Singularity Signals

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Singularity signals

Singularity signals are a set of signals generated by recursion using a recursive generator and a base case (like n!=n(n-1)! and base case 0!=1).

The Integral generator is given by $u_{-k} = \int_{-\infty}^{t} u_{-k+1}(\sigma)d\sigma$

The derivative generator is given by $u_k = \frac{du_{k-1}(t)}{dt}$

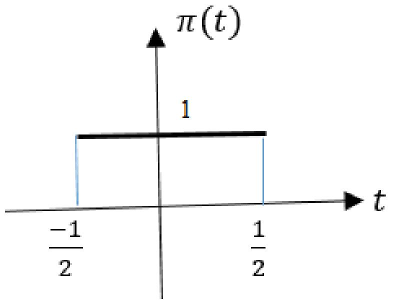
The base case is a special functional called Dirac impulse $\delta(t)$ with $\begin{cases} \delta(t) = 0 \quad for \ \ t \neq 0 \\ \int_{-\infty}^{\infty} \delta(t)dt = 1 \end{cases}$

Derivation of $\delta(t)$

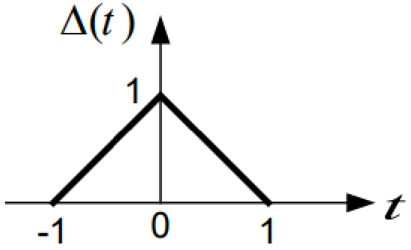
$\delta(t)$ is not a function, it is a functional signal that can be derived as the limit of different types of signals.

Consider the following important signals in the signals and systems domain with area =1:

- Finite impulse $\pi(t) = \begin{cases} 1 \quad for \ |t| < \frac{1}{2} \\ 0 \quad for \ |t| > \frac{1}{2} \end{cases}$



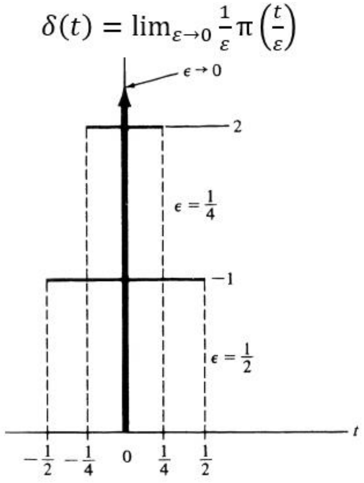
- Triangular signal $\Delta(t) \begin{cases} 1 - |t| \quad for \ |t| \leq 1 \\ 0 \quad for \ |t| > 1 \end{cases} =$ `

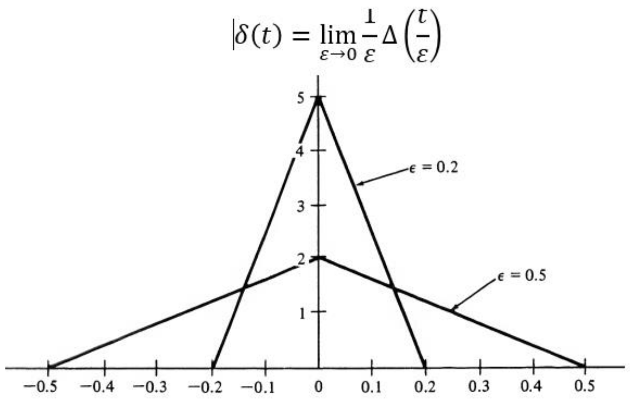


$\delta(t)$ can be obtained as the limit of a scaled version of both $\pi(t)$ and $\Delta(t)$:

$$\delta(t) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\pi\left(\frac{t}{\varepsilon}\right)$$

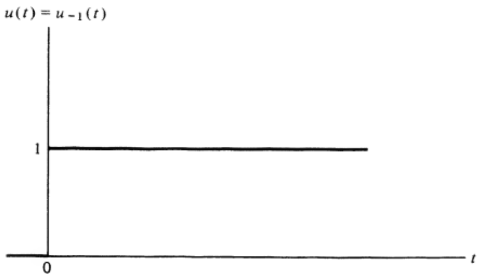
$$\delta(t) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Delta\left(\frac{t}{\varepsilon}\right)$$





