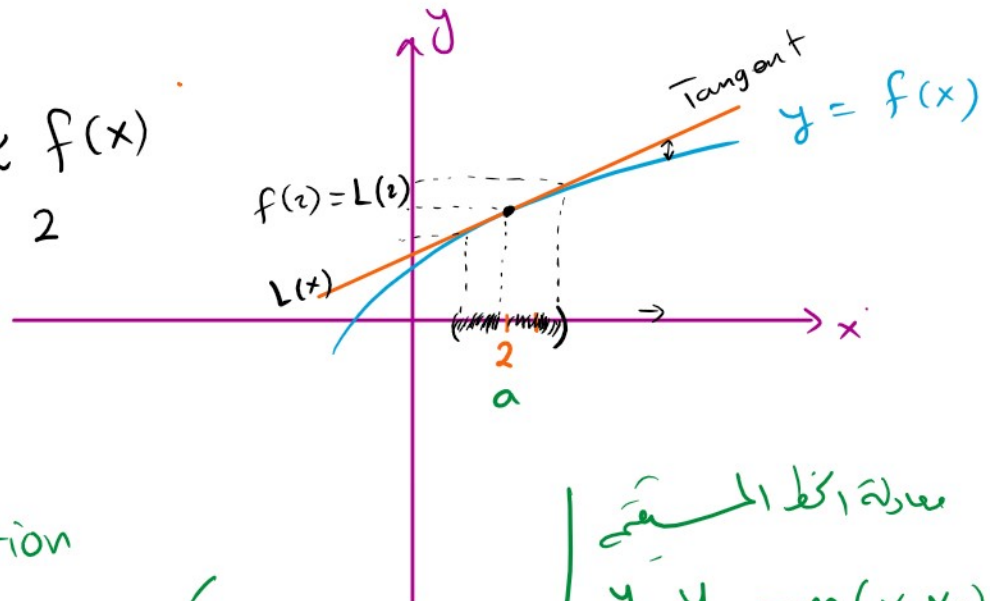


linearization $f(2)$

$$L(x) \approx f(x) \\ \text{near } 2$$

 $L(x)$
 $L(x)$: linearization

$$L(x) = f(a) + f'(a)(x-a)$$

معادلة الخط المماس

$$y - y_0 = m(x - x_0)$$

$$(x_0, y_0) = (a, f(a))$$

$$m = f'(a)$$

$$y - f(a) = f'(a)(x - a)$$

$$L(x) \approx f(x)$$

 $L(x)$ approximates $f(x)$ near a

Standard linear approximation of f at $x=a$

Ex: Let $f(x) = \sqrt{x+1}$

Find linearization of f at $x=0$ $\downarrow a=0$

$$L(x) = f(a) + f'(a)(x-a)$$

$$= f(0) + f'(0)(x-0)$$

$$= \sqrt{0+1} + \frac{1}{2\sqrt{0+1}}x$$

$$= 1 + \frac{1}{2}x$$

$$f'(x) = \frac{1}{2\sqrt{x+1}}$$

$$f'(0) = \frac{1}{2\sqrt{0+1}}$$

$$f(a) = f(0)$$

$$= \sqrt{0+1}$$

$$= \sqrt{1}$$

$$= 1$$

$$= 1 + \left[\frac{1}{2} \right] x$$

$$\left| \frac{1}{2\sqrt{0+1}} \right| = 1$$

$$= \frac{1}{2(1)}$$

$$= \frac{1}{2}$$

$$L(x) = 1 + \frac{x}{2}$$

