3.4 The Derivateves as a Rate of Change

Definition: The instantaneous rate of change of f w.r.t X at X_0 is

$$f'(X_0) = \lim_{h\to 0} \frac{f(X_0+h)-f(X_0)}{h}$$

"limit exists"

Example: Let $A = \frac{\pi}{4} D^2$

A: area of a circle
D: diameter of circle

Find the instantaneous rate of change of the area w.r.t the diameter when $D = 10$ m

$$\left.\frac{dA}{dD}\right|_{D=10} = \left.\frac{\pi}{4}\, 2D\right|_{D=10} = \left.\frac{\pi}{2} D\right|_{D=10} = \frac{\pi}{2}(10) = 5\pi \ m^2/m$$

Suppose an object is moving accordingly to

$$s = f(t)$$

t: time
s: distance

- The displacement of the object over the time interval $[t_1, t_2]$ is

$\Delta t = t_2 - t_1$

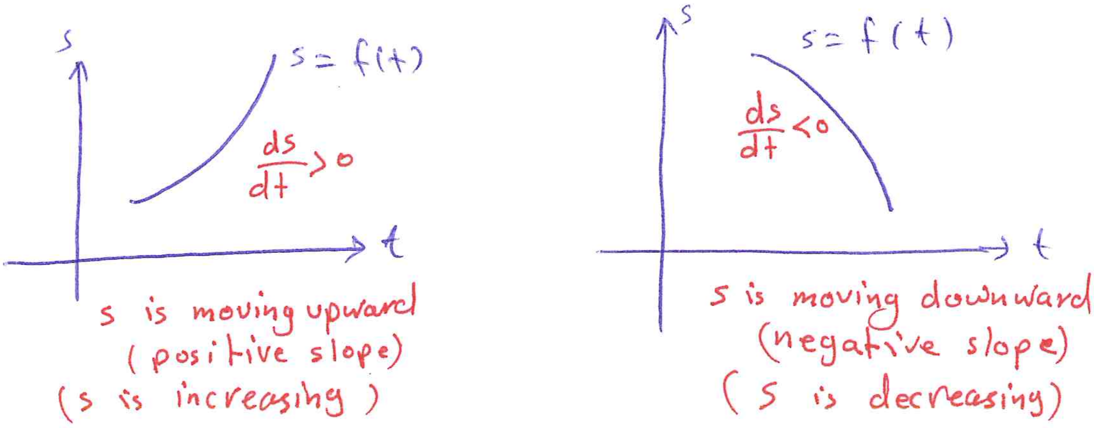
$$\Delta s = f(t_2) - f(t_1) = f(t_1+\Delta t) - f(t_1)$$

- The average velocity of the object over the time interval $[t_1, t_2]$ is

$$V_{av} = \frac{\text{displacement}}{\text{travel time}} = \frac{\Delta s}{\Delta t} = \frac{f(t_1+\Delta t) - f(t_1)}{\Delta t}$$

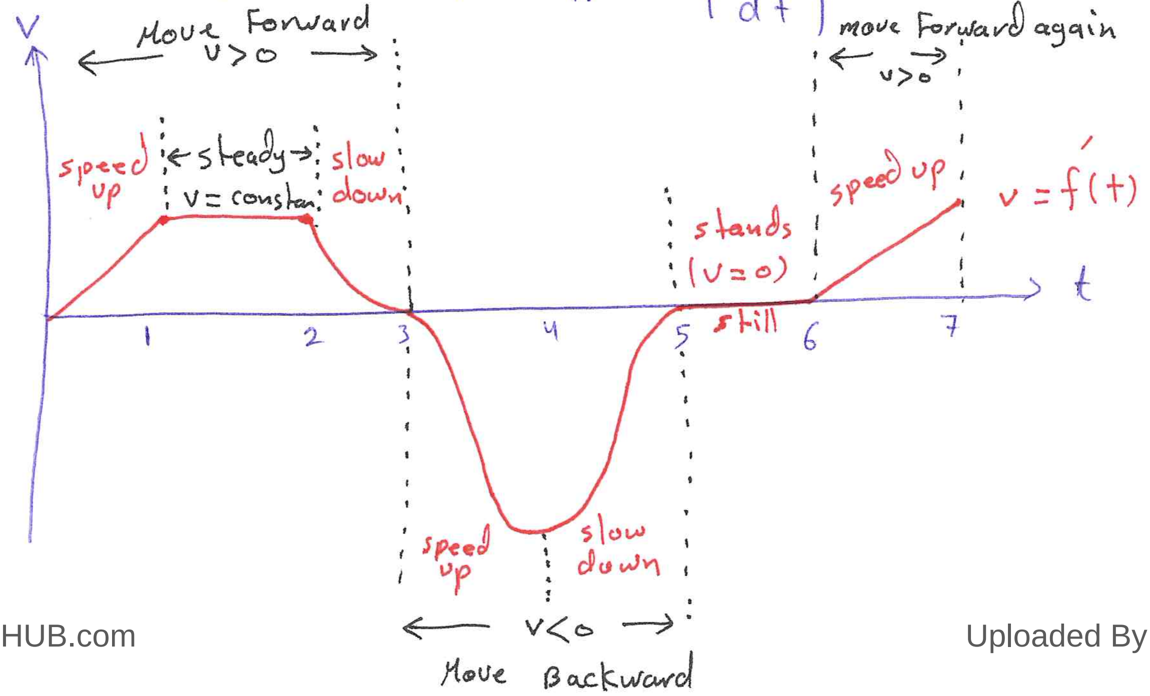
- The velocity (instantaneous velocity) at time t is

