

6.1 ⇒ Volume Using Cross-Sections

* The Disk Method

• Graph

• Volume

→ About x-axis
→ About Horizontal line

$$V = \int_a^b A(x) dx$$

→ About y-axis
→ About Vertical line

$$V = \int_c^d A(y) dy$$

• $R(x) / R(y) \longrightarrow A(x) = \pi R^2(x)$ (Disk)

* The Washer Method

we use this method if the solid of revolution has a hole in it.

• Graph

• Volume

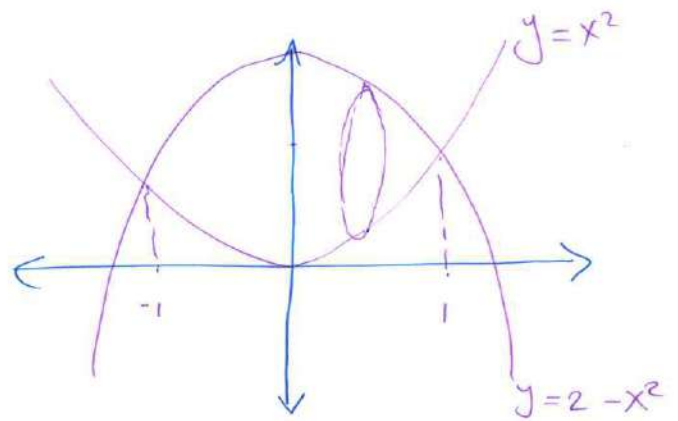
→ about x-axis
→ about horizontal line

$$V = \int_a^b A(x) dx = \int_a^b \pi [R^2(x) - r^2(x)] dx$$

→ about y-axis
→ about vertical line

$$V = \int_c^d A(y) dy = \int_c^d \pi [R^2(y) - r^2(y)] dy$$

Q2 The solid lies between planes perpendicular to the x -axis at $x=-1$ & $x=1$. The cross sections perpendicular to the x -axis are circular disk whose diameters run from the parabola $y=x^2$ to the parabola $y=2-x^2$



Solution:-

Cross section (Disk) $\perp$ x -axis

$$V = \int_{-1}^1 A(x) dx$$

$$V = \int_{-1}^1 \pi R^2(x) dx$$

$$= \int_{-1}^1 \pi [1 - 2x^2 + x^4] dx$$

$$= \pi \left[x - \frac{2x^3}{3} + \frac{x^5}{5} \right]_{-1}^1$$

$$= \pi \left[1 - \frac{2}{3} + \frac{1}{5} + 1 - \frac{2}{3} + \frac{1}{5} \right]$$

$$= \boxed{\frac{16\pi}{15}}$$

$$D = 2 - x^2 - x^2$$

$$D = 2 - 2x^2$$

$$R = \frac{1}{2} D = 1 - x^2$$

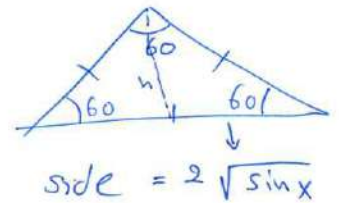
$$R^2 = (1 - x^2)^2 = 1 - 2x^2 + x^4$$

Q15 a) Equilateral triangle :- مثلث متساوي الاضلاع

Cross section [equilateral triangle]  x-axis

$$A = \int_{\boxed{0}}^{\boxed{\pi}} A(x) dx$$

$$A(x) = \text{Area} = \frac{1}{2} (\text{side})(h) \\ = \frac{1}{2} (\text{side})(\text{side})(\sin 60)$$



$$= \int_0^{\pi} \frac{1}{2} (\text{side})^2 (\sin 60) dx$$

$$= \int_0^{\pi} \frac{1}{2} \cdot 4 \sin x \cdot \left(\frac{\sqrt{3}}{2}\right) dx$$

$$= \int_0^{\pi} \sqrt{8} \sin x dx = -\sqrt{3} [\cos x]_0^{\pi} = -\sqrt{3} [-2] = \boxed{2\sqrt{3}}$$

b) Cross section (Square)  x-axis

$$V = \int_{\boxed{0}}^{\boxed{\pi}} A(x) dx = \int_0^{\pi} (\text{side})^2 dx$$

$$= \int_0^{\pi} 4 \sin x dx = -4 \cos x \Big|_0^{\pi} \\ = -4 \cdot [-2] = \boxed{8}$$

