6.2 Volumes Using Cylindrical Shells

4) Shell-method.

The volume of the solid generated by revolving the region between the x-axis the graph of a continuous function $y = f(x) \geq 0$, $L \leq a \leq x \leq b$ about a vertical line $x = L$ is

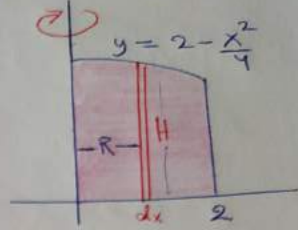
$$V = \int_a^b 2\pi \begin{pmatrix} \text{shell} \\ \text{radius} \end{pmatrix} \begin{pmatrix} \text{shell} \\ \text{height} \end{pmatrix} dx$$

about a horizontal line $y = L$

$$V = \int_c^d 2\pi \begin{pmatrix} \text{shell} \\ \text{radius} \end{pmatrix} \begin{pmatrix} \text{shell} \\ \text{height} \end{pmatrix} dy$$

Question 2.
Page 324

Use the shell method to find the volume of the solid generated by revolving the shaded region about y-axis.



$$V = \int_0^2 2\pi \left(\text{shell radius}\right)\left(\text{shell height}\right) dx$$

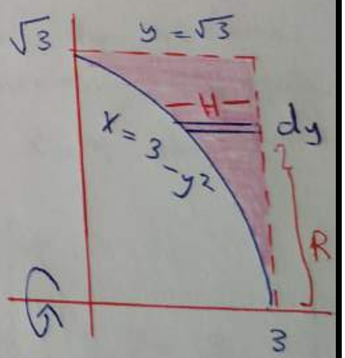
$$= \int_0^2 2\pi\,(x)\left(2 - \frac{x^2}{4}\right) dx$$

$$= 2\pi \int_0^2 2x - \frac{x^3}{4}\, dx = 2\pi \left[\frac{\cancel{2}x^2}{\cancel{2}} - \frac{x^4}{16}\right]\Big|_0^2$$

$$= 2\pi \left[(2)^2 - \frac{(2)^4}{16} - \left((0)^2 - \frac{(0)^4}{16}\right)\right] = 2\pi(3) = 6\pi$$

Question 4.
Page 324

Use the shell method to find the volume of the solid generated by revolving the shaded region about x-axis.



$$V = \int_0^{\sqrt{3}} 2\pi \left(\text{shell radius}\right)\left(\text{shell height}\right) dy$$

$$= \int_0^{\sqrt{3}} 2\pi\,(y)\left(3 - (3 - y^2)\right) dy$$

$$= 2\pi \int_0^{\sqrt{3}} y\,(y^2)\, dy$$

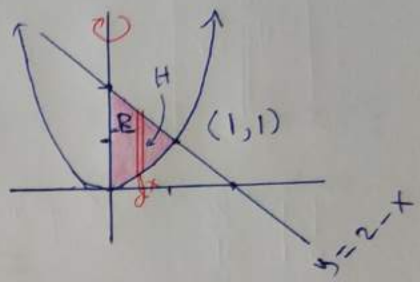
$$= 2\pi \int_0^{\sqrt{3}} y^3\, dy = \cancel{2}\pi\, \frac{y^4}{\cancel{2}4}\Big|_0^{\sqrt{3}}$$

$$= \frac{\pi}{2}\left[(\sqrt{3})^4 - (0)^4\right] = \frac{\pi}{2}\left(3^{\frac{1}{2}}\right)^4 = \frac{9\pi}{2}$$

Question 9. Page 324

Use the shell method to find the volume of the solid generated by revolving the shaded region bounded by $y=x^2$, $y=2-x$, $x=0$ for $x \geq 0$ about the y-axis.

$$V=\int_a^b 2\pi \left(\begin{matrix}\text{shell}\\ \text{radius}\end{matrix}\right)\left(\begin{matrix}\text{shell}\\ \text{height}\end{matrix}\right) dx$$