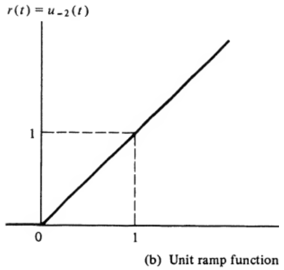
<u>Important Singularity Signals</u>:

- Unit step signal $u(t) = u_{-1} = \int_{-\infty}^{t} u_0(\sigma)d\sigma = \int_{-\infty}^{t}\delta(\sigma)d\sigma = \begin{cases}1 & for\ t > 0\\ 0 & for\ t < 0\end{cases}$



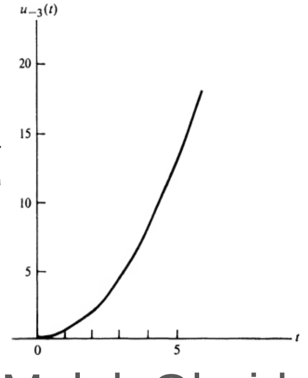
(a) Unit step function

- Ramp signal $r(t) = u_{-2} = \int_{-\infty}^{t} u_{-1}(\sigma)d\sigma = \int_{-\infty}^{t} u(\sigma)d\sigma = tu(t) = \begin{cases}t & for\ t \geq 0\\ 0 & for\ t \leq 0\end{cases}$



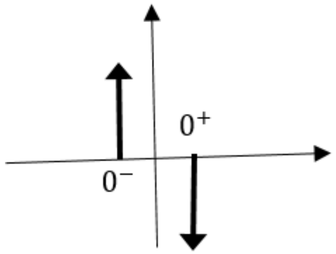
(b) Unit ramp function

- Parabolic signal $p(t) = u_{-3} = \int_{-\infty}^{t} u_{-2}(\sigma)d\sigma = \int^{t} r(\sigma)d\sigma = \frac{1}{2}t^2u(t) = \begin{cases}\frac{1}{2}t^2 & for\ t \geq 0\\ 0 & for\ t \leq 0\end{cases}$



(c) Unit parabolic function

- Kronocker impulse ck(t) = $\delta'(t)$



Properties of $\delta(t)$

1. Point property of $\delta(t)$: if $x(t)$ is continuous at $t = t_0, then \quad x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$
2. Sampling property of $\delta(t)$: if $x(t)$ is continuous at $t = t_0 then \int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = x(t_0)$
3. Scaling property of $\delta(t)$: $\delta(\alpha t) = \frac{1}{|\alpha|}\delta(t)$

Proof: $\int_{-\infty}^{\infty} \delta(\alpha t)dt$, by substitution of variables $t' = \alpha t \rightarrow \begin{cases} \alpha > 0 \quad t \rightarrow \pm\infty, t' \rightarrow \pm\infty \\ \frac{1}{|\alpha|}dt' = dt \\ \alpha < 0, \alpha = -|\alpha|, \quad t \rightarrow \infty, t' \rightarrow -\infty \\ 0 \quad t \rightarrow -\infty, t' \rightarrow \infty \\ -\frac{1}{|\alpha|}dt' = dt \end{cases}$

$\int_{=\infty}^{\infty} \delta(\alpha t)dt$=$\int_{-\infty}^{\infty} \delta(t')\frac{1}{|\alpha|}dt' = \int_{+\infty}^{-\infty} \delta(t') \cdot -\frac{1}{|\alpha|}dt' \rightarrow \delta(\alpha t) = \frac{1}{|\alpha|}\delta(t)$

4. *even symmetry property of* $\delta(t)$: applying $\alpha = -1$ *in the scaling property we obtain* $\delta(-t) = \frac{1}{|-1|}\delta(t) = \delta(t)$.
5. Convolution property of $\int_{-\infty}^{\infty} x(\sigma)\delta(t-\sigma)d\sigma = x(t)$

an important implication is that applying the Dirac impulse at the input of a system with unkown transfer relation *we can obtain* the transfer relation at the output of the system.