$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}$$



- Speed is the absolute value of velocity

$$\text{Speed} = |v(t)| = \left|\frac{ds}{dt}\right|$$



- Acceleration is the derivative of velocity w.r.t time.

$$a(t) = \frac{dv}{dt} = \frac{d^2 s}{dt^2}$$

- Jerk is the derivative of acceleration w.r.t. time

$$j(t) = \frac{da}{dt} = \frac{d^2 v}{dv^2} = \frac{d^3 s}{ds^3}$$

Example The free falling equation for an object falls from a rest is $s = \frac{1}{2} g t^2$ where g is the gravity acceleration $= 9.8\ m/s^2$

$$s = \frac{1}{2}(9.8) t^2 = 4.9 t^2$$

(a) How many meters does the object fall in the first 2s

$$s(2) = 4.9(2)^2 = 4.9(4) = 19.6\ m$$

(b) What is its velocity at $t=2$?

$$v(t) = s'(t) = 2(4.9)t = 9.8t$$

$$v(2) = 9.8(2) = 19.6\ m/s$$

(c) what is its speed at $t=2$?

$$\text{Speed}\Big|_{t=2} = |v(2)| = 19.6\ m/s$$

(d) what is its acceleration at $t=2$?

$$a(t) = \frac{dv}{dt} = 9.8$$

$$a(2) = 9.8\ m/s^2$$

(e) what is its jerk at $t=2$

$j(t) = a'(t) = 0$. Thus $j(2) = 0\ m/s^3$

Example: A rock straight up with a lunch velocity of 50 m/s. It reaches a height of $s = 49t - 4.9t^2$ m after t second.

(a) How high does the rock go?

velocity $= 0$

$v(t) = s'(t) = 49 - 9.8t = 0 \Leftrightarrow t = 5$ second

$\Rightarrow$ The rock's height at $t=5$ is $s(5) = 49(5) - 4.9(5)^2$
$= 122.5$ m

[b] what is the velocity and speed when the rock is 78.4 m above the ground on the way above and on the way down?

$$s(t) = 49t - 4.9t^2 = 78.4$$

$$4.9t^2 - 49t + 78.4 = 0$$

$$4.9(t^2 - 10t + 16) = 0$$

$$(t-2)(t-8) = 0 \qquad t = 2 \text{ seconds}, \quad t = 8 \text{ seconds}$$

$$v(t) = 49 - 9.8t$$

$$v(2) = 49 - 9.8(2) = 29.4 \text{ m/s}$$ " The rock is moving up"

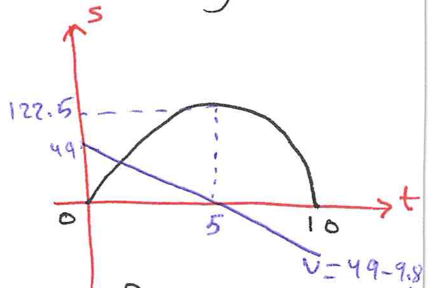
$$v(8) = 49 - 9.8(8) = -29.4 \text{ m/s}$$ " " " " " down"

The speed is $|v(t)| = 29.4$ m/s

[c] what is the acceleration of the rock at any time?

$$a(t) = v'(t) = -9.8 \text{ m/s}^2$$

The acceleration is always downward.
As the rock rises, it slows down.
As the rock falls, it speeds up.



[d] When does the rock hit the ground again? when $s(t) = 0$

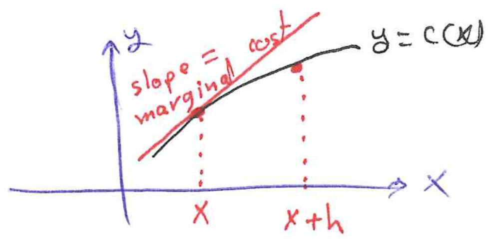
$$s(t) = 49t - 4.9t^2 = 0$$

$$4.9t(10 - t) = 0 \iff t = 0 \text{ sec}, \quad t = 10 \text{ sec.}$$

Derivative in Economics:

- $c(x)$: cost of production " to produce x"

- Marginal cost of production: $c'(x) = \lim_{h \to 0} \frac{c(x+h) - c(x)}{h}$



- $f'(x)$ measures the sensitivity
"small change in x produce large change in the value of a function f(x)"