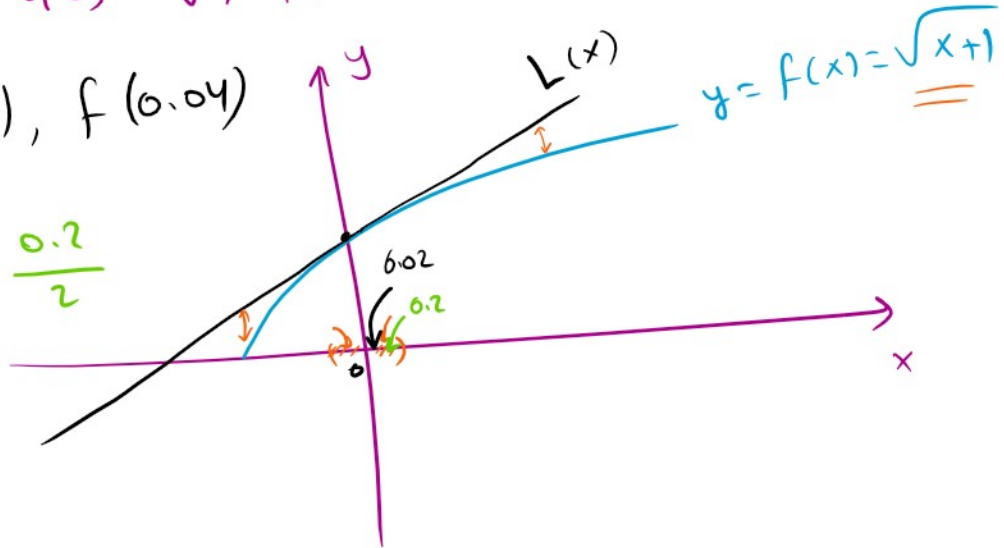
$L(x)$ approximates $\sqrt{x+1}$ near $x=0$

2) Estimate $f(0.2)$, $f(0.04)$

$$f(0.2) \approx L(0.2) = 1 + \frac{0.2}{2}$$

$$= 1 + 0.1$$

$$= 1.1$$



$$f(0.2) \approx 1.1$$

$$f(0.04) \approx L(0.04) = 1 + \frac{0.04}{2} = 1 + 0.02 = 1.02$$

$$f(0.04) \approx 1.02$$

3) Find true values of $f(0.2)$, $f(0.04)$

$$f(0.2) = \sqrt{0.2 + 1} = \sqrt{1.2} = 1.095445115$$

$$f(0.04) = \sqrt{0.04 + 1} = \sqrt{1.04} = 1.0198039027$$

4) Find Error

④ Find Error

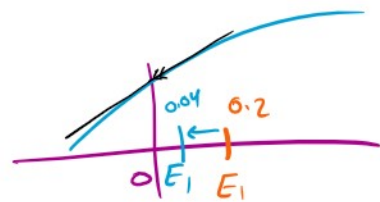
$$\text{Error} = E = |\text{True value} - \text{Estimated value}|$$

$$E_1 = |f(0.2) - L(0.2)| = |1.09544515 - 1.1| \approx 0.00455$$

$$E_2 = |f(0.04) - L(0.04)| = |1.0198039027 - 1.02| \approx 0.000196$$

$$E_2 < E_1$$

as $x \rightarrow 0$



as $x \rightarrow 0 \Rightarrow E \downarrow \Rightarrow L(x)$ is better approximation

