Linear Quadratic Regulator (LQR)

LQR: is an optimal approach which is used to minimize a cost function to compute the optimal response for the system.

Definitions:

-Negative Definite Matrix (NDM): it is a Hermitian matrix all of whose eigenvalues are negative.

-Semi-Negative Definite Matrix (SNDM): it is a Hermitian matrix all of whose eigenvalues are nonpositive.

-Positive Definite Matrix (PDM): it is a Hermitian matrix all of whose eigenvalues are positive.

-Semi-Positive Definite Matrix (SPDM): it is a Hermitian matrix all of whose eigenvalues are nonnegative

Design a Regulator by using LQR:

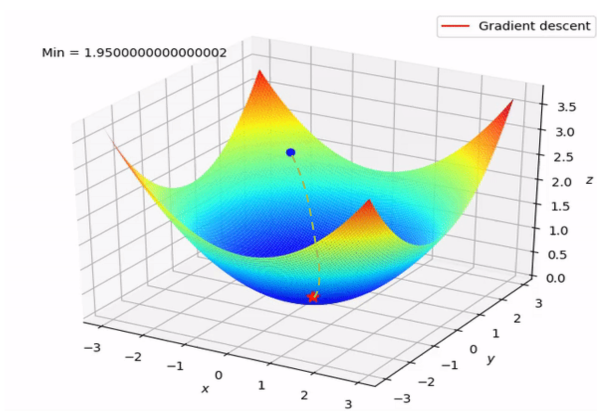


Fig. 1: plot for the cost function

Consider the following system:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$
$$y = \mathbf{C}\mathbf{x} \qquad 1$$

The goal of the LQR is to minimize the cost function or it is called the performance index.

$$J = \int_0^\infty (\boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} + \boldsymbol{u}^T\boldsymbol{R}\boldsymbol{u})dt \qquad 2$$
$$\text{sub to}: \dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$$

Where:

$\boldsymbol{Q} \in \boldsymbol{R}^{n*n}$**:** is a positive-definite (or positive-semidefinite) and a real symmetric matrix.

R∈ $\boldsymbol{R^{r*r}}$**:** is a positive-definite and real symmetric matrix.

The optimal solution is computed by using the following equation:

$$\mathbf{u}(t) = -\mathbf{Kx}(t) \tag{3}$$

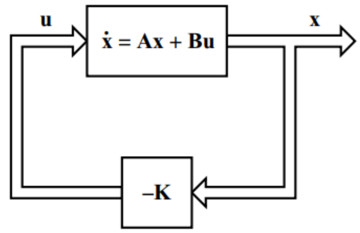


Fig 2: Quadratic Optimal Regulator System

Now let us solve the optimization problem. Substituting Equation (3) into Equation (1), we obtain:

$$\dot{\mathbf{x}} = \mathbf{Ax} - \mathbf{BKx} = (\mathbf{A} - \mathbf{BK})\mathbf{x} \tag{4}$$

In the following derivations, we assume that the matrix $(\boldsymbol{A}-\boldsymbol{BK})$ is asymptotically stable i.e. the eigenvalues of $(\boldsymbol{A}-\boldsymbol{BK})$ have negative real parts.

Substituting Equation (3) into Equation (2) yields:

$$\begin{aligned} J &= \int_0^\infty (\mathbf{x}^*\mathbf{Qx} + \mathbf{x}^*\mathbf{K}^*\mathbf{RKx})\,dt \\ &= \int_0^\infty \mathbf{x}^*(\mathbf{Q} + \mathbf{K}^*\mathbf{RK})\mathbf{x}\,dt \end{aligned} \tag{5}$$

Let us set

$$\mathbf{x}^*(\mathbf{Q} + \mathbf{K}^*\mathbf{RK})\mathbf{x} = -\frac{d}{dt}(\mathbf{x}^*\mathbf{Px}) \tag{6}$$

where **P** is called a co- 1`state matrix and it is a positive-definite and a real symmetric matrix.