Proof: consider the sampling property and then substitute t by σ and t_0 by t we obtain $\int_{-\infty}^{\infty} x(\sigma)\delta(\sigma - t)d\sigma = x(t)$

Now applying the even property of $\delta(t)$ we obtain the result $\int_{-\infty}^{\infty} x(\sigma)\delta(t-\sigma)d\sigma = x(t)$

6. Interval property of $\delta(t)$: $\int_{t_1}^{t_2} x(t)\delta(t-t_0)dt = \begin{cases} 0 \; if \quad t_0 \notin \;]t_1, \; t_2[\\ x(t_0) if \quad t_0 \in \;]t_1, \; t_2[\end{cases}$ the sampling property is valid for any interval that includes $\delta(t)$

Example: $\int_{-1}^{5} t^2\delta(t-6)dt = 0, \int_{-1}^{5} t^2\delta(t-2)dt = 2^2 = 4$

8. Point differentiation property of $\delta(t-t_0)$: the n[th] derivatives of $\delta(t-t_0)$derives an n-times differentiable signal at the point t_0 , that is $\int_{t_1}^{t_2} x(t)\delta^{(n)}(t-t_0)dt$

$$= \begin{cases} 0 \;\; if \quad t_0 \notin \;\;]t_1, \quad t_2[\\ (-1)^n \frac{d^n x(t)}{dt^n}\Big|_{t=\;\;t_0} \;\; if \quad t_0 \in \;\;]t_1, \quad t_2[\end{cases}$$

<u>Example:</u> $\int_1^7 t^3\delta^{(2)}(t-10)dt = 0,\;\; \int_1^7 t^3\delta^{(2)}(t-3)dt. = 6t|_{t=3},\; = 18$

<u>Exercise: Prove the differentiation theorem by induction.</u>

<u>Induction Method:</u>

1. Prove true for k=1.
2. Assume true for k=n-1.
3. Prove true for k=n-> true statement.

<u>Generalized property of polynomials for the singularity signals:</u>

The following identity equality holds if and only if $\alpha_k = \beta_k \quad \forall k$

$$\sum_{-n}^{n} \alpha_k\, u_k(t) = \sum_{-n}^{n} \beta_k\, u_k(t)$$

<u>Example:</u> $Ar(t) + 4u(t) - 2\delta(t) + B\delta'(t) + 5\delta^{(2)}(t) = 7r(t) + Cu(t) - D\delta(t) + E\delta^{(2)}(t) \leftrightarrow$

A=7, B=0, C=4, D=2, and E=5

Power and Energy Signals

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Energy and Power Signals: In electrical circuits, a signal may represent voltage or a current. Consider a voltage $v(t)$ developed across a resistor R , producing a current $i(t)$.

1. The instantaneous power $p(t)$ dissipated in R is defined by

$$p(t) = |v(t)|^2 / R$$
$$= R\,|i(t)|^2$$

For $R = 1$, we can write $p(t) = |v(t)|^2 = |i(t)|^2$.

Therefore, regardless whether a given signal $x(t)$ represents a voltage or a current, we may express $p(t)$ associated the signal $x(t)$ as

$$p(t) = x^2(t).$$

Definition (Energy Signal) : A signal $x(t)$ is said to be an energy signal with energy $\leftrightarrow E = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt < \infty$

Definition (Power Signal) : A signal $x(t)$ is said to be a power signal with average power $P_{av} \leftrightarrow P_{av} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt < \infty$

Important facts

- A signal that is not energy or power is said to be neither energy nor power.
- $E \text{ and } P_{av}$ are positive because they are the integration of positive functions (magnitude squared).
- An energy signal has $P_{av} = 0$ (infinite of order zero with respect to T) $>$
- A power signal has $E \to \infty$ (infinite of order one with respect to T)