$$L(x) = f(a) + f'(a)(x-a)$$

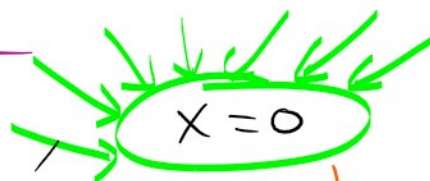
Tangent line
 $(x_0, y_0), f'(a)$

$$y - y_0 = m(x - x_0)$$

$$y - f(a) = f'(a)(x - a)$$

$$y = f(a) + f'(a)(x - a)$$

Ex $f(x) = \sqrt{x+1}$



$$f(x) = (1+x)^{\frac{1}{2}}$$

=

$$L(x) = 1 + \frac{x}{2}$$

$$\Rightarrow f(x) = (1+x)^{\frac{1}{2}}$$

$$L(x) = 1 + \frac{1}{2}x$$

Remark $f(x) = (x+1)^m$, $x=0$

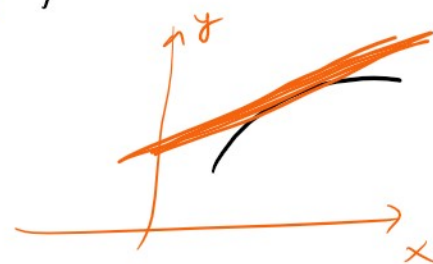
The linearisation about $x=0$ is

$$L(x) = 1 + mx$$

$\approx$
approximation
تقريب

Exp Find $L(x)$ at $x=0$ for

$$① f(x) = (1-x)^{\frac{3}{2}}$$



$$L(x) = 1 - \frac{3}{2}x$$

$$f(0) = \dots$$

$$f'(0) = \dots$$

$$② g(x) = \frac{1}{\sqrt[3]{1+x}}$$

$$= \frac{1}{(1+x)^{\frac{1}{3}}}$$

- $\frac{1}{3}$

$$= (1+x)^{-\frac{1}{3}}$$

$$L(x) = 1 + (-\frac{1}{3})x \\ = 1 - \frac{x}{3}$$

Exp Find linearization of $f(x) = \cos x + 1$
at $x = \frac{\pi}{3}$

$$a = \frac{\pi}{3} \Rightarrow f(a) = f\left(\frac{\pi}{3}\right) = \cos \frac{\pi}{3} + 1$$

$$= \frac{1}{2} + 1 = \frac{3}{2}$$

$$\left. \begin{array}{l} f'(x) = -\sin x \\ f'\left(\frac{\pi}{3}\right) = -\sin \frac{\pi}{3} \\ = -\frac{\sqrt{3}}{2} \end{array} \right\}$$

$$f'(a) = f'\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

$$L(x) = f\left(\frac{\pi}{3}\right) + f'\left(\frac{\pi}{3}\right) \left(x - \frac{\pi}{3}\right)$$

$$= \frac{3}{2} + -\frac{\sqrt{3}}{2} \left(x - \frac{\pi}{3}\right)$$

$$= \frac{3}{2} - \frac{\sqrt{3}}{2}x + \frac{\sqrt{3}\pi}{6}$$

$$= \left(\frac{3}{2} + \frac{\sqrt{3}\pi}{6}\right) - \frac{\sqrt{3}}{2}x \quad \checkmark$$

Differential

$y = f(x) \rightarrow$ differentiable

$$\dot{y} = \dot{f}(x)$$

$$\frac{dy}{dx} = \dot{f}(x)$$

$$\left. \frac{dy}{dx} \right|_{x=a} = \dot{f}(a)$$

$$dy = \dot{f}(a) dx$$

differential
(Estimated Value)
 $dy \approx \Delta y$
True Value

$$\begin{aligned} \Delta y &= y_2 - y_1 \\ &= f(x_2) - f(x_1) = \Delta f \end{aligned}$$

$$\begin{aligned} \Delta x &= x_2 - x_1 \\ &= dx \end{aligned}$$

$$\text{Error} = E = \left| \frac{\text{True Value} - \text{Estimated Value}}{\text{True Value}} \right|$$

$$= |\Delta y - dy|$$

$$\Delta L \approx f(x)$$

$$dy \approx f(x)$$

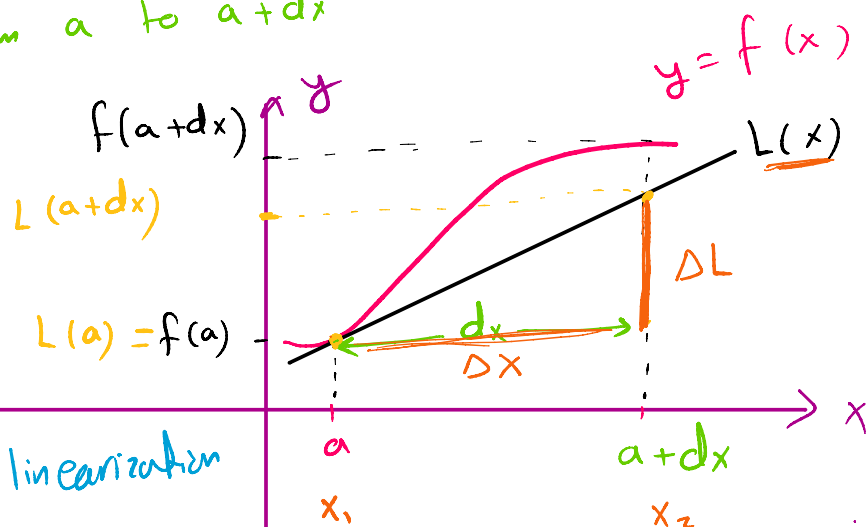
Exp show that $dy = \Delta L$

Assume x changes from a to $a+dx$

$$\Delta x = x_2 - x_1$$

$$= (a+dx) - a$$

$$= dx$$



Assume $L(x)$ is the linearization of f at $x=a$

True change is $\Delta f = f(x_2) - f(x_1) = f(a+dx) - f(a) = \Delta y$

Estimated change "differential" $dy = f'(a) dx$

$$= \frac{\Delta L}{\Delta x} dx$$

$$= \Delta L$$

Hence, we can use differential dy to estimate the true change Δy

$$\begin{array}{l|l} dy \approx \Delta y & \Delta L \approx \Delta y \\ dy \approx \Delta f & \Delta L \approx \Delta f \end{array}$$

