Def Let $f(x)$ be any function defined on open interval containing c (or except c).

Then we say the limit of $f(x)$ is L as x approaches (تقترب) c means

$$\lim_{x \to c} f(x) = L \quad , \quad L \text{ finite number}$$

المجال صورة المدى

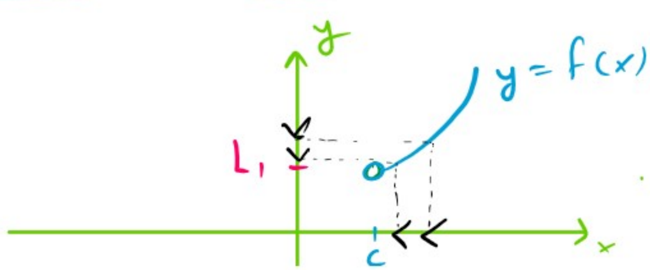
- $\lim_{x \to c} f(x) = L$ means

$$\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = L$$

- If $\lim_{x \to c^+} f(x) \neq \lim_{x \to c^-} f(x)$ then

$\lim_{x \to c} f(x)$ does not exist (or DNE)

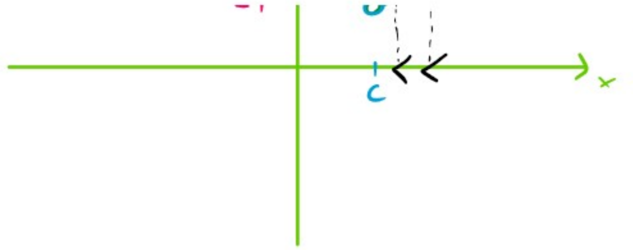
One-Sided limits



limit from right $\Rightarrow$

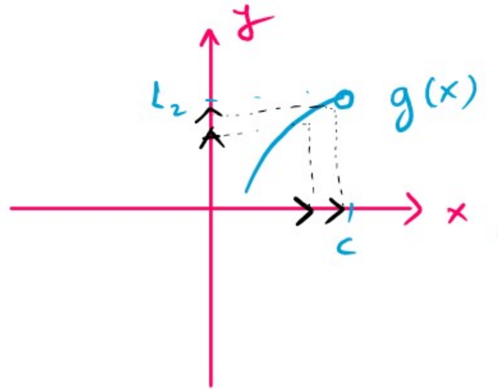
limit from right $\Rightarrow$

$$\lim_{x \to c^+} f(x) = L_1$$



limit from left $\Rightarrow$

$$\lim_{x \to c^-} g(x) = L_2$$



Remark: To find the limit $\Rightarrow$

we use Direct Substitution تعويض مباشر

$\Downarrow$

$\frac{0}{\text{رقم غير الصفر}} = 0$

$\frac{0}{2} = 0$ ✓

$\frac{0}{-3} = 0$ ✓

✓ $\frac{\text{رقم غير الصفر}}{0}$: DNE

undefined

$\frac{2}{0}$: DNE

$\frac{-\sqrt{2}}{0}$: DNE

$\frac{0}{0}$

يلزم تحليل وتبسيط

أو

اشتقاق البسط والمقام

Exp

① $\lim_{x \to 1} (2x+3) = 2\boxed{1} + 3 = 2 + 3 = 5$

② $\lim_{x \to -1} \frac{x^2 - 4}{3x+2} = \frac{\boxed{-1}^2 - 4}{3\boxed{-1} + 2} = \frac{1-4}{-3+2} = \frac{-3}{-1} = 3$

③ $\lim\limits_{x \to 2} \dfrac{x^2 - 4}{x - 2} = \dfrac{\boxed{2}^2 - 4}{\boxed{2} - 2} = \dfrac{4 - 4}{2 - 2} = \dfrac{0}{0}$

يلزم تحليل

$\lim\limits_{x \to 2} \dfrac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}} = \lim\limits_{x \to 2} (x+2) = \boxed{2} + 2 = 4$

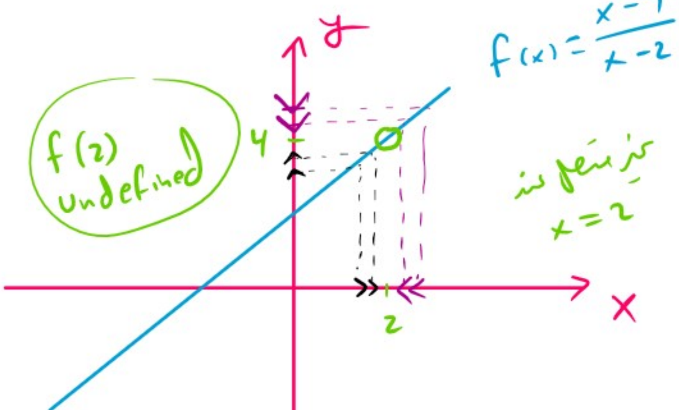
أو
نشتق البسط والمقام

$\lim\limits_{x \to 2} \dfrac{2x - 0}{1 - 0} = \lim\limits_{x \to 2} 2x = 2\,\boxed{2} = 4$

