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MATH1321

Calculus 2

Chapter 10.7

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10.7 Power Series

are infinite sum of poly's.

$$\sum_{n=0}^{\infty} a_n (x-a)^n = \underbrace{a_0}_{} + \underline{a_1}(x-a) + \underline{a_2}(x-a)^2 + \dots$$

a_i: coefficients a: center

$\longrightarrow$ $|x-a| < R$ IC: interval of conv.

$-R < x-a < R$

R: radius of convergence. $a-R < x < a+R$

Note: To find R and IC we apply RT.

Ex. $\sum_{n=0}^{\infty} x^n$ $(a=0)$ geometric since $r=x$

$= 1 + x + x^2 + x^3 + \dots$

If $|x|<1$ then $\sum_{n=0}^{\infty} x^n$ converges to $\frac{a}{1-r}$.

$\longrightarrow \sum_{n=0}^{\infty} x^n$ conv. to $\frac{1}{1-x}$.

IC $= (-1, 1)$ $R = 1$

$f(x) = \frac{1}{1-x}$ for $|x|<1 \longrightarrow$ we can approximate

$f(x)$ by $P_0(x) = 1$ $P_1(x) = 1+x$

$P_2(x) = 1+x+x^2$ $P_3(x) = 1+x+x^2+x^3$

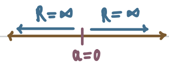
Ex. Find R and IC for?

1) $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$ Apply RT: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{\cancel{n}+1}}{(n+1)\cancel{n!}} \cdot \frac{\cancel{n!}}{\cancel{x^n}}\right| = \left|\frac{x}{n+1}\right|$$

$|x| \lim_{n\to\infty} \left|\frac{1}{n+1}\right| = 0 < 1$

so IC $= (-\infty, \infty)$ or $\mathbb{R}$



and $R = \infty$.

$\longrightarrow$ This means that $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$ conv. Abs. $\forall x$.

2) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ Apply RT: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$

$$\frac{a_{n+1}}{a_n} = \frac{x^{\cancel{n}+1}}{n+1} \cdot \frac{n}{\cancel{x^n}} = \frac{xn}{n+1}$$

$|x| \lim_{n\to\infty} \frac{n}{n+1} = |x|$.

If $|x|<1$ then $\sum a_n$ conv. abs.

Check endpoints for conv. condit.:

$x=-1 \longrightarrow \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(-1)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{2n-1}}{n}$

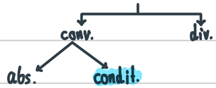
$= \sum_{n=1}^{\infty} (-1)^{2n}(-1)^{-1} \frac{1}{n} = -\sum_{n=1}^{\infty} \frac{1}{n}$. (HS) div.

$x=1 \longrightarrow \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$. (AHS) conv. but not conv. abs. since $\sum |a_n| = \underbrace{\sum \frac{1}{n}}_{\text{div.}}$

so conv. abs. $= (-1, 1)$ and $R = 1$.   at $x=1$ conv. condit.

IC $= (-1, 1]$.

Summury



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3) $\sum_{n=1}^{\infty} (\ln n)\, x^n$ Apply RT: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$

$$\frac{a_{n+1}}{a_n} = \frac{\ln(n+1)\, x^{n+1}}{\ln n \;\cancel{x^n}}$$

$|x| \lim \frac{\ln n+1}{\ln n}$ (L' Hopital) $= |x| < 1$

conv. abs. $= (-1, 1)$.

$R = 1$

Check for conv. condit.:

at $x = 1$ $\sum_{n=1}^{\infty} \ln n$ div. by n^{th} term test.

at $x = -1$ $\sum_{n=1}^{\infty} \ln n\, (-1)^n$ div. by n^{th} term test.

IC also $(-1, 1)$.

$\nexists\, x$ s.t $\sum_{n=1}^{\infty} \ln n\, x^n$ conv. condit.

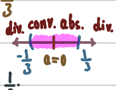
Ex. Find IC, R, conv. abs. and conv. condit ?

1) $\sum_{n=0}^{\infty} 3^n x^n$ Apply RT: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$

$$\frac{a_{n+1}}{a_n} = \frac{3^{n+1} x^{n+1}}{\cancel{3^n} \cancel{x^n}} = 3|x|.$$

$\lim_{n\to\infty} 3|x| = 3|x| < 1$ $\quad |x| < \frac{1}{3}$

so $-\frac{1}{3} < x < \frac{1}{3}$



$\therefore$ conv. abs. $= (-\frac{1}{3}, \frac{1}{3})$ and $R = \frac{1}{3}$.

Check endpoints for conv. condit.:

at $x = \frac{1}{3}$ $\sum_{n=0}^{\infty} 3^n (\frac{1}{3})^n = \sum_{n=0}^{\infty} 1$ div. by n^{th} term test.

at $x = \frac{-1}{3}$ $\sum_{n=0}^{\infty} 3^n (\frac{-1}{3})^n = \sum_{n=0}^{\infty} (-1)^n$ div. by n^{th} term test.

since both endpoints div. $IC = (-\frac{1}{3}, \frac{1}{3})$.