Q9 The solid lies between planes perpendicular to the y-axis at $y=0$ & $y=2$. The cross section perpendicular to the y-axis are circular disk with diameter running from the y-axis to the parabola $x = \sqrt{5} y^2$

Solution

Cross section (Disk) $\perp$ y-axis

$$V = \int_{\boxed{0}}^{\boxed{2}} A(y) dy$$

$$D = \sqrt{5} y^2 - 0$$

$$D = \sqrt{5} y^2$$

$$R = \frac{1}{2} D = \frac{\sqrt{5}}{2} y^2$$

$$= \int_0^2 \pi R^2(y) dy = \int_0^2 \pi \left(\frac{5}{4} y^4 \right) dy \quad R^2(y) = \frac{5}{4} y^4$$

$$= \frac{5\pi}{4} \left[\frac{y^5}{5} \right]_0^2 = \frac{\pi}{4} [32] = \boxed{8\pi}$$

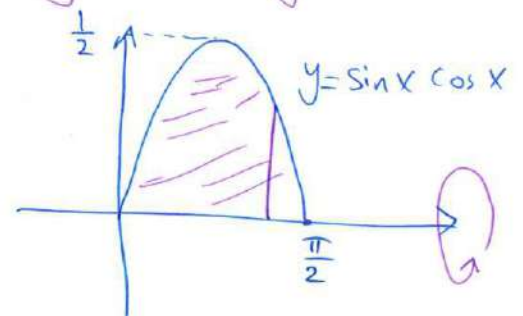
Q18 Find Volume of the solid generated by revolving the region about the x-axis

$$V = \int_{\boxed{0}}^{\boxed{\pi/2}} A(x) dx$$

$$= \int_0^{\pi/2} \pi (\sin x \cos x)^2 dx$$

$$= \pi \int_0^{\pi/2} \left(\frac{\sin 2x}{2} \right)^2 dx$$

$$= \frac{\pi}{4} \int_0^{\pi/2} \sin^2(2x) dx$$



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$$= \frac{\pi}{4} \int_0^{\pi/2} \sin^2(2x) dx$$

$$= \frac{\pi}{4} \int_0^{\pi/2} \frac{1 - \cos(4x)}{2} dx$$

$$= \frac{\pi}{4} \left[\frac{1}{2}x - \frac{\sin(4x)}{8} \right]_0^{\pi/2}$$

$$= \frac{\pi}{4} \left[\frac{\pi}{4} - 0 - 0 \right] = \boxed{\frac{\pi^2}{16}}$$

