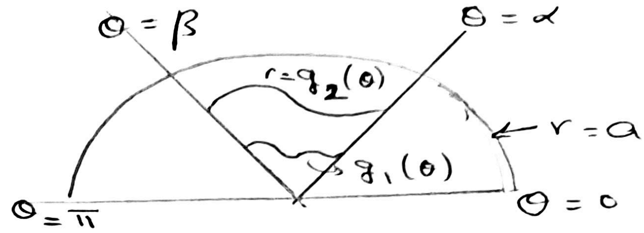
15.4 Double Integrals in Polar Form

$x = r\cos\theta$

$y = r\sin\theta$

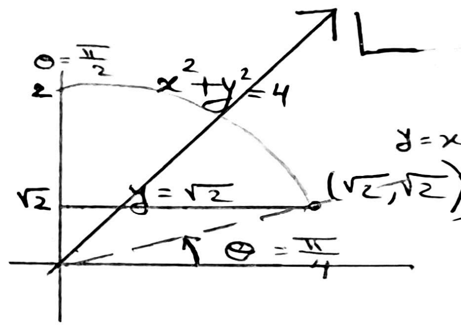
$x^2 + y^2 = r^2$, $\tan\theta = \frac{y}{x}$



$dA = dx\,dy$ or $dy\,dx$ or $r\,dr\,d\theta$

$$\therefore \quad V = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=g_1(\theta)}^{r=g_2(\theta)} f(r,\theta)\, r\, dr\, d\theta$$

How to find the limits of Integration



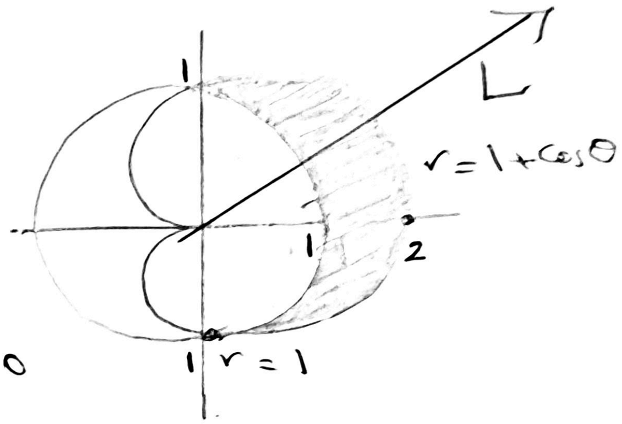
1. Sketch the region and label the bounding curves.

2. Find the r-limits of Integration. Imagine a ray L from the origin cutting through R in the direction of Increasing r. Mark the r values where L enters & leaves R

3. Find the smallest & largest θ-values that bound R.

$$\Rightarrow \iint_R f(r,\theta)\, dA = \int_{\pi/4}^{\pi/2} \int_{\sqrt{2}\csc\theta}^{r=2} f(r,\theta)\, r\, dr\, d\theta.$$

Example 1. Find the limits of integration for integrating $f(r,\theta)$ over the region R that lies inside the Cardioid $r = 1+\cos\theta$ and outside the circle $r=1$



Sol: r: $1 \le r \le 1+\cos\theta$

θ: $1+\cos\theta = 1 \Rightarrow \cos\theta = 0$

$\therefore \theta = \pm\frac{\pi}{2}$

$\therefore -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$$\therefore \iint_R f(x,y)\,dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{1}^{1+\cos\theta} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

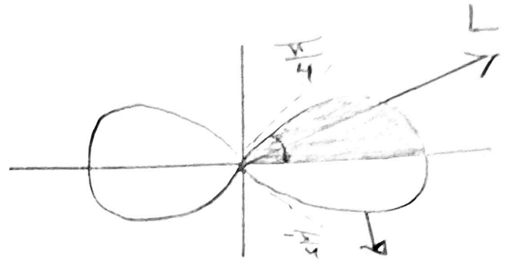
Area in Polar Coordinates

The area of a closed and bounded region R in the polar coordinate plane is

$$\boxed{A = \iint_R dA = \iint r\,dr\,d\theta}$$

Example 2: Find the area enclosed by the lemniscate $r^2 = 4\cos 2\theta$

$$A = 4\int_0^{\frac{\pi}{4}}\int_0^{\sqrt{4\cos 2\theta}} r\,dr\,d\theta$$



$r = \sqrt{4\cos 2\theta}$

$$= 4\int_0^{\frac{\pi}{4}}\left[\left(\frac{r^2}{2}\right)\Big|_0^{\sqrt{4\cos 2\theta}}\right]d\theta = 4\int_0^{\frac{\pi}{4}} 2\cos 2\theta\,d\theta = \boxed{4}$$