$y = x + 2$

$f(x) = \dfrac{x^2 - 4}{x - 2} = \dfrac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}}$

Domain $= \mathbb{R} \setminus \{2\}$



$\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} (x+2) = \boxed{2} + 2 = 4$

$\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} (x+2) = \boxed{2} + 2 = 4$

$\boxed{\lim\limits_{x \to 2} f(x) = 4}$

but $f(2)$ undefined

④ $\lim\limits_{x \to 1} \dfrac{x^2 - 3x + 2}{x^2 + 2x} = \dfrac{\boxed{1}^2 - 3\boxed{1} + 2}{\boxed{1}^2 + 2\boxed{1}} = \dfrac{1 - 3 + 2}{1 + 2}$

$= \dfrac{-2 + 2}{3} = \dfrac{0}{3} = 0$

(5) $$\lim_{x \to 3} \frac{x^2+2x-3}{x-3} = \frac{\boxed{3}^2 + 2\boxed{3} - 3}{\boxed{3} - 3} = \frac{9+6-3}{0} = \frac{12}{0}$$

$\Leftarrow$ DNE $\Leftarrow$

2.99999 3.00001

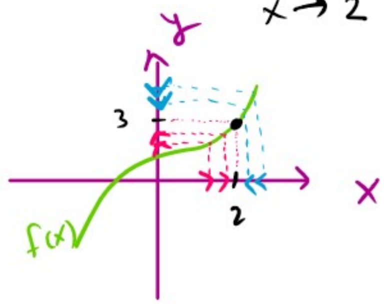
3

$$\lim_{x \to 3^+} \frac{x^2+2x-3}{x-3} = \frac{12}{\boxed{3.00001} - 3} = \frac{12}{\text{small} +} = \infty$$

$$\lim_{x \to 3^-} \frac{x^2+2x-3}{x-3} = \frac{12}{\boxed{2.99999} - 3} = \frac{12}{\text{small} -} = -\infty$$

Exp Find $\lim_{x \to 2} f(x)$ and $f(2)$ for

(1)



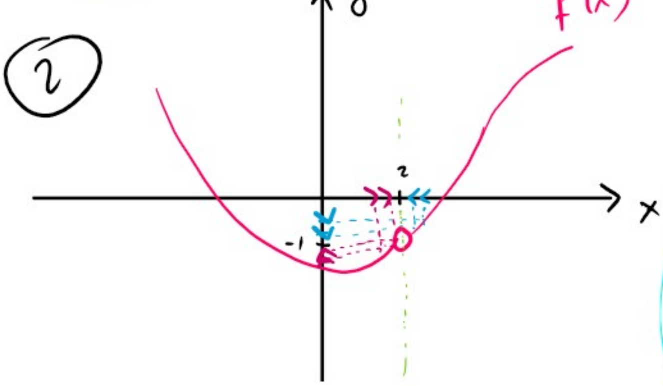
$f(2) = 3$

الاقتران متصل عند $x = 2$

$$\lim_{x \to 2^+} f(x) = 3 = \lim_{x \to 2^-} f(x) \Rightarrow \lim_{x \to 2} f(x) = 3$$

---

(2)