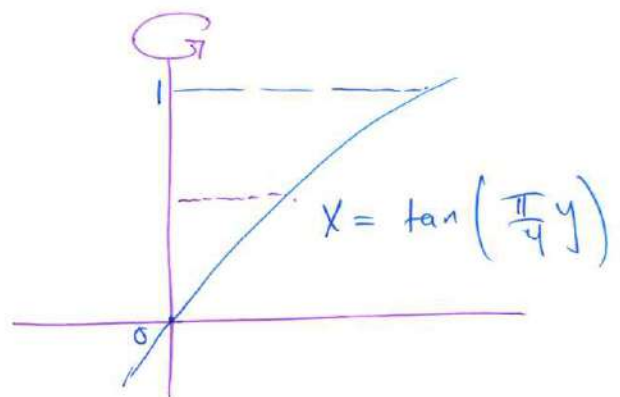
Q17 About the y-axis

$$V = \int_0^1 A(y) dy$$

$$= \int_0^1 \pi \tan^2\left(\frac{\pi}{4}y\right) dy$$

$$= \pi \int_0^1 \sec^2\left(\frac{\pi}{4}y\right) - 1 dy$$

$$= \pi \left[\frac{\tan(\pi/4 y)}{\pi/4} - y \right]_0^1$$



$$R(y) = \tan\left(\frac{\pi}{4}y\right) - 0$$

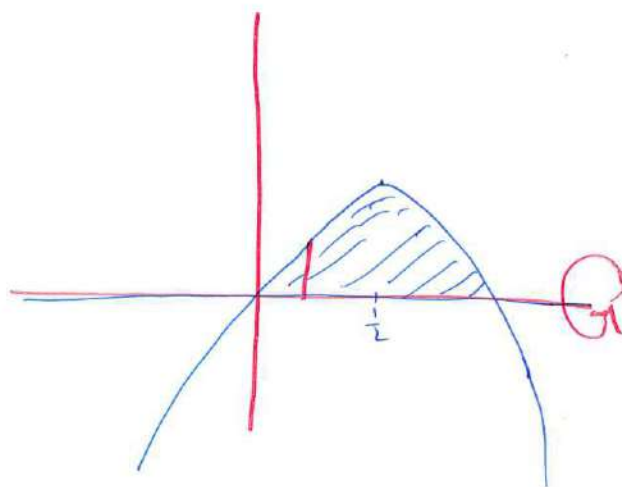
$$R^2(y) = \tan^2\left(\frac{\pi}{4}y\right)$$

$$= \pi \left[\frac{4}{\pi} \tan\left(\frac{\pi}{4}\right) - 1 \right]$$

$$= \pi \left[\frac{4}{\pi} - 1 \right]$$

Q22 Find the volumes of the solids generated by revolving the region bounded by the line and curves in Exercises 19-24 about the x-axis

22. $y = x - x^2$, $y = 0$



$$V = \int_{\boxed{0}}^{\boxed{1}} A(x) \, dx$$

$$= \int_0^1 \pi R^2(x) \, dx$$

$$= \pi \int_0^1 x^2 - 2x^3 + x^4 \, dx$$

$$= \pi \left[\frac{x^3}{3} - \frac{x^4}{2} + \frac{x^5}{5} \right]_0^1$$

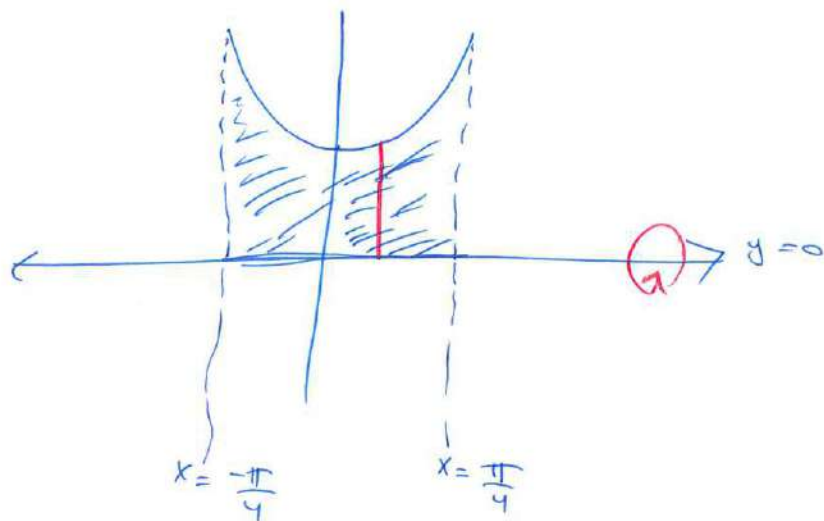
$$= \pi \left[\frac{1}{3} - \frac{1}{2} + \frac{1}{5} \right] = \boxed{\frac{\pi}{30}}$$

$$R(x) = x - x^2 - 0$$
$$R^2(x) = x^2 - 2x^3 + x^4$$

Q₂₁

$y = \sec x$, $y = 0$, $x = -\frac{\pi}{4}$, $x = \frac{\pi}{4}$
about x -axis

$$V = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} A(x) dx$$



$$= \int_{-\pi/4}^{\pi/4} \pi R^2(x) dx$$

$$= \int_{-\pi/4}^{\pi/4} \pi \sec^2 x dx$$

$$= 2\pi \int_0^{\pi/4} \sec^2 x dx = 2\pi [\tan x]_0^{\pi/4}$$

$$= 2\pi [1 - 0] = \boxed{2\pi}$$

$$R(x) = \sec x - 0$$

$$R^2(x) = \sec^2 x$$

Q₂₅ Find the volume of the solid generated by revolving the region in the first quadrant bounded above by $y = \sqrt{2}$, below by $y = \sec x \tan x$, and on the left by y -axis about the line $y = \sqrt{2}$