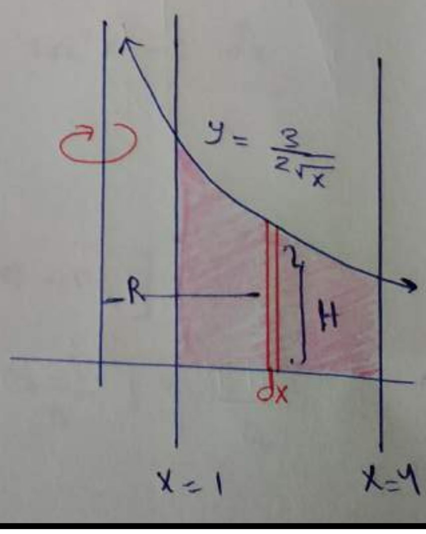
$$x^2 = 2-x \rightarrow x^2+x-2=0$$
$$(x+2)(x-1)=0$$
$$x=-2,1$$

$$V=\int_0^1 2\pi\,(x)(2-x-x^2)\,dx$$
$$=2\pi\int_0^1 (2x-x^2-x^3)\,dx = 2\pi\left[\frac{\cancel{2}x^2}{\cancel{2}}-\frac{x^3}{3}-\frac{x^4}{4}\right]\Big|_0^1$$
$$=2\pi\left[(1)^2-\frac{(1)^3}{3}-\frac{(1)^4}{4}-(0-0-0)\right]=2\pi\left[1-\frac{1}{3}-\frac{1}{4}\right]$$
$$=2\pi\left[\frac{12-4-3}{12}\right]=\frac{10\pi}{12}$$

Question 12. Page 325

Use the shell method to find the volume of the region generated by revolving the region bounded by $y=\frac{3}{2\sqrt{x}}$, $y=0$, $x=1$, $x=4$ about y-axis

$$V=\int_1^4 2\pi \left(\begin{matrix}\text{shell}\\ \text{radius}\end{matrix}\right)\left(\begin{matrix}\text{shell}\\ \text{height}\end{matrix}\right) dx$$



$$=\cancel{2}\pi\int_1^4 (x)\left(\frac{3}{\cancel{2}\sqrt{x}}\right)dx$$
$$=3\pi\int_1^4 \sqrt{x}\,dx = 3\pi\,\frac{x^{3/2}}{3/2}\Big|_1^4$$
$$=\cancel{3}\cdot\frac{2}{\cancel{3}}\pi\left[(4)^{3/2}-(1)^{3/2}\right]$$
$$=2\pi[8-1]=14\pi$$

Question 14.
Page 325

Let $g(x) = \begin{cases} \frac{(\tan x)^2}{x}, & 0 < x \le \frac{\pi}{4} \\ 0, & x = 0 \end{cases}$

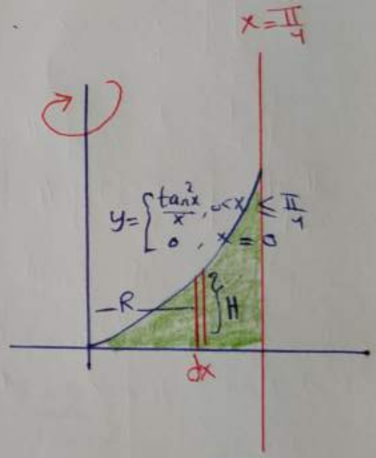
a) Show that $x g(x) = (\tan x)^2, \quad 0 \le x \le \frac{\pi}{4}$

at $x = 0 \rightarrow g(x) = 0 \rightarrow$
$x g(x) = 0 = (\tan 0)^2$

$$x g(x) = \begin{cases} (\tan x)^2, & 0 < x < \frac{\pi}{4} \\ 0, & x = 0 \end{cases} = \begin{cases} (\tan x)^2, & 0 < x < \frac{\pi}{4} \\ (\tan x)^2, & x = 0 \end{cases}$$

so $x g(x) = (\tan x)^2, \quad 0 \le x \le \frac{\pi}{4}$

b) Find the volume of the solid generated by revolving the shaded region about the y-axis.



$$V = \int_0^{\frac{\pi}{4}} 2\pi \left(\begin{matrix}\text{shell}\\ \text{radius}\end{matrix}\right)\left(\begin{matrix}\text{shell}\\ \text{height}\end{matrix}\right) dx$$

$$= 2\pi \int_0^{\frac{\pi}{4}} (x) \left(\frac{(\tan x)^2}{x} \right) dx$$

but $x \cdot \frac{(\tan x)^2}{x} = (\tan x)^2$

$$V = 2\pi \int_0^{4} (\tan x)^2 \, dx = 2\pi \int_0^{\frac{\pi}{4}} \sec^2 x - 1 \, dx$$

$$= 2\pi \left[\tan x - x \right] \Big|_0^{\frac{\pi}{4}}$$

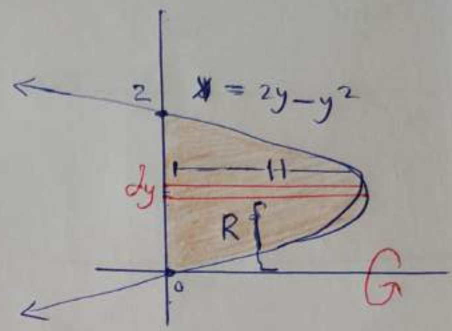
$$= 2\pi \left[\tan\left(\frac{\pi}{4}\right) - \frac{\pi}{4} - \left(\tan(0) - 0\right) \right]$$

$$= 2\pi \left[1 - \frac{\pi}{4} \right] = \cancel{2}\pi \left[\frac{4 - \pi}{\cancel{4}\,2} \right] = \frac{4\pi - \pi^2}{2}$$

