

11.3 Polar Coordinates

* $P(r, \theta)$

r : Directed distance from O origin (pole) to P

θ : Directed angle from initial ray to OP

* Equation Graph

$r = a$ circle of radius $|a|$ centered at O

$\theta = \theta_0$ line through O making an angle θ_0 with the initial ray.

* Equations Relating Polar and Cartesian Coordinates.

$$x = r \cos \theta, \quad y = r \sin \theta, \quad x^2 + y^2 = r^2$$

$$\tan \theta = \frac{y}{x}$$

1] Which polar coordinate pairs label the same point?

a) $(3, 0)$ b) $(-3, 0)$ c) $(2, \frac{2\pi}{3})$

d) $(2, \frac{7\pi}{3})$ e) $(-3, \pi)$ f) $(2, \frac{\pi}{3})$

g) $(-3, 2\pi)$ h) $(-2, -\frac{\pi}{3})$

$$(3, 0) \leftrightarrow (-3, \pi)$$

$$(-3, 0) \leftrightarrow (-3, 2\pi)$$

$$(2, \frac{2\pi}{3}) \leftrightarrow (-2, -\frac{\pi}{3})$$

$$(2, \frac{7\pi}{3}) \leftrightarrow (2, \frac{\pi}{3})$$

6] Find the Cartesian coordinates of the following points

a) $(\sqrt{2}, \frac{\pi}{4})$

$$\left. \begin{aligned} x &= \sqrt{2} \cos \frac{\pi}{4} = \sqrt{2} \cdot \frac{1}{\sqrt{2}} = 1 \\ y &= \sqrt{2} \sin \frac{\pi}{4} = \sqrt{2} \cdot \frac{1}{\sqrt{2}} = 1 \end{aligned} \right\} \rightarrow (1, 1)$$

b) $(1, 0)$

$$\left. \begin{aligned} x &= 1 \cos 0 = 1 \\ y &= 1 \sin 0 = 0 \end{aligned} \right\} \rightarrow (1, 0)$$

c) $(0, \frac{\pi}{2})$

$$\left. \begin{aligned} x &= 0 \cos(\frac{\pi}{2}) = 0 \\ y &= 0 \sin(\frac{\pi}{2}) = 0 \end{aligned} \right\} \rightarrow (0, 0)$$

d) $(-\sqrt{2}, \frac{\pi}{4})$

$$\left. \begin{aligned} x &= -\sqrt{2} \cos \frac{\pi}{4} = -1 \\ y &= -\sqrt{2} \sin \frac{\pi}{4} = -1 \end{aligned} \right\} \rightarrow (-1, -1)$$

e) $(-3, \frac{5\pi}{6})$

$$\left. \begin{aligned} x &= -3 \cos(\frac{5\pi}{6}) = -3 \cdot \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2} \\ y &= -3 \sin(\frac{5\pi}{6}) = -3 \cdot \frac{1}{2} = -\frac{3}{2} \end{aligned} \right\} \rightarrow (\frac{3\sqrt{3}}{2}, -\frac{3}{2})$$

f) $(5, \tan^{-1}(\frac{4}{3}))$ ↙
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$$\left. \begin{aligned} x &= 5 \cos(\tan^{-1} \frac{4}{3}) = 5(0.6) = 3 \\ y &= 5 \sin(\tan^{-1} \frac{4}{3}) = 5(0.79968) \approx 4 \end{aligned} \right\} (3, 4)$$

g) $(-1, 7\pi)$

$$\begin{aligned} x &= -\cos(7\pi) = -(-1) = 1 \\ y &= -\sin(7\pi) = 0 \end{aligned} \quad \left. \vphantom{\begin{aligned} x &= -\cos(7\pi) = -(-1) = 1 \\ y &= -\sin(7\pi) = 0 \end{aligned}} \right\} \rightarrow (1, 0)$$

b) $(2\sqrt{3}, \frac{2\pi}{3})$

$$\begin{aligned} x &= 2\sqrt{3} \cos\left(\frac{2\pi}{3}\right) = 2\sqrt{3} \left(-\frac{1}{2}\right) = -\sqrt{3} \\ y &= 2\sqrt{3} \sin\left(\frac{2\pi}{3}\right) = 2\sqrt{3} \left(\frac{\sqrt{3}}{2}\right) = 3 \end{aligned} \quad \left. \vphantom{\begin{aligned} x &= 2\sqrt{3} \cos\left(\frac{2\pi}{3}\right) = 2\sqrt{3} \left(-\frac{1}{2}\right) = -\sqrt{3} \\ y &= 2\sqrt{3} \sin\left(\frac{2\pi}{3}\right) = 2\sqrt{3} \left(\frac{\sqrt{3}}{2}\right) = 3 \end{aligned}} \right\} \rightarrow (-\sqrt{3}, 3)$$

[7] Find the polar coordinates, $0 \leq \theta < 2\pi$, $r \geq 0$ of the following points

a) $(1, 1)$

$$r = \sqrt{1^2 + 1^2} = \sqrt{2}$$

~~$\tan \theta = \frac{1}{1} \rightarrow \theta = \frac{\pi}{4}$~~

$$\sin \theta = \frac{1}{\sqrt{2}}, \quad \cos \theta = \frac{1}{\sqrt{2}} \rightarrow \theta = \frac{\pi}{4}$$

$$\rightarrow \left(\sqrt{2}, \frac{\pi}{4}\right)$$

b) $(-3, 0)$, $r = \sqrt{(-3)^2 + 0^2} = 3$

$$\sin \theta = 0, \quad \cos \theta = -1 \rightarrow \theta = \pi$$

$$\rightarrow (3, \pi)$$

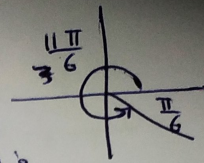
c) $(\sqrt{3}, -1)$

$$r = \sqrt{(\sqrt{3})^2 + (-1)^2} = 2$$

$$\sin \theta = -\frac{1}{2}, \quad \cos \theta = \frac{\sqrt{3}}{2} \rightarrow \theta = \frac{11\pi}{6}$$