$$V = \int_{\boxed{0}}^{\boxed{\pi/4}} A(x) dx$$

$$V = \int_0^{\pi/4} \pi R^2(x) dx$$

$$V = \pi \int_0^{\pi/4} 2 - \sqrt{8} \sec x \tan x$$

$$+ \pi \int_0^{\pi/4} \sec^2 x \tan^2 x dx$$

$$V = \pi \left[2x - \sqrt{8} \sec x \right]_0^{\pi/4}$$

$$+ \pi \int_{\boxed{0}}^{\boxed{\pi/4}} \sec^2 x \frac{U^2 dU}{\sec^2 x}$$

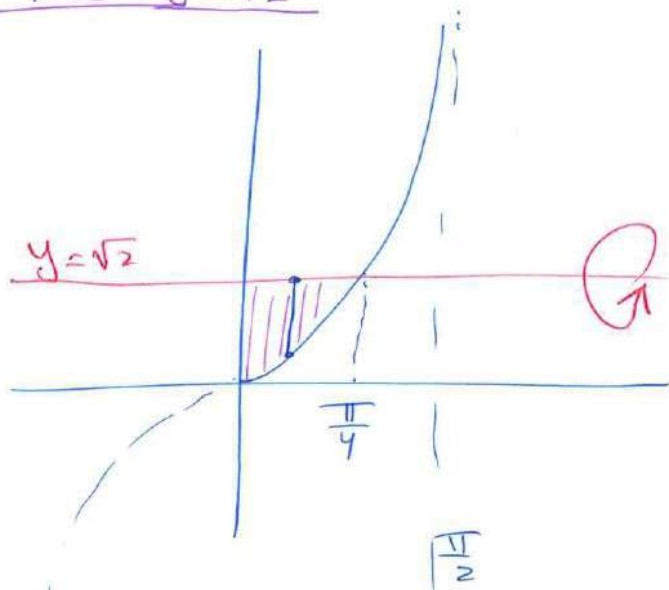
let $U = \tan x$
 $dU = \sec^2 x dx$

$x = 0 \rightarrow U = 0$

$x = \frac{\pi}{4} \rightarrow U = 1$

$$= \pi \left[\frac{\pi}{2} - 4 + \sqrt{8} \right] + \pi \left[\frac{U^3}{3} \right]_0^1$$

$$= \pi \left[\frac{\pi}{2} - 4 + \sqrt{8} + \frac{1}{3} \right] = \pi \left[\frac{\pi}{2} + 2\sqrt{2} - \frac{11}{3} \right]$$



$$R(x) = \sqrt{2} - \sec x \tan x$$

$$R^2(x) = 2 - 2\sqrt{2} \sec x \tan x + \sec^2 x \tan^2 x$$

Q₂₇ Find the volume of the solid generated by revolving the regions bounded by the line and curve about y-axis