Question 17. Page 325 Use the shell method to find the volume of the solid generated by revolving the region bounded by $x = 2y - y^2$, $x = 0$ about x-axis.

$x = 2y - y^2$

$2y - y^2 = 0 \Rightarrow y(2-y) = 0 \Rightarrow y = 0, 2$



$$V = \int_0^2 2\pi \begin{pmatrix}\text{shell}\\ \text{height}\end{pmatrix}\begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix} dy$$

$$V = 2\pi \int_0^2 (2y - y^2)\, y \; dy$$

$$= 2\pi \int_0^2 2y^2 - y^3 \; dy$$

$$= 2\pi \left[\frac{2y^3}{3} - \frac{y^4}{4} \right] \Big|_0^2$$

$$= 2\pi \left[\frac{2(2)^3}{3} - \frac{(2)^4}{4} - \left(\frac{2(0)^3}{3} - \frac{(0)^4}{4} \right) \right]$$

$$= 2\pi \left[\frac{16}{3} - \frac{16}{4} \right]$$

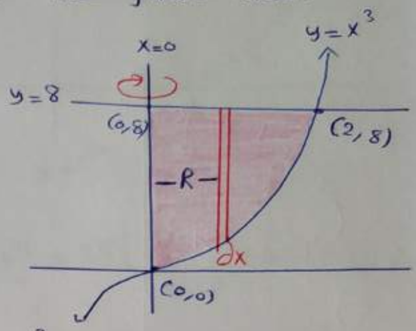
$$= 2\pi (16) \left[\frac{1}{3} - \frac{1}{4} \right] = 32\pi \left[\frac{4-3}{12} \right]$$

$$= \frac{32\pi}{12} = \frac{8\pi}{3}$$

Question 24. use the shell method to find the volume of
Page 325 the solids generated by revolving the regions bounded by the given curves about the given axis.

$y = x^3,\quad y = 8,\quad x = 0$

a) The y-axis.



$V = \int 2\pi\,(R)(H)\,dx$

$x^3 = 8 \rightarrow x = 2$

$V = \int_0^2 2\pi\,(x)(8 - x^3)\,dx = 2\pi \int_0^2 8x - x^4\,dx$

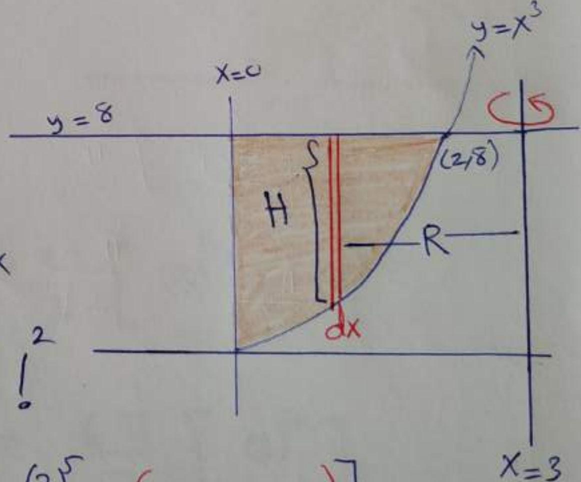
$= 2\pi\left[\frac{\cancel{8}^{4}x^2}{\cancel{2}} - \frac{x^5}{5}\right]\Big|_0^2 = 2\pi\left[4(2)^2 - \frac{(2)^5}{5} - (0 - 0)\right]$

$= 2\pi\left[16 - \frac{32}{5}\right] = 2\pi(16)\left[1 - \frac{2}{5}\right]$

$= \frac{96\pi}{5}$

b) The line $x = 3$

$V = \int_0^2 2\pi\,(R)\,(H)\,dx$



$= 2\pi \int_0^2 (3 - x)(8 - x^3)\,dx$

$= 2\pi \int_0^2 24 - 8x - 3x^3 + x^4\,dx$

$2\pi\left[24x - \frac{\cancel{8}^{4}x^2}{\cancel{2}} - \frac{3x^4}{4} + \frac{x^5}{5}\right]\Big|_0^2$