$$\mathbf{x}^*(\mathbf{Q} + \mathbf{K}^*\mathbf{RK})\mathbf{x} = -\dot{\mathbf{x}}^*\mathbf{Px} - \mathbf{x}^*\mathbf{P}\dot{\mathbf{x}} = -\mathbf{x}^*\left[(\mathbf{A} - \mathbf{BK})^*\mathbf{P} + \mathbf{P}(\mathbf{A} - \mathbf{BK})\right]\mathbf{x} \tag{7}$$

Comparing both sides of this last equation and noting that this equation must hold true for any **x**, we require that:

$$(\mathbf{A} - \mathbf{BK})^*\mathbf{P} + \mathbf{P}(\mathbf{A} - \mathbf{BK}) = -(\mathbf{Q} + \mathbf{K}^*\mathbf{RK}) \qquad 8$$

It can be proved that if $(\boldsymbol{A} - \boldsymbol{BK})$ is a asymptotically stable matrix, there exists a positive-definite matrix **P** that satisfies Equation (8). While Equation (8) is called the algebraic Riccati Equation. This Equation can be reduced to another formula which is shown in Equation (9) and it is called the **Reduced Algebraic Riccati Equation:**

$$\mathbf{A}^*\mathbf{P} + \mathbf{PA} - \mathbf{PBR}^{-1}\mathbf{B}^*\mathbf{P} + \mathbf{Q} = \mathbf{0} \qquad 9$$

The optimal control gain matrix **K** can be written as shown in Equation (10):

$$\boldsymbol{K} = \mathbf{R}^{-1}\mathbf{B}^*\mathbf{P} \qquad 10$$

Therefore the optimal control action can be written as shown in Equation (11):

$$\mathbf{u}(t) = -\mathbf{Kx}(t) = -\mathbf{R}^{-1}\mathbf{B}^*\mathbf{Px}(t) \qquad 11$$

The performance index J can be evaluated as shown in Equation (12).

$$J = \int_0^\infty \mathbf{x}^*(\mathbf{Q} + \mathbf{K}^*\mathbf{RK})\mathbf{x}\,dt = -\mathbf{x}^*\mathbf{Px}\Big|_0^\infty = -\mathbf{x}^*(\infty)\mathbf{Px}(\infty) + \mathbf{x}^*(0)\mathbf{Px}(0) \qquad 12$$

Since all eigenvalues of $(\boldsymbol{A} - \boldsymbol{BK})$ are assumed to have negative real parts, we have $x(\infty) \to 0$. Therefore, we obtain the performance index J is given by Equation (13).

$$J = \mathbf{x}^*(0)\mathbf{Px}(0) \qquad 13$$

Thus, the performance index J can be obtained in terms of the initial condition $\mathbf{x}(0)$ and **P**.

The design steps may be stated as follows:

1. Select the design weighting matrices **Q** and **R**.

   Q: is a positive-definite (or positive-semidefinite) and a real symmetric matrix.

   R: is a positive-definite and real symmetric matrix.

2. Solve Equation (9), the reduced-matrix Riccati equation, for the matrix **P**. If a positive-definite matrix **P** exists then the system is asymptomatically stable system.

3. Substitute this matrix **P** into Equation (10). The resulting matrix **K** is the optimal gain matrix.

4. The input control action is computed by using Equation (11).

5. To compute the performance index J is calculated by Equation (13).

6. The closed loop eigenvalues of the system are computed by using Equation (4).

Note:

The relationship between the **Q** matrix and the location of eigenvalues is summarized by: increasing the **Q** matrix is shifted the eigenvalues more to the lift on the s-plane and the vice versa is correct. On the other hand, increasing the ***R*** matrix is forced the eigenvalues to go more to the right of the s-plane and the vice versa is correct.

In Matlab use the following command:

[K, P, lamda] = lqr (A,B,Q,R);

K: gain matrix.

P: co-state matrix.

lamda: closed loop eigenvalues.