$$x = \sqrt{5} y^2$$

$$x=0$$

$$y=1$$

$$y=-1$$

$$V = \int_{-1}^1 A(y) dy$$

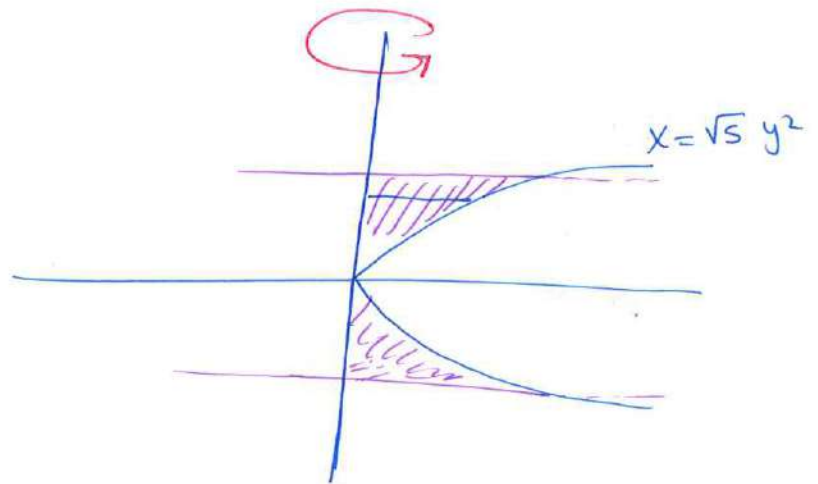
$$V = \int_{-1}^1 \pi R^2(y) dy$$

$$= \int_{-1}^1 \pi [\sqrt{5} y^2]^2 dy$$

$$= \int_{-1}^1 5\pi y^4 dy$$

$$= 5\pi \left[\frac{y^5}{5} \right]_{-1}^1$$

$$= \pi [1+1] = 2\pi$$



for the graph

$$x = \sqrt{5} y^2 \rightarrow |y| = \frac{1}{\sqrt{5}} \sqrt{x}$$

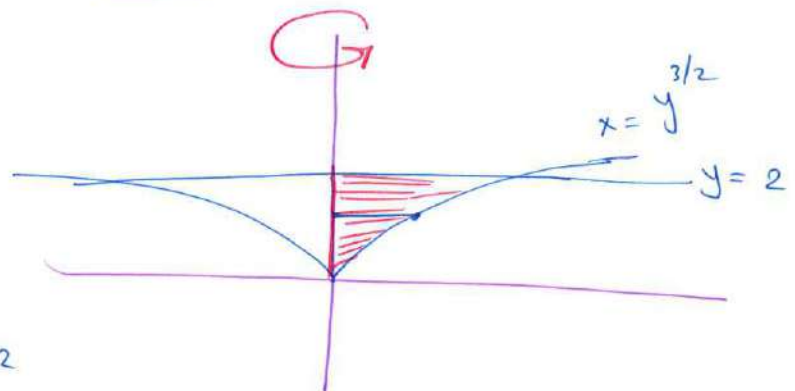
Cross section (Disk Method)

Q₂₈ $x = y^{3/2}$, $x=0$, $y=2$ about y-axis

$$V = \int_0^2 A(y) dy$$

$$V = \int_0^2 \pi y^3 dy = \pi \left[\frac{y^4}{4} \right]_0^2$$

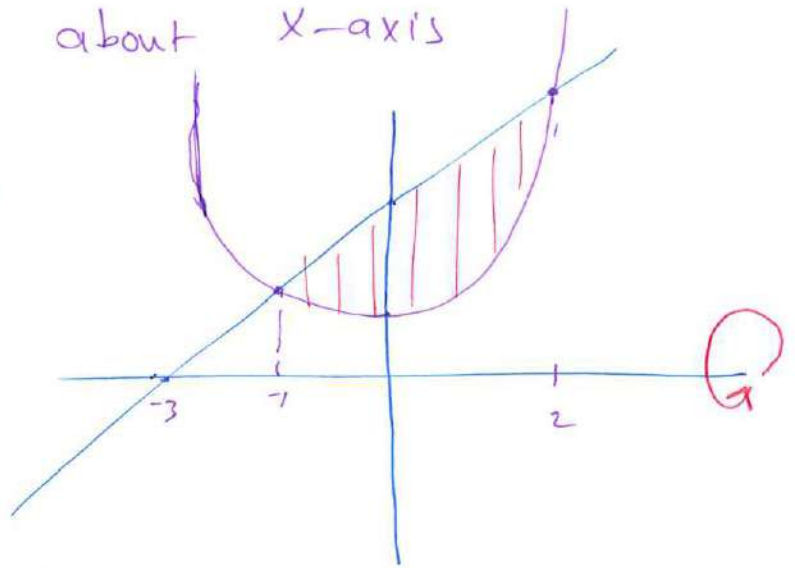
$$4\pi$$



Q37 Find the volume of the solid generated by revolving the region bounded by the line and curve

$$y = x^2 + 1, \quad y = x + 3 \quad \text{about } x\text{-axis}$$

Using Washer Method:



$$V = \int_{-1}^2 A(x) dx$$

$$= \int_{-1}^2 \pi [R^2(x) - r^2(x)] dx$$

$$r(x) = x^2 + 1$$

$$R(x) = x + 3$$

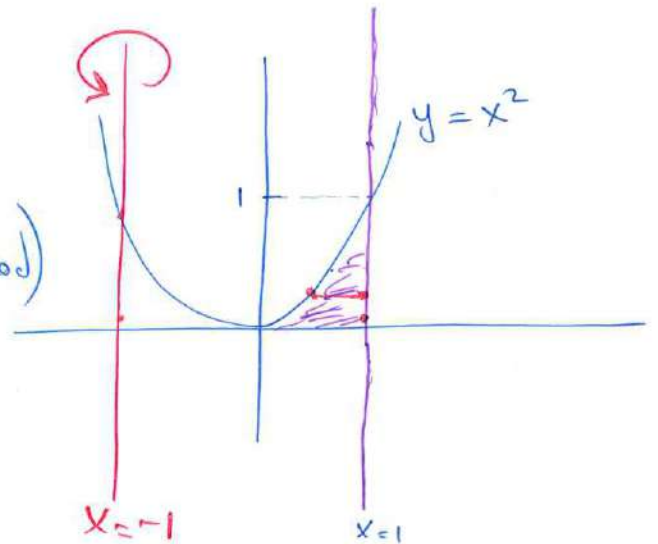
$$= \int_{-1}^2 \pi [x^2 + 6x + 9 - x^4 - 2x^2 - 1] dx$$

$$= \pi \int_{-1}^2 -x^2 + 6x - x^4 + 8 dx$$

$$\pi \left[-\frac{x^3}{3} + \frac{6x^2}{2} - \frac{x^5}{5} + 8x \right]_{-1}^2$$

$$\pi \left[-\frac{8}{3} + 12 - \frac{32}{5} + 16 - \frac{1}{3} - 3 - \frac{1}{5} + 8 \right] = \boxed{\frac{117\pi}{5}}$$

Q45 The region in the first quadrant bounded above by the curve $y=x^2$, below by the x-axis, on the right by $x=1$ about $x=-1$



Solution (By Using Washer Method)
about Vertical line

$$V = \int_{[0]}^{[1]} A(y) dy$$

$$= \int_0^1 \pi [R^2(y) - r^2(y)] dy$$

$$= \int_0^1 \pi [4 - (\sqrt{y} + 1)^2] dy$$

$$= \pi \int_0^1 4 - y - 2\sqrt{y} - 1 dy$$

$$= \pi \left[3y - \frac{y^2}{2} - \frac{4}{3} y^{3/2} \right]_0^1$$

$$= \pi \left[3 - \frac{1}{2} - \frac{4}{3} \right] = \boxed{\frac{7\pi}{6}}$$

$$R(y) = 1 - (-1) = 2$$

$$r(y) = \sqrt{y} - (-1)$$

$$r(y) = \sqrt{y} + 1$$

Q 47

Find the volume of the solid generated by the region bounded by $y = \sqrt{x}$ and the line $y = 2$ & $x = 0$ about

- (a) the x-axis
- (b) the y-axis
- (c) the line $y = 2$ [horizontal line]
- (d) the line $x = 4$ [vertical line]