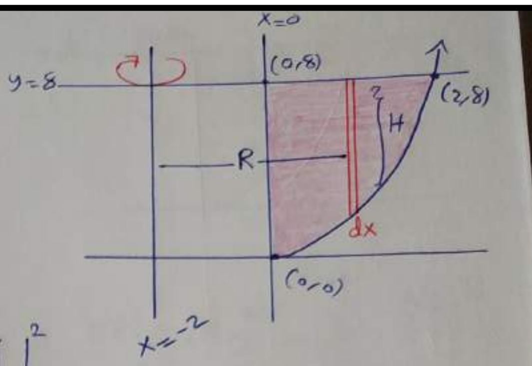
$2\pi\left[24(2) - 4(2)^2 - \frac{3(2)^4}{4} + \frac{(2)^5}{5} - (0 - 0 - 0 + 0)\right]$

$2\pi\left[48 - 16 - \frac{3(\cancel{16})^{4}}{\cancel{4}} + \frac{32}{5}\right] = 2\pi\left[20 + \frac{32}{5}\right]$

$= 2\pi\left[\frac{132}{5}\right] = \frac{264\pi}{5}$

c) The line $x = -2$

$$V = \int_0^2 2\pi\,(R)(H)\,dx$$



$$= 2\pi \int_0^2 (x+2)(8-x^3)\,dx$$

$$= 2\pi \int_0^2 8x - x^4 + 16 - 2x^3\,dx$$

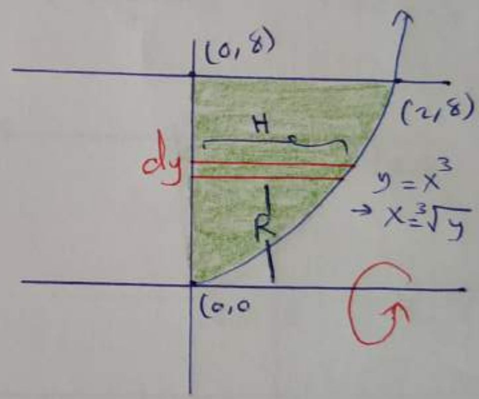
$$= 2\pi \left[\frac{\overset{4}{\cancel{8}}x^2}{\cancel{2}} - \frac{x^5}{5} + 16x - \frac{\cancel{2}x^4}{\cancel{4}\,2} \right]_0^2$$

$$= 2\pi \left[4(2)^2 - \frac{(2)^5}{5} + 16(2) - \frac{(2)^4}{2} - 0 \right] = 2\pi \left[16 - \frac{32}{5} + 32 - \frac{16}{2} \right]$$

$$= 2\pi \left[40 - \frac{32}{5} \right] = 2\pi \left[\frac{168}{5} \right] = \frac{336\,\pi}{5}$$

d) The x-axis.

$$V = \int_0^8 2\pi\,(R)(H)\,dy$$



$$= 2\pi \int_0^8 y\,(\sqrt[3]{y})\,dy$$

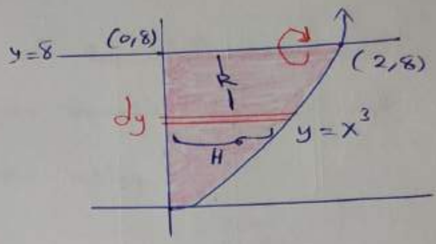
$$= 2\pi \int_0^8 y^{4/3}\,dy$$

$$= 2\pi \left. \frac{y^{7/3}}{7/3} \right|_0^8 = 2\pi \cdot \frac{3}{7} \left[(8)^{7/3} - (0)^{7/3} \right]$$

$$= \frac{6\pi}{7} \cdot \left[(\sqrt[3]{8})^7 - 0 \right] = \frac{6\pi}{7} \left[(2)^7 \right]$$