<u>Design a Tracking System based on LQR:</u>

There are two cases for the tracking system:

Case 1: Design of Type 1 Servo System when the Plant Has an Integrator:

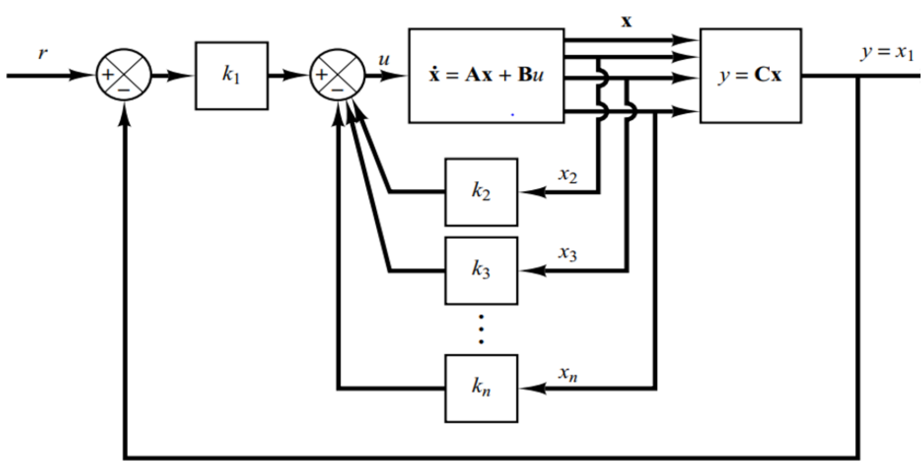


Fig 3. Tracking system when the plant has an integrator.

The closed loop dynamical matrix for the tracking system when the plant has an integrator is given by:

$$\dot{\mathbf{x}}(t) - \dot{\mathbf{x}}(\infty) = (\mathbf{A} - \mathbf{BK})[\mathbf{x}(t) - \mathbf{x}(\infty)] \qquad 14$$

Define

$$\mathbf{x}(t) - \mathbf{x}(\infty) = \mathbf{e}(t)$$

Then Equation (14) becomes

$$\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{BK})\mathbf{e} \qquad 15$$

Therefore: the cost function is modified to Equation (16).

$$J = \int_0^\infty (\boldsymbol{e}^T \boldsymbol{Q} \boldsymbol{e} + \boldsymbol{u}^T \boldsymbol{R} \boldsymbol{u})dt \qquad 16$$
$$\text{sub to}: \dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{Bu}$$

The same procedures are used to solve the case 1 for the tracking system by LQR.

Case 2: Design of Type 1 Servo System when the Plant Has No Integrator:

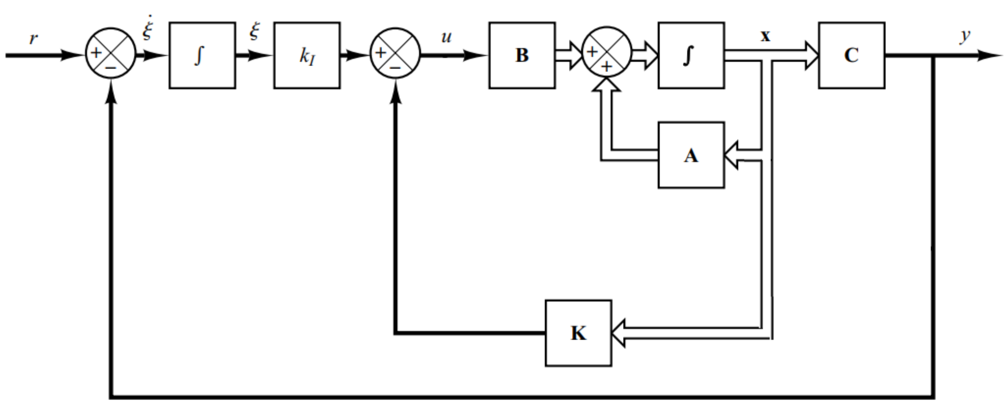


Fig 4. Tracking system when the plant has No integrator.

