

\* A sequence is a list of numbers  $a_1, a_2, \dots, a_n, \dots$

- where  $a_i$  are numbers with index  $i$  "order"
- it can be finite or infinite
- it is a function that sends 1 to  $a_1$ ,  
2 to  $a_2$ ,  
...  
 $n$  to  $a_n$  "the  $n^{\text{th}}$  term"

Exp  $a_n = \sqrt{n}$

$$a_1 = 1$$

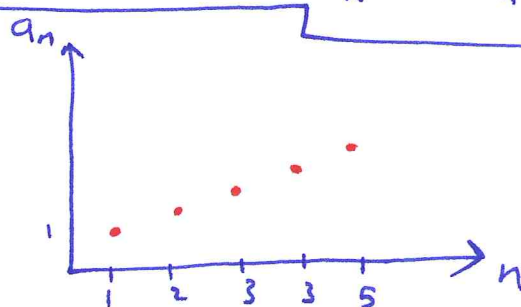
$$a_2 = \sqrt{2}$$

$$a_3 = \sqrt{3}$$

$$\vdots$$

$$a_n = \sqrt{n}$$

$$\vdots$$



$$a_n \rightarrow \infty \text{ as } n \rightarrow \infty$$

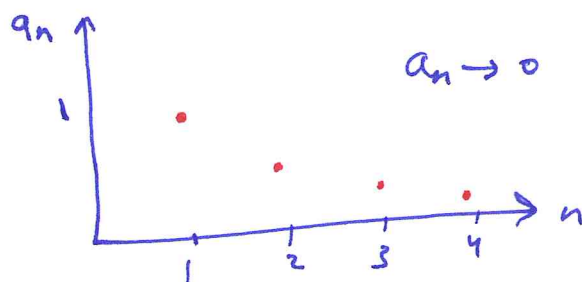
Exp  $a_n = \frac{1}{n}$

$$a_1 = 1$$

$$a_2 = \frac{1}{2}$$

$$a_3 = \frac{1}{3}$$

$$\vdots$$



$$a_n \rightarrow 0 \text{ as } n \rightarrow \infty$$

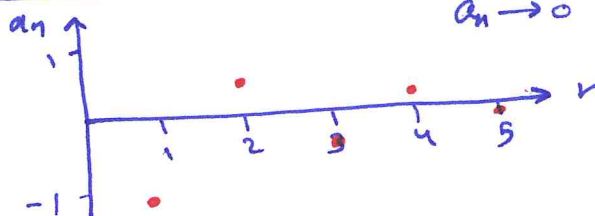
Exp  $a_n = (-1)^n \frac{1}{n}$

$$a_1 = -1$$

$$a_2 = +\frac{1}{2}$$

$$a_3 = -\frac{1}{3}$$

$$\vdots$$



$$a_n \rightarrow 0 \text{ as } n \rightarrow \infty$$

By Sandwich Th.

$$-\frac{1}{n} \leq (-1)^n \frac{1}{n} \leq \frac{1}{n}$$

Both  $-\frac{1}{n}$  and  $\frac{1}{n}$  approach 0 as  $n \rightarrow \infty$ .

Def: The sequence  $\{a_n\}$  converges to the number  $L$  if for every number  $\epsilon > 0$ , there exists an integer  $N$  s.t. for all  $n > N \Rightarrow |a_n - L| < \epsilon$ .

If such number  $L$  does not exist, we say the sequence  $\{a_n\}$  diverges.

$\lim_{n \rightarrow \infty} a_n = \infty$   
 $\lim_{n \rightarrow \infty} a_n = -\infty$

Exp ① Fine  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$

(49)

② show that  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$  Let  $\epsilon > 0$ , we need to show that there exist an integer  $N$  such that for all

$$n > N \Rightarrow |a_n - L| < \epsilon$$

$$|\frac{1}{n} - 0| < \epsilon \Leftrightarrow |\frac{1}{n}| < \epsilon$$

This implication will hold if  $\frac{1}{n} < \epsilon \Leftrightarrow n > \frac{1}{\epsilon}$

Take  $N$  to be any integer greater than  $\frac{1}{\epsilon}$

③ Find  $\lim_{n \rightarrow \infty} K = K$

④ show that  $\lim_{n \rightarrow \infty} K = K$  Let  $\epsilon > 0$ , we need to show that  $\exists$  an integer  $N$  s.t for all

$$n > N \Rightarrow |a_n - L| < \epsilon$$

$$|K - K| < \epsilon \Leftrightarrow 0 < \epsilon$$

$N$  can be any positive integer.