$$= \frac{6\pi}{7}(128) = \frac{768\,\pi}{7}$$

e) The line $y = 8$



$V = \int_0^8 2\pi (R)(H)\, dy$

$= 2\pi \int_0^8 (8-y)(\sqrt[3]{y})\, dy$

$= 2\pi \int_0^8 8y^{1/3} - y^{4/3}\, dy$

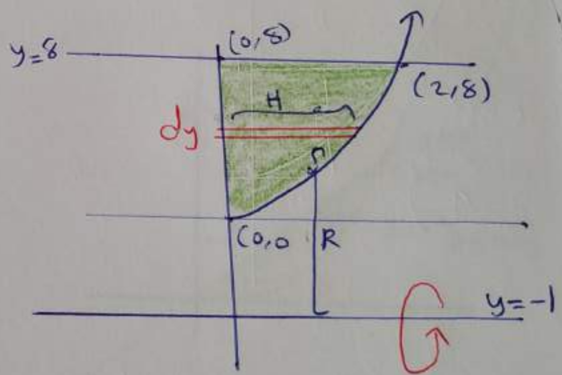
$= 2\pi \left[\frac{8y^{4/3}}{4/3} - \frac{y^{7/3}}{7/3} \right]\Big|_0^8 = 2\pi \left[\frac{3}{\cancel{4}} \cdot \overset{2}{\cancel{8}} y^{4/3} - \frac{3}{7} \cdot y^{7/3} \right]\Big|_0^8$

$= 2\pi \left[6y^{4/3} - \frac{3}{7} y^{7/3} \right]\Big|_0^8 = 2\pi \left[6(8)^{4/3} - \frac{3}{7}(8)^{7/3} - 0 \right]$

$= 2\pi \left[6(2)^4 - \frac{3}{7}(2)^7 \right] = 2\pi \left[6(16) - \frac{3(128)}{7} \right]$

$= 2\pi (16) \left[\cancel{3}6 - \frac{3(8)}{7} \right) = 32\pi \left[\frac{42 - 24}{7} \right] = 32\pi \left(\frac{18}{7} \right) = \frac{576\pi}{7}$

f) The line $y = -1$



$V = \int_0^8 2\pi (R)(H)\, dy$

$= 2\pi \int_0^8 (1+y)(\sqrt[3]{y})\, dy$

$= 2\pi \int_0^8 y^{1/3} + y^{4/3}\, dy$

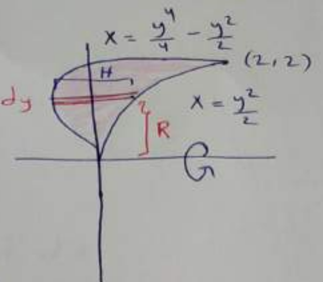
$= 2\pi \left[\frac{y^{4/3}}{4/3} + \frac{y^{7/3}}{7/3} \right]\Big|_0^8 = 2\pi \left[\frac{3}{4} y^{4/3} + \frac{3}{7} y^{7/3} \right]\Big|_0^8$

$= 2\pi \left[\frac{3}{4}(8)^{4/3} + \frac{3}{7}(8)^{7/3} \right] = 2\pi \left[\frac{3}{4}(2)^4 + \frac{3}{7}(2)^7 \right]$

$= 2\pi \left[3(4) + \frac{3(128)}{7} \right] = 2\pi \left[\frac{84 + 384}{7} \right] = \frac{936\pi}{7}$

28 use the shell method.

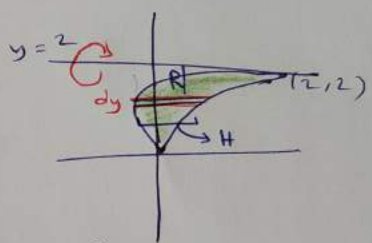
a) The x-axis



$$V = \int_0^2 2\pi \; R \, H \, dy$$

$$= 2\pi \int_0^2 (y)\left(\frac{y^2}{2} - \left[\frac{y^4}{4} - \frac{y^2}{2}\right]\right) dy$$

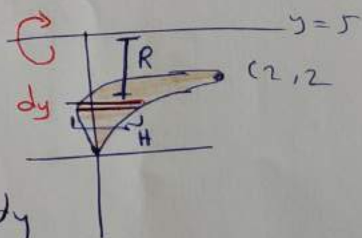
b) the line $y = 2$



$$V = \int_0^2 2\pi \; R \; H \; dy$$

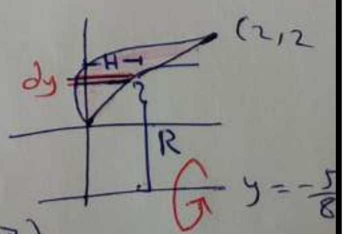
$$= 2\pi \int_0^2 (2-y)\left(\frac{y^2}{2} - \left[\frac{y^4}{4} - \frac{y^2}{2}\right]\right) dy$$

