1.3: The probability set function.

let $C \subset \mathcal{C}$

$\uparrow$ Event $\quad \uparrow$ sample.

$\rightarrow p(C) =$ prob. of event C.

Def 7: The probability set function P is defined as follows, $P: \mathcal{C} \rightarrow [0,1]$.

$\rightarrow$ P has the following properties:

1. $p(C) \geq 0$ for all $C \subset \mathcal{C}$
2. $p(C_1 \cup C_2 \cup \dots) = p(C_1) + p(C_2) + \dots$ for all $C_1, C_2 \dots \subset \mathcal{C}$ such that $C_i \cap C_j = \phi$ «disjoint», $i \neq j$.
3. $p(\mathcal{C}) = 1$.

Theorem 1: For each $C \subset \mathcal{C}$, $p(C) = 1 - p(C^*)$.

Theorem 2: The probability of the null set is zero, i.e $p(\phi) = 0$.

Theorem 3: If C_1, C_2 subset from $\mathcal{C}$ s.t $C_1 \subset C_2$ then $p(C_1) \leq p(C_2)$.

Theorem 4: For each subset C from $\mathcal{C}$, $0 \leq p(C) \leq 1$.

Theorem 5: If C_1 and C_2 are subsets from $\mathcal{C}$ then

$$p(C_1 \cup C_2) = p(C_1) + p(C_2) - p(C_1 \cap C_2).$$

Proofs:

Thm 1

$\mathcal{C} = C \cup C^*$

$\emptyset = C \cap C^*$

$P(\mathcal{C}) = P(C \cup C^*) = P(C) + P(C^*)$

$1 = P(C) + P(C^*)$

$P(C) = 1 - P(C^*)$ ∎

Thm 2

$\mathcal{C}^* = \emptyset$

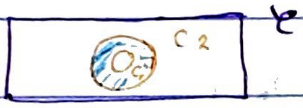
$P(\mathcal{C}) = 1 - P(\mathcal{C}^*)$ By Thm 1

$1 = 1 - P(\emptyset)$

$P(\emptyset) = 0$.

∎

Thm 3



$C_1 \subset C_2$

$C_2 = C_1 \cup (C_1^* \cap C_2)$

$\emptyset = C_1 \cap (C_1^* \cap C_2)$

$P(C_2) = P(C_1 \cup (C_1^* \cap C_2))$

$P(C_2) = P(C_1) + P(C_1^* \cap C_2)$.

$P(C_2) - P(C_1) = P(C_1^* \cap C_2)$

$\rightarrow P(C_1^* \cap C_2) \geq 0$

$\Rightarrow P(C_2) - P(C_1) \geq 0$

$\Rightarrow P(C_2) \geq P(C_1)$ ∎

Thm 4

since $\emptyset \subset C \subset \mathcal{C}$

By Thm 3: $P(\emptyset) \leq P(C) \leq P(\mathcal{C})$

But $\underbrace{0}_{\text{By Thm 2}} \leq P(C) \leq \underbrace{1}_{\text{By def}}$

So $0 \leq P(C) \leq 1$ ∎

Thm 5:

$C_1 \cup C_2 = C_1 \cup (C_2 \cap C_1^*)$

$C_2 = (C_1 \cap C_2) \cup (C_2 \cap C_1^*)$

$\rightarrow C_1 \cup C_2 = C_1 \cup (C_2 \cap C_1^*) + C_1 \cap (C_2 \cap C_1^*)$

$\rightarrow$

$P(C_1 \cup C_2) = P(C_1) + P(C_2 \cap C_1^*)$

and $P(C_2) = P(C_1 \cap C_2) + P(C_1^* \cap C_2)$

$\Rightarrow P(C_1^* \cap C_2) = P(C_2) - P(C_1^* \cap C_2)$

So $P(C_1 \cup C_2) = P(C_1) + P(C_2) - P(C_1^* \cap C_2)$.