$f(2)$ is undefined

$$\lim_{x \to 2^+} f(x) = -1$$

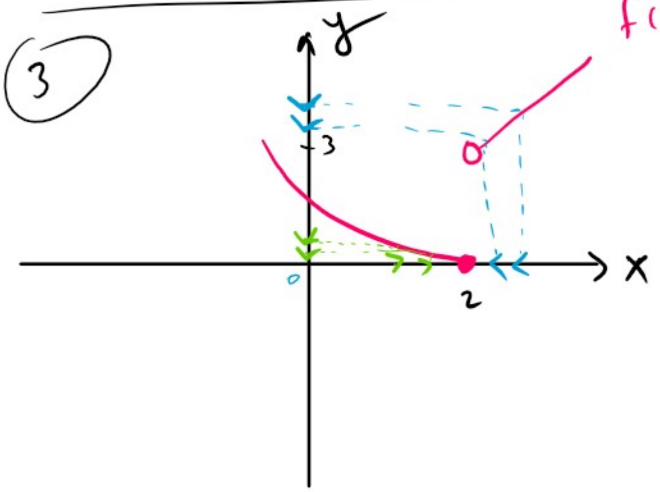
$$\lim_{x \to 2^-} f(x) = -1$$

$$\lim_{x \to 2} f(x) = -1$$

$\lim_{x \to 2^-} f(x) = $

الاقتران غير متصل عند $x=2$

③



$f(2) = 0$

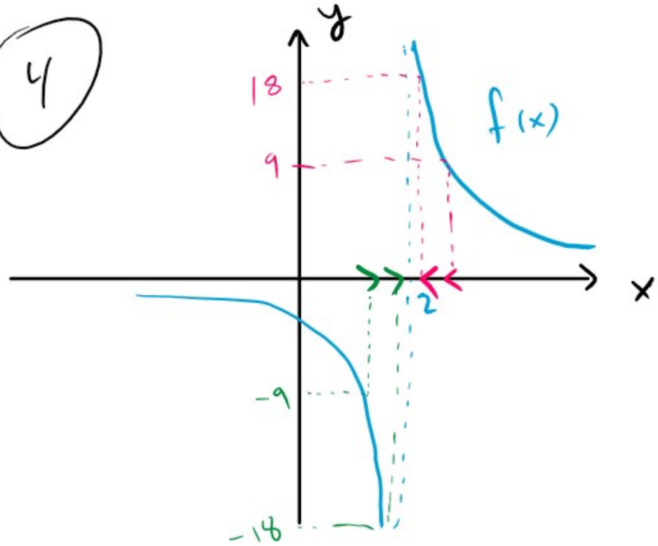
$\lim_{x \to 2^-} f(x) = 0$

$\lim_{x \to 2^+} f(x) = 3$

$\lim_{x \to 2} f(x)$ DNE

الاقتران غير متصل عند $x=2$

④



$f(2)$ is undefined

$\lim_{x \to 2^+} f(x) = +\infty$

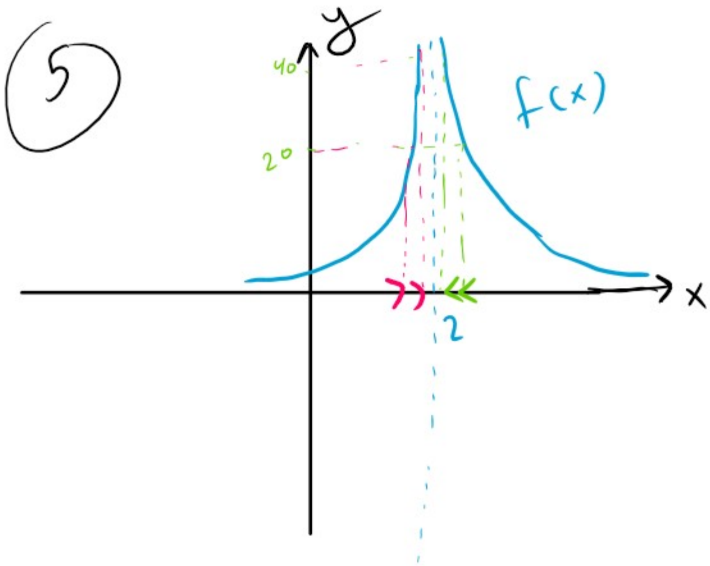
$\lim_{x \to 2^-} f(x) = -\infty$

$\lim_{x \to 2} f(x)$ DNE

الاقتران غير متصل عند $x=2$

x^-

5



f(2) is undefined

$\lim_{x\to 2^+} f(x) = +\infty$

$\lim_{x\to 2^-} f(x) = +\infty$

$\lim_{x\to 2} f(x)$ DNE

f is not cont. at $x=2$

Exp A) Can $\lim_{x\to c^-} f(x)$ exists if $f(c)$ undefined?

Yes See $\boxed{2}$ above

B) Does $\lim_{x\to c} f(x)$ exist if $f(c)=0$?