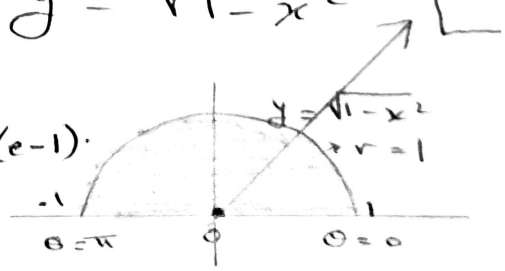
Changing Cartesian Integrals into Polar Integrals:

$$\iint_R f(x,y)\,dx\,dy = \iint f(r\cos\theta,\ r\sin\theta)\,r\,dr\,d\theta$$

Example 3. Evaluate $I = \iint_R e^{x^2+y^2}\,dy\,dx$

where R is the semicircular region bounded by the x-axis and the curve $y = \sqrt{1-x^2}$

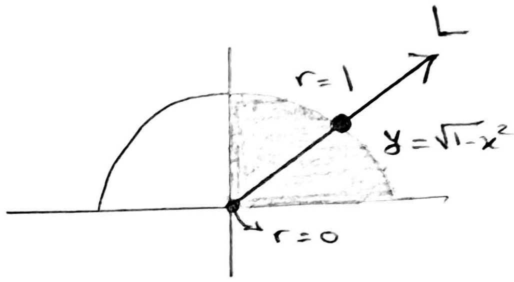
$$I = \int_0^{\pi}\int_0^1 \underbrace{e^{r^2}\cdot r\,dr}_{u=r^2}\,d\theta = \int_0^{\pi}\left(\frac{1}{2}e^{r^2}\Big|_{r=0}^{r=1}\right)d\theta = \frac{\pi}{2}(e-1).$$



Example 4: Evaluate the Integral

$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} (x^2+y^2)\,dy\,dx$$

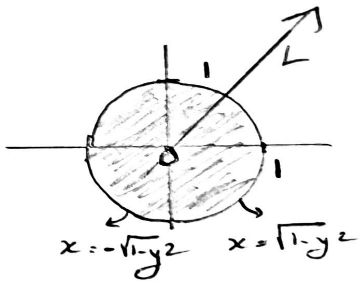
$$= \int_0^{\frac{\pi}{2}} \int_0^1 r^2 (r)\,dr\,d\theta =$$



$$= \int_0^{\frac{\pi}{2}} \left[\frac{r^4}{4} \Big|_0^1 \right] d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{4}\,d\theta = \frac{1}{4}\left(\frac{\pi}{2}\right) = \boxed{\frac{\pi}{8}}$$

Example.5. Find the Volume of the solid region bounded above by the paraboloid $z = 9 - x^2 - y^2$ and below by the unit circle ($x^2+y^2=1$) in the xy-plane

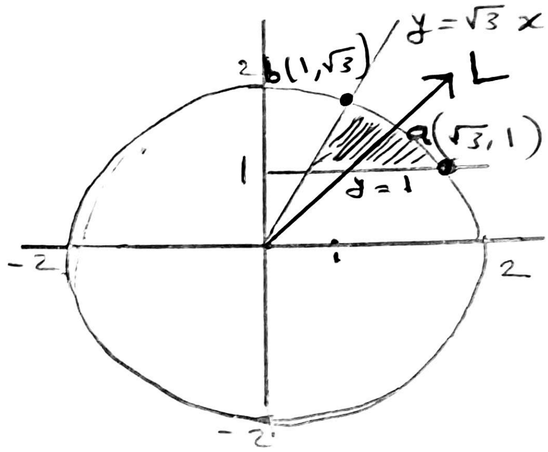
Sol: Volume $= \displaystyle\int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (9-x^2-y^2)\,dx\,dy$



$$= \int_0^{2\pi} \int_0^1 (9-r^2)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi} \left[\left(\frac{9r^2}{2} - \frac{9r^4}{4} \right) \Big|_0^1 \right] d\theta = \int_0^{2\pi} \frac{9}{2} - \frac{1}{4}\,d\theta = \boxed{\frac{17\pi}{2}}$$

