7 Curvilinear coordinates

Read: Boas sec. 5.4, 10.8, 10.9.

7.1 Review of spherical and cylindrical coords.

First I'll review spherical and cylindrical coordinate systems so you can have them in mind when we discuss more general cases.

7.1.1 Spherical coordinates

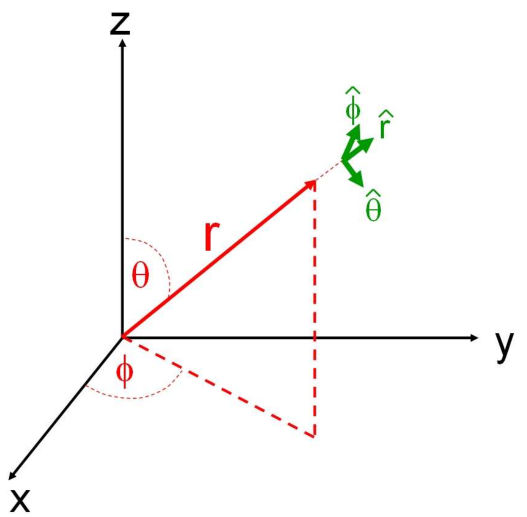

Figure 1: Spherical coordinate system.

The conventional choice of coordinates is shown in Fig. 1. θ is called the "polar angle", ϕ the "azimuthal angle". The transformation from Cartesian coords. is

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta. \tag{1}$$

In the figure the unit vectors pointing in the directions of the changes of the three spherical coordinates r, θ, ϕ are also shown. Any vector can be expressed in terms of them:

$$\begin{aligned}\vec{A} &= A_x\hat{x} + A_y\hat{y} + A_z\hat{z} \\ &= A_r\hat{r} + A_\theta\hat{\theta} + A_\phi\hat{\phi}.\end{aligned} \tag{2}$$

Note the qualitatively new element here: while both $\hat{x}, \hat{y}, \hat{z}$ and $\hat{r}, \hat{\theta}, \hat{\phi}$ are three mutually orthogonal unit vectors, $\hat{x}, \hat{y}, \hat{z}$ are fixed in space but $\hat{r}, \hat{\theta}, \hat{\phi}$ point in different directions according to the direction of vector $\vec{r}$. We now ask by how large a distance ds the head of the vector $\hat{r}$ changes if infinitesimal changes $dr, d\theta, d\phi$ are made in the three spherical directions:

$$ds_r = dr \quad , \qquad ds_\theta = r d\theta \quad , \quad ds_\phi = r \sin\theta d\phi, \tag{3}$$

as seen from figure 2 (only the $\hat{\theta}$ and $\hat{\phi}$ displacements are shown).

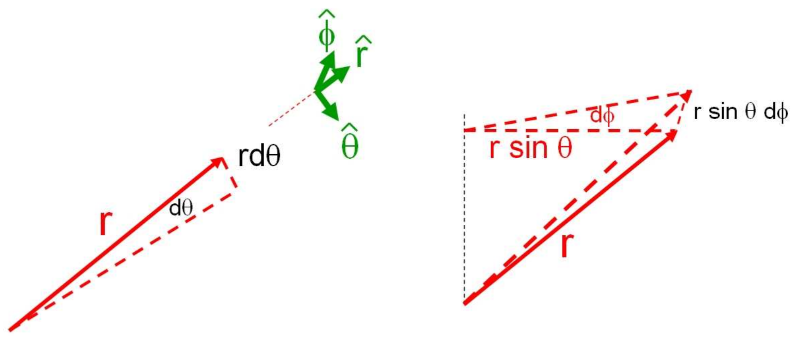

Figure 2: Geometry of infinitesimal changes of $\vec{r}$.

