4.6 Exercises :

Q57 : let X_1, X_2, X_3 be a Random sample from a distribution of the continuous type have p.d.f $f(x) = \begin{cases} 2X , & 0<X<1 \\ 0 , & \text{elsewhere} \end{cases}$

a. compute the probability that the smallest of these X_i exceeds the median of the dist.

$$F(X) = Pr(X_i < x) = \int_0^1 2x\,dx = X^2 \Rightarrow F(x) = \begin{cases} 0 , & x \le 0 \\ x^2 , & 0 < x \le 1 \\ 1 , & x \ge 1 \end{cases}$$

median $\rightarrow \int_0^t 2x dx = \frac{1}{2} \rightarrow t^2 = \frac{1}{2} \rightarrow t = \frac{1}{\sqrt{2}}$

$\rightarrow$ smallest $X_1 = y_1$

- $g_1(y_1) = \frac{3!}{0!\,2!} \left(\cancel{F(y_1)}\right)^{\cancel{0}} \left(1 - F(y_1)\right)^2 f(y_1) , \quad 0<y_1<1$

$$= 3\,(1-y_1^2)^2\, 2y_1 , \quad 0<y_1<1$$

$$= 6y_1 [1-2y_1^2 + y_1^4] , \quad 0<y_1<1$$

$$= \begin{cases} 6y_1 - 12y_1^3 + 6y_1^5 , & 0<y_1<1 \\ 0 , & \text{elsewhere} \end{cases}$$

- $Pr(y_1 > \frac{1}{\sqrt{2}}) = \int_{\frac{1}{\sqrt{2}}}^{1} 6y_1 - 12y_1^3 + 6y_1^5 \; dy_1$

$$= 3y_1^2 - 3y_1^4 + y_1^6 \Big|_{\frac{1}{\sqrt{2}}}^{1}$$

$$= \cancel{3-3}+1 - \left(\frac{3}{2} - \frac{3}{4} + \frac{1}{8}\right)$$

$$= \frac{8\times 1}{8} - \frac{4\times 3}{4\times 2} + \frac{2\times 3}{2\times 4} - \frac{1}{8}$$

$$= \frac{8-12+6-1}{8} = \underline{\underline{\frac{1}{8}}}$$

probability that the smallest (X_1) exceeds median $= \frac{1}{8}$.

b. if $Y_1 < Y_2 < Y_3$ are the order statistics, find the correlation between Y_2 and Y_3.

• $g(y_2, y_3) = \frac{3!}{1!\,0!\,0!} \left(F(y_2)\right)^1 \left(F(y_3) - F(y_2)\right)^0 \left(1 - F(y_3)\right)^0 f(y_2)\, f(y_3), \quad 0<y<y_3<1$

$= 6\, y_2^2\, (2y_2)(2y_3)$

$= \begin{cases} 24\, y_2^3\, y_3 & , \; 0<y_2<y_3<1 \\ 0 & , \; \text{elsewhere} \end{cases}$

• $E(y_2 y_3) = \int_0^1 \int_{y_3}^1 y_2 y_3 \,(24\, y_2^3 y_3)\; dy_2\, dy_3$

$= \int_0^1 \int_{y_3}^1 24\, y_2^4\, y_3^2 \; dy_2\, dy_3$

$= \int_0^1 \frac{24\, y_2^5\, y_3^2}{5} \Big|_{y_3}^1 \; dy_3$

$= \int_0^1 \frac{24}{5}\left(y_3^2 - y_3^7\right) dy_3$

$= \frac{24}{5}\left(\frac{y_3^3}{3} - \frac{y_3^8}{8} \Big|_0^1\right) = \frac{24}{5}\left(\frac{1}{3} - \frac{1}{8}\right) = \frac{8}{5} - \frac{3}{5} = \frac{5}{5} = 1$