Example: determine if the signal $x(t) = Ae^{\alpha t}u(t)$ *with* $\alpha \in R$ *is energy, power, or neither energy nor power*.
The signal has 3 cases (has 3 different behaviors) with respect to the parameter α:

- $\alpha < 0$

  Energy Test: $E = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt = \lim_{T\to\infty} \int_{-T}^{T} |Ae^{\alpha t}u(t)|^2 dt$

  $= \lim_{T\to\infty} \int_{0}^{T} |Ae^{\alpha t}|^2 dt = \lim_{T\to\infty} \int_{0}^{T} A^2 e^{2\alpha t} dt = A^2 \lim_{T\to\infty} \frac{e^{2\alpha t}}{2\alpha} \Big|_0^T = -\frac{1}{2\alpha} < \infty$, Therefore the signal is Energy signal

- $\alpha = 0$

  $E = \lim_{T\to\infty} \int_{0}^{T} A^2 dt = A^2 \lim_{T\to\infty} T \to \infty$, Therefore the signal is not an Energy signal, then we have to test if the signal is a power signal (we can not say that it is a power signal if it is not energy)

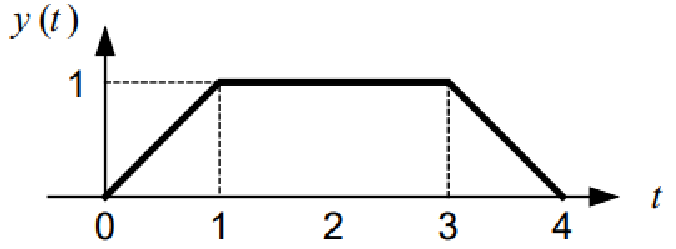
  $P_{av} = \lim_{T\to\infty} \frac{1}{2T} \int_{0}^{T} A^2 dt = A^2 \frac{T}{2T} = \frac{A^2}{2}$, Therefore the signal is a power signal.

- $\alpha > 0$

  $P_{av} = \lim_{T\to\infty} \frac{1}{2T} \int_{0}^{T} A^2 e^{2\alpha t} dt = A^2 \lim_{T\to\infty} \frac{1}{2T} \frac{e^{2\alpha t}}{2\alpha} \Big|_0^T \to \infty$, neither power nor energy, *because the exponential with positive exponent* is *the highest infinite*.

Theorem I: A time limited and bounded signal is an energy signal (a sufficient but not necessary condition).

Example: The total energy can be computed as sum of the **three orthogonal signals**



$x(t) = r(t)\pi\left(t - \frac{1}{2}\right) + \pi\left(\frac{t-2}{2}\right) + r(-t+4)\pi\left(t + \frac{7}{2}\right)$, *hence* its energy is

$$\text{E} = \int_0^1 t^2 dt + \int_1^3 1^2 dt + \int_3^4 (-(t-4))^2 dt$$

Exercise: compute the integration.

Theorem II: A periodic bounded signal with fundamental period T_0 is a power signal with $P_{av} = \frac{1}{T_0}\int_{T_0} |x(t)|^2 dt$ (a sufficient but not necessary condition). That is the power is computed as $\frac{E_p}{T_0}$ with E_p the energy of one period.

Proof: A periodic signal can be written as sum of an infinite number of energy signals that is $x(t) = \sum_{k=-\infty}^{\infty} x_p(t + kT_0)$ hence, the energy of N periods is NE_p and the average power is can be computed as $\lim_{N\to\infty} \frac{NE_p}{NT_0} = \frac{E_p}{T_0}$

Example: determine if the sinusoidal signal $x(t) = Acos(\omega_0 t + \varphi)$ is energy, power, or neither energy nor power, and in case it is energy or power compute its value.

