Limits

Def • Let $f(x)$ be a function defined on an open interval containing c, or except perhaps at c. Then we say the limit of $f(x)$ is L as x approaches c means

$$\lim_{x \to c} f(x) = L \quad \text{where } L \text{ is finite number}$$

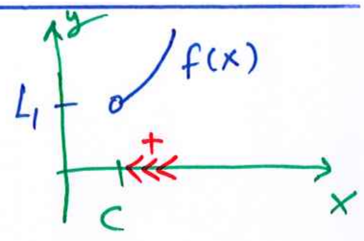
- $\lim_{x \to c} f(x) = L$ also means $\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = L$

- If $\lim_{x \to c^+} f(x) \neq \lim_{x \to c^-} f(x)$ then we say

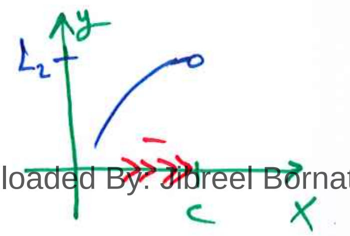
$$\lim_{x \to c} f(x) \text{ does not exist (or we write DNE)}$$

One-Sided Limits:

limit from right $\Rightarrow$ $\lim_{x \to c^+} f(x) = L_1$



limit from left $\Rightarrow$ $\lim_{x \to c^-} f(x) = L_2$



Exp Find ① $\lim_{x \to 3} (2x+1) = 2(3)+1 = 6+1 = 7$

② $\lim_{x \to -1} \dfrac{x^2-4}{3x+2} = \dfrac{(-1)^2-4}{3(-1)+2} = \dfrac{1-4}{-3+2} = \dfrac{-3}{-1} = 3$

③ $\lim\limits_{x \to 2} \dfrac{x^2 - 4}{x - 2}$ $\boxed{\frac{0}{0}}$ → يلزم تحليل

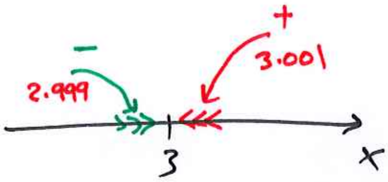
$$\lim_{x \to 2} \frac{\cancel{(x-2)}(x+2)}{\cancel{x-2}} = \lim_{x \to 2} (x+2) = (2)+2 = 4$$

④ $\lim\limits_{x \to 1} \dfrac{x^2 - 3x + 2}{x^2 + 2x}$

$$\frac{(1)^2 - 3(1) + 2}{(1)^2 + 2(1)} = \frac{1-3+2}{1+2} = \boxed{\frac{0}{3} = 0}$$

⑤ $\lim\limits_{x \to 3} \dfrac{x^2 + 2x - 3}{x - 3}$ $\boxed{\frac{12}{0}}$ → يلزم من اليمين ومن الشمال

دائماً → DNE



$\lim\limits_{x \to 3} \dfrac{x^2 + 2x - 3}{x - 3}$ DNE since

$$\lim_{x \to 3^+} \frac{x^2 + 2x - 3}{x - 3} = \frac{(3)^2 + 2(3) - 3}{\text{small positive}} = \frac{12}{\text{small} +} = +\infty$$

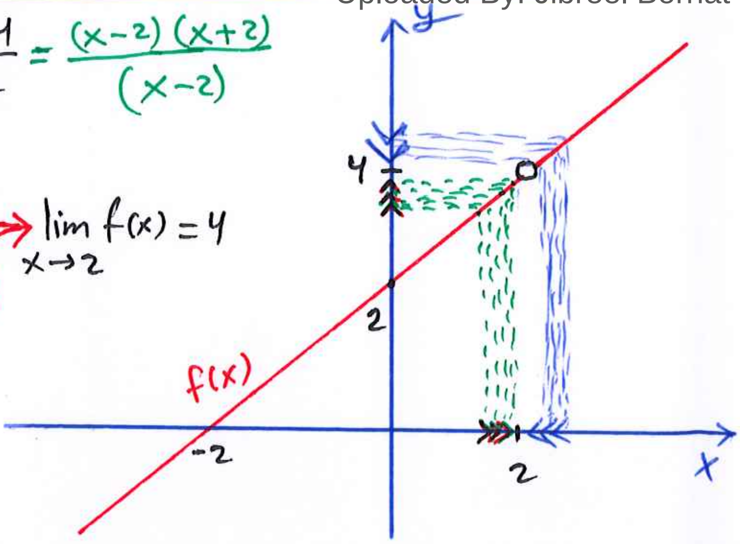
$$\lim_{x \to 3^-} \frac{x^2 + 2x - 3}{x - 3} = \frac{(3)^2 + 2(3) - 3}{\text{small negative}} = \frac{12}{\text{small} -} = -\infty$$

Exp Sketch ③ $\Rightarrow f(x) = \dfrac{x^2 - 4}{x - 2} = \dfrac{(x-2)(x+2)}{(x-2)}$