Not necessarily $\Rightarrow$ see $\boxed{3}$ above

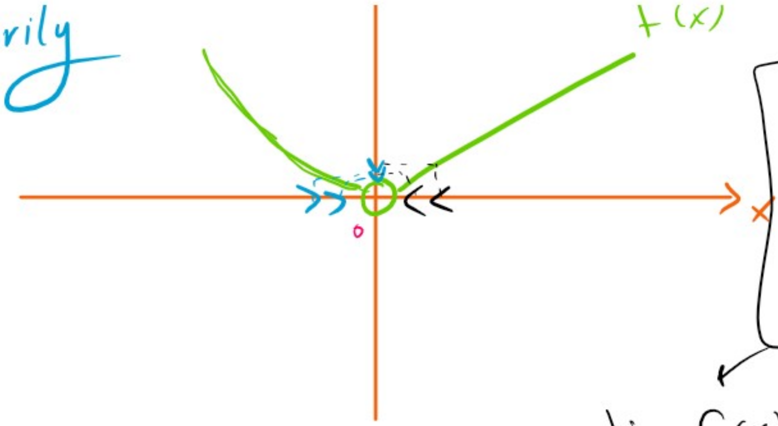
c) Does $f(c)=0$ if $\lim_{x\to c} f(x)=0$?

Not necessarily

$f(0)$ is undefined

$\lim f(x) = 0$

necessarily



$\lim_{x \to 0^-} f(x) = 0$

$\lim_{x \to 0^+} f(x) = 0$

$\lim_{x \to 0} f(x) = 0$

Remark : we may write

$f(x) \longrightarrow L$ as $x \longrightarrow c$

instead of

$\boxed{\lim_{x \to c} f(x) = L}$

Properties of limits

Assume K constant , $\lim_{x \to c} f(x) = L$, $\lim_{x \to c} g(x) = M$

Then ① $\lim_{x \to c} K = K$

Exp $\lim_{x \to -2} \sqrt{5} = \sqrt{5}$, $\lim_{x \to e} \pi = \pi$

② $\lim_{x \to c} x = c$ Exp $\lim_{x \to -\sqrt{3}} x = \boxed{-\sqrt{3}}$

③ $\lim (f(x) \pm g(x)) = \lim f(x) \pm \lim g(x) = L \pm M$

$$③ \quad \lim_{x \to c} \left(f(x) \pm g(x) \right) = \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x) = L \pm M$$

$$④ \quad \lim_{x \to c} \left(f(x)\, g(x) \right) = \left(\lim_{x \to c} f(x) \right) \left(\lim_{x \to c} g(x) \right) = L M$$

$$⑤ \quad \lim_{x \to c} \left(\frac{f(x)}{g(x)} \right) = \frac{L}{M} \quad , \ M \neq 0$$

$$⑥ \quad \lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to c} f(x)} = \sqrt[n]{L}$$

where $L > 0$ if n even

Exp ① $\displaystyle \lim_{x \to 2} \left(x^3 - x^2 + 1 \right) = \boxed{2}^3 - \boxed{2}^2 + 1$

$= 8 - 4 + 1$

$= 5$

polynomial of degree 3 — كثير الحدود من الدرجة 3

Def: The function $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ is called polynomial function of degree n where n is positive integer

---

② $\displaystyle \lim_{x \to 0} \frac{x}{x^2 - 2x} = \frac{\boxed{0}}{\boxed{0}^2 - 2\boxed{0}} = \frac{0}{0-0} = \frac{0}{0}$

rational function — اقتران نسبي

تحليل أو اشتقاق

استعاضة $\lim_{x \to 0} \frac{1}{2x-2} = \frac{1}{2\boxed{0}-2} = \frac{1}{0-2} = \frac{1}{-2}$

تحليل $\lim_{x \to 0} \frac{\cancel{x}}{\cancel{x}(x-2)} = \lim_{x \to 0} \frac{1}{x-2} = \frac{1}{\boxed{0}-2} = \frac{1}{-2}$

Def The fuction $h(x) = \frac{f(x)}{g(x)}$ where f and g are polynomials is called Rational function

---

Remark • If $f(x)$ is polynomial then $\lim_{x \to c} f(x) = f(c)$ exists for all values of c

كثيرات الحدود تكون متصلة دائماً

• If $h(x) = \frac{f(x)}{g(x)}$ is rational function

Then $\lim_{x \to c} h(x) = \frac{f(c)}{g(c)}, \quad g(c) \neq 0$

الاقترانات النسبية متصلة على جميع الأعداد ما عدا أصفار المقام

Exp ① $\lim \frac{2x-5}{\ldots} = \frac{2(1)-5}{(1)^2+(1)^3+3} = \frac{2-5}{1+1+3} = \frac{-3}{5}$

Exp ① $\lim\limits_{x\to 1} \dfrac{2x-5}{x^2+x^3+3} = \dfrac{-3}{(1)^2+(1)^3+3} = \dfrac{-3}{1+1+3} = -\dfrac{3}{5}$

