Exercises 1.2 :

Q7: If $A_1, A_2, A_3, \dots$ are sets s.t $A_K \subset A_{K+1}$, $K = 1,2,3,\dots$, $\lim_{K\to\infty} A_K$ is defined as the union $A_1 \cup A_2 \cup A_3 \cup \dots$. Find $\lim_{K\to\infty} A_K$ if :

a. $A_K = \{ x : \frac{1}{K} \le x \le 3 - \frac{1}{K} \}$ $\rightsquigarrow$ $A_K = \left[\frac{1}{K}, 3 - \frac{1}{K} \right]$, $K = 1,2,3,\dots$

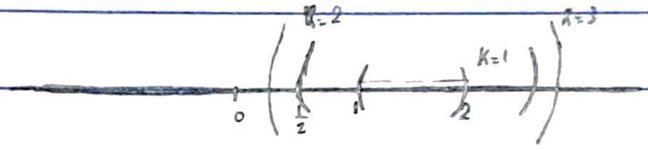
$A_1 = [1, 2]$

$A_2 = [\frac{1}{2}, \frac{5}{2}]$

$A_3 = [\frac{1}{3}, \frac{8}{3}]$

$\lim_{K\to\infty} A_K = A_1 \cup A_2 \cup A_3 \cup \dots =$

3,0 بنابراین بین



$\therefore \lim_{K\to\infty} A_K = \{ x : 0 < x < 3 \}$.

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b. $A_K = \{ (x,y) : \frac{1}{K} \le x^2 + y^2 \le 4 - \frac{1}{K} \}$.

?

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Q8. --- Find $\lim_{K\to\infty} A_K$ is defined as intersection $A_1 \cap A_2 \cap \dots$?

a. $A_K = \{x : 2 - \frac{1}{K} < x \le 2\}$

$\therefore \lim_{K\to\infty} A_K = \{x : x = 2\}$

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b. $A_K = \{x : 2 < x \le 2 + \frac{1}{K}\}$

$\lim_{K\to\infty} A_K = \emptyset$

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c. $A_K = \{(x,y) : 0 \le x^2 + y^2 \le \frac{1}{K}\}$

Q9. For every one-dimensional set A, let $Q(A) = \sum_A f(x)$, where

$$f(x) = \begin{cases} (\frac{2}{3})(\frac{1}{3})^x, & x = 0, 1, 2, \dots \\ 0, & \text{else where} \end{cases}$$ if $A_1 = \{x : x = 0, 1, 2, 3\}$ and $A_2 = \{x : x = 0, 1, 2, \dots\}$

Find $Q(A_1)$ and $Q(A_2)$:

→ $Q(A_1) = \sum_0^3 (\frac{2}{3})\left((\frac{1}{3})^0 + (\frac{1}{3})^1 + (\frac{1}{3})^2 + (\frac{1}{3})^3\right) = \frac{2}{3} \times \frac{40}{27} = \frac{80}{81}$

→ $Q(A_2) = \frac{2}{3} \sum_0^\infty (\frac{1}{3})^x$ → geometric seq. , sum $= \frac{a}{1-r}$

$a = (\frac{1}{3})^0 = 1$

$r = \frac{1}{3} \div 1 = \frac{1}{3}$

$1 - r = \frac{2}{3}$

$= \frac{2}{3}\left(\frac{1}{\frac{2}{3}}\right) = \frac{2}{3} \times \frac{3}{2}$

$= 1$ ✓

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Q10: $Q(A) = \int_A f(x)\,dx$, $f(x) = \begin{cases} 6x(1-x), & 0<x<1 \\ 0, & \text{elsewhere} \end{cases}$.