Example 6: Using polar integration, find the area of the region R in the xy-plane enclosed by the circle $x^2+y^2=4$, above by the line $y=1$, and below by the line $y=\sqrt{3}\,x$

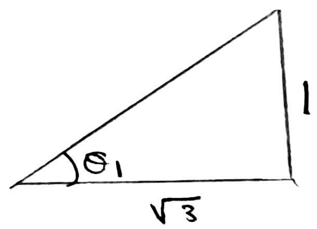


Sol: $\text{Area} = \iint_R 1\, dA$

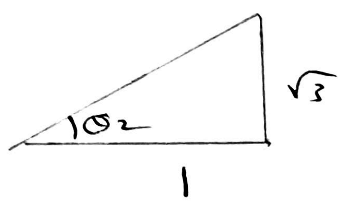
$$= \int_{\pi/6}^{\pi/3} \int_{\csc\theta}^{2} r\,dr\,d\theta$$

$$= \int_{\pi/6}^{\pi/3} \left(\frac{r^2}{2}\right)\Big|_{\csc\theta}^{2} d\theta = \int_{\pi/6}^{\pi/3} \left(2 - \frac{\csc^2\theta}{2}\right) d\theta = \boxed{\frac{\pi - \sqrt{3}}{3}}$$

$\Rightarrow \theta_1 = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$



$\theta_2 = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$



for a: $x^2+y^2=4$ & $y=1 \Rightarrow x=\sqrt{3} \Rightarrow (\sqrt{3}, 1)$

b: $x^2+y^2=4$ & $y=\sqrt{3}\,x \Rightarrow x^2+3x^2=4$

$\Rightarrow x^2 = 1$

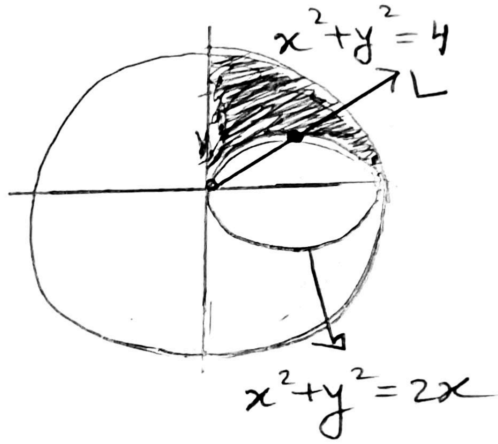
$\Rightarrow x = 1$

$\Rightarrow (1, \sqrt{3})$

Example 7. Evaluate $\iint_R y \, dA$, where

R lies in the first quadrant lies between

$x^2 + y^2 = 4$ & $x^2 + y^2 = 2x$.



Sol: $x^2 + y^2 = 2x$ ← Inner

$\Rightarrow \quad r^2 = 2r\cos\theta$

$\Rightarrow \quad \boxed{r = 2\cos\theta}$ or $\boxed{r = 0}$

$x^2 + y^2 = 4 \Rightarrow \boxed{r = 2}$

There fore $2\cos\theta \le r \le 2$, $0 \le \theta \le \frac{\pi}{2}$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \left[\int_{2\cos\theta}^{2} (r\sin\theta)\, r\, dr \right] d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{r^3}{3} \sin\theta \Big|_{2\cos\theta}^{2} \right) d\theta$$

Let $u = \cos\theta$
$du = -\sin\theta \, d\theta$
$\theta = 0, u = 1$
$\theta = \frac{\pi}{2}, u = 0$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{8}{3}\sin\theta - \frac{8\cos^3\theta \sin\theta}{3} \right) d\theta$$