$$= 2\pi \int_0^2 (2-y)\left(y^2 - \frac{y^4}{4}\right) dy$$



c) the line $y = 5$

$$V = 2\pi \int_0^2 (5-y)\left(\frac{y^2}{2} - \left[\frac{y^4}{4} - \frac{y^2}{2}\right]\right) dy$$



d) the line $y = -\frac{5}{8}$

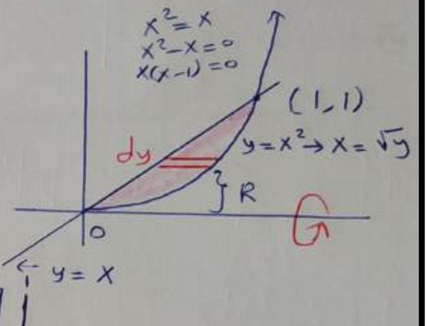
$$V = 2\pi \int_0^2 \left(y + \frac{5}{8}\right)\left(\frac{y^2}{2} - \left[\frac{y^4}{4} - \frac{y^2}{2}\right]\right) dy$$

Question 29 Page 325

Compute the volume of the solid generated by revolving the region bounded by $y=x$ and $y=x^2$ about each coordinate axis using:

a) The shell method:

About x-axis



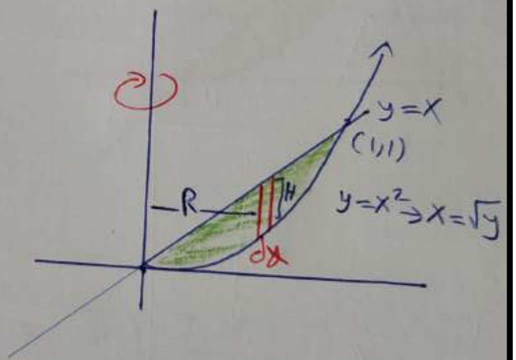
$$V=\int_0^1 2\pi (R)(H)\,dy = 2\pi\int_0^1 (y)(\sqrt{y}-y)\,dy$$

$$= 2\pi\int_0^1 y^{3/2}-y^2\,dy = 2\pi\left[\frac{y^{5/2}}{5/2}-\frac{y^3}{3}\right]_0^1$$

$$= 2\pi\left[\frac{2}{5}(1)^{5/2}-\frac{(1)^3}{3}-(0-0)\right] = 2\pi\left[\frac{2}{5}-\frac{1}{3}\right]$$

$$= 2\pi\left[\frac{6-5}{15}\right] = \frac{2\pi}{15}$$

About y-axis



$$V=\int_0^1 2\pi (R)(H)\,dx$$

$$= 2\pi\int_0^1 (x)(x-x^2)\,dx$$

$$= 2\pi\int_0^1 x^2-x^3\,dx$$

$$= 2\pi\left[\frac{x^3}{3}-\frac{x^4}{4}\right]_0^1$$

$$= 2\pi\left[\frac{1}{3}-\frac{1}{4}-(0-0)\right]$$

$$= 2\pi\left[\frac{4-3}{12}\right] = \frac{2\pi}{12} = \frac{\pi}{6}$$

b) The washer method.

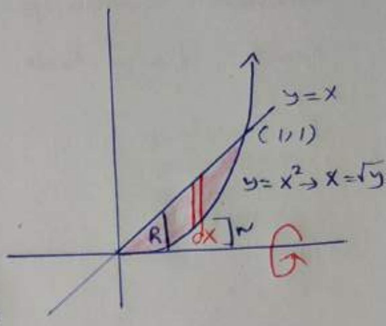
About x - axis

$$V = \int_0^1 \pi \left(R^2(x) - r^2(x) \right) dx$$

$$= \pi \int_0^1 (x)^2 - (x^2)^2 \, dx$$

$$= \pi \int_0^1 x^2 - x^4 \, dx = \pi \left[\frac{x^3}{3} - \frac{x^5}{5} \right] \Big|_0^1$$

$$= \pi \left[\frac{1}{3} - \frac{1}{5} - (0 - 0) \right] = \pi \left[\frac{5-3}{15} \right] = \frac{2\pi}{15}$$