Th Assume that  $\{a_n\}$  and  $\{b_n\}$  are sequences of real #'s, and let  $A$  and  $B$  be real #'s.

If  $\lim_{n \rightarrow \infty} a_n = A$  and  $\lim_{n \rightarrow \infty} b_n = B$ , then

①  $\lim_{n \rightarrow \infty} (a_n + b_n) = A + B$

..... Sum Rule

②  $\lim_{n \rightarrow \infty} (a_n - b_n) = A - B$

..... Difference Rule

③  $\lim_{n \rightarrow \infty} (K b_n) = K B$ , ( $K$  is any number) --- Constant Multiple Rule

④  $\lim_{n \rightarrow \infty} (a_n b_n) = A B$

..... Product Rule

⑤  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{A}{B}$  " $B \neq 0$ "

..... Quotient Rule

Exp ①  $\lim_{n \rightarrow \infty} \left( \frac{-\sqrt{3}}{n} \right) = -\sqrt{3} \lim_{n \rightarrow \infty} \frac{1}{n} = -\sqrt{3} \cdot 0 = 0$  (50)

②  $\lim_{n \rightarrow \infty} \left( \frac{2n+5}{3n} \right) = \lim_{n \rightarrow \infty} \left( \frac{2}{3} + \frac{5}{n} \right) = \frac{2}{3}$

③  $\lim_{n \rightarrow \infty} \frac{n-2n^3}{n^3} = \lim_{n \rightarrow \infty} \left( \frac{1}{n^2} - 2 \right) = -2 + \lim_{n \rightarrow \infty} \frac{1}{n^2} = -2 + 0 = -2$

④  $\lim_{n \rightarrow \infty} \frac{3 + \sqrt{8} n^5}{3 + \sqrt{2} n^5} = \lim_{n \rightarrow \infty} \frac{\frac{3}{n^5} + \sqrt{8}}{\frac{3}{n^5} + \sqrt{2}} = \frac{0 + \sqrt{8}}{0 + \sqrt{2}} = 2$

### Th (Sandwich Th.)

Let  $\{a_n\}$ ,  $\{b_n\}$  and  $\{c_n\}$  be sequences of real numbers with  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = L$ . If  $a_n \leq b_n \leq c_n$  for all  $n$  beyond some number  $N$ , then  $\lim_{n \rightarrow \infty} b_n = L$ .

Exp ①  $\lim_{n \rightarrow \infty} \frac{\sin n}{n} \rightarrow 0$  because  $-\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}$

②  $(-1)^n \frac{1}{n} \rightarrow 0$  because  $-\frac{1}{n} \leq (-1)^n \frac{1}{n} \leq \frac{1}{n}$

Th 5 ①  $\lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0$

$\lim_{n \rightarrow \infty} \frac{1/n}{1} = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$

②  $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$

$\lim_{n \rightarrow \infty} e^{\ln n^{1/n}} = \lim_{n \rightarrow \infty} e^{\frac{\ln n}{n}} = e^0 = 1$

③  $\lim_{n \rightarrow \infty} x^{1/n} = 1 \quad (x > 0)$

$\lim_{n \rightarrow \infty} e^{\ln x^{1/n}} = \lim_{n \rightarrow \infty} e^{\frac{\ln x}{n}} = e^0 = 1$

④  $\lim_{n \rightarrow \infty} x^n = 0 \quad (|x| < 1)$

$(\frac{1}{2})^n \rightarrow 0$  as  $n \rightarrow \infty$  /  $(-\frac{1}{2})^n = (-1)^n \frac{1}{2^n} \rightarrow 0$  by Sandwich  
 $\rightarrow x$  is fixed as  $n \rightarrow \infty$