(a) About x-axis

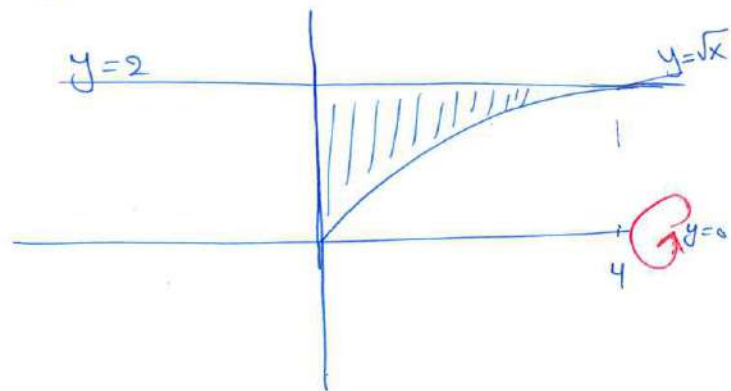
By using washer Method

$$V = \int_0^4 A(x) dx$$

$$= \int_0^4 \pi [R^2(x) - r^2(x)] dx$$

$$= \int_0^4 \pi [4 - x] dx$$

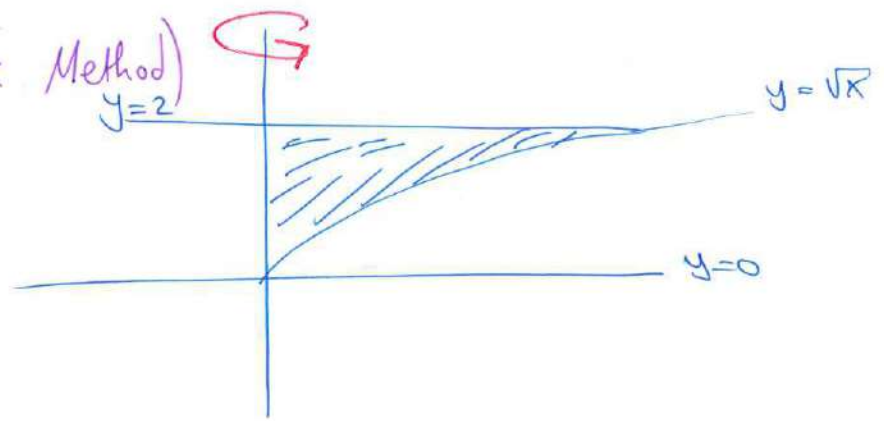
$$= \pi \left[4x - \frac{x^2}{2} \right]_0^4 = 8\pi$$



$$r(x) = \sqrt{x}$$

$$R(x) = 2$$

⑥ About y-axis (Disk Method)



$$V = \int_{\boxed{0}}^{\boxed{2}} A(y) dy$$

$$= \int_0^2 \pi y^4 dy$$

$$= \pi \left[\frac{y^5}{5} \right]_0^2$$

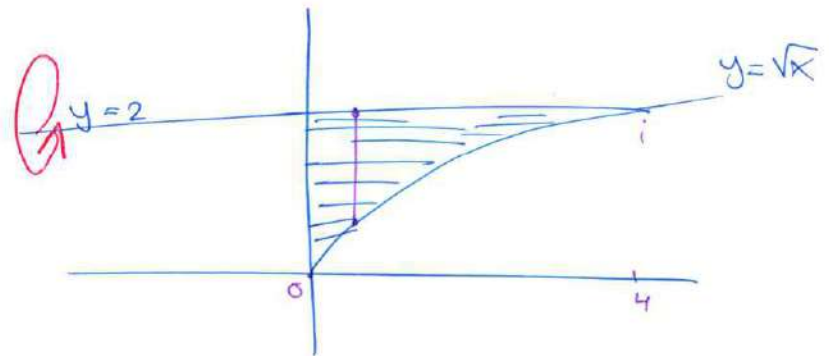
$$= \boxed{\frac{32}{5} \pi}$$

$$R(y) = y^2 - 0$$

$$R(y) = y^2$$

$$R^2(y) = y^4$$

⑦ About y=2 (Disk Method)



$$V = \int_{\boxed{0}}^{\boxed{4}} A(x) dx$$

$$V = \int_0^4 \pi (4 - 4\sqrt{x} + x) dx$$

$$\pi \left[4x - \frac{8x^{3/2}}{3} + \frac{x^2}{2} \right]_0^4 = \boxed{\frac{8\pi}{3}}$$

$$R(x) = 2 - \sqrt{x}$$

$$R^2(x) = 4 - 4\sqrt{x} + x$$

③ about $X=4$
(By using washer Method)