So the total change is

$$d\vec{s} = dr\hat{r} + r d\theta\hat{\theta} + r\sin\theta d\phi\hat{\phi}. \tag{4}$$

The volume element will be

$$d\tau = ds_r ds_\theta ds_\phi = r^2 \sin\theta \, dr \, d\theta \, d\phi, \tag{5}$$

and the surface measure at constant r will be

$$d\vec{a} = ds_\theta \, ds_\phi = \hat{r} = r^2 \sin\theta d\theta d\phi \, \hat{r}. \tag{6}$$

Ex. 1: Volume of sphere of radius R:

$$\int_{\text{sphere}} d\tau = \int_{r=0}^{R} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} r^2 \sin\theta dr \, d\theta \, d\phi = (R^3/3)(2)(2\pi) = \frac{4}{3}\pi R^3. \tag{7}$$

More interesting: gradient, etc. in spherical coordinates:

$$\begin{aligned} \vec{\nabla}\psi &= \frac{\partial\psi}{\partial x}\hat{i} + \frac{\partial\psi}{\partial y}\hat{j} + \frac{\partial\psi}{\partial z}\hat{k} \\ \frac{\partial\psi}{\partial x} &= \frac{\partial\psi}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial\psi}{\partial\theta}\frac{\partial\theta}{\partial x} + \frac{\partial\psi}{\partial\phi}\frac{\partial\phi}{\partial x}, \text{ etc.,} \end{aligned} \tag{8}$$

and $\hat{i}$, $\hat{j}$, and $\hat{k}$ can be replaced by

$$\begin{aligned} \hat{i} &= \sin\theta\cos\phi\,\hat{r} + \cos\theta\cos\phi\,\hat{\theta} - \sin\phi\,\hat{\phi} \\ \hat{j} &= \sin\theta\sin\phi\,\hat{r} + \cos\theta\sin\phi\,\hat{\theta} + \cos\phi\,\hat{\phi} \\ \hat{k} &= \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}. \end{aligned} \tag{9}$$

Combining all these we find

$$\vec{\nabla}\psi = \frac{\partial\psi}{\partial r}\,\hat{r} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\,\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\,\hat{\phi}. \tag{10}$$

Similarly, we find

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi} \tag{11}$$

and

$$\vec{\nabla}\times\vec{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\,\hat{\theta} & r\sin\theta\,\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix} \tag{12}$$

and

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} \tag{13}$$

7.1.2 Cylindrical coordinates

I won't belabor the cylindrical coordinates, but just give you the results to have handy. I've written here the cylindrical radial coordinate as called r, the angle variable θ, like Boas, but keep in mind that a lot of books use ρ and ϕ.

$$\begin{aligned}
x &= r\cos\theta \quad ; \quad y = r\sin\theta \quad ; \quad z = z \\
ds_r &= dr \quad ; \quad ds_\theta = rd\theta \quad ; \quad ds_z = dz \\
d\vec{\ell} &= dr\,\hat{r} + rd\theta\,\hat{\theta} + dz\,\hat{z} \\
d\tau &= rdrd\theta dz \\
\vec{\nabla}\psi &= \frac{\partial\psi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{\theta} + \frac{\partial\psi}{\partial z}\hat{z} \\
\vec{\nabla}\cdot\vec{A} &= \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial\theta} + \frac{\partial A_z}{\partial z} \\
\vec{\nabla}\times\vec{A} &= \frac{1}{r}\begin{vmatrix} \hat{r} & r\hat{\theta} & \hat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ A_r & rA_\theta & A_z \end{vmatrix}
\end{aligned} \tag{14}$$

7.1.3 General coordinate systems

With these specific examples in mind, let's go back the the general case, and see where all the factors come from. We can pick a new set of coordinates q_1, q_2, q_3, which have isosurfaces which need not be planes nor parallel to each other. Let's just assume that among x, y, z and q_1, q_2, q_3 there are some relations

$$x = x(q_1, q_2, q_3) \quad ,; \quad y = y(q_1, q_2, q_3) \quad ; \quad z = z(q_1, q_2, q_3) \tag{15}$$

which we can find and invert to get

$$q_1 = q_1(x, y, z) \;\; ; \quad q_2 = q_2(x, y, z) \;\; ; \quad q_3 = q_3(x, y, z) \tag{16}$$

The differentials are then

$$dx = \frac{\partial x}{\partial q_1} dq_1 + \frac{\partial x}{\partial q_2} dq_2 + \frac{\partial x}{\partial q_3} dq_3 \tag{17}$$