Solution: According to theorem II the signal is a power signal because it is a periodic bounded signal. Therefore,

$$P_{av} = \frac{1}{T_0}\int_{T_0} A^2 cos^2(\omega_0 t + \varphi)dt = \frac{1}{T_0}\int_{T_0} \frac{A^2}{2}(1 + cos(2(\omega_0 t + \varphi)))dt = \frac{1}{T_0}\int_{T_0} \frac{A^2}{2} dt = \frac{A^2}{2}$$

(note: $cos(2(\omega_0 t + \varphi)$ is an alternating signal with period $\frac{T_0}{2}$, therefore its integration over integer number of periods =0

Theorem III power/energy of orthogonal signals:

Let $x(t)$ be a power/energy signal defined as the combination of n power/energy **orthogonal** signals, then the average power/energy of $x(t)$ is obtained as the sum of the average power/energy of the signals that compose $x(t)$ that is:

$P_{av\_x} = \sum_{i=1}^{n} P_{av\_x_i}$ (*power signal*) $E_x = \sum_{i=1}^{n} E_{x_i}$ (Energy signal)

Definition: Two real signals $x_1(t), and\ x_2(t)$ are said to be orthogonal on the interval $[a, b] \leftrightarrow \int_a^b x_1(t) \cdot x_2(t)dt = 0$

Example1: determine if $x_1(t) = r(t)\pi\left(t - \frac{1}{2}\right)$ and $\mathbf{2}\pi\left(\frac{t-2}{2}\right)$ are energy signals and if they are orthogonal (plot the signals).

The signals are energy signals because they are time limited and bounded, moreover they are orthogonal on $[-\infty, \infty]$ because $\int_{-\infty}^{\infty} x_1(t) \cdot x_2(t)dt = \int_{-\infty}^{\infty} 0dt = 0$

Example2: determine if the signals $x_1(t) = cos(n\omega_0 t)$ and $x_2(t) = cos(m\omega_0 t)$.
$with\ n\ and\ m\ integers\ are\ orthogonal.$

The signals are power signals since they are periodic and bounded.

Solution: orthogonal on$[a, b]\ if\ \exists$ an interval $[a, b] on\ which\ \int_a^b x_1(t) \cdot x_2(t)dt = 0$

$$\int_{T_0} cos(n\omega_0 t) \cdot cos(m\omega_0 t)dt = \frac{1}{2}\int_{T_0} \cos\big((n+m)\omega_0 t\big)\, dt + \frac{1}{2}\int_{T_0} \cos\big((n-m)\omega_0 t\big)\, dt =$$

$$\begin{cases} 0 & for\ n \neq m\ \ both\ signals\ are\ alternating \\ \frac{T_0}{2} & when\ n = m\ the\ first\ signal\ is\ alternating\ and\ the\ second = 1 \end{cases}$$

Therefore the signals are orthogonal on $\boldsymbol{T_0}$.

Exercise1: Determine if the signals $x_1(t) = sin(n\omega_0 t)$ and $x_2(t) = sin(m\omega_0 t)$ $with\ n\ and\ m\ integers\ are\ orthogonal.$

Determine if the signals $x_1(t) = sin(n\omega_0 t)$ and $x_2(t) = cos(m\omega_0 t)$ $with\ n\ and\ m\ integers\ are\ orthogonal.$

Hint: use the relative trigonometric formulas on the sinusoidal functions product.

Exercise2: determine **if the signals** $x_1(t) = cos(8\pi t)$ and $x_2(t) = cos(12\pi t)\ are\ orthogonal.$

determine **if the signals** $x_1(t) = cos(8\pi t)$ and $x_2(t) = cos\left(12\pi t + \frac{\pi}{3}\right)\ are\ orthogonal.$

Exercise3: Determine the average power of the signal $x(t) = x_1(t) + x_2(t)$ for both cases of Exercise2.

Energy and power spectral density functions: (more detailed inspection will be done in the Fourier Series and Transform chapters)

Definition: a positive spectral function $G(f)$ is said to be the **energy spectral density function** of the energy signal $x(t) \leftrightarrow E = \int_{-\infty}^{\infty} G(f)df$

Definition: a positive spectral function $S(f)$ is said to be the **power spectral density function** of the power signal $x(t) \leftrightarrow P_{av} = \int_{-\infty}^{\infty} S(f)df$

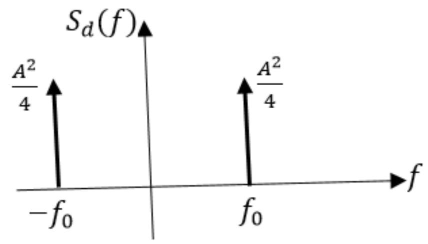
Example: write the power spectral density function of the sinusoidal signal $x(t) = Acos(\omega_0 t + \varphi)$.