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$$\rightarrow (2, \frac{11\pi}{6})$$



d) $(-3, 4)$

$$r = \sqrt{(-3)^2 + (4)^2} = 5$$

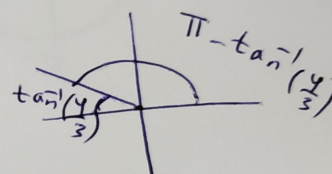
$$\sin \theta = \frac{4}{5}, \quad \cos \theta = -\frac{3}{5}$$

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$$\tan \theta^* = \frac{4}{3} \rightarrow \theta^* = \tan^{-1}\left(\frac{4}{3}\right)$$

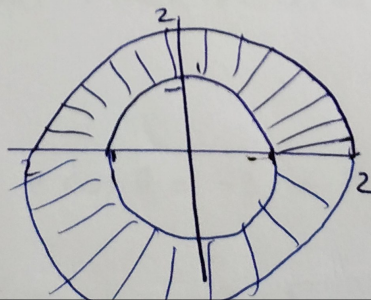
$$\rightarrow (5, \pi - \tan^{-1}\left(\frac{4}{3}\right))$$

note that $\tan^{-1}\left(\frac{4}{3}\right) \approx 59^\circ$

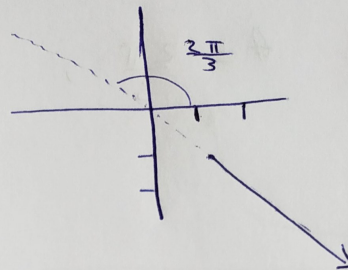


Graph the sets of points whose polar coordinates satisfy the equations and inequalities:-

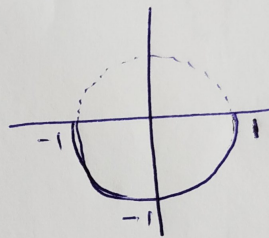
14) $1 \leq r \leq 2$



16 $\theta = \frac{2\pi}{3}, r \leq -2$

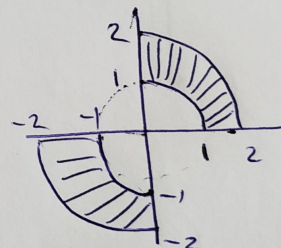


22 $0 \leq \theta \leq \pi, r = -1$



26 $0 \leq \theta \leq \frac{\pi}{2}, 1 \leq |r| \leq 2$

$1 \leq r \leq 2$ or $-2 \leq r \leq -1$

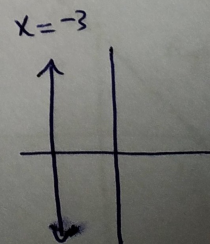


32 Replace the polar equations with equivalent Cartesian equations. Then describe the graph.

$r = -3 \sec \theta$

$r = \frac{-3}{\cos \theta} \rightarrow r \cos \theta = -3 = x$

vertical line through $(-3, 0)$

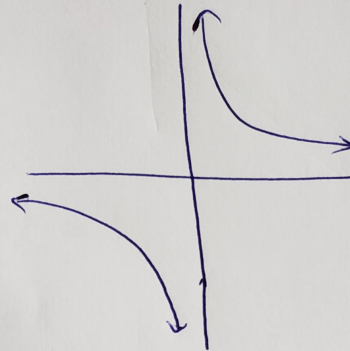


38 $r^2 \sin 2\theta = 2$

$$2r^2 \sin\theta \cos\theta = 2$$

$$(r \sin\theta)(r \cos\theta) = 1$$

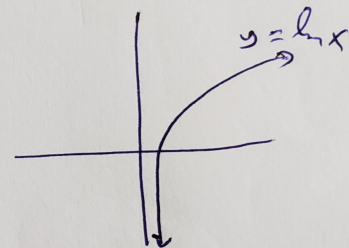
$$xy = 1 \rightarrow y = \frac{1}{x}$$



42 $r \sin\theta = \ln r + \ln \cos\theta$

$$r \sin\theta = \ln r \cos\theta$$

$$y = \ln x$$



52 $r \sin\left(\frac{2\pi}{3} - \theta\right) = 5$

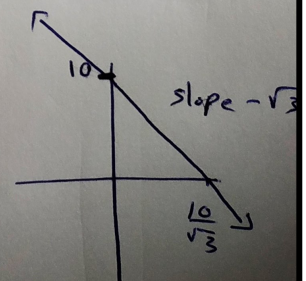
$$r \left[\sin\left(\frac{2\pi}{3}\right) \cos\theta - \cos\left(\frac{2\pi}{3}\right) \sin\theta \right] = 5$$

$$r \left[\frac{\sqrt{3}}{2} \cos\theta + \frac{1}{2} \sin\theta \right] = 5$$

$$\frac{\sqrt{3}}{2} r \cos\theta + \frac{1}{2} r \sin\theta = 5$$

$$\frac{\sqrt{3}}{2} x + \frac{1}{2} y = 5$$

$$\sqrt{3} x + y = 10$$



$$\boxed{62} \quad x^2 + xy + y^2 = 1$$

$$\underbrace{x^2 + y^2} + xy = 1$$

$$r^2 + (r \cos \theta)(r \sin \theta) = 1$$

$$r^2 + r^2 \cos \theta \sin \theta = 1$$

$$r^2(1 + \cos \theta \sin \theta) = 1$$