Note that $\sum_{n=0}^{\infty} 3^n x^n = \sum_{n=0}^{\infty} (3x)^n = 1 + 3x + (3x)^2 + \ldots$

is conv. geometric series since $|x| < \frac{1}{3}$ so $|r| < 1$.

$\longrightarrow \frac{1}{1-3x}$.

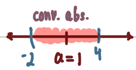
2) $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n^3\, 3^n}$ Apply RT: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$

$$\frac{a_{n+1}}{a_n} = \frac{(x-1)^{n+1}}{(n+1)^3\, 3^{n+1}} \cdot \frac{n^3\, \cancel{3^n}}{\cancel{(x-1)^n}} = \frac{n^3 |x-1|}{3(n+1)^3}$$

$\frac{|x-1|}{3} \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^3 = \frac{1}{3}|x-1| < 1$

$\longrightarrow |x-1| < 3$ so $-3 < x-1 < 3$ therefore $-2 < x < 4$

$\therefore$ conv. abs. $= (-2, 4)$ and $R = 3$.



Check endpoints for conv. condit.:

at $x = -2$ $\sum_{n=1}^{\infty} \frac{(-3)^n}{n^3 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$ conv. by AST.

at $x = 4$ $\sum_{n=1}^{\infty} \frac{(3)^n}{n^3 3^n} = \sum_{n=1}^{\infty} \frac{1}{n^3}$ conv. by P-series test.

$IC = [-2, 4]$.

At $x = -2$ and $x = 4$ does the series conv. Abs. ?

No → $x = -2$ and $x = 4$ conv. condit.

Yes → $x = -2$ and $x = 4$ conv. abs.

$\nexists$ no point (x) where the power series conv. condit.

Hence the power series conv. abs. $\forall x \in [-2, 4]$.

31) $\sum_{n=0}^{\infty} n!\, x^n$ Apply RT: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$

$\frac{a_{n+1}}{a_n} = \frac{(n+1)\cancel{n!}\, x^{n+1}}{\cancel{n!}\, \cancel{x^n}} = (n+1)x$

$|x| \lim_{n\to\infty} n+1 = |x| \cdot \infty > 1$

This infinite series diverges for every x

except $x=0$ since $\sum_{n=0}^{\infty} n!\, x^0 = \sum_{n=0}^{\infty} 0 = 0.$

$R=0.$

div. conv. div.

$a=0$

Theorem: Assume $\sum_{n=0}^{\infty} a_n x^n = A(x)$ and $\sum_{n=0}^{\infty} b_n x^n = B(x)$ converges Abs. on $|x|<R$

Then $(\sum a_n x^n)(\sum b_n x^n)$ converges Abs. to $A(x).B(x)$ on $|x|<R$.

Theorem: If $\sum a_n x^n$ conv. abs. on $|x|<R$

Then $\sum a_n (f(x))^n$ conv. abs. on $|f(x)|<R$

for any cont. function f.

Ex. $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots = \frac{1}{1-x}$ if $|x|<1$

This means $\sum_{n=0}^{\infty} x^n$ conv. abs. to $\frac{1}{1-x}$ on $|x|<1$

$\sum_{n=0}^{\infty} (4x^2)^n = 1 + 4x^2 + (4x^2)^2 + \dots = \frac{1}{1-4x^2}$ if $|4x^2|<1$

so $4x^2<1 \longrightarrow x^2 < \frac{1}{4} \longrightarrow |x| < \frac{1}{2}$

$f(x) = 4x^2$ cont.

Theorem: Term by term Differentiation:

If $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + \ldots$

Assume this power series conv. abs. on $|x-a| < R$

and If $f(x)$ has all derivatives on $|x-c| < R$ then

$f'(x) = \sum_{n=1}^{\infty} n a_n (x-c)^{n-1} = 0 + a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \ldots$

$f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x-c)^{n-2} = 0 + 2a_2 + 6a_3(x-c) + 12a_4(x-c)^2 + \ldots$

Theorem: Term by term Integration:

Assume $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ conv. abs. on $|x-c| < R$

$= a_0 + a_1(x-c) + a_2(x-c)^2 + \ldots$

Then $\int f(x)\,dx = \sum_{n=0}^{\infty} a_n \frac{(x-c)^{n+1}}{n+1} + C$ on $|x-c| < R$.

Ex. Identify this function:

$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \quad |x| \leq 1.$

a) $\sin^{-1}x$ b) $\cos^{-1}x$ c) $\tan^{-1}x$ d) $\sec^{-1}x$?

compare with $\sum_{n=0}^{\infty} a_n (x-a)^n \longrightarrow a = 0$

$f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$ on $|x| \leq 1$ so $f(0) = 0$

$f'(x) = 1 - x^2 + x^4 - x^6 + \ldots$ on $|x| < 1$

conv. geometric series since $r = |x^2| = x^2 < 1$ so $|x| < 1$

$f'(x) = \frac{1}{1-x^2} = \frac{1}{1+x^2}$

$\int f'(x)\,dx = \int \frac{1}{1+x^2}\,dx \longrightarrow f(x) = \tan^{-1}x + C$ (we need to find C)

but $f(0) = 0$ so $C = 0$.

$\therefore f(x) = \tan^{-1}x$.