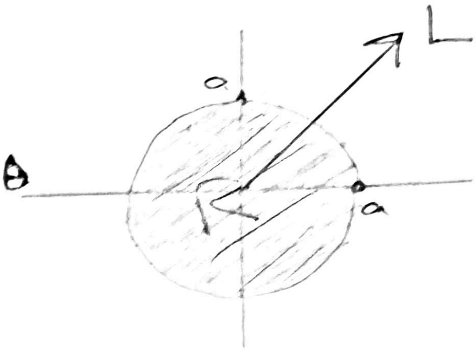
$$= -\frac{8}{3}\cos\theta \Big]_0^{\frac{\pi}{2}} - \frac{8}{3}\int_0^1 u^3\, du = \frac{8}{3} - \frac{2}{3}(1-0) = \boxed{2}$$

Example 8: Find the average value of

$f(x,y)=\sqrt{x^2+y^2}$ above the disk $x^2+y^2\le a^2$

in the xy-plane.

Sol: $\text{av}(f)=\dfrac{1}{\text{Area}(R)}\displaystyle\iint_R f(x,y)\,dA$



$$\text{Area}(R)=\iint_R dA=\int_0^{2\pi}\int_0^a r\,dr\,d\theta$$

$$=\int_0^{2\pi}\left.\frac{r^2}{2}\right]_0^a d\theta=\int_0^{2\pi}\frac{a^2}{2}\,d\theta=\boxed{\pi a^2}$$

$$\iint_R f(x,y)\,dA=\iint_R\sqrt{x^2+y^2}\,dA$$

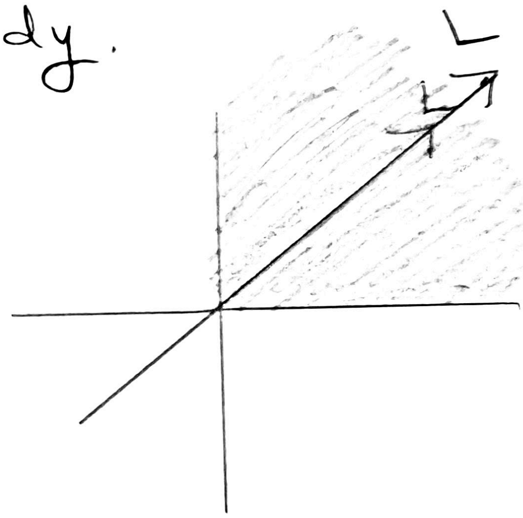
$$=\int_0^{2\pi}\int_0^a\sqrt{r^2}\cdot r\,dr\,d\theta=\int_0^{2\pi}\left(\left.\frac{r^3}{3}\right|_0^a\right)d\theta=\boxed{\frac{2\pi}{3}a^3}$$

$$\therefore\ \text{av}(f)=\frac{1}{\pi a^2}\left(\frac{2\pi}{3}a^3\right)=\boxed{\frac{2}{3}a}$$

Example 9. Use your knowledge in section 15.4 to prove that $I = \int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$.

Sol: $I^2 = \left(\int_0^\infty e^{-x^2}\,dx\right)\left(\int_0^\infty e^{-y^2}\,dy\right)$

$$= \int_0^\infty\int_0^\infty e^{-x^2-y^2}\,dx\,dy.$$



$$I^2 = \int_0^{\frac{\pi}{2}}\int_0^\infty e^{-r^2}\, r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}}\left(\lim_{A\to\infty}\int_0^A e^{-r^2}\, r\,dr\right)d\theta$$

Let $u = -r^2$

$du = -2r\,dr$

$-\frac{1}{2}du = r\,dr$

$$= \int_0^{\frac{\pi}{2}}\left(\lim_{A\to\infty} -\frac{1}{2}e^{-r^2}\Big|_0^A\right)d\theta$$

$$= \int_0^{\frac{\pi}{2}}\left(\lim_{A\to\infty} \cancel{-\frac{1}{2}e^{-A^2}} + \frac{1}{2}\right)d\theta = \int_0^{\frac{\pi}{2}}\frac{1}{2}\,d\theta = \boxed{\frac{\pi}{4} = I^2}$$

$$\therefore\ I = \frac{\sqrt{\pi}}{2}$$