② $\lim\limits_{x\to 3} \dfrac{4}{x-3} = \dfrac{4}{(3)-3} = \dfrac{4}{0}$ DNE

$\lim\limits_{x\to 3^+} \dfrac{4}{x-3} = \dfrac{4}{\boxed{3.00..1}-3} = \dfrac{4}{\text{small}+} = +\infty$

3.00001, 2.9999, 3

$\lim\limits_{x\to 3^-} \dfrac{4}{x-3} = \dfrac{4}{\boxed{2.9999}-3} = \dfrac{4}{\text{small}-} = -\infty$

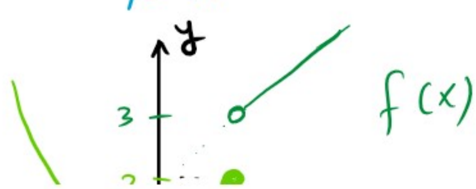
Exp $f(x) = \begin{cases} x^2+1, & \text{if } x \le 1 \\ x+2, & \text{if } x > 1 \end{cases}$

Find

① $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} (x^2+1) = (1)^2+1 = 1+1 = 2$

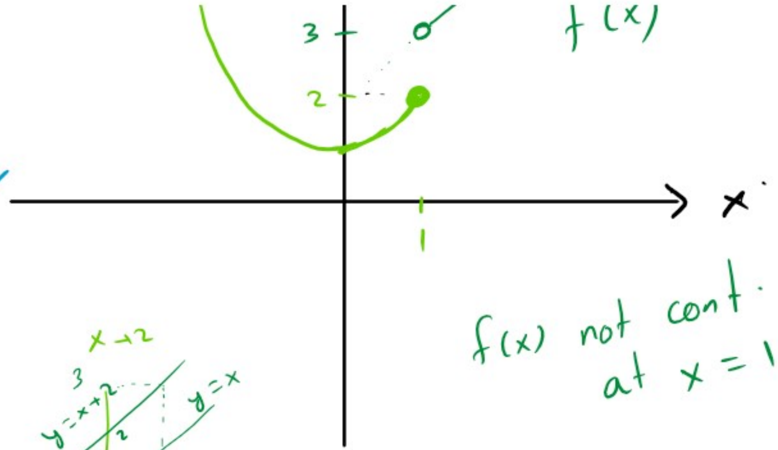
② $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} (x+2) = (1)+2 = 3$

③ $\lim\limits_{x\to 1} f(x)$ DNE since $\lim\limits_{x\to 1^-} f(x) \neq \lim\limits_{x\to 1^+} f(x)$



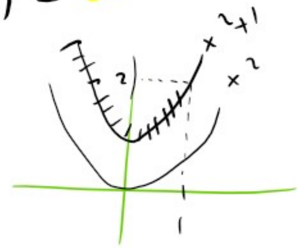
④ sketch $f(x)$

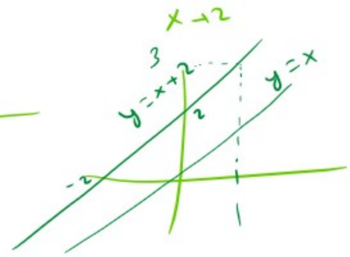
④ Sketch $f(x)$



$f(x)$ not cont. at $x = 1$

⑤ $f(1) = 2 = \boxed{1}^2 + 1$ ✓





Exp $\lim_{x \to 2} g(x) = 3$, $\lim_{x \to 2} \left(f(x) + 2g(x) \right) = 7$

① Find $\lim_{x \to 2} f(x)$

$\lim_{x \to 2} f(x) + 2 \boxed{\lim_{x \to 2} g(x)} = 7$

$\lim_{x \to 2} f(x) + 2(3) = 7$

$\lim_{x \to 2} f(x) + 6 = 7$

$\boxed{\lim_{x \to 2} f(x) = 1}$

② $\lim_{x \to 2} \left[(f(x))^2 - 5g(x) \right]$

$(\quad)^2 - 5 \left(\lim g(x) \right)$

$x \to 2$

$$\lim_{x\to 2} \left(f(x)\right)^2 - 5 \boxed{\lim_{x\to 2} g(x)}$$

$$\lim_{x\to 2} \left(f(x)\right)\left(f(x)\right) - 5(3)$$

$$\left(\lim_{x\to 2} f(x)\right)\left(\lim_{x\to 2} f(x)\right) - 15$$

$$(1)\ (1) - 15$$

$$1 - 15$$

$$-14$$