About y- axis

$$V = \pi \int_0^1 R^2(y) - r^2(y) \, dy$$

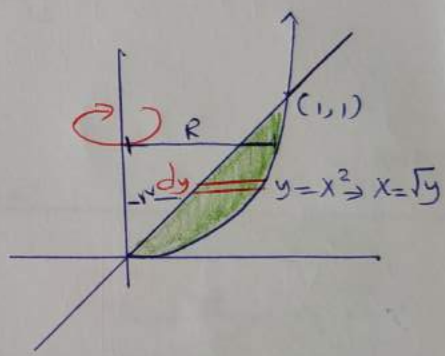
$$= \pi \int_0^1 (\sqrt{y})^2 - (y)^2 \, dy$$

$$= \pi \int_0^1 y - y^2 \, dy$$

$$= \pi \left[\frac{y^2}{2} - \frac{y^3}{3} \right] \Big|_0^1$$

$$= \pi \left[\frac{1}{2} - \frac{1}{3} - (0 - 0) \right] = \pi \left[\frac{3-2}{6} \right]$$

$$= \frac{\pi}{6}$$



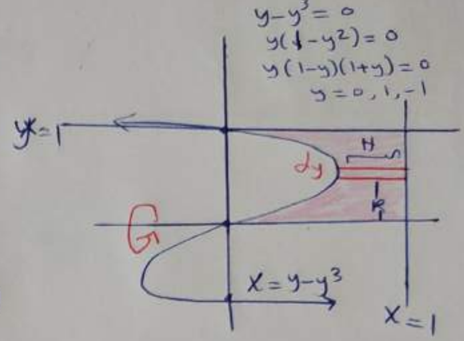
Question 34 The region in the first quadrant bounded

Page 326 by $x = y - y^3$, $x = 1$ and $y = 1$ about

a) The x-axis

الاسهل أخذ شريحة موازية لمحور الدوران

(shell method)



shell method:

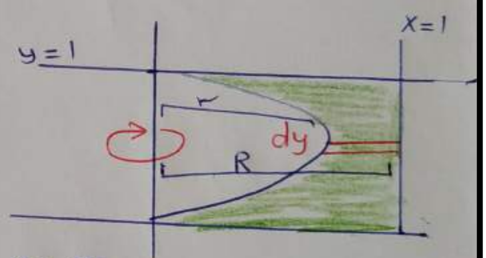
$$V = \int_0^1 2\pi \, (R)\,(H)\, dy$$

$$= 2\pi \int_0^1 (y)\left(1 - (y - y^3)\right) dy$$

$$= 2\pi \int_0^1 y - y^2 + y^4 \, dy = \frac{11\pi}{15}$$

b) The y-axis.

Washer method:



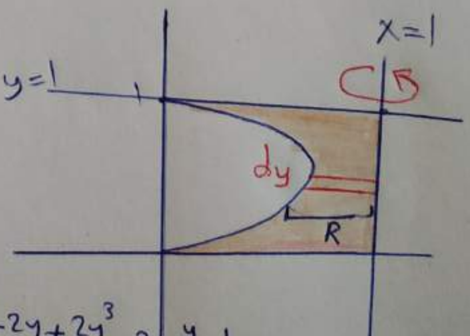
$$V = \pi \int_0^1 R^2(y) - r^2(y)\, dy$$

$$= \pi \int_0^1 (1)^2 - (y - y^3)^2 \, dy$$

$$= \pi \int_0^1 1 - y^2 + 2y^4 - y^6 \, dy = \frac{97\pi}{105}$$

c) The line $x = 1$

Washer method

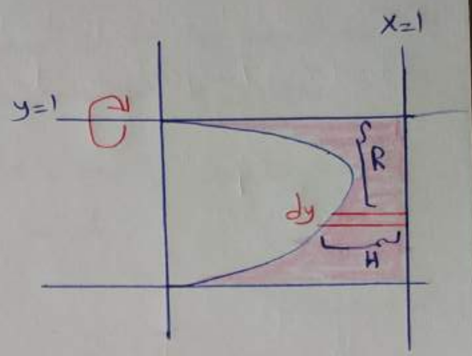


$$V = \pi \int_0^1 R^2(y) - r^2(y)\, dy$$

$$= \pi \int_0^1 \left(1 - (y - y^3)\right)^2 - (0)^2 \, dy$$

$$= \pi \int_0^1 (1 - y + y^3)^2 dy = \pi \int_0^1 1 + y^2 + y^6 - 2y + 2y^3 - 2y^4 \, dy = \frac{121\pi}{210}$$

d) The line $y=1$ [illegible] (shell)