∎

example 1 :

$\mathcal{E} = \{(1,1),(1,2), \ldots\ldots, (6,6)\}$

probability of any sample point $= \frac{1}{36}$

$C_1 = \{(1,1),(2,1),(3,1),(4,1),(5,1)\}$.

$C_2 = \{(1,2),(2,2),(3,2)\}$.

Find :

1. $p(C_1) = \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} = \frac{5}{36}$

2. $p(C_2) = \frac{1}{36} + \frac{1}{36} + \frac{1}{36} = \frac{3}{36}$.

3. $p(C_1 \cap C_2) =$

$\hookrightarrow C_1 \cap C_2 = \emptyset$ so $p(C_1 \cap C_2) = p(\emptyset) = 0$.

4. $p(C_1 \cup C_2) = p(C_1) + p(C_2) - p(C_1 \cap C_2)$

$= \frac{5}{36} + \frac{3}{36} - 0 = \frac{8}{36}$

---

example 2 : Fair ✓

$\mathcal{E} = \{(H,H),(H,T),(T,H),(T,T)\}$

probability of each sample point $= \frac{1}{4}$

$C_1 = \{(H,H),(H,T)\}$

$C_2 = \{(H,H),(T,H)\}$. Find.

1. $p(C_1) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

2. $p(C_2) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

3. $p(C_1 \cap C_2) \rightarrow C_1 \cap C_2 = \{(H,H)\}$ so $p(C_1 \cap C_2) = \frac{1}{4}$.

4. $p(C_1 \cup C_2) = p(C_1) + p(C_2) - p(C_1 \cap C_2)$

$= \frac{1}{2} + \frac{1}{2} - \frac{1}{4}$

$= \frac{3}{4}$ ✓

example 3: ordinary deck of 52 cards. أوراق اللعب مكونة من 52 بطاقة

كل وحدة منهم فيها 13 بطاقة

- ♡ : hearts A 2 3 4 .... 10 J Q K
- ◇ : Diamonds A 2 3 4 .... 10 J Q K. نسحب بطاقات بشكل عشوائي Random
- ♤ : spade A 2 3 4 .... 10 J Q K
- ♧ : clubs A 2 3 4 .... 10 J Q K.

→ E_1 : event of having a spade (13) when drawing 1 cards at random:

$$P(E_1) = \frac{13}{52} = \frac{1}{4}.$$

→ E_2 : event of having a King when drawing 1 cards at random:

$$P(E_2) = \frac{4}{52} = \frac{1}{13}.$$

→ E_3 : event of having 3 Kings and 2 Queens when drawing 5 card at random?

$$P(E_3) = \frac{\binom{4}{3}\binom{4}{2}\binom{44}{0}}{\binom{52}{5}} \qquad \binom{44}{0} = 1$$

$4+4+44=52$

$3+2+0=5$

4 : عدد K بكل البطاقات

4 : عدد Q بكل البطاقات

52 → عدد البطاقات كلهم

5 → عدد البطاقات إلي بين ايديهم

لحد ما نسحب من 52 بطاقة 5 بطاقات بحيث انه 3 منهم K و 2 منهم Q

$$= \frac{24}{2598960} = 9.2 \times 10^{-6}.$$

→ E_4: event of having 2 Kings and 2 Queens and 1 Jack when drawing 5 cards at Random:

$$P(E_4) = \frac{\binom{4}{2}\binom{4}{2}\binom{4}{1}\binom{40}{0}}{\binom{52}{5}} = 5.5 \times 10^{-5}.$$

Remark of exp3 :

$$p(E_1) = \frac{\binom{13}{1}\binom{39}{0}}{\binom{52}{1}} = \frac{13}{52} = \frac{1}{4}$$

$$p(E_2) = \frac{\binom{4}{1}\binom{48}{0}}{\binom{52}{1}} = \frac{4}{52} = \frac{1}{13}.$$