$$V = \int_{\square}^{\square} A(y) dy$$

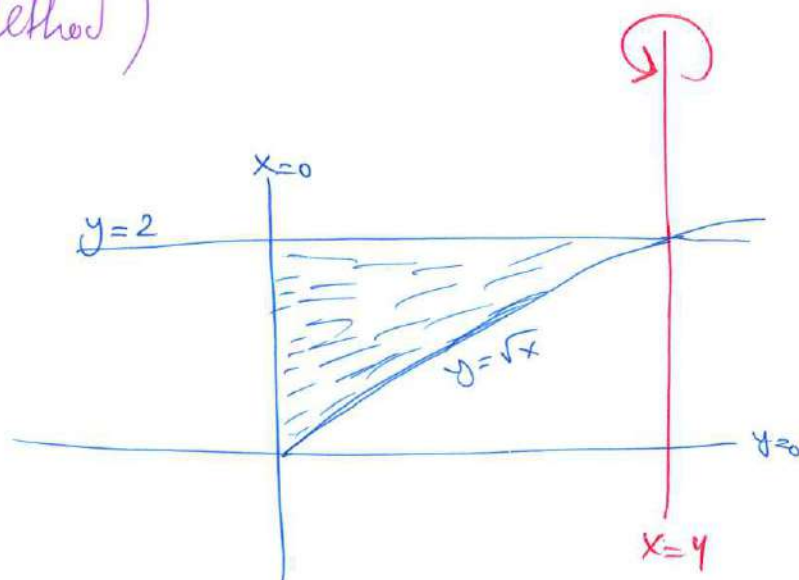
$$V = \int_0^2 A(y) dy$$

$$= \int_0^2 \pi [R^2(y) - r^2(y)] dy$$

$$= \int_0^2 \pi [16 - (4 - y^2)^2] dy$$

$$= \pi \int_0^2 16 - 16 + 8y^2 - y^4 dy = \pi \left[\frac{8y^3}{3} - \frac{y^5}{5} \right]_0^2$$

$$= \pi \left[\frac{64}{3} - \frac{32}{5} \right] = \frac{224\pi}{15}$$



$$R(y) = 4 - 0 = 4$$

$$r(y) = 4 - y^2$$

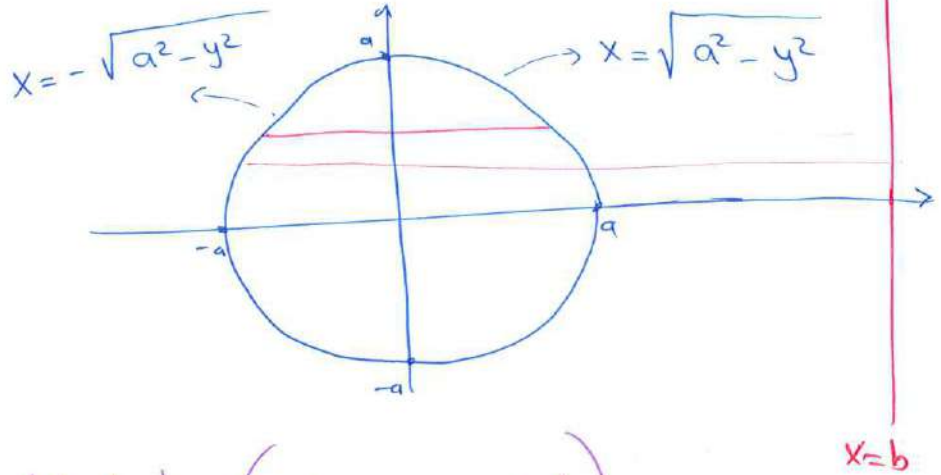
Q51

The disk $x^2 + y^2 \leq a^2$ is revolved about the line $x=b$, $b > a$ to generate a solid shaped like a doughnut and called a torus. Find its volume

$$x^2 + y^2 = a^2$$

$$x^2 = a^2 - y^2$$

$$x = \pm \sqrt{a^2 - y^2}$$



By using Washer Method (about $x=b$)
vertical line

$$\begin{aligned} V &= \int_{-a}^a A(y) dy = \int_{-a}^a \pi [R^2(y) - r^2(y)] dy \\ &= \pi \int_{-a}^a \left[\left[b + \sqrt{a^2 - y^2} \right]^2 - \left[b - \sqrt{a^2 - y^2} \right]^2 \right] dy \\ &= \pi \int_{-a}^a \left(\cancel{b^2} + 2b\sqrt{a^2 - y^2} + \cancel{a^2 - y^2} - \cancel{b^2} + 2b\sqrt{a^2 - y^2} - \cancel{a^2 - y^2} \right) dy \\ &= \pi \int_{-a}^a 4b\sqrt{a^2 - y^2} dy \\ &= 4\pi b \int_{-a}^a \sqrt{a^2 - y^2} dy \\ &= 4\pi b \cdot \frac{\pi a^2}{2} = \boxed{2\pi^2 b a^2} \end{aligned}$$

hint : $\int_{-a}^a \sqrt{a^2 - y^2} dy = \frac{\pi a^2}{2}$