Exp 1 Find differential of $y = x^3 + 2x$ at $x = 1$, $dx = \frac{1}{5}$

estimated change

$$\begin{aligned} dy &= f'(a) dx \\ &= f'(1) dx \\ &= \boxed{5} \boxed{\frac{1}{5}} \end{aligned}$$

$$\begin{aligned} y' &= 3x^2 + 2 \\ y'(1) &= 3(1)^2 + 2 \\ &= 3 + 2 \\ &= 5 \end{aligned}$$

✓ = 1 ✓

2

Find True change

$$\begin{aligned} \Delta f &= f(x_2) - f(x_1) \\ &= f\left(\frac{6}{5}\right) - f(1) \\ &= f(1.2) - f(1) \end{aligned}$$

$$\begin{array}{l} y = x^3 + 2x \\ f(1) = 1^3 + 2 \\ \quad = 1 + 2 = 3 \\ \hline f(1.2) = (1.2)^3 + 2(1.2) \\ \quad = 4.128 \end{array}$$

$$\Delta x = dx = x_2 - x_1$$

$$\frac{1}{5} = x_2 - 1$$

$$x_2 = 1 + \frac{1}{5}$$

$$x_2 = \frac{6}{5}$$

$$= f(1.2) - f(1) = 4.128$$

$$x_2 = \frac{6}{5}$$

$$= 4.128 - 3 = 1.128$$

True change

$dy = 1$
estimated change

(3)

Find error $\Rightarrow E = |TV - EV|$
 $= |1.128 - 1| = \underline{\underline{0.128}}$

Ex Find differential for $y = \tan(2x)$
 at $x = 0$ if $dx = \frac{1}{4}$

$$\begin{aligned} dy &= f'(a) dx \\ &= f'(0) dx \\ &= (2) \left(\frac{1}{4}\right) \\ &= \frac{1}{2} \end{aligned}$$

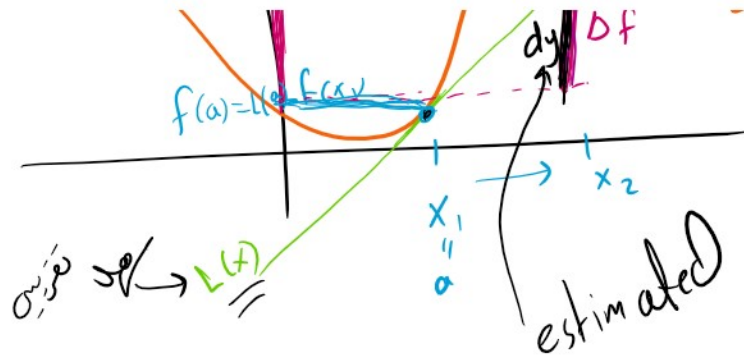
$$\begin{aligned} f'(x) &= 2 \sec^2(2x) \\ f'(0) &= 2 \sec^2 0 \\ &= 2 (1)^2 \\ &= 2 \end{aligned}$$

$$\begin{aligned} y &= 5 \\ y(3) &= 5 \\ y(3.001) &= 5 \end{aligned}$$

$f(x)$



$dy \approx \Delta f$
 ↓
 differential



$L(x) = f(x)$ only at $a \rightarrow E = 0$

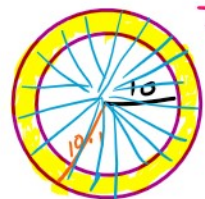
$E = |\Delta f - dy| \Rightarrow 0$

Exp r : radius of circle
 Assume r changes from 10 cm to 10.1 cm

$r_1 = 10$
 $r_2 = 10.1$
 $\Delta r = dr = r_2 - r_1$
 $= 10.1 - 10$
 $= 0.1$

① Estimate the change in the circle's area

estimated change in area of circle
 dA



$A = r^2 \pi$

$dA = 2r\pi dr$
 $= 2(10)\pi (0.1)$

$dA = 2\pi$
 EV

② Find true change in the area

$\Delta A = A(r_2) - A(r_1)$
 $= A(10.1) - A(10)$
 $= (10.1)^2 \pi - 10^2 \pi$
 $= 102.01 \pi - 100 \pi$