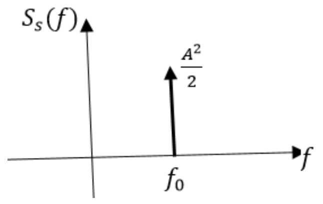
Solution: The signal is a power signal with $\boldsymbol{P_{av} = \frac{A^2}{2}}$, therefore $S(f)$ should include a Dirac impulse because it is the only signal which has a finite integration value for a spectral function defined at one point.

The single sided power spectral density function $S_s(f)$ of $x(t)$ is defined as : $S_s(f) = \frac{A^2}{2}\delta(f - f_0)$

The double sided power spectral density function $S_d(f)$ of $x(t)$ is defined as : $S_d(f) = \frac{A^2}{4}\delta(f - f_0) + \frac{A^2}{4}\delta(f + f_0)$

Exercise: Plot the single and double sided power spectral representations of $x(t) = 10cos\left(20\pi t + \frac{\pi}{3}\right) + 15cos\left(40\pi t + \frac{\pi}{4}\right)$





Signal Expression and plot

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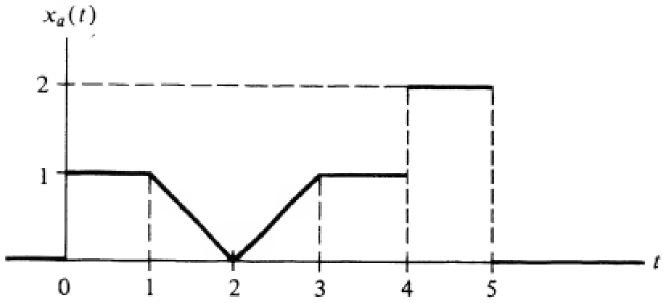
Signal Expression:

- **Windowing:** dividing the signal into its different parts using a finite pulse function based on the breaking points at which the signal changes its behavior. The window $\pi(t) = \mathrm{u}\left(\mathrm{t}+\frac{1}{2}\right) - u\left(t-\frac{1}{2}\right) = \mathrm{u}\left(\mathrm{t}+\frac{1}{2}\right) \cdot u(-t+\frac{1}{2})$ expressed by the singularity signal $u(t)$ is shifted and scaled based on the breaking points.
- **Combination:** Signals are written as the combination of different signals that combined at the breaking points generate the signal segment between the considered breaking points.

Example1: Express the following signal using elementary signals:



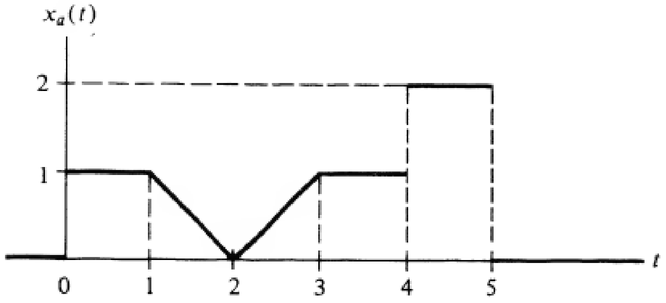
- Windowing: the signal is divided into five segment based on the breaking poi

$t \in [0,1] \to \pi\left(t-\frac{1}{2}\right)$, $t \in [1,2] \to \pi\left(t-\frac{3}{2}\right)$, $t \in [2,3] \to \pi\left(t-\frac{5}{2}\right)$, $t \in [3,4] \to \pi\left(t-\frac{7}{2}\right)$, $t \in [4,5] \to \pi\left(t-\frac{9}{2}\right)$

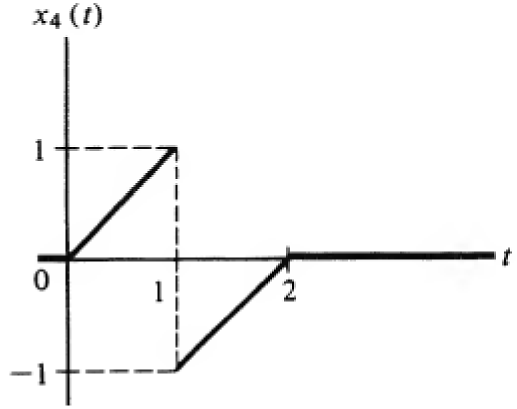
The first segment function is $s_1 = 1 \cdot \pi\left(t-\frac{1}{2}\right)$, the second segment is part of the folded-shifted ramp signal (slope =-1) $s_2 = r(-t+2) \cdot \pi\left(t-\frac{3}{2}\right)$, the third segment is part of the shifted ramp signal (slope =1) $s_3 = r(t-2) \cdot \pi\left(t-\frac{5}{2}\right)$, the fourth signal is $s_4 = 1 \cdot \pi\left(t-\frac{7}{2}\right)$, and the fifth signal is $s_5 = 2 \cdot \pi\left(t-\frac{9}{2}\right)$. Thus the signals $x_a(t)$ is expressed as:

$$x_a(t) = \pi\left(t-\frac{1}{2}\right) + r(-t+2) \cdot \pi\left(t-\frac{3}{2}\right) + r(t-2) \cdot \pi\left(t-\frac{5}{2}\right) + \pi\left(t-\frac{7}{2}\right) + 2 \cdot \pi\left(t-\frac{9}{2}\right)$$

- Combination: The first segment $s_1 = u(t)$, the second segment a negative slope ramp shifted at $t = 1$ and combined with is s_1, thus
  $s_2 = s_1 - r(t-1)$, the third segment is s_2 combined with the ramp of slope 2 (difference of slopes 1-(-1)) shifted at $t = 2$
  $s_3 = s_2 + 2r(t-2)$, the result is a ramp with slope 1, the forth segment is s_3 added to a ramp with slope (0-1=-1) shifted at $t = 3$, thus
  $s_4 = s_3 - r(t-3)$, which results in a constant signal $u(t-3)$ with height =1, the fifth segment s_5 is s_4 to which a constant signal $u(t-4)$ with height =1 is added to achieve a u(t) signal with height =2, the last signal s_6 is s_5 to which a negative $-2u(t-5)$ with height =-2 is added to achieve the zero signal after the last breaking point at $t = 5$, thus
  $x_a(t) = \mathrm{u(t) - r(t-1) + 2r(t-2) - r(t-3) + u(t-4) - 2u(t-5)}$

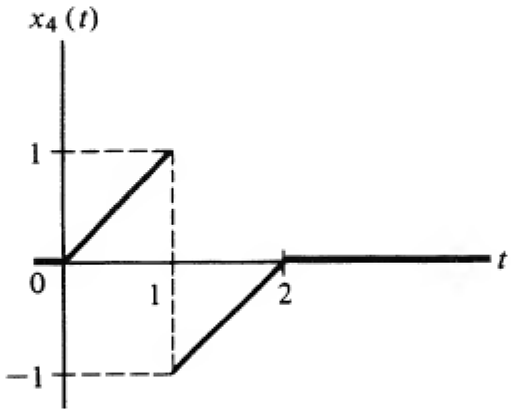


Example2: Express the following signal using elementary signals:



- Windowing: the signal is divided into two segment based on the breaking point using the following windows: $t \in [0,1] \rightarrow \pi\left(t - \frac{1}{2}\right)$, $t \in [1,2] \rightarrow \pi\left(t - \frac{3}{2}\right)$
  The first segment function is $s_1 = r(t) \cdot \pi\left(t - \frac{1}{2}\right)$, the second segment is a negative slope folded ramp signal shifted at $t = 2$, $s_2 = -r(-t+2) \cdot \pi\left(t - \frac{3}{2}\right)$, thus
  $x_4 = r(t) \cdot \pi\left(t - \frac{1}{2}\right) - r(-t+2) \cdot \pi\left(t - \frac{3}{2}\right)$
  Note: the form is not unique, thus you can express the signal in a different form

- <u>Combination:</u> The first segment is generated by a ramp signal with slope 1 starting at $t = 0$. At $t = 1$, the signal is broken, shifting it down by a unit step signal with height = -2, that is $-2u(t-2)$ which becomes a line that continues beyond $t = 2$. However, the signal should become zero after $t = 2$, thus a ramp with slope (0-1) should be added to cancel the previous ramp extension after t =2.   Hence, $x_4(t)$ is expressed as:
  $x_4(t) = r(t) - 2u(t-1) - r(t-2)$



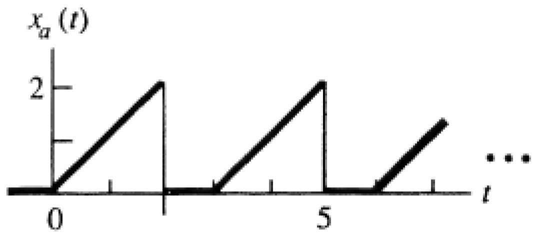
<u>Expression of periodic signals</u>

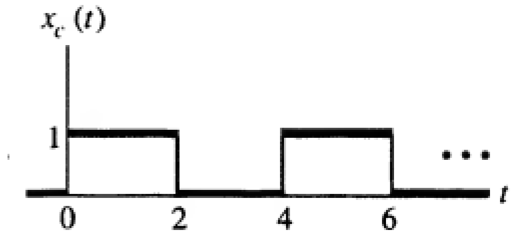
Let $x(t)$be a periodic signal with period $T_{0,}$ to express the signal using elementary signal firstly, express the basic period of the signal $x_p(t)$ (around $t = 0$) and then transform the it into $x(t)$ using the expression: $x(t) = \sum_{n=-\infty}^{\infty} x_p(t - nT_0)$ for a double sided periodic signal or $\sum_{n=0}^{\infty} x_p(t - nT_0)$ for a single sided periodic signal>

<u>Example 3:</u> Express the following periodic signals using elementary signals

$$x_{a\_p}(t) = r(t) \cdot \pi(\frac{t-1}{2}) \rightarrow x_a(t) = \sum_{n=0}^{\infty} r(t-3n)\pi(\frac{t-1-3n)}{2}$$

$$x_{c\_p}(t) = \pi(\frac{t-1}{2}) \rightarrow x_c(t) = \sum_{n=0}^{\infty} \pi\ (\frac{t-1-4n)}{2}$$

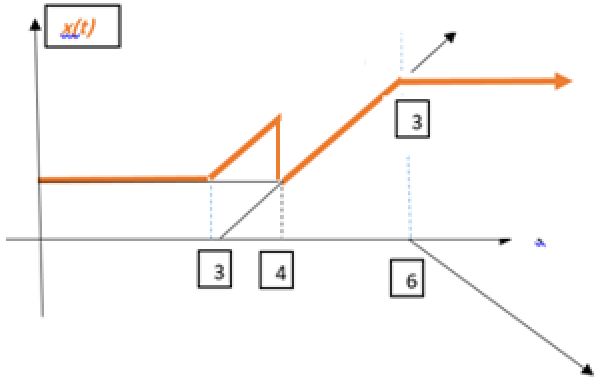




Signal Plot:

Plot each term of the signal expression and combine the these plots based on the critical points at which each single signal changes occur.

Example: Plot the signal $x_1(t) = \pi\left(\frac{t-2}{4}\right) + r(t-3) - r(t-6)$



Example: Plot the signal $x_1(t) = \pi\left(\frac{t-2}{4}\right) + r(-t+2) + 2\pi(\frac{t-4}{2})$