$$V = \int_0^1 2\pi (R)(H)\,dy$$

$$= 2\pi \int_0^1 (1-y)(1-(y-y^3))\,dy$$

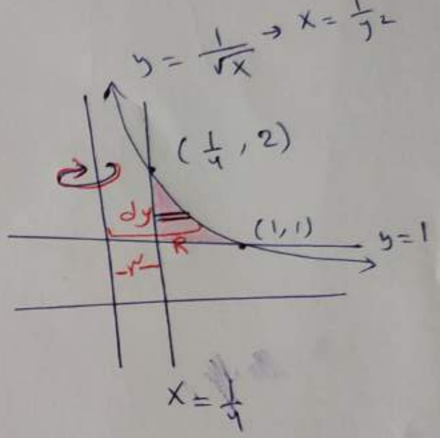
$$= 2\pi \int_0^1 (1-y)(1-y+y^3)\,dy$$

$$= 2\pi \int_0^1 1-y+y^3-y+y^2-y^4\,dy = \frac{23\pi}{30}$$

38 The region in the first quadrant that is bounded above by $y = \frac{1}{\sqrt{x}}$, on the left by $x = \frac{1}{4}$ and below by $y = 1$ is revolved ~~by~~ about y-axis to generate a solid.

Find the volume by:

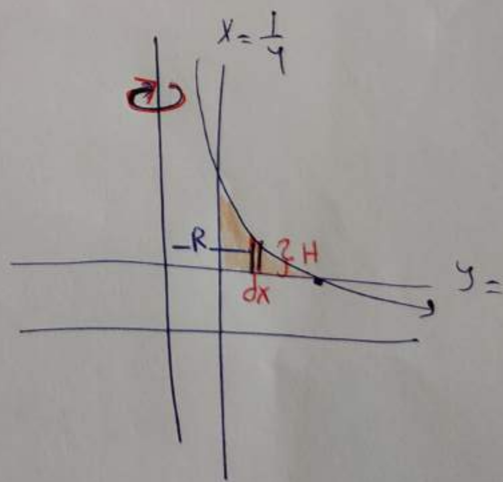
a) the washer method.



$$V = \int_{1}^{2} \pi \left[R^2(y) - r^2(y) \right] dy$$

$$= \pi \int_{1}^{2} \left[\left(\frac{1}{y^2}\right)^2 - \left(\frac{1}{4}\right)^2 \right] dy$$

) The shell - method



$$V = \int_{\frac{1}{4}}^{1} 2\pi \; R \; H \; dx$$

$$V = 2\pi \int_{\frac{1}{4}}^{1} (X)\left(\frac{1}{\sqrt{x}} - 1\right)^2 dx$$

Question 39. Page 326

The region shown here is to be revolved about the x-axis to generated a solid.

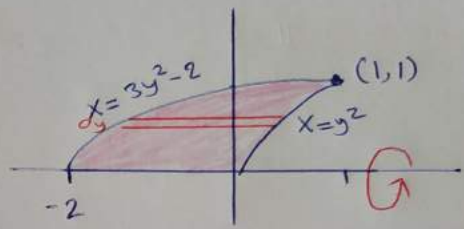
Which of the methods (disk, washer, shell) could you use to find the volume of the solid?

a) Shell:

$$V = \int_0^1 2\pi (R)(H)\, dy$$

$$= 2\pi \int_0^1 (y)\left(y^2 - (3y^2 - 2)\right) dy$$

$$= \pi$$



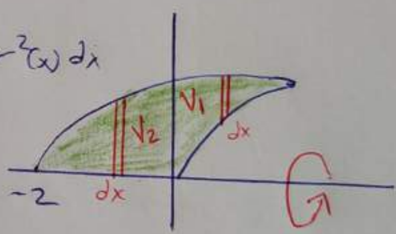
b) ~~Disk~~ ~~cshes~~ washer

$$V = V_1 + V_2$$

$$= \pi \int_{-2}^{0} R^2(x) - r^2(x)\, dx + \pi \int_0^1 R^2(x) - r^2(x)\, dx$$

$$= \pi \int_{-2}^{0} \left(\sqrt{\frac{x+2}{3}}\right)^2 - 0^2 \; dx$$

$$+ \pi \int_0^1 \left(\sqrt{\frac{x+2}{3}}\right)^2 - (\sqrt{x})^2\, dx$$



$x = 3y^2 - 2$

$\rightarrow y = \sqrt{\frac{x+2}{3}}$

$x = y^2 \rightarrow y = \sqrt{x}$

c) Disk: $V = V_1 - V_2$ المساحة باللون الأخضر

$$= \pi \int_{-2}^{1} \left(\sqrt{\frac{x+2}{3}}\right)^2 dx - \pi \int_0^1 (\sqrt{x})^2\, dx$$

نجد المساحة تحت المنحنى الاعلى

ونطرح منها المساحة تحت المنحنى $y = x^2$.