$dA = dy$
 $dx = dr$

error

(3) Find error

$$E = |TV - EV|$$

$$= |2.01\pi - 2\pi| = \underline{\underline{0.01\pi}}$$

$$TV = 2.01\pi$$

(4) Estimate the area of the large circle

$$A(r_1) + dA$$

$$= 100\pi + 2\pi$$

$$= 102\pi$$

True area of large circle is
 $A(r_2) = A(10.1)$
 $= 102.01\pi$

Exp (1) $f(x) = c \Rightarrow f'(x) = 0$

h...n

≡ ∪ T(x) - - -

$$f(x) = \sqrt{9} \Rightarrow f'(x) = 0$$

$$(2) f(x) = a x^n \Rightarrow f'(x) = a n x^{n-1}$$

$$f(x) = 3x^2 - 5x^3 + 1$$

$$f'(x) = 6x - 15x^2$$

$$(3) f(x) = (x^3 + 4x)^5 \quad \text{Find } f'(1)$$

$$f'(x) = 5(x^3 + 4x)^4 (3x^2 + 4)$$

$$f'(1) = 5(1 + 4)^4 (3 + 4)$$

$$= 5(5)^4 (7)$$

$$= (5)(7) \dots$$

Exp

show that

$$\text{if } f(x) = \cot x \\ \text{then } f'(x) = -\csc^2 x$$

$$f(x) = \cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$$

$$f'(x) = \frac{(\sin x)(-\sin x) - (\cos x)(\cos x)}{\sin^2 x}$$

$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}$$

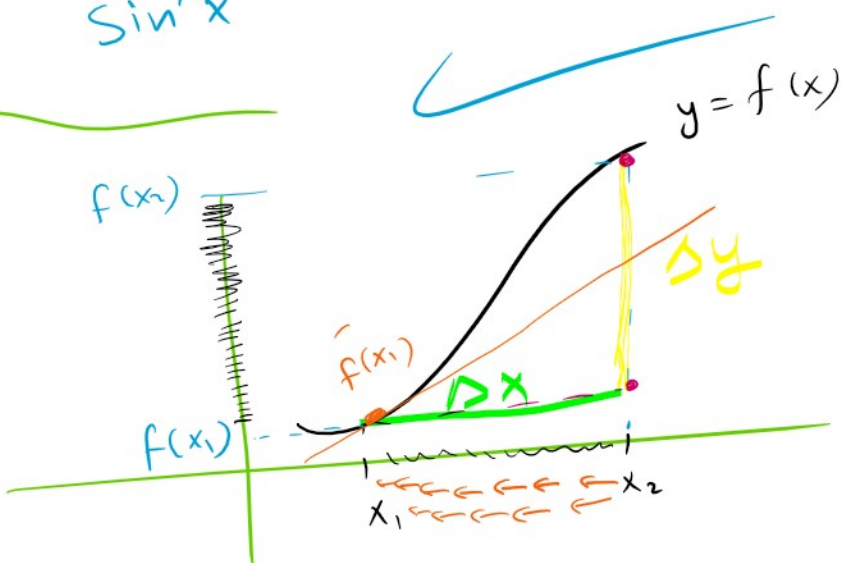
$$= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x}$$

$$= \frac{-1}{\sin^2 x} = -\csc^2 x$$

$$y = f(x)$$

$$\Delta x = x_2 - x_1$$

$$\begin{aligned} \Delta y &= y_2 - y_1 \\ &= f(x_2) - f(x_1) \end{aligned}$$



$$= f(x_2) - f(x_1)$$

$$\text{average change} = \frac{\Delta y}{\Delta x} =$$

$$\text{avg? } 1.7 \text{ up } 50 \text{ down } 50 \text{ } \leftarrow 1.72 = \frac{1.7 + 1.6 + 1.75 + \dots}{50}$$