It's very useful to know what the measure of distance, or *metric*, is in a given coordinate system. Of course in Cartesian coordinates, the distance between two points whose coordinates differ by dx, dy, dz is ds, where

$$ds^2 = dx^2 + dy^2 + dz^2. \tag{18}$$

Your book calls ds the *arc length*. Now if you imagine squaring an equation like (17), you'll get terms like dq_1^2, but also terms like $dq_1 dq_2$, etc. So in general, plugging into (18) we expect

$$ds^2 = g_{11} dq_1^2 + g_{12} dq_1 dq_2 + ... = \sum_{ij} g_{ij} dq_i dq_j, \tag{19}$$

and the g_{ij} are called the *metric components* (and g itself is the *metric tensor*). In Einstein's theory of general relativity, the metric components depend on the amount of mass nearby!

Most of the coordinate systems we are interested in are *orthogonal*, i.e. $g_{ij} \propto \delta_{ij}$. Thus we can write

$$ds^2 = (h_1\, dq_1)^2 + (h_2\, dq_2)^2 + (h_3\, dq_3)^2. \tag{20}$$

The h_i's are called *scale factors*, and are 1 for Cartesian coordinates.

Now let's look at the change of the position vector $\vec{r}$, in our new coordinate system, when we change the coordinates q_i by a small amount. We have

$$d\vec{r} = d\vec{s} = h_1 dq_1\, \hat{q}_1 + h_2 dq_2\, \hat{q}_2 + h_3 dq_3\, \hat{q}_3. \tag{21}$$

We can define the distance changes s_1, s_2, and s_3 by $s_i \equiv h_i q_i$. Let's make contact with something concrete by comparing with, say, spherical coordinates. The infinitesimal change in the position vector is what's given in (4), so we can identify the scale factors for spherical coordinates as $h_r = 1$, $h_\theta = r$, and $h_\phi = r\sin\theta$.

Since we know how to express $d\vec{r}$ now, we can immediately say how to do line elements for line integrals,

$$\int \vec{v} \cdot d\vec{r} = \sum_i \int v_i h_i dq_i, \tag{22}$$

as well as surface and volume integrals:

$$\begin{aligned}\int \vec{v} \cdot d\vec{a} &= \int v_1 ds_2 ds_3 + \int v_2 ds_1 ds_3 + \int v_3 ds_1 ds_2 \\ &= \int v_1 h_2 h_3 dq_2 dq_3 + \int v_2 h_1 h_3 dq_1 dq_3 + \int v_3 h_1 h_2 dq_1 dq_2 \end{aligned} \tag{23}$$

and

$$\int d\tau ... = \int h_1 h_2 h_3 dq_1 dq_2 dq_3 ..., \tag{24}$$

where the ... stands for the integrand.

Differential operators in curvilinear coordinates. I am not going to develop all of this here; it's pretty tedious, and is discussed in Boas secs. 9.8 and 9.9. However the basic idea comes from noting that the gradient is the fastest change of a scalar field, so the q_1 component is obtained by dotting into $\hat{q}_1$, i.e.

$$\hat{q}_1 \cdot \vec{\nabla}\psi = \frac{\partial \psi}{\partial s_1} = \frac{1}{h_1}\frac{\partial \psi}{\partial q_1}, \tag{25}$$

etc. Note that we are allowed to do the last step because h_1 is a function of q_2 and q_3, but these are held constant during the partial differentiation. Therefore:

$$\vec{\nabla}\psi = \sum_i \hat{q}_i \frac{1}{h_i}\frac{\partial \psi}{\partial q_i}. \tag{26}$$

Similarly

$$\vec{\nabla} \cdot \vec{v} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}(v_1 h_2 h_3) + \frac{\partial}{\partial q_2}(v_2 h_1 h_3) + \frac{\partial}{\partial q_3}(v_3 h_1 h_2)\right] \tag{27}$$

and

$$\vec{\nabla} \times \vec{v} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} \hat{q}_1 h_1 & \hat{q}_2 h_2 & \hat{q}_3 h_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1 v_1 & h_2 v_2 & h_3 v_3 \end{vmatrix} \tag{28}$$