 & \lambda - 1 \end{bmatrix} \right| = (\lambda + 1)(\lambda - 1) = 0$$

$\lambda_1 = 1$

$\lambda_2 = -1$



unstable

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x in the sense of BIBO Stability

$$\dot{x}(t) = \begin{bmatrix} -1 & 10 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} -2 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} -2 & 3 \end{bmatrix} x(t) - 2u(t)$$

$$[sI - A] = \begin{bmatrix} s+1 & -10 \\ 0 & s-1 \end{bmatrix}$$

$$(sI-A)^{-1} = \begin{bmatrix} s-1 & 10 \\ 0 & s+1 \end{bmatrix} \frac{1}{s^2-1}$$

$$C(sI-A)^{-1} = \begin{bmatrix} -2 & 3 \end{bmatrix} \begin{bmatrix} \frac{s-1}{\Delta} & \frac{10}{\Delta} \\ \frac{0}{\Delta} & \frac{s+1}{\Delta} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-2(s-1)}{\Delta} & \frac{(3s-17)}{\Delta} \end{bmatrix}$$

$$C(sI-A)^{-1}B = \begin{bmatrix} \frac{-2(s-1)}{\Delta} & \frac{(3s-17)}{\Delta} \end{bmatrix} \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

$$= \frac{+4(s-1)}{s^2-1} = \frac{4\cancel{(s-1)}}{\cancel{(s-1)}(s+1)}$$

$s = -1$ (stable) in the sense of BIBO