① $A_1 = \{x : \frac{1}{4} < x < \frac{3}{4}\}$ Find $Q(A)$.

$$Q(A) = \int_{\frac{1}{4}}^{\frac{3}{4}} (6x - 6x^2)\,dx = 6\int_{\frac{1}{4}}^{\frac{3}{4}} (x - x^2)\,dx = 6\left(\frac{x^2}{2} - \frac{x^3}{3}\Big|_{\frac{1}{4}}^{\frac{3}{4}}\right)$$

$$= 6\left(\frac{11}{96}\right) = \frac{11}{16}$$ ✓

② $A_2 = \{\frac{1}{2}\}$

$$Q(A_2) = \int_{\frac{1}{2}}^{\frac{1}{2}} 0\,dx = 0$$

③ $A_3 = \{x : 0 < x < 10\}$

$$Q(A_3) = \int_0^1 (6x - 6x^2)\,dx + \int_1^{10} 0\,dx = 3x^2 - 2x^3\Big|_0^1$$

$$= 3 - 2$$

$$= 1$$

Q11. $Q(A) = \iint_A (x^2 + y^2)\,dx\,dy$ for every two-dimensional set A

① $A_1 = \{(x,y) : -1 \le x \le 1,\ -1 \le y \le 1\}$

$$Q(A_1) = \int_{-1}^{1}\int_{-1}^{1} (x^2+y^2)\,dx\,dy = \int_{-1}^{1} \frac{x^3}{3} + y^2 x\Big|_{-1}^{1}\,dy \rightarrow$$

(y: -1 to 1; x: -1 to 1)

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11.1 : $Q(A) = \int_{-1}^{1} \left(\frac{x^3}{3} + y^2 x\right)\Big|_{-1}^{1} dy$ $\left(\frac{-1}{3} - y^2 - \frac{1}{3} - y^2\right) = \frac{-2}{3} - 2y^2$

$$= \int_{-1}^{1} -\frac{2}{3} - 2y^2 \, dy$$

$$= -\frac{2}{3}y - \frac{2}{3}y^3 \Big|_{-1}^{1} = \left(-\frac{2}{3}\right)(1) - \left(\frac{2}{3}\right)(-1)^3 - \left(-\frac{2}{3} - \frac{2}{3}\right)$$

$$= \frac{2}{3} + \frac{2}{3} + \frac{2}{3} + \frac{2}{3}$$

$$Q(A_1) = \frac{8}{3}$$

2. $A_2 = \{(x,y) : -1 \le x = y \le 1\}$

+3

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Q12: let A denote the set of points that are interior to, or on the boundary of, a square with opposite vertices at the points $(0,0)$ and $(1,1)$. let

$$Q(A) = \iint_A dy\,dx .$$

a. If $A \subset A$ is the set $\{(x,y) : 0 < x < y < 1\}$, compute $Q(A)$.

$$Q(A) = \int_0^1 \int_0^y dx\,dy = \int_0^1 x \Big|_0^y dy = \int_0^1 y\,dy$$

$$= \frac{y^2}{2}\Big|_0^1$$

$$= \frac{1}{2} .$$

b. $A = \{(x,y) : 0 < x = y < 1\}$, compute $Q(A)$.

this is a line, Area $= 0$.

$$Q(A) = \iint 0\,dx\,dy = 0 .$$

c. $A = \{(x,y) : 0 < \frac{x}{2} \le y \le \frac{3x}{2} < 1\}$, compute $Q(A)$.

$$Q(A) = \int_0^{\frac{2}{3}} \int_{\frac{x}{2}}^{\frac{3x}{2}} dy\,dx = \int_0^{\frac{2}{3}} y \Big|_{\frac{x}{2}}^{\frac{3x}{2}} dx$$

$$= \int_0^{\frac{2}{3}} \left(\frac{3x}{2} - \frac{x}{2}\right) dx$$

$$= \int_0^{\frac{2}{3}} x\,dx = \frac{x^2}{2}\Big|_0^{\frac{2}{3}} = \frac{1}{2}\left(\frac{4}{9}\right) = \frac{2}{9}$$

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