$\Rightarrow \quad E(y_1 y_2) = 1$

• $g(y_2) = \frac{3!}{1!\,1!} \left(F(y_2)\right)^1 \left(1 - F(y_2)\right)^1 f(y_2) \quad = 6\, y_2^2\,(1 - y_2^2)\; 2y_2$

$= 12\, y_2^3 - 12\, y_2^5$

$\rightsquigarrow g(y_2) = \begin{cases} 12y_2^3 - 12\, y_2^5 & , \; 0<y_2<1 \\ 0 & , \; \text{else} \end{cases}$

$\bullet\ E(y_2) = \int_0^1 12y_2^3 - 12y_2^5 \, dy_2 = 3y_2^4 - 2y_2^6 \Big|_0^1$

$= 3 - 2$

$= 1$

$\rightarrow E(y_2) = 1$

$\bullet\ g(y_3) = \frac{3!}{2!\,0!} \left(F(y_3)\right)^2 \cancel{\left(1 - F(y_3)\right)^0} \, f(y_3)$

$= 3 \, y_3^4 \, 2y_3$

$= 6 \, y_3^5$

$$g(y_3) = \begin{cases} 6y_3^5 , & 0 < y_3 < 1 \\ 0 , & \text{else} \end{cases}$$

$\bullet\ E(y_3) = \int_0^1 6y_3^5 \, dy_3 = y_3^6 \Big|_0^1 = 1$

$\Rightarrow \rho = \frac{E(y_2 y_3) - E(y_2)E(y_3)}{\sigma_2 \sigma_3} = \frac{1 - 1 \cdot 1}{\underline{\sigma_2 \sigma_3}} = 0$

عشان البسط 0
ما في داعي أوجد σ_2 , σ_3

so Y_2 , Y_3 independent

Q59: let $Y_1 < Y_2 < Y_3 < Y_4 < Y_5$ denote the order of statistics of a Random sample of size 5 from distribution having p.d.f $f(x) = \begin{cases} e^{-x} , & 0 < x < \infty \\ 0 , & \text{elsewhere} \end{cases}$

show $Z_1 = Y_2$ and $Z_2 = Y_2 - Y_4$ are independent.

$$Pr(M \leq m) = \int_0^m e^{-x}\, dx = -e^{-x}\Big|_0^m = e^{-x}\Big|_m^0 = 1 - e^{-m}$$

$$F(x) = \begin{cases} 0 , & x \leq 0 \\ 1 - e^{-x} , & 0 \leq x < \infty \end{cases}$$

$$g(y_2, y_4) = \frac{5!}{1!\,1!\,1!} (F(y_2))^1 \left(F(y_4) - F(y_2) \right)^1 (1 - F(y_4))^1 f(y_2) f(y_4).$$

$$= 120 \, (1 - e^{-y_2})(1 - e^{-y_4} - (1 - e^{-y_2}))(1 - (1 - e^{-y_4}))\, e^{-y_2} e^{-y_4}$$

$$= 120 \, (1 - e^{-y_2})(-e^{-y_4} + e^{-y_2})(e^{-y_4})\, e^{-y_2} e^{-y_4}$$

$$= 120 \, e^{-y_2} e^{-2y_4} \left(-e^{-y_4} + e^{-y_2} + e^{-y_2} e^{-y_4} - e^{-2y_2} \right)$$

$$= -120 e^{-y_2} e^{-3y_4} + 120 e^{-2y_2} e^{-2y_4} + 120 e^{-2y_2} e^{-3y_4} - 120 e^{-3y_2} e^{-y_4}$$

$$\Rightarrow g(y_2, y_4) = \begin{cases} 120 e^{-y_2} e^{-2y_4} \left(-e^{-y_4} + e^{-y_2} + e^{-y_2} e^{-y_4} - e^{-2y_2} \right) , & 0 < y_2 < y_4 < 1 \\ 0 , & \text{else} \end{cases}$$

$$Z_1 = Y_2 \rightarrow Y_2 = Z_1$$

$$Z_2 = Y_2 - Y_4 \rightarrow Y_4 = Z_2 + Z_1$$

$$J = \begin{vmatrix} \frac{dy_2}{dz_1} & \frac{dy_2}{dz_2} \\ \frac{dy_4}{dz_1} & \frac{dy_4}{dz_2} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$$