- $\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} (x+2) = (2)+2 = 4$
- $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} (x+2) = (2)+2 = 4$

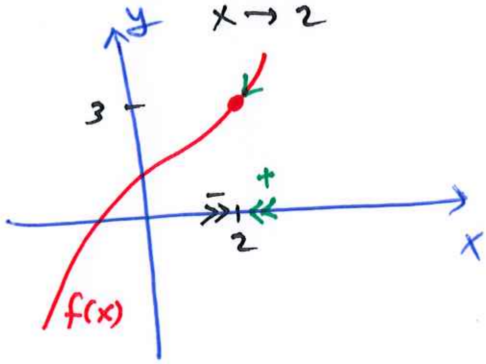
→ $\lim\limits_{x \to 2} f(x) = 4$

- Note that $f(2) \Rightarrow$ DNE



Exp Find $\lim\limits_{x\to 2} f(x)$ and $f(2)$ for the following

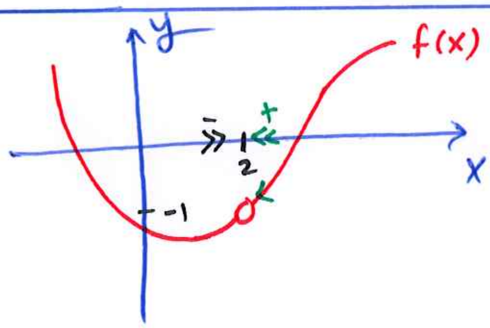
A)



- $f(2) = 3$
- $\lim\limits_{x\to 2^+} f(x) = 3 = \lim\limits_{x\to 2^-} f(x)$ so

$$\lim_{x\to 2} f(x) = 3$$

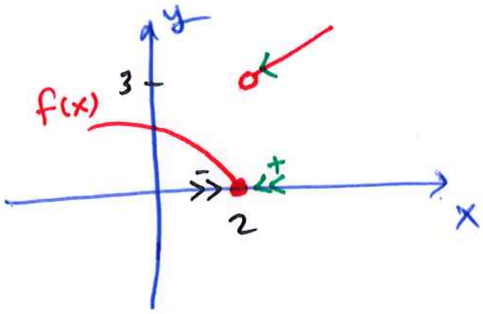
B)



- $f(2)$ undefined
- $\lim\limits_{x\to 2^-} f(x) = -1 = \lim\limits_{x\to 2^+} f(x)$ so

$$\lim_{x\to 2} f(x) = -1$$

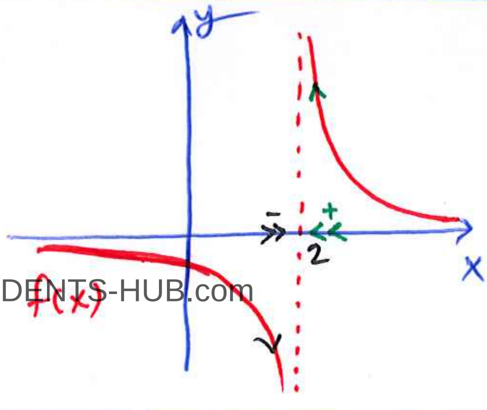
C)



- $f(2) = 0$
- $\lim\limits_{x\to 2^-} f(x) = 0$ but $\lim\limits_{x\to 2^+} f(x) = 3$

so $\lim\limits_{x\to 2} f(x)$ DNE

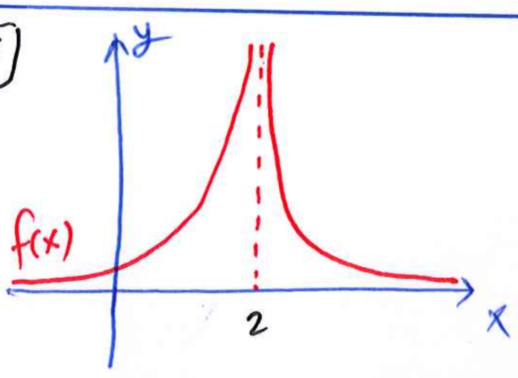
D)



- $f(2)$ undefined
- $\lim\limits_{x\to 2^+} f(x) = \infty$ but $\lim\limits_{x\to 2^-} f(x) = -\infty$

so $\lim\limits_{x\to 2} f(x)$ DNE

E)



- $f(2)$ undefined
- $\lim\limits_{x\to 2^-} f(x) = \infty = \lim\limits_{x\to 2^+} f(x)$ but

$\lim\limits_{x\to 2} f(x)$ DNE since ∞ is not finite

Exp [i] Can $\lim_{x\to c^-} f(x)$ exists if $f(c)$ is undefined?

Yes see $\boxed{B}$

[ii] Does $\lim_{x\to c} f(x)$ exist if $f(c)=0$?

Not necessarily $\Rightarrow$ see $\boxed{C}$

[iii] Does $f(c)=0$ if $\lim_{x\to c} f(x)=0$?