The closed loop dynamical matrix for the tracking system when the plant has an integrator is given by:

$$\dot{\mathbf{e}} = (\hat{\mathbf{A}} - \hat{\mathbf{B}}\hat{\mathbf{K}})\mathbf{e} \qquad 17$$

where

$$\hat{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & 0 \end{bmatrix}, \qquad \hat{\mathbf{B}} = \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix}$$

$$\hat{\mathbf{K}} = [\mathbf{K} \;\vdots\; -k_I] \qquad \mathbf{e}(t) = \begin{bmatrix} \mathbf{x}_e(t) \\ \xi_e(t) \end{bmatrix} = (n+1)\text{-vector}$$

Therefore: the cost function is modified to Equation (17).

$$J = \int_0^{\infty} (\boldsymbol{e}^T \boldsymbol{Q} \boldsymbol{e} + \boldsymbol{u}^T \boldsymbol{R} \boldsymbol{u}) dt \qquad 18$$

$$\text{sub to} : \dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$$

The same procedures are used to solve the case 2 for the tracking system by LQR.

Design an Observer based on LQR:

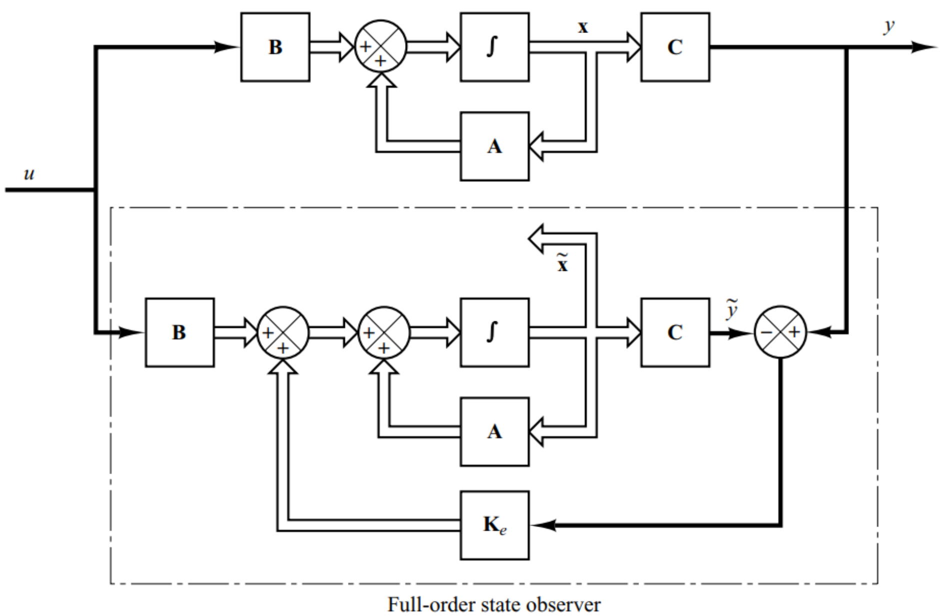


Fig 5. A regulator system with an integration with an observer.

$$\mathbf{e} = \mathbf{x} - \tilde{\mathbf{x}} \qquad 19$$

The closed loop dynamical matrix for the observer is given by:

$$\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{K}_e \mathbf{C})\mathbf{e} \qquad 20$$

Therefore: the cost function is modified to Equation (20).

$$J = \int_0^{\infty} (\boldsymbol{e}^T \boldsymbol{Q} \boldsymbol{e} + \boldsymbol{u}^T \boldsymbol{R} \boldsymbol{u}) dt \qquad 21$$

$$\text{sub to} : \dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$$

The same procedures are used to solve the observer system by LQR.

Consider the system shown in Figure 10–36. Assuming the control signal to be

$$u(t) = -\mathbf{K}\mathbf{x}(t)$$

determine the optimal feedback gain matrix **K** such that the following performance index is minimized:

$$J = \int_0^\infty (\mathbf{x}^T \mathbf{Q} \mathbf{x} + u^2)\, dt$$

where

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 \\ 0 & \mu \end{bmatrix} \qquad (\mu \geq 0)$$

From Figure 10–36, we find that the state equation for the plant is

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

We shall demonstrate the use of the reduced-matrix Riccati equation in the design of the optimal control system. Let us solve Equation (10–118), rewritten as

$$\mathbf{A}^*\mathbf{P} + \mathbf{P}\mathbf{A} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^*\mathbf{P} + \mathbf{Q} = \mathbf{0}$$

Noting that matrix **A** is real and matrix **Q** is real symmetric, we see that matrix **P** is a real symmetric matrix. Hence, this last equation can be written as