Cont. 59.

$$h(z_1, z_2) = \begin{cases} g(z_1, z_2 + z_1) \ \underset{\sim}{|J|} = 1, & 0 < z_1 < z_2 < 1 \\ 0, & \text{else} \end{cases}$$

$$= \begin{cases} 120 e^{-z_1} e^{-2(z_1+z_2)} \left(-e^{-2(z_1+z_2)} + e^{-z_1} + e^{-z_1} e^{-(z_1+z_2)} - e^{-2z_1}\right), & 0 < z_1 < z_2 < 1 \\ 0, & \text{else} \end{cases}$$

to show its independent :

$\exists h_1(z_1) =$

Q62 : Find the probability that the Range of a Random sample of size 4 from the uniform distribution having p.d.f $f(x) = \begin{cases} 1 & , 0<x<1 \\ 0 & , \text{else} \end{cases}$ is less than $\frac{1}{2}$

$\rightarrow F(x) = \begin{cases} 0 & , x \leq 0 \\ x & , 0 \leq x \leq 1 \\ 1 & , x \geq 1 \end{cases}$

$\rightarrow Z_1 = Y_4 - Y_1 \qquad \rightarrow Y_1 = Z_2 - Z_1$

$Z_2 = Y_4 \qquad \rightarrow Y_4 = Z_2$

$\rightarrow g(y_1, y_4) = \frac{4!}{2!\,0!\,0!} \left(\cancel{F(y_1)}\right)^0 \left(F(y_4) - F(y_1)\right)^2 \left(\cancel{1 - F(y_4)}\right)^0 f(y_1)\, f(y_2)$

$= 12\,(y_4 - y_2)^2\,(1)(1)$

$= 12\,(y_4^2 - 2y_2 y_4 + y_2^2)$

$g(y_1, y_4) = \begin{cases} 12(y_4^2 - 2y_2 y_4 + y_2^2) & , 0 < y_1 < y_4 < 1 \\ 0 & , \text{else} \end{cases}$

$\rightarrow J = \begin{vmatrix} \frac{dy_1}{dz_1} & \frac{dy_1}{dz_2} \\ \frac{dy_4}{dz_1} & \frac{dy_4}{dz_2} \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} = -1$

$\rightarrow h(z_1, z_2) = \begin{cases} g(z_2 - z_1\, , z_2)\ |J| & , 0 < z_1 < z_2 < 1 \\ 0 & , \text{else} \end{cases}$

$= \begin{cases} 12\,(z_2 - z_2 + z_1)^2\ |-1| & , 0 < z_1 < z_2 < 1 \\ 0 & , \text{else} \end{cases}$

$$h(z_1, z_2) = \begin{cases} 12 z_1^2 & , \quad 0 < z_1 < z_2 < 1 \\ 0 & , \quad \text{else} \end{cases}$$

$$\rightarrow h(z_1) = \int_{z_1}^{1} 12 z_1^2 \, dz_2 = 12 z_1^2 z_2 \Big|_{z_1}^{1} = 12 z_1^2 - 12 z_1^3$$

$$h(z_1) = \begin{cases} 12 z_1^2 (1 - z_1) & , \quad 0 < z_1 < z_2 < 1 \\ 0 & , \quad \text{else} \end{cases}$$

$$\rightarrow Pr(z_1 < \tfrac{1}{2}) = \int_{0}^{\frac{1}{2}} 12 z_1^2 - 12 z_1^3 \, dz_1$$

$$= 4 z_1^3 - 3 z_1^4 \Big|_{0}^{\frac{1}{2}}$$

$$= \frac{2 \times 4}{2 \times 8} - \frac{3}{16}$$

$$= \frac{5}{16}$$