⑤  $\lim_{n \rightarrow \infty} \left( 1 + \frac{x}{n} \right)^n = e^x$

$= \lim_{n \rightarrow \infty} e^{\ln \left( 1 + \frac{x}{n} \right)^n} = \lim_{n \rightarrow \infty} e^{\frac{\ln \left( 1 + \frac{x}{n} \right)}{1/n}} = \lim_{n \rightarrow \infty} e^{\frac{-\frac{x}{n^2}}{1 - \frac{x}{n}}} = e^x$

⑥  $\lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0, \text{ (any } x \text{)}$

$\frac{\infty}{\infty} \Rightarrow \frac{n x^{n-1}}{(n!)}, \frac{n(n-1)x^{n-2}}{(n!)}, \dots, \frac{\text{constant}}{(n!)^{(n-1)}} = 0$   
 or  $0 \leq \frac{x x \dots x}{n(n-1) \dots 1} \leq \frac{x}{n}$  or  $\frac{x}{n} \leq \frac{x x \dots x}{n(n-1) \dots 1} \leq 0$  Sandwich

Exp ①  $\lim_{n \rightarrow \infty} \frac{\ln n^3}{3n} = \lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0$  [1] (51)

②  $\lim_{n \rightarrow \infty} \sqrt[n]{n^3} = \lim_{n \rightarrow \infty} n^{\frac{3}{n}} = \lim_{n \rightarrow \infty} \left(n^{\frac{1}{n}}\right)^3 = 1^3 = 1$  [2]

③  $\lim_{n \rightarrow \infty} \sqrt[n]{\pi n} = \lim_{n \rightarrow \infty} \pi^{\frac{1}{n}} \lim_{n \rightarrow \infty} n^{\frac{1}{n}} = (1)(1) = 1$  [3] [2]

④  $\lim_{n \rightarrow \infty} \frac{\pi^{-n}}{e^n} = \lim_{n \rightarrow \infty} \left(\frac{\pi}{e}\right)^{-n} = \lim_{n \rightarrow \infty} \left(\frac{e}{\pi}\right)^n = 0$  [4]

⑤  $\lim_{n \rightarrow \infty} \left(\frac{n+1}{n-1}\right)^n = \lim_{n \rightarrow \infty} \left(\frac{n-1+2}{n-1}\right)^n = \lim_{n \rightarrow \infty} \left(1 + \frac{2}{n-1}\right)^n$   
 $= \lim_{u \rightarrow \infty} \left(1 + \frac{2}{u}\right)^{u+1}$   $u = n-1$   
 $1+u = n$   
 $= \lim_{u \rightarrow \infty} \left(1 + \frac{2}{u}\right) \lim_{u \rightarrow \infty} \left(1 + \frac{2}{u}\right)^u$  [5]  
 $= (1)(e^2) = e^2$  (see the book for another way.)

Exp Find a formula for the  $n^{\text{th}}$  term of the sequence

①  $1, -4, 9, -16, 25, \dots$   $a_n = (-1)^{n+1} n^2, n=1, 2, 3, \dots$   
 $n \geq 1$

②  $0, 3, 8, 15, 24, \dots$   $a_n = n^2 - 1, n \geq 1$

Exp (Recursive Defined Sequence) Assume the following sequence converges, find its limit.

$a_1 = 1, a_{n+1} = \frac{1}{2} a_n$   
 $a_2 = \frac{1}{2} a_1 = \frac{1}{2}$   
 $a_3 = \frac{1}{2} a_2 = \frac{1}{2} \left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2$   
 $a_4 = \frac{1}{2} a_3 = \frac{1}{2} \left(\frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^3$   
 $a_5 = \frac{1}{2} a_4 = \frac{1}{2} \left(\frac{1}{2}\right)^3 = \left(\frac{1}{2}\right)^4$   
 $\vdots$   
 $a_n = \left(\frac{1}{2}\right)^{n-1}$

$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left(\frac{1}{2}\right)^{n-1}$   
 $= 2 \lim_{n \rightarrow \infty} \left(\frac{1}{2}\right)^n$   
 $= 2 \cdot 0$   
 $= 0$