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$- \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}[1][0 \quad 1]\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & \mu \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

This equation can be simplified to

$$\begin{bmatrix} 0 & 0 \\ p_{11} & p_{12} \end{bmatrix} + \begin{bmatrix} 0 & p_{11} \\ 0 & p_{12} \end{bmatrix} - \begin{bmatrix} p_{12}^2 & p_{12}p_{22} \\ p_{12}p_{22} & p_{22}^2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & \mu \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

from which we obtain the following three equations:

$$1 - p_{12}^2 = 0$$

$$p_{11} - p_{12}p_{22} = 0$$

$$\mu + 2p_{12} - p_{22}^2 = 0$$

Solving these three simultaneous equations for p_{11}, p_{12}, and p_{22}, requiring $\mathbf{P}$ to be positive definite, we obtain

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} = \begin{bmatrix} \sqrt{\mu + 2} & 1 \\ 1 & \sqrt{\mu + 2} \end{bmatrix}$$

Referring to Equation (10–117), the optimal feedback gain matrix $\mathbf{K}$ is obtained as

Referring to Equation (10–117), the optimal feedback gain matrix $\mathbf{K}$ is obtained as

$$\begin{aligned} \mathbf{K} &= \mathbf{R}^{-1}\mathbf{B}^*\mathbf{P} \\ &= [1][0 \quad 1]\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \\ &= [p_{12} \quad p_{22}] \\ &= [1 \quad \sqrt{\mu + 2}] \end{aligned}$$

Thus, the optimal control signal is

$$u = -\mathbf{K}\mathbf{x} = -x_1 - \sqrt{\mu + 2}\, x_2 \tag{10–120}$$

Note that the control law given by Equation (10–120) yields an optimal result for any initial state under the given performance index. Figure 10–37 is the block diagram for this system.

Since the characteristic equation is

$$|s\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K}| = s^2 + \sqrt{\mu + 2}\, s + 1 = 0$$

if $\mu = 1$, the two closed-loop poles are located at

$$s = -0.866 + j\,0.5, \qquad s = -0.866 - j\,0.5$$

These correspond to the desired closed-loop poles when $\mu = 1$.

EXAMPLE 10–4 Design a type 1 servo system when the plant transfer function has an integrator. Assume that the plant transfer function is given by

The reference input r is a step function and the required steady state error is equal to zero.

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$
$$y = \mathbf{C}\mathbf{x} + Du$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{C} = [1 \quad 0 \quad 0], \quad D = [0]$$

Let us determine the state-feedback gain matrix $\mathbf{K}$, where

$$\mathbf{K} = [k_1 \quad k_2 \quad k_3]$$

such that the following performance index is minimized:

$$J = \int_0^\infty (\mathbf{x}'\mathbf{Q}\mathbf{x} + u'Ru)\, dt$$

where

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, R = 10$$

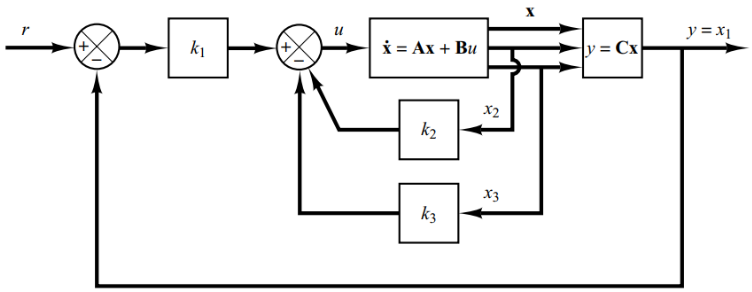
b) Compute the performance index J if $x(0) = \begin{bmatrix} 0 \\ 0 \\ 0.1 \end{bmatrix}$.

c) Compute the closed loop dynamic matrix for the tracking system.