Not necessarily $\Rightarrow$ see $\boxed{C}$

Remark: we may write $f(x)\to L$ as $x\to c$ instead of

$$\lim_{x\to c} f(x) = L$$

Properties of Limits: Assume K constant, $\lim_{x\to c} f(x)=L$, $\lim_{x\to c} g(x)=M$

Then ① $\lim_{x\to c} K = K$

② $\lim_{x\to c} x = c$

③ $\lim_{x\to c}\left(f(x)\pm g(x)\right) = L\pm M$

④ $\lim_{x\to c}\left(f(x)\,g(x)\right) = LM$

⑤ $\lim_{x\to c}\dfrac{f(x)}{g(x)} = \dfrac{L}{M}$, $M\neq 0$

⑥ $\lim_{x\to c}\sqrt[n]{f(x)} = \sqrt[n]{L}$ where $L>0$ if n is even.

Def • The function $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, $a_n \neq 0$ is called polynomial function of degree n where n is positive integer

• The function $h(x) = \dfrac{f(x)}{g(x)}$, where f and g are polynomials, is called rational function

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Remark • If $f(x)$ is polynomial then $\lim\limits_{x \to c} f(x) = f(c)$ for all values of c

• If $h(x) = \dfrac{f(x)}{g(x)}$ is rational function then

$$\lim_{x \to c} h(x) = \frac{f(c)}{g(c)} \quad \text{where} \quad g(c) \neq 0$$

Exp ① $\lim\limits_{x \to 0} \dfrac{x}{x^2 - 2x}$ $\quad \dfrac{0}{0} \to$ تحليل

$$\lim_{x \to 0} \frac{\cancel{x}}{\cancel{x}(x-2)}$$

$$\lim_{x \to 0} \frac{1}{x-2} = \frac{1}{0-2} = -\frac{1}{2}$$

② $\lim\limits_{x \to 2} \left(x^3 - x^2 + 1 \right) = (2)^3 - (2)^2 + 1 = 8 - 4 + 1 = 5$

③ $\lim\limits_{x \to 1} \dfrac{3}{x-1}$ $\quad \dfrac{3}{0} \to$ DNE since

$$\lim_{x \to 1^+} \frac{3}{x-1} = \frac{3}{\text{small}+} = \infty$$

$$\lim_{x \to 1^-} \frac{3}{x-1} = \frac{3}{\text{small}-} = -\infty$$

$\Rightarrow \lim\limits_{x \to 1} \dfrac{3}{x-1}$ DNE

Exp Let $f(x) = \begin{cases} x^2+1 & \text{if } x \le 1 \\ x+2 & \text{if } x > 1 \end{cases}$

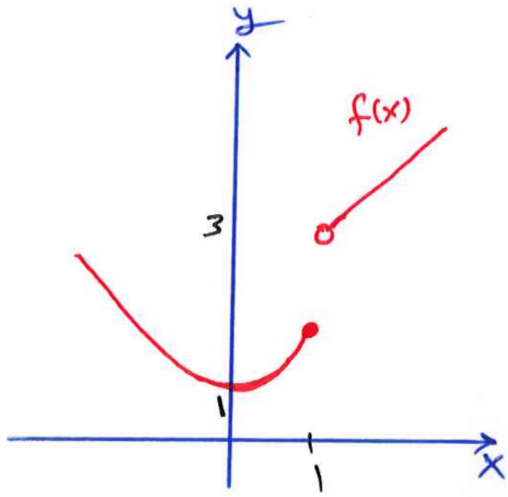
Find

① $\lim\limits_{x \to 1^-} f(x) = (1)^2 + 1 = 2$

② $\lim\limits_{x \to 1^+} f(x) = (1) + 2 = 3$

③ $\lim\limits_{x \to 1} f(x)$ DNE since $\lim\limits_{x \to 1^-} f(x) \neq \lim\limits_{x \to 1^+} f(x)$

④ sketch $f(x)$



Exp Assume $\lim\limits_{x \to 2} g(x) = 3$ and $\lim\limits_{x \to 2} \left(f(x) + 2g(x)\right) = 7$

Find ① $\lim\limits_{x \to 2} f(x)$

since $\lim\limits_{x \to 2} \left(f(x) + 2g(x)\right) = 7$

$\lim\limits_{x \to 2} f(x) + 2 \lim\limits_{x \to 2} g(x) = 7$

$\lim\limits_{x \to 2} f(x) + 2(3) = 7 \Rightarrow \lim\limits_{x \to 2} f(x) = 7 - 6 = 1$

② $\lim\limits_{x \to 2} \left[[f(x)]^2 - 5g(x) \right]$

$\lim\limits_{x \to 2} [f(x)]^2 - 5 \lim\limits_{x \to 2} g(x)$

$\left(\lim\limits_{x \to 2} f(x)\right)\left(\lim\limits_{x \to 2} f(x)\right) - 5 \lim\limits_{x \to 2} g(x)$

$(1)(1) - 5(3) = 1 - 15 = -14$