$$f'(x_1) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$$

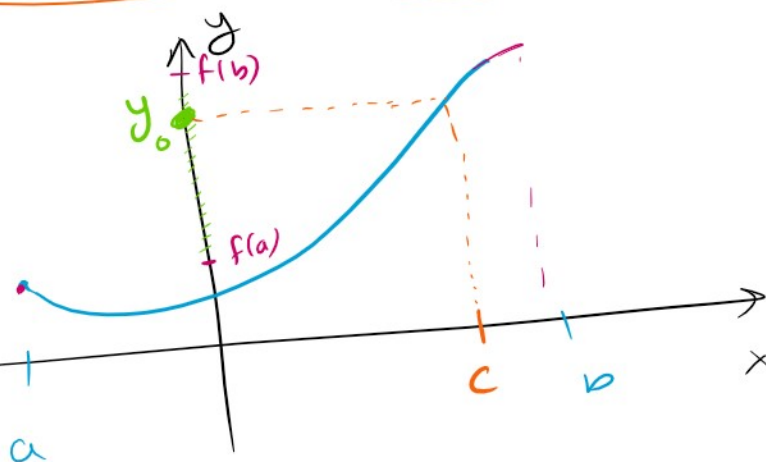
IVT

f cont $[a, b]$ ✓

$$y_0 \in [f(a), f(b)]$$

Then $\exists$ at least
number $c \in (a, b)$

$$\text{s.t. } f(c) = y_0$$

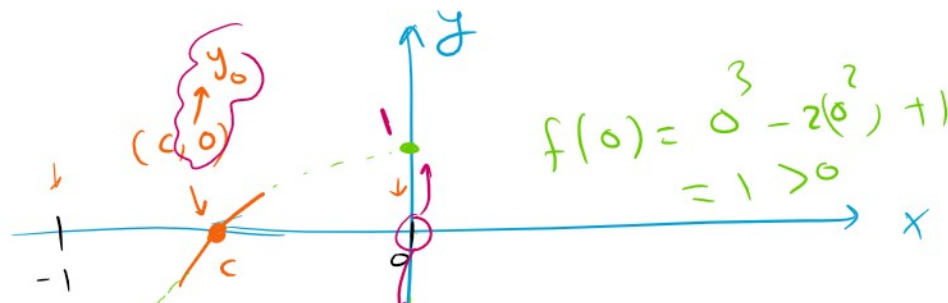


Ex show that $f(x) = x^3 - 2x^2 + 1$ has root on $[-1, 0]$

(SI) $f(x)$ poly $\Rightarrow f$ cont on $\mathbb{R} \Rightarrow f$ cont on $[-1, 0]$

$$f(c) = 0$$

$$f(-1) = (-1)^3 - 2(-1)^2 + 1$$



$$f(0) = 0^3 - 2(0)^2 + 1 = 1 > 0$$

$$f(-1) = (-1)^3 - 2(-1) + 1$$

$$= -1 - 2 + 1$$

$$= -3 + 1 = -2 < 0$$

$$f(-1) = -2$$

Since $y_0 = 0 \in [f(-1), f(0)] = [-2, 1]$

Then by INT $\Rightarrow \exists c \in (-1, 0)$ st $f(c) = y_0$
 $f(c) = 0$
 $\downarrow$
 c is root

Bolzano Th : Special case of IVT
 when $y_0 = 0$

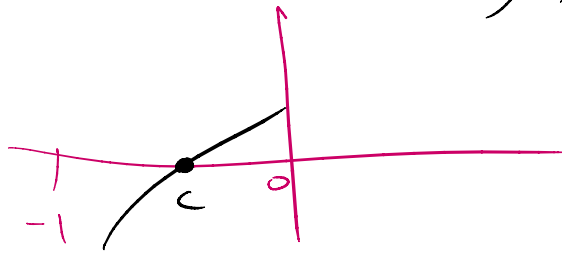
(S2) f cont. on $[-1, 0]$

$$f(-1) f(0) \stackrel{??}{<} 0$$

$$(-2)(1) = -2 < 0$$

$\Rightarrow f$ crosses x -axis
 on $[-1, 0]$

$$\Rightarrow \exists c \in (-1, 0)$$



$$\Rightarrow \exists c \in (-1, 0)$$

$$\text{s.t. } f(c) = 0$$

$\nearrow$
 y_0