$$|sI - A| = \left|\begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix}\right| = \left|\begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 0 & 2 & s+3 \end{bmatrix}\right|$$

$$|sI - A| = s^3 + 3s^2 + 2\,s = s(\,s^2 + 3s + s)$$

$$s_1 = 0, s_2 = -1, s_3 = -2 \text{ so it is case 1}$$



First: solve the reduced Riccati equation

So let $P = \begin{bmatrix} p_1 & p_2 & p_3 \\ p_2 & p_4 & p_5 \\ p_3 & p_5 & p_6 \end{bmatrix}$

$$\mathbf{A}^*\mathbf{P} + \mathbf{P}\mathbf{A} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^*\mathbf{P} + \mathbf{Q} = \mathbf{0}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -2 \\ 0 & 1 & -3 \end{bmatrix}\begin{bmatrix} p_1 & p_2 & p_3 \\ p_2 & p_4 & p_5 \\ p_3 & p_5 & p_6 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 & p_3 \\ p_2 & p_4 & p_5 \\ p_3 & p_5 & p_6 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix}$$
$$- \begin{bmatrix} p_1 & p_2 & p_3 \\ p_2 & p_4 & p_5 \\ p_3 & p_5 & p_6 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\frac{1}{10}\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} p_1 & p_2 & p_3 \\ p_2 & p_4 & p_5 \\ p_3 & p_5 & p_6 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Consequently:

$$\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathrm{a} = 1 - \frac{\mathrm{p}_3^2}{10} = 0$$

$$b = p_1 - 2p_3 - \frac{(p_3 p_5)}{10} = 0$$

$$c = p_2 - 3p_3 - \frac{(p_3 p_6)}{10} = 0$$

$$d = -\frac{\mathrm{p}_5^2}{10} - 4p_5 + 2p_2 + 1 = 0$$

$$e = p_3 + p_4 - 3p_5 - 2p_6 - \frac{(p_5 p_6)}{10} = 0$$

$$f = -\frac{\mathrm{p}_6^2}{10} - 6p_6 + 2p_5 + 1 = 0$$

So, $\boldsymbol{P} = \begin{bmatrix} p_1 & p_2 & p_3 \\ p_2 & p_4 & p_5 \\ p_3 & p_5 & p_6 \end{bmatrix} = \begin{bmatrix} 7.8129 & 10.0206 & 3.1623 \\ 10.0206 & 15.1278 & 4.7065 \\ 3.1623 & 4.7065 & 1.6880 \end{bmatrix}$

$$\Rightarrow |s\boldsymbol{I} - \boldsymbol{P}| = s^3 - 24.63s^2 + 24.35s - 3.948 = 0$$

$$s_1 = 0.2037$$

$$s_2 = 0.8209$$

$$s_3 = 23.6041$$

Second, compute the gain matrix ($\boldsymbol{K}$)

$$\boldsymbol{K} = \mathbf{R}^{-1}\mathbf{B}^*\mathbf{P} = \frac{1}{10}[0 \quad 0 \quad 1] \begin{bmatrix} 7.8129 & 10.0206 & 3.1623 \\ 10.0206 & 15.1278 & 4.7065 \\ 3.1623 & 4.7065 & 1.6880 \end{bmatrix}$$

$$\boldsymbol{K} = [0.3162 \quad 0.4707 \quad 0.1688]$$

Third, compute the control action u(t)

$$u(t) = -\boldsymbol{K}\boldsymbol{x} = -[0.3162 \quad 0.4707 \quad 0.1688] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$u(t) = -0.3162x_1 + 0.4707x_2 + 0.1688x_3$$

b) The performance index is computed by

$$J = \mathbf{x}^*(0)\mathbf{P}\mathbf{x}(0)$$

$$J = [0 \quad 0 \quad 0.1] \begin{bmatrix} 7.8129 & 10.0206 & 3.1623 \\ 10.0206 & 15.1278 & 4.7065 \\ 3.1623 & 4.7065 & 1.6880 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0.1 \end{bmatrix} = 0.0169$$

c) The closed loop eigenvalues of the system are computed by using Equation

$$\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{e}$$

$$|s\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{B}\boldsymbol{K}| = \left| \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} [0.3162 \quad 0.4707 \quad 0.1688] \right|$$

$$|s\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{B}\boldsymbol{K}| = s^3 + 3.17s^2 + 2.47s + 0.316 = 0$$

$$s_1 = -0.1587$$

$$s_2 = -2.0268$$